ABELIAN HYPERCOMPLEX 8-DIMENSIONAL NILMANIFOLDS

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Abstract. We study invariant abelian hypercomplex structures on 8-dimensional nilpotent Lie groups. We prove that a group N admitting such a structure is either abelian or an abelian extension of a group of type H. We determine the Poincaré polynomials of the associated nilmanifolds and study the existence of symplectic and quaternionic structures on such spaces.

1. Introduction

Throughout this paper we will concentrate ourselves on the case of a nilmanifold $M = \Gamma\backslash N$, N a nilpotent Lie group of dimension 8, Γ a discrete subgroup, such that N is endowed with an abelian invariant hypercomplex structure, i.e an invariant hypercomplex structure $\{J_i\}_{i=1,2}$ such that for each $\{J_i\}_{i=1,2}$ any two $(1,0)$-vector fields commute.

Our purpose is two-fold. On the one hand we prove that a nilpotent 8-dimensional nilpotent Lie group admitting an invariant abelian hypercomplex structure is either euclidean space or a trivial extension of a group of type H. Moreover, any invariant metric compatible with the whole sphere of complex structure is a central modification of the type H metric (see Theorem 4.1).

On the other hand, we study some topological and geometrical properties of the associated compact nilmanifolds (see Sections 5 and 6). More precisely, using Nomizu's theorem we compute the real cohomology and we study the existence of symplectic and quaternionic structures on such spaces.

2. Abelian complex structures on nilpotent groups

Let N be a nilpotent Lie group. An *invariant almost complex structure* J on N is an endomorphism of $\mathfrak{n}$, the Lie algebra of N, satisfying $J^2 = -I$. The endomorphism J extends to the complexification $\mathfrak{n}^{\mathbb{C}} = \mathfrak{n} \oplus i\mathfrak{n}$ giving a splitting

$$\mathfrak{n}^C = \mathfrak{n}^{1,0} \oplus \mathfrak{n}^{0,1}$$

where

$$\mathfrak{n}^{1,0} = \{X - iJX : X \in \mathfrak{n}\} \text{ and } \mathfrak{n}^{0,1} = \{X + iJX : X \in \mathfrak{n}\}.$$

Similarly if $\mathfrak{n}^*$ denotes the dual Lie algebra of $\mathfrak{n}$, the induced J in $\mathfrak{n}^*$ produces a splitting

$$\mathfrak{n}^{*C} = \Lambda^{1,0}\mathfrak{n}^* \oplus \Lambda^{0,1}\mathfrak{n}^*$$

where

$$\Lambda^{1,0}\mathfrak{n}^* = \{X^* - iJX^* : X^* \in \mathfrak{n}^*\} \text{ and } \Lambda^{0,1}\mathfrak{n}^* = \{X^* + iJX^* : X^* \in \mathfrak{n}^*\}.$$

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If τ denotes complex conjugation in $\mathfrak{n}$ or $\mathfrak{n}^*$, then it follows that

$$\mathfrak{n}^{0,1} = \tau(\mathfrak{n}^{1,0}) \text{ and } \Lambda^{0,1}\mathfrak{n}^* = \tau(\Lambda^{1,0}\mathfrak{n}^*).$$

Furthermore, the invariant almost complex structure is integrable, that is, it gives complex coordinates on N, if $\mathfrak{n}^{1,0}$ is a complex subalgebra of $\mathfrak{n}^C$ or, equivalently, if the Nijehuis tensor

$$N(X,Y) = J([X,Y] - [JX,JY]) - ([JX,Y] + [X,JY])$$

vanishes identically, for any X, Y vector fields on N. In terms of invariant differential forms, the integrability condition for J is equivalent to

$$d\Lambda^{1,0}\mathfrak{n}^* \subset \Lambda^{2,0}\mathfrak{n}^* \oplus \Lambda^{1,1}\mathfrak{n}^*.$$

A special class of invariant complex structures on N appears when we require that the subalgebra $\mathfrak{n}^{1,0}$ be abelian. We will refer to them as *abelian* complex structures. Such a J is characterized by the condition $[X,Y] = [JX,JY]$ for $X, Y \in \mathfrak{n}$. This class has several convenient properties. For instance, J satisfies the additional condition

$$d\Lambda^{1,0}\mathfrak{n}^* \subset \Lambda^{1,1}\mathfrak{n}^*,$$

since $d\omega(X,Y) = -\omega([X,Y]) = 0$, if ω is a left invariant form$(1,0)$ and X, Y are left invariant $(1,0)$ vector fields. Furthermore, J preserves the center $\mathfrak{z}$ of $\mathfrak{n}$, hence it induces a complex structure on $\mathfrak{n}/\mathfrak{z}$ which is abelian. In particular J preserves the ascending central series of $\mathfrak{n}$.

An *invariant hypercomplex structure* on N is a pair of anticommuting invariant complex structures J_1, J_2 on M. The hypercomplex structure will be called *abelian*, if J_1, J_2 are so. In this case, there is a sphere of abelian complex structures J_q, $q \in S^2$ on M and, for each q, a decomposition

$$\mathfrak{n}^{\mathbb{C}} = \mathfrak{n}_q^{1,0} \oplus \mathfrak{n}_q^{0,1}$$

where each summand is an abelian complex subalgebra.

We note that the condition $[X,Y] = [JX,JY]$ (X, Y left invariant vector fields) for an invariant complex structure J on a Lie group G has been already considered in [2] (see Proposition 4.1) and in [3, Definition 2.1.2'].

3. Groups of type H

In this section we shall collect some basic facts on groups of type H which will be needed to formulate our main result.

These groups are natural generalizations of the Iwasawa N-groups associated to semisimple Lie groups of real rank one. They were introduced in [15] and studied by many authors in connection with a number of questions in geometry and analysis. We start by recalling their definition.

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra endowed with an inner product $\langle\, ,\, \rangle$. Then $\mathfrak{n}$ has an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$. Define a linear mapping $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ by

$$\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle \tag{1}$$

(note that J_Z is skew-symmetric). Now $\mathfrak{n}$ is said to be an algebra of type H if for any $Z \in \mathfrak{z}$

$$J_Z^2 = -\langle Z, Z\rangle I \tag{2}$$

The corresponding group of type H is the simply connected Lie group N with Lie algebra $\mathfrak{n}$, endowed with the left invariant metric induced by the inner product $\langle\, ,\, \rangle$ on $\mathfrak{n}$.

Groups (algebras) of type H have been classified ([15]). It follows from this classification that for each $i, 1 \leq i \leq 3$, there is only one algebra of type H with i-dimensional center and dimension ≤ 8. They can be described as follows.

Let $\mathfrak{h}_i, i = 1, 2, 3$ denote the following two step nilpotent Lie algebras. The underlying vector space is $\mathbb{R}^i \times \mathbb{C}^2, i = 1, 2, 3$ and the brackets $[\, ,\,]_i$ on $\mathfrak{h}_i$ are given by $\psi_i : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}^i$

$$\psi_1((x_1, x_2)(x_1', x_2')) = \mathrm{Im}(\overline{x_1}x_1' + \overline{x_2}x_2')$$

$$\psi_2((x_1, x_2)(x_1', x_2')) = x_1x_2' - x_1'x_2$$

$$\psi_3((x_1, x_2)(x_1', x_2')) = (\mathrm{Im}(\overline{x_1}x_1' + x_2x_2'), \overline{x_1}x_2' - x_2x_1').$$

Then we set, for any $z, z' \in \mathbb{R}^i$ and $(x_1, x_2), (x_1', x_2') \in \mathbb{C}^2$

$$[z + (x_1, x_2), z' + (x_1', x_2')]_i = \psi_i((x_1, x_2)(x_1', x_2')), \quad i = 1, 2, 3. \tag{3}$$

We denote $\mathfrak{h}_i = (\mathfrak{h}, [\, ,\,]_i)$, and we let H_i be the simply connected Lie group with Lie algebra $\mathfrak{h}_i$.

Let $\langle\, ,\, \rangle_i$ on $\mathfrak{h}_i$, $i = 1, 2$ be defined as follows

$$\langle (x_1, x_2), (x_1', x_2') \rangle_i = \mathrm{Re}\,(\overline{x_1}x_1' + x_2x_2'), \quad \langle\, z_1, z_2\, \rangle_i = \mathrm{Re}\,\overline{z_1}z_2$$

for any (x_1, x_2), $(x_1', x_2') \in \mathbb{C}^2$ and $z_1, z_2 \in \mathbb{C}$. (Here we identify $\mathbb{R}^1$ with $\mathrm{Im}\,\mathbb{C} \subset \mathbb{C}$ in the first case and $\mathbb{R}^2$ with $\mathbb{C}$ in the second case). When $i = 3$, we observe that ψ_3 can be given by $\psi_3(q_1, q_2) = \mathrm{Im}(\overline{q_1}q_2)$, $q_1, q_2 \in \mathbb{H}$, where $\mathbb{H}$ denotes the quaternions, canonically identified with $\mathbb{C}^2$. In this case we set

$$\langle\, q_1, q_2\, \rangle_3 = \mathrm{Re}\,\overline{q_1}q_2$$

for any $q_1, q_2 \in$ either $\mathbb{C}^2 = \mathbb{H}$ or $\mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$.

Endowed with the inner products $\langle\, ,\, \rangle_i$, the 2-step nilpotent algebras $\mathfrak{h}_i, i = 1, 2, 3$ are of type H. Indeed, we have that J_z is given by right multiplication by $z \in \mathrm{Im}\,\mathbb{C}$, if $i = 1$, $J_z(x_1, x_2) = (-z\overline{x_2}, \overline{x_1}z)$, if $i = 2$ and J_q is right multiplication by $q \in \mathrm{Im}\,\mathbb{H}$, if $i = 3$.

Remark. Groups of type H with centers of dimension 1, 2 and 3 can be viewed as group of matrices, generalizing the case of the 3-dimensional real Heisenberg group.

We recall two realizations of the $2n+1$ dimensional real Heisenberg group: firstly, as a subgroup of $GL(n + 2, \mathbb{R})$

$$H(n, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & I & b^t \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}^n, c \in \mathbb{R} \right\}$$

or alternately, as a subgroup of $GL(n + 2, \mathbb{C})$

$$H_1(n) = \left\{ \begin{pmatrix} 1 & z & \mathrm{Im}\,w \\ 0 & I & -\overline{z}^t \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{C}^n, w \in \mathbb{C} \right\}.$$

Generalizations of the $2n+1$-dimensional real Heisenberg group to the complex numbers $\mathbb{C}$, in the first case, and to the quaternions $\mathbb{H}$, in the second case, give the complex Heisenberg group of matrices

$$H(n,\mathbb{C}) = \left\{ \begin{pmatrix} 1 & \zeta & z \\ 0 & I & \omega^t \\ 0 & 0 & 1 \end{pmatrix} : \zeta, \omega \in \mathbb{C}^n, z \in \mathbb{C} \right\}$$

and

$$H_3(n) = \left\{ \begin{pmatrix} 1 & q & \operatorname{Im} h \\ 0 & I & -\overline{q}^t \\ 0 & 0 & 1 \end{pmatrix} : q \in \mathbb{H}^n, h \in \mathbb{H} \right\}.$$

The three families so obtained are two step nilpotent groups with $1, 2$ and 3 dimensional centers. They are groups of type H. The cases of $H_1(n)$ and $H_3(n)$ correspond to the nilpotent part in the Iwasawa decomposition of the isometry group of the complex hyperbolic space and quaternionic hyperbolic space respectively.

The groups of type H denoted above by H_1, H_2 and H_3 are isomorphic to $H(2,\mathbb{R}) = H_1(2), H(1,\mathbb{C})$ and $H_3(1)$ respectively.

4. 8-dimensional abelian hypercomplex nilpotent groups

We now get into the main theme of this paper. From now on we shall assume that N is an 8-dimensional nilpotent Lie group, endowed with an abelian hypercomplex structure. The next lemma gives a first restriction on N.

Lemma 4.1. *Let N be an 8-dimensional nilpotent Lie group with an abelian hypercomplex structure. Then N is either abelian, or 2-step nilpotent with a 4-dimensional center.*

Proof. Let J_1, J_2 be two anticommuting almost complex structures on the Lie algebra $\mathfrak{n}$ of N satisfying $[J_iX, J_iY] = [X,Y],\ i = 1,2$. Then $J_i, i = 1,2$ preserves the center $\mathfrak{z}$ of $\mathfrak{n}$, hence $\mathfrak{z}$ is either 4 or 8-dimensional. Assume $\mathfrak{z}$ is 4-dimensional. Then $\mathfrak{n}/\mathfrak{z}$ is a 4-dimensional nilpotent Lie algebra admitting a pair of anticommuting complex structures which preserve the center. Thus $\mathfrak{n}/\mathfrak{z}$ is abelian or equivalently $\mathfrak{n}$ is 2-step nilpotent. If $\mathfrak{z}$ is 8-dimensional, then $\mathfrak{n}$ is abelian and the lemma follows.

Remark The same proof as in Lemma 4.1 shows in general that a nilpotent $4k$-dimensional Lie group with an abelian hypercomplex structure is at most k-step nilpotent.

We may now state the classification result.

Theorem 4.1. *Let N be an 8-dimensional nilpotent Lie group. Then N carries an abelian hypercomplex structure if and only if N is either abelian or isomorphic to N_i, a trivial extension of a group of type H with center of* dimension *i =1, 2* or *3.*

Proof. Let J_1, J_2 be a pair of abelian, anticommuting complex structures on N. According to the previous lemma N is either abelian or 2-step nilpotent. If N is 2-step nilpotent, let $\mathfrak{n}$ be the Lie algebra of N endowed with an inner product $\langle\, ,\rangle$ such that J_1 and J_2 are orthogonal endomorphisms. Then $\mathfrak{n}$ has a (non trivial) orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$, dim $\mathfrak{z} = 4$, and $[\mathfrak{v},\mathfrak{v}] \subset \mathfrak{z}$. Define a linear map $K : \mathfrak{z} \to \operatorname{End}(\mathfrak{v})$ by

$$\langle K_Z X, Y\rangle = \langle Z, [X,Y]\rangle \tag{4}$$

It is clear that K_Z is skew-symmetric and $K_Z = 0$ if and only if $Z \perp [\mathfrak{v}, \mathfrak{v}]$. Moreover, since J_1, J_2 are abelian complex structures it follows that

$$\langle K_Z J_i X, J_i Y\rangle = \langle Z, [J_i X, J_i Y]\rangle = \langle Z, [X, Y]\rangle = \langle K_Z X, Y\rangle. \tag{5}$$

hence K_Z commutes with $J_i, i = 1, 2$ (that is, the K_Z are quaternionic linear).

In particular, $K_Z = 0$ or K_Z is an isomorphism. It then follows that the Lie algebra $\mathfrak{h} = [\mathfrak{v}, \mathfrak{v}] \oplus \mathfrak{v}$ is a nondegenerate ([10]) two step nilpotent Lie algebra, that is, K_Z is an isomorphism for every $Z \in [\mathfrak{h}, \mathfrak{h}]$. Moreover $\{K_Z, Z \in [\mathfrak{h}, \mathfrak{h}]\}$ give as many linearly independent vector fields as the dimension of $[\mathfrak{h}, \mathfrak{h}]$. Hence the possibilities for this dimension are $1, 2$ or 3. We shall next characterize the 2-step nilpotent Lie algebra $\mathfrak{h}$. If $Z \in [\mathfrak{h}, \mathfrak{h}] = [\mathfrak{v}, \mathfrak{v}]$, $Z \neq 0$, the map K_Z is skew symmetric and non singular hence K_Z^2 is self adjoint and non singular. Since $K_Z^2 J_i = J_i K_Z^2$ it follows that $K_Z^2 = -\lambda(Z) I$, $\lambda(Z) > 0$ if $Z \neq 0$.

Now to complete the proof it is sufficient to verify that it is possible to change the inner product on $[\mathfrak{h}, \mathfrak{h}]$ in such a way that $\lambda(Z) = ||Z||^2$, hence $\mathfrak{h} = [\mathfrak{v}, \mathfrak{v}] \oplus \mathfrak{v}$ becomes an H-type algebra. This can either be seen by a direct computation or, for a proof of a more general result, see [17].

Thus, $\mathfrak{n} = \mathfrak{z}_s \oplus \mathfrak{h}$, is a trivial extension of an H-type algebra, where $\mathfrak{z}_s \simeq \mathbb{R}^s$ with $s = 1, 2, 3$.

To complete the proof we need to show that the nilpotent Lie algebras of dimension 8 in question, carry a pair of natural anticommuting abelian complex structures.

Let $\mathfrak{n}_i = \mathbb{R}^{4-i} \oplus \mathfrak{h}_i$ be the trivial extensions of the algebras of type H with i-dimensional center, $i = 1, 2, 3$, given in Section 3 and let N_i denote the corresponding simply connected nilpotent Lie groups. Consider $J_1, J_2 : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$J_1(x_1, x_2) = (ix_1, -ix_2) \quad J_2(x_1, x_2) = (-x_2, x_1).$$

Clearly $J_1 J_2 = -J_2 J_1$. Moreover it is easily checked that

$$\psi_j(J_i(x_1, x_2), J_i(x_1', x_2')) = \psi_j((x_1, x_2), (x_1', x_2')), \ i, j = 1, 2.$$

Thus, the extension of J_1, J_2 to a pair of anticommuting complex structures of $\mathbb{R}^4 \times \mathbb{C}^2$, preserving $\mathbb{R}^4$ defines an abelian hypercomplex structure on $\mathfrak{n}_1$ and $\mathfrak{n}_2$ respectively. In the case of $\mathfrak{n}_3$, as observed in Section 3, ψ_3 can be given by $\psi_3(q_1, q_2) = \mathrm{Im}\,(\overline{q_1} q_2)$, $q_1, q_2 \in \mathbb{H}$, where $\mathbb{H}$ denotes the quaternions, canonically identified with $\mathbb{C}^2$. Let L_p denote left multiplication in $\mathbb{H}$ by a unit quaternion p. Since

$$\psi_3(L_p(q_1), L_p(q_2)) = \mathrm{Im}\,(\overline{pq_1} pq_2) = \psi_3(q_1, q_2),$$

the extension of L_p to an almost complex structure of $\mathbb{R}^4 \times \mathbb{C}^2$ preserving $\mathbb{R}^4$, is an abelian complex structure on $\mathfrak{n}_3$. Taking any pair of anticommuting quaternions p_1, p_2 of length 1, we obtain the desired abelian hypercomplex structure on $\mathfrak{n}_3$.

Remark. The groups N_1, N_2 can only carry abelian hypercomplex structures since their commutator subgroups N_1', N_2' are 1-dimensional and 2-dimensional respectively ([2]). On the other hand N_3 does admit non abelian (integrable) hypercomplex structures. For example a non abelian hypercomplex structure on N_3 is given by right multiplication by i and by j in $\mathbb{R}^4$.

5. Topological properties of associated nilmanifolds

It follows from Theorem 4.1 that any 8-dimensional manifold covered by either $\mathbb{R}^8$ or by N_i, a trivial extension of the groups of type H, with i-dimensional center, $i = 1, 2, 3$, carries an abelian hypercomplex structure. Now, any group of type H admits a discrete co-compact subgroup Γ (see [16]). If N_i, $i = 1, 2, 3$, is the simply connected nilpotent Lie group defined in the previous section, we shall fix a discrete co-compact subgroup, Γ_i, of N_i and we let $M_i = \Gamma_i \backslash N_i$. In this section we shall study cohomological properties of the nilmanifolds $M_i, i = 1, 2, 3$.

To compute the real cohomology groups of the nilmanifolds covered by the nilpotent groups N_i, it will be useful to describe the Lie algebras n_i in terms of 1-forms. We identify $\mathbb{R}^4 \times \mathbb{C}^2$ with $\mathfrak{z} \oplus \mathfrak{v}$ and denote the standard basis $(1, 0), (i, 0), (0, 1), (0, i)$ of $\mathbb{C}^2$ by $e_i, i = 1, 2, 3, 4$, respectively. Moreover $\mathbb{R}^4$ is identified with $\mathfrak{z}$, spanned by $e_i, i = 5, 6, 7, 8$. Using the above identification it is not hard to show that
(i) $\mathfrak{n}_1 = \mathfrak{z}_1 \oplus \mathfrak{v}_1$, where $\mathfrak{z}_1 = span\{e_5, e_6, e_7, e_8\}$ is the center of $\mathfrak{n}_1$ and $\mathfrak{v}_1 = \text{span}\{e_1, e_2, e_3, e_4\}$, with non zero brackets

$$[e_1, e_2] = [e_3, e_4] = e_8.$$

(ii) $\mathfrak{n}_2 = \mathfrak{z}_2 \oplus \mathfrak{v}_2$, where $\mathfrak{z}_2 = \text{span}\{e_5, e_6, e_7, e_8\}$ is the center of $\mathfrak{n}_2$ and $\mathfrak{v}_2 = span\{e_1, e_2, e_3, e_4\}$, with non zero brackets

$$[e_1, e_3] = -[e_2, e_4] = e_7; \quad [e_1, e_4] = [e_2, e_3] = e_8.$$

(iii) $\mathfrak{n}_3 = \mathfrak{z}_3 \oplus \mathfrak{v}_3$, where $\mathfrak{z}_3 = \text{span}\{e_5, e_6, e_7, e_8\}$ is the center of $\mathfrak{n}_3$ and $\mathfrak{v}_3 = span\{e_1, e_2, e_3, e_4\}$, with non zero brackets

$$[e_1, e_2] = -[e_3, e_4] = e_6; \quad [e_1, e_3] = [e_2, e_4] = e_7; \quad [e_1, e_4] = -[e_2, e_3] = e_8.$$

We denote by $\{e^k, k = 1, \dots, 8\}$ the basis of $\mathfrak{n}_i^*$, dual to the basis $\{e_k, k = 1, \dots, 8\}$. Since these forms e^k are left-invariant by Γ_i, $i = 1, 2, 3$, they push down to differential forms on the quotient manifolds M_i. We shall denote the corresponding 1-forms on M_i by the same symbols e^k.

Nomizu's Theorem [18] asserts that the natural mapping from

$$H^k(\mathfrak{n}_i) = Ker(d| \wedge^k \mathfrak{n}_i^*)/d(\wedge^{k-1}\mathfrak{n}_i^*)$$

to the de Rham cohomology group $H^k(M_i, \mathbb{R})$ is an isomorphism. In terms of the basis of n_i^*, we have the following descriptions:
(i) for $\mathfrak{n}_1$

$$\begin{aligned} de^i &= 0, \quad i = 1, \dots 7, \\ de^8 &= -e^1 \wedge e^2 - e^3 \wedge e^4; \end{aligned}$$

(ii) for $\mathfrak{n}_2$

$$\begin{aligned} de^i &= 0, \quad i = 1, \dots 6, \\ de^7 &= -e^1 \wedge e^3 + e^2 \wedge e^4, \\ de^8 &= -e^1 \wedge e^4 - e^2 \wedge e^3; \end{aligned}$$

(iii) for $\mathfrak{n}_3$

$$\begin{aligned} de^i &= 0, \quad i = 1, \dots 5, \\ de^6 &= -e^1 \wedge e^2 + e^3 \wedge e^4, \\ de^7 &= -e^1 \wedge e^3 - e^2 \wedge e^4, \\ de^8 &= -e^1 \wedge e^4 + e^2 \wedge e^3. \end{aligned}$$

The Betti numbers $b_k(M_i) = \dim H^k(M_i, \mathbb{R})$ are then given by the tables

k	0	1	2	3
nullity$(d\| \wedge^k \mathfrak{g}^*)$	1	7	21	40
rank$(d\| \wedge^{k-1} \mathfrak{g}^*)$	0	0	1	7
$b_k(M_1)$	1	7	20	33

k	0	1	2	3
nullity$(d\| \wedge^k \mathfrak{g}^*)$	1	6	19	39
rank$(d\| \wedge^{k-1} \mathfrak{g}^*)$	0	0	2	9
$b_k(M_2)$	1	6	17	30

k	0	1	2	3
nullity$(d\| \wedge^k \mathfrak{g}^*)$	1	5	18	35
rank$(d\| \wedge^{k-1} \mathfrak{g}^*)$	0	0	3	10
$b_k(M_3)$	1	5	15	25

and by the vanishing of the Euler characteristic $\chi = 2 - 2b_1 + 2b_2 - 2b_3 + b_4$.

Thus we obtain the following expressions for the Poincaré polymonials:

$$\begin{aligned} P(M_1, t) &= 1 + 7t + 20t^2 + 33t^3 + 38t^4 + 33t^5 + 20t^6 + 7t^7 + t^8 \\ &= (t+1)^4(t^2+1)(t^2+3t+1). \end{aligned}$$

$$\begin{aligned} P(M_2, t) &= 1 + 6t + 17t^2 + 30t^3 + 36t^4 + 30t^5 + 17t^6 + 6t^7 + t^8 \\ &= (t+1)^4(t^2+t+1)^2. \end{aligned}$$

$$\begin{aligned} P(M_3, t) &= 1 + 5t + 15t^2 + 25t^3 + 28t^4 + 25t^5 + 15t^6 + 5t^7 + t^8 \\ &= (t+1)^2(t^2+1)(t^4+3t^3+7t^2+3t+1). \end{aligned}$$

It is known (see [5]) that a compact nilmanifold admits a Kähler metric if and only if it is a torus. An alternative approach to the proof consists of showing that the minimal model $\mathcal{M}_M$ of the nilmanifold M is not formal in the sense of [13],(see also [24]) i.e. there exists no quasi-isomorphism between $(\mathcal{M}_M, d)$ and $(H^*(M), 0)$. This can often be done simply by constructing a nonzero Massey triple product [7]. Indeed, it was proved in [9] that the minimal model of a compact Kähler manifold is formal and this implies in particular that all Massey triple products on the manifold must be zero. The minimal model of nilmanifolds associated with a given rational nilpotent Lie algebras was determined in [14].

We recall the definition of Massey triple products in terms of differential forms on M. Suppose that there are cohomology classes $[\alpha_1] \in H^p(M)$, $[\alpha_2] \in H^q(M)$ and $[\alpha_3] \in H^r(M)$ (represented by differential forms α_i, $i = 1, 2, 3$) such that $[\alpha_1] \cdot [\alpha_2] = [\alpha_2] \cdot [\alpha_3] = 0$. Then

$$\alpha_1 \wedge \alpha_2 = d\beta_1, \qquad \alpha_2 \wedge \alpha_3 = d\beta_2,$$

for some choice of differential forms β_1 and β_2. Let $\gamma = \alpha_1 \wedge \beta_2 + (-1)^{p-1}\beta_1 \wedge \alpha_3$, then γ is a closed form of degree $p+q+r-1$ and its cohomology class $[\gamma]$, which depends on the choice of β_i, $i = 1, 2$, is well-defined modulo $[\alpha_1] \cdot H^{q+r-1}(M) \oplus [\alpha_3] \cdot H^{p+q-1}(M)$. Indeed, the class $[[\gamma]]$ in the quotient $\frac{H^{p+q+r-1}(M)}{[\alpha_1] \cdot H^{q+r-1}(M) \oplus [\alpha_3] \cdot H^{p+q-1}(M)}$ is independent of the choice of $\beta_i, i = 1, 2$ and it is called the triple Massey product of $[\alpha_1], [\alpha_2]$ and $[\alpha_3]$. We will denote it by $< [\alpha_1], [\alpha_2], [\alpha_3] >$.

Proposition 5.1. *The compact nilmanifolds M_i, $i = 1, 2, 3$, have a nonzero Massey triple product.*

Proof. The nonzero Massey triple products are given in each case by

$$
\begin{array}{lll}
(i) & <[e^2\wedge e^4],[e^3],[e^3]>=[[e^2\wedge e^3\wedge e^8]]\in \frac{H^3(M_1)}{[e^2\wedge e^4]\cdot H^1(M_1)\oplus[e^3]\cdot H^2(M_1)} & \text{for } M_1;\\
(ii) & <[e^3\wedge e^2],[e^4],[e^4]>=[[e^3\wedge e^4\wedge e^7]]\in \frac{H^3(M_2)}{[e^3\wedge e^2]\cdot H^1(M_2)\oplus[e^4]\cdot H^2(M_2)} & \text{for } M_2;\\
(iii) & <[e^2\wedge e^3],[e^4],[e^4]>=[[e^2\wedge e^4\wedge e^6]]\in \frac{H^3(M_3)}{[e^2\wedge e^3]\cdot H^1(M_3)\oplus[e^4]\cdot H^2(M_3)} & \text{for } M_3.
\end{array}
$$

To verify the assertion we observe
(i) $e^2 \wedge e^4 \wedge e^3 = -d(e^2 \wedge e^8)$ so that $[e^2 \wedge e^4] \cdot [e^3] = 0$ and the Massey product is represented by $e^2 \wedge e^3 \wedge e^8$ whose cohomology class $[e^2 \wedge e^3 \wedge e^8]$ is nonzero since N_1 is 2-step and it is also nonzero in the quotient $\frac{H^3(M_1)}{[e^2\wedge e^4]\cdot H^1(M_1)\oplus[e^3]\cdot H^2(M_1)}$.
(ii) $e^3 \wedge e^2 \wedge e^4 = -d(e^3 \wedge e^7)$, so that $[e^3 \wedge e^2] \cdot [e^4] = 0$ and one can proceed as before.
(iii) $e^2 \wedge e^3 \wedge e^4 = -d(e^2 \wedge e^6)$ and so that $[e^2 \wedge e^3] \cdot [e^4] = 0$.

To conclude this section, we study the existence of symplectic structures on $M_i, i = 1, 2, 3$.

(i) M_1 is not symplectic. Indeed, if $M_1 = N_1/\Gamma_1$ admitted a symplectic structure, by Nomizu's theorem, the symplectic form would be cohomologous to a left invariant form $\omega_1 \in \Lambda^2\mathfrak{n}_1^*$ satisfying $d\omega_1 = 0$ and ${\omega_1}^4 \neq 0$. Thus $\omega_1 = e^8 \wedge f^1 + \eta_1$ with $f^1 \in \ker d$ and $\eta_1 \in \Lambda^2(\ker d)$. From $0 = de^8 \wedge f^1 = -(e^1 \wedge e^2 + e^3 \wedge e^4) \wedge f^1$ one obtains that $f^1 = 0$, hence ${\omega_1}^4 = 0$ (compare with Theorem 2 in [8], also see Example 3.9 in [24].)

(ii) M_2 is a symplectic manifold (compare with [1], where a sphere of symplectic structures is constructed on the Iwasawa manifold). We exhibit below a left invariant symplectic form on N_2.

Let $\omega_2 = e^8 \wedge f^1 + e^7 \wedge Jf^1 + e^6 \wedge Kf^1 + e^5 \wedge JKf^1$ where $f^1 \in span\{e^i, i = 1, 2, 3, 4\}$, and J, K are the maps given by right multiplication by j, k in $\mathbb{R}^4$. This form is clearly non degenerate. To show it is closed one substitutes in $d\omega_2$, the expressions for de^i, $i = 5, 6, 7, 8$ obtained above.

(iii) M_3 is a symplectic manifold. Let $\omega_3 = e^8 \wedge f^1 + e^7 \wedge Jf^1 + e^6 \wedge (J + KJ)f^1 + e^5 \wedge Kf^1$ where $f^1 \in span\{e^i, i = 1, 2, 3, 4\}$, and J, K are the maps given by right multiplication by j, $-k$ in R^4. This form is non degenerate. To show it is closed one computes $d\omega_3$, using the expressions for de^i, $i = 5, 6, 7, 8$ obtained previously and observing that $span\{e^i,\ i = 1, 2, 3, 4\} \in \ker d$.

6. More general structures

It is known (see [19]) that every hypercomplex structure $\{J_i\}_{i=1,2}$ on a $4k$-dimensional differentiable manifold M uniquely determines an affine, torsion free connection (called the Obata connection), with respect to which the complex structures $J_i, i = 1, 2$ are parallel. In other words, the $GL(k, \mathbb{H})$-structure they determine admits a torsion free connection. Moreover, if the connection is flat, then M is quaternionic in the sense of Sommese ([22]), that is M can be covered by coordinate neighborhoods such that the transition functions are quaternionic. In particular, the $GL(k, \mathbb{H})$-structure is integrable, hence a flat affine structure exists on M.

As a consequence of the next proposition it will follow that the nilmanifolds $M_i,\ i = 1, 2, 3$ are quaternionic in the sense of Sommese.

Proposition 6.1. *Every abelian hypercomplex structure on a 2-step nilpotent Lie group is flat.*

Proof. The Obata connection associated to an invariant hypercomplex structure $J_1, J_2, J_3 = J_1J_2$ on a nilpotent Lie group N is given by

$$\nabla_X Y = \frac{1}{2}[X,Y] + \frac{1}{12}\sigma([J_1X, J_3Y] + [J_2X, J_3Y]) + \frac{1}{6}\sum_{i=1}^{i=3}[J_iX,Y] + [J_iY,X]$$

where X, Y lie in $\mathfrak{n}$ and σ denotes the cyclic sum with respect to the indices $1, 2, 3$.

If the hypercomplex structure preserves the center $\mathfrak{z}$ then $\nabla_X = 0$ for $X \in \mathfrak{z}$. Moreover, if the group is 2-step nilpotent then $\nabla_X Y \in \mathfrak{z}$ for every $X, Y \in \mathfrak{n}$. Hence $R = 0$, R the curvature tensor associated to ∇.

For any abelian hypercomplex structure, (M_i, J_1, J_2), $i = 1, 2, 3$, is a quaternionic manifold in the sense of [4, Section 14.62]. Indeed by [19] the $GL(2, \mathbb{H})$-structure determined by the hypercomplex structure admits a torsion-free connection, hence $M_i,\ i = 1, 2, 3$ admits a $GL(2, \mathbb{H})\mathbb{H}^*$-structure with a torsion-free connection and so it is quaternionic.

The selection of a special metric among those which are hermitian with respect to a given complex structure on a manifold, is often a useful tool to study the manifold. We should remark that the proof of Theorem 3.1 implies that any invariant metric compatible with an abelian hypercomplex structure on an 8-dimensional nilpotent group is essentially the standard metric (i.e. a trivial extension of a modified metric of type H). Moreover, the abelian hypercomplex structures constructed on the groups $N_i, i = 1, 2, 3$ are all compatible with the metric of type H on the H_i factor of N_i. In the rest of the section we use on the nilmanifolds M_i, the metric $\langle\,,\,\rangle$ induced from N_i.

An almost complex structure J and $\langle\,,\,\rangle$, a riemannian metric on a differentiable manifold M^{2n} of dimension $2n$, compatible with J is called an almost hermitian structure and it represents a $U(n)$-structure. It determines a 'fundamental 2-form '$f_J \in \wedge^2\mathfrak{n}_i^*$ by means of the standard formula

$$f_J(X,Y) = \langle JX, Y\rangle.$$

In general, classes of almost Hermitian structures were defined systematically by Gray-Hervella [12] by decomposing the tensor ∇f_J (or equivalently ∇J) into four $U(n)$-irreducible classes lying in invariant spaces denoted by W_k, $k = 1, 2, 3, 4$. The component of ∇f_J in $W_1 \oplus W_2$ can be identified with the Nijehuis tensor and that in $W_3 \oplus W_4$ can be identified with Re $(df)^{1,2}$. The component of ∇f_J in W_4 is represented by the 1-form

$$\theta = \frac{1}{n-1}J\delta f_J = -\frac{1}{n-1}J * d * f_J = -\frac{1}{n-1}J * d(f_J^{n-1})$$

known as the Lie form (see for example [11]).

If J is an abelian complex structure on M_i, $i = 1, 2, 3$, compatible with the metric $\langle\,,\,\rangle_i$ and f_J is the fundamental associated 2-form, then $\nabla f_J \in W_3$, i.e. the 2-form f_J is co-closed ($\delta f_J = 0$). This follows from the fact that, in this case, $*d * f_J = 0$ since f_J is perpendicular to the image of d.

The existence of a pair J_1, J_2 on M_i, $i = 1, 2, 3$, of anti-commuting almost-complex structures compatible with the metric $<,>_i$ implies that M_i has an $Sp(2)$-structure, defined by the intersection of the two corresponding $U(4)$-structures. The holonomy groups of M_i are not contained in $Sp(2)$, i.e. the M_i are not hyperkähler. We can only assert that $(M_i, \langle\,,\,\rangle_i, J_1, J_2)$, $i = 1, 2, 3$, is a hyperhermitian manifold of dimension 8 such that $\delta f_{J_1} = \delta f_{J_2} = \delta f_{J_1 J_2} = 0$, where $f_{J_1}, f_{J_2}, f_{J_1 J_2}$ denote respectively the 2-forms associated to J_1, J_2, J_3.

It is possible to generalize the definition of hyperkähler manifold. Indeed in general the group $Sp(k)$ is not a maximal subgroup of $SO(4k)$, since it commutes with the action of the group $Sp(1)$ of unit quaternions. Denoting by $Sp(k)Sp(1)$ the proper subgroup $Sp(k) \times_{\mathbb{Z}_2} Sp(1)$ of $SO(4k)$, it is possible to study also the $Sp(2)Sp(1)$-structure on M_i, $i = 1, 2, 3$ (see [21, 23]). If the holonomy group of a Riemannian manifold M of dimension $4k \geq 8$ is contained in $Sp(k)Sp(1)$, M is called a *quaternionic Kähler manifold*. The $Sp(k)Sp(1)$-structure is characterized by the existence of a 4-form Ω, linearly equivalent at each point to

$$f_{J_1} \wedge f_{J_1} + f_{J_2} \wedge f_{J_2} + f_{J_1 J_2} \wedge f_{J_1 J_2}.$$

The covariant derivative $\nabla\Omega$ can be identified with the structure function of the $Sp(k)Sp(1)$-structure. In general $\nabla\Omega$ belongs to the space $[\lambda_0^1 \otimes \sigma^1] \otimes [\lambda_0^2 \otimes \sigma^2]$, where σ^r denotes the $(r+1)$-dimensional symmetric tensor product of the basic representation of $Sp(1) = SU(2)$ on $\mathbb{C}^k$ and λ_s^r is the $Sp(k)$-module with dominant weight [6, 23]

$$(\underbrace{2, \ldots, 2}_{s}, \underbrace{1, \ldots, 1}_{r-2s}, 0, \ldots, 0), \quad 0 \leq r - 2s \leq k.$$

For dimension 8 we have that

$$[\lambda_0^1 \otimes \sigma^1] \otimes [\lambda_0^2 \otimes \sigma^2] \cong [\lambda_1^3 \otimes \sigma^3] \oplus [\lambda_0^1 \otimes \sigma^3] \oplus [\lambda_1^3 \otimes \sigma^1] \oplus [\lambda_0^1 \otimes \sigma^1].$$

For the nilmanifolds M_i, $i = 1, 2, 3$ it is possible to prove by integrability of J_1 and J_2 that $\nabla\Omega$ and $d\Omega$ belong to the space $[\lambda_1^3 \otimes \sigma^1] \oplus [\lambda_0^1 \otimes \sigma^1]$, since the $GL(2, \mathbb{H})\mathbb{H}^*$-structure function belongs to the submodules involving σ^3. Moreover, one can show that $d\Omega \neq 0$, hence M_i, $i = 1, 2, 3$, is not quaternionic Kähler. This also follows from the fact that the induced metric is not Einstein.

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