

# Lifetime of a target in the presence of $N$ independent walkers

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## Abstract

We study the survival probability of an immobile target in presence of  $N$  independent diffusing walkers. We address the problem of the Mean Target Lifetime and its dependence on the number and initial distribution of the walkers when the trapping is perfect or imperfect. We consider the diffusion on lattices and in the continuous space and we address the bulk limit corresponding to a density of diffusing particles and only one isolated trap. Also, we use intermittent motion for optimization of search strategies.

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## 1. Introduction

Several models of trapping of random walkers by a target have been extensively discussed in the literature [1–4]. In most cases, in the analysis of the target lifetime, the initial distribution of walkers is taken at random (spatially homogeneous distribution) and the trapping is assumed “perfect” [5–7]. On the other hand, some results concerning to the *order statistics* of a set of random walkers are known when they are initially placed onto a given site of an Euclidean lattice [8, 9] or a fractal structure [2]. Thus, the question that arises naturally is how different initial configurations of the walkers, number of walkers, and trapping mechanisms affect on the survival probability of the target.

In this paper we address the problem of trapping by a fixed target at the origin in the presence of a set of  $N$  independent random walkers. The quantity of our interest is the lifetime of the trap, which ends when *any* walker reaches the target under the appropriate circumstances. Our approach not only provides a unified framework that comprises several situations of trapping scattered in the literature, but also enables us to

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compute (analytical and numerically) the lifetime of the trap in an exact way for a vast number of practical situations. Thus, we consider that the wandering of the walkers in the space may be normal or anomalous diffusion, and that the trapping may have particular characteristics: It may be perfect, in which case the lifetime of the trap reckons the time spend by the first walker to reach the origin, or imperfect [6, 10]. The last situation includes the cases in which a walker passing by the origin is not trapped with certainty. Also, the present formalism could be applied even though we have dynamical trapping, i.e., the state of trap change in time like gated trapping [11, 12].

A novel application of our concepts comes from the hand of intermittent motion phenomena, which has recently motivated numerous studies in physics, chemistry, and biology [13–17]. In this piece of work, we show that the analysis of the trap’s lifetime, as a function of the transition rates among internal states of the walker, allow us to optimize the intermittent search strategy for a hidden target.

The paper is organized as follows. Section 2 presents the general formalism and define the concepts of survival of the target, lifetime density, and mean quantities; and establishes the connection with the problem of only one walker. Particularly, the last issue or first-passage time problem [18], is reviewed in Sec. 3, whereas in Sec. 4 we reconsider the effects of dimensionality and number of walkers on our problem. This section also reviews the basic about the continuous–time random walk (CTRW). In Sec. 5 the effects due to initial spatial distribution of the set walkers are analyzed and the bulk limit is constructed. Section 6 presents several assorted illustrations for discrete and continuous systems with different types of initial distributions and considering the effects of finite size of the space and imperfection in the trapping mechanism. Last, in Sec. 7, we discuss the usefulness of our approach for searching targets with stochastic intermittent motion. Finally in Sec.8 we give our conclusions.

## 2. Lifetime of the Target

The major objective of this contribution is the study of the effects of the initial distribution of independent walkers and the characteristics of the trapping process in the survival of the target. For this task, we begin reviewing and generalizing the formalism developed in Ref. [5, 6]. The survival probability at time  $t$ ,  $\Phi_N(t)$ , of the static target (trap) at the origin in presence of  $N$  independent walkers that diffuse on a lattice can be written as [5]

$$\Phi_N(t) = \sum_{\vec{s}_1} \dots \sum_{\vec{s}_N} u(\vec{s}_1, \dots, \vec{s}_N) \prod_{i=1}^N \Phi_1(\vec{s}_i, t), \quad (1)$$

where  $u(\vec{s}_1, \dots, \vec{s}_N)$  denote the joint probability distribution of *initially* finding the first walker at a position  $\vec{s}_1$ , the second at  $\vec{s}_2$  and so on.  $\Phi_1(\vec{s}_i, t)$  is the survival probability of the target at time  $t$  in the presence of only one walker initially at position  $\vec{s}_i$ . In Eq. (1), the sums run over all the lattices sites, and become in integrals over the space in the case of diffusion in the continuous space.

From the survival probability, we define the target lifetime density (TLD),  $F_N(t)$ , in the presence of  $N$  walkers, in the standard way by

$$F_N(t) = -\frac{d}{dt}\Phi_N(t). \quad (2)$$

Then, we can write [5, 6]

$$F_N(t) = \sum_{\vec{s}_1} \dots \sum_{\vec{s}_N} u(\vec{s}_1, \dots, \vec{s}_N) \sum_{i=1}^N F_1(\vec{s}_i, t) \prod_{j \neq i}^N \Phi_1(\vec{s}_j, t), \quad (3)$$

where  $F_1(\vec{s}_i, t)$  is the target lifetime density in the presence of only one walker, initially at position  $\vec{s}_i$ . This quantity is defined from  $\Phi_1(\vec{s}_i, t)$  in an analogous way to Eq. (2),

$$F_1(\vec{s}_i, t) = -\frac{d}{dt} \Phi_1(\vec{s}_i, t). \quad (4)$$

In the case of perfect trapping,  $F_1(\vec{s}_i, t)$  is the first-passage time density of the walker. When  $F_1(\vec{s}_i, t)$  is normalized, trapping is certain and the process is called *recurrent* in the sense proposed by Hughes [19]. On the other hand, if

$$f_1(\vec{s}_i) = \int_0^\infty F_1(\vec{s}_i, t) dt < 1, \quad (5)$$

then the process is called *transient* [19].  $f_1(\vec{s}_i)$  is the probability that a walker starting from site  $\vec{s}_i$  will ever reach the origin.

Now, using TLD, we introduce the Mean Target Lifetime (MTL) [2, 20]

$$T_N = \int_0^\infty t F_N(t) dt. \quad (6)$$

If  $t \Phi_N(t) \rightarrow 0$  for  $t \rightarrow \infty$ , then we can also write

$$T_N = \int_0^\infty \Phi_N(t) dt. \quad (7)$$

### 3. First-Passage time

A general expression for  $\Phi_1(\vec{s}_i, t)$  can be constructed in terms of the conditional probability  $q(\vec{s}, t | \vec{s}_i, t = 0)$ , corresponding to a walker be in  $\vec{s}$  at time  $t$ , given that it was at  $\vec{s}_i$  at  $t = 0$ , restricted by the presence of a trap at the origin

$$\Phi_1(\vec{s}_i, t) = \sum_{\vec{s}} q(\vec{s}, t | \vec{s}_i, t = 0), \quad (8)$$

where the sum runs over all lattice sites and must be replaced by an integral in the continuous case. This expression is valid for any kind of trap, allowing for example imperfect trapping or dynamical gated trapping.

For Markov processes, in the perfect trapping case, we can additionally exploit the connection between the probability density of first arrival at the origin at time  $t$  from the initial site  $\vec{s}_i$ ,  $F_1(\vec{s}_i, t)$ , and the conditional probability of finding an unrestricted walker at site  $\vec{s}$  at time  $t$ , given that it was initially at  $\vec{s}_i$ ,  $P(\vec{s}, t | \vec{s}_i, t = 0)$ , [21]

$$P(\vec{0}, t | \vec{s}_i, t = 0) = \Psi(\vec{s}_i, t) \delta_{\vec{s}_i, \vec{0}} + \int_0^t P(\vec{0}, t | \vec{0}, t') F_1(\vec{s}_i, t') dt', \quad (9)$$

where  $\Psi(\vec{s}_i, \tau)$  is the sojourn probability, i.e., the probability that the walker remains on the site  $\vec{s}_i$  a time lag  $\tau$  without a transition. Moreover, for an stationary process we have  $P(\vec{s}, t | \vec{0}, t') = P(\vec{s}, t - t' | \vec{0}, t = 0)$  and the integral in Eq. (9) becomes in a convolution. Thus, the Laplace transform of Eq. (9) lead us to the Laplace transform of the first-passage time density

$$\hat{F}_1(\vec{s}_i, u) = \frac{\hat{P}(\vec{0}, u | \vec{s}_i, t = 0) - \hat{\Psi}(\vec{s}_i, u) \delta_{\vec{s}_i, \vec{0}}}{\hat{P}(\vec{0}, u | \vec{0}, t = 0)}, \quad (10)$$

where the caret denotes the Laplace transform of the corresponding function. Therefore, using Eqs. (4) and (10), the initial condition  $\Phi_1(\vec{s}_i, t = 0) = 1$ , and taking  $\vec{s}_i \neq \vec{0}$ , we finally get the Laplace transform of the survival probability of the target in presence of only one walker,

$$\hat{\Phi}_1(\vec{s}_i, u) = \frac{\hat{P}(\vec{0}, u | \vec{0}, t = 0) - \hat{P}(\vec{0}, u | \vec{s}_i, t = 0)}{u \hat{P}(\vec{0}, u | \vec{0}, t = 0)}. \quad (11)$$

Finally, in a similar way as in Eq.(7), the mean first-passage time (MFPT) for the walker can be written as

$$T = T_1 = \int_0^\infty \Phi_1(t) dt. \quad (12)$$

#### 4. MTL in $d$ -dimensions

If the trapping is perfect, MTL is also the MFPT for the first of the set of  $N$  random walkers in reach the target. An interesting problem is presented when the MFPT for only one walker diverges, i.e., when the integral of Eq. (12) diverges. In this situation, the interesting question that arises is to find the minimum number of walkers such that MTL becomes finite [8], independently of the initial positions of the walkers. That is, from Eqs. (1) and (7), to find  $N$  for which the integral

$$\int_0^\infty \prod_{i=1}^N \Phi_1(\vec{s}_i, t) dt, \quad (13)$$

converges. Using our formalism, we can directly rederive the known results [8] for this problem.

For concreteness, in this section, we use the CTRW [22, 23] for the walker's dynamics. This allows us compute analytically the survival probability of the trap in presence of one walker. Hence, we can write

$$\hat{P}(\vec{s}, u | \vec{0}, t = 0) = \frac{1 - \hat{\psi}(u)}{u} G(\vec{s}, \hat{\psi}(u)), \quad (14)$$

where  $\hat{\psi}(u)$  is the Laplace transform of the pausing time probability density,  $\psi(t)$ , and  $G(\vec{s}, z)$  is the lattice Green's function. In  $d$ -dimensions, it is given by

$$G(\vec{s}, z) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \frac{\exp(-i\vec{s} \cdot \vec{k})}{1 - z \Lambda(\vec{k})} d^d k, \quad (15)$$

where  $\Lambda(\vec{k})$  is the structure function of the lattice. For a symmetrical walk on a simple cubic  $d$ -dimensional lattice we get  $\Lambda(\vec{k}) = (\cos k_1 + \dots + \cos k_d)/d$ , where  $k_i$  is the  $i$ -th component of  $\vec{k}$ .

Assuming that  $\hat{\psi}(u) \rightarrow 1$  for  $u \rightarrow 0$ , the behavior of  $G(\vec{s}, \hat{\psi}(u))$  is given by the values of  $\vec{k}$  such that  $\Lambda(\vec{k}) \approx 1$ , i.e.,  $|\vec{k}| \ll 1$ . Thus,  $\cos k_i \approx 1 - k_i^2/2$  and  $\Lambda(\vec{k}) = 1 - |\vec{k}|^2/(2d)$ . Therefore, we only need to consider the expansion  $\exp(-i\vec{s} \cdot \vec{k}) \approx 1 - i\vec{s} \cdot \vec{k} - (\vec{s} \cdot \vec{k})^2/2$  in the numerator of Eq. (15). Hence, for  $u \rightarrow 0$ , we get  $G(\vec{s}, \hat{\psi}(u)) \approx G(\vec{0}, \hat{\psi}(u)) - K_d(\vec{s}, \hat{\psi}(u))$ , where

$$K_d(\vec{s}, \hat{\psi}(u)) = \frac{1}{2(2\pi)^d} \times \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \frac{(\vec{s} \cdot \vec{k})^2 d^d k}{1 - \hat{\psi}(u) + \hat{\psi}(u) k^2/(2d)}. \quad (16)$$

Given that we are dealing with unrestricted and spatially homogeneous walks, we get  $P(\vec{s}, t|\vec{s}_i, t=0) = P(\vec{s} - \vec{s}_i, t|\vec{0}, t=0)$ . Hence, using these results in Eq. (11), we obtain

$$\hat{\Phi}_1(\vec{s}_i, u) \approx \frac{K_d(\vec{s}_i, \hat{\psi}(u))}{u G(\vec{0}, \hat{\psi}(u))}, \quad (17)$$

where we have used Eq. (14) and the translational invariance of a CTRW on unbounded lattices ( $P(\vec{0}, t|\vec{s}, t=0) = P(\vec{s}, t|\vec{0}, t=0)$ ).

$K_d(\vec{s}, z)$  remains finite when  $z = 1$  since the  $k^2$  in the numerator just cancels the singularity in the denominator. On the other hand, the singularity at  $z = 1$  in  $G(\vec{0}, z)$  depends on the dimensionality of the lattice. For studying this singularity, a good approximate expression of Eq. (15) is obtained taking the integrals over the  $d$ -ball of radius  $\pi$  inscribed into the first Brillouin zone. Thus, for  $u \rightarrow 0$ , we get

$$G(\vec{0}, \hat{\psi}(u)) \approx \frac{d C_d}{(2\pi)^d} \int_0^\pi \frac{k^{d-1} dk}{1 - \hat{\psi}(u) + \hat{\psi}(u) k^2/(2d)}, \quad (18)$$

where  $C_d = \pi^{d/2}/\Gamma(d/2 + 1)$ .

For normal diffusion, the pausing time density is  $\psi(t) = \lambda \exp(-\lambda t)$  [22]. Hence,

$$\hat{\psi}(u) = \left(1 + \frac{u}{\lambda}\right)^{-1}, \quad (19)$$

and we get  $\hat{\psi}(u) \approx 1 - u/\lambda$ , for  $u \rightarrow 0$ . For anomalous diffusion, this asymptotic behavior is generalized by  $\hat{\psi}(u) \approx 1 - \kappa u^\alpha$ , with  $0 < \alpha < 1$ . Hence, normal diffusion is the most severe case when we analyse the divergence of  $G(\vec{0}, z)$  at  $z = 0$ .

#### 4.1. One dimension

For  $d = 1$ ,  $C_1 = 2$ , and by direct integration of Eq. (18) we obtain

$$G(\vec{0}, \hat{\psi}(u)) \approx \frac{1}{\pi} \frac{1}{\sqrt{(1 - \hat{\psi}(u))\hat{\psi}(u)/2}} \arctan\left(\pi \sqrt{\frac{\hat{\psi}(u)/2}{1 - \hat{\psi}(u)}}\right). \quad (20)$$

For  $u \rightarrow 0$ ,  $\hat{\psi}(u)/(1 - \hat{\psi}(u)) \rightarrow \infty$  and the last factor of Eq. (20) goes to  $\pi/2$ . Thus, for normal diffusion,

$$G(\vec{0}, \hat{\psi}(u)) \approx \frac{1}{\sqrt{2(1 - \hat{\psi}(u))}} \propto u^{-1/2}. \quad (21)$$

In this manner, from Eq. (17), we obtain  $\hat{\Phi}_1(\vec{s}_i, u) \propto u^{-1/2}$  and using a Tauberian theorem [20, 23], we immediately get  $\Phi_1(\vec{s}_i, t) \propto t^{-1/2}$ . Therefore, Eq. (12) diverges and the convergence of Eq. (13) is obtained for  $N > 2$ . This result is also valid for imperfect traps [6, 10], and for dynamical gated trapping [11, 12] too.

#### 4.2. Two dimensions

For  $d = 2$ ,  $C_2 = \pi$ , and by direct integration of Eq. (18) we obtain

$$G(\vec{0}, \hat{\psi}(u)) \approx \frac{1}{\pi \hat{\psi}(u)} \ln \left( 1 + \frac{\pi^2 \hat{\psi}(u)}{4(1 - \hat{\psi}(u))} \right). \quad (22)$$

Thus, for  $u \rightarrow 0$ ,  $G(\vec{0}, \hat{\psi}(u)) \approx -\ln(1 - \hat{\psi}(u))/\pi \propto -\ln u$ ,  $\hat{\Phi}_1(\vec{s}_i, u) \propto -1/(u \ln u)$  and  $\Phi_1(\vec{s}_i, t) \propto \ln t$ . Therefore, the convergence of Eq. (13) is not reached for any value of  $N$ .

#### 4.3. $d \geq 3$

For  $d > 2$ ,  $G(\vec{0}, \hat{\psi}(u))$  remains finite for  $u \rightarrow 0$  since the  $k^{d-1}$  in the numerator just cancels the singularity in the denominator of Eq. (18). Thus, from Eqs. (17),  $\hat{\Phi}_1(\vec{s}_i, u) \propto 1/u$ . Alternatively this behavior can be seen from Eq. (4), and taking into account Eq. (5). Thus, the asymptotic behavior of the survival probability results

$$\Phi_1(\vec{s}_i, t \rightarrow \infty) = 1 - \int_0^{t \rightarrow \infty} F_1(s_i, t') dt' = 1 - f_1(\vec{s}_i). \quad (23)$$

Therefore, the long time behavior of the survival probability is constant, i.e., time independent. For  $d \geq 3$ , the process results transient [19] ( $0 < f_1(\vec{s}_i) < 1$ ) so each factor in Eq. (13) results  $0 < 1 - f_1(s_i) < 1$ , and then, the convergence of Eq. (13) is not reached for any value of  $N$ .

On the other hand, although the MFPT diverges for  $d \geq 3$ , the asymptotic limit of the survival probability of the target, which is proportional to  $\prod_{i=1}^N (1 - f_1(\vec{s}_i))$  with each factor less than one, decreases monotonously with  $N$ , independently of the initial distribution of the walkers.

Alternatively, we can proceed as in Ref. [8] choosing only the random walkers that will ever reach the target. For this purpose, it is necessary to define the conditional probability density of first arrival at the origin at time  $t$  from the initial site  $\vec{s}_i$ , given that the walker will eventually arrive there:  $F_1(\vec{s}_i, t)/f_1(\vec{s}_i)$ . Working with this quantity, we can find in  $d = 3$  an expression equivalent to Eq. (21). Thus, in  $d = 3$ , MTL is finite if at least three walkers eventually reach the origin (target).

## 5. Initial joint distribution

Now, we consider the effects on TLD and MTL due to different initial probability distributions,  $u(\vec{s}_1, \dots, \vec{s}_N)$ . Most of cases in the literature [24–26] belong to the following kinds of distributions:

### 5.1. Concentrated

All walkers can begin at the same point of the space,

$$u(\vec{s}_1, \dots, \vec{s}_N) = \prod_{i=1}^N \delta_{\vec{s}_i, \vec{s}_0}, \quad (\vec{s}_0 \neq \vec{0}). \quad (24)$$

Thus, from Eq. (1) we obtain

$$\Phi_N^{con}(t) = (\Phi_1(\vec{s}_0, t))^N, \quad (25)$$

and using Eq. (3) results

$$F_N^{con}(t) = N F_1(\vec{s}_0, t) (\Phi_1(\vec{s}_0, t))^{N-1}. \quad (26)$$

Recalling that  $\Phi_1(\vec{s}_0, t) = 1 - \int_0^t F_1(\vec{s}_0, \tau) d\tau$ , the last equation may be compared with Eq. (19) in Ref.[26].

### 5.2. Equally likely sites

Alternatively, the initial site of each walker can be chosen by chance among the  $M$  sites of a given set  $\mathcal{S}$  with equal probability,

$$u(\vec{s}_1, \dots, \vec{s}_N) = \begin{cases} M^{-N} & \text{if all } \vec{s}_i \in \mathcal{S}, \\ 0 & \text{otherwise,} \end{cases} \quad (27)$$

where  $\mathcal{S}$  is such that  $\vec{0} \notin \mathcal{S}$ . Thus, from Eq. (1) results

$$\Phi_N^{els}(t) = \left( \frac{1}{M} \sum_{\vec{s} \in \mathcal{S}} \Phi_1(\vec{s}, t) \right)^N. \quad (28)$$

Also, we can write Eq. (28) as  $\Phi_N^{els}(t) = (\langle \Phi_1(\vec{s}, t) \rangle)^N$ , where  $\langle \dots \rangle$  denotes the spacial average taken in the set  $\mathcal{S}$ . Alternatively, we can recast Eq. (28) as

$$\Phi_N^{els}(t) = \left( 1 - \frac{1}{M} \sum_{i=1}^M (1 - \Phi_1(\vec{s}_i, t)) \right)^N, \quad (29)$$

where  $\vec{s}_i$  ( $i = 1, \dots, M$ ) are the positions of the  $M$  sites of set  $\mathcal{S}$ . Note that there is not any restriction between  $N$  and  $M$ . Eq. (29) allows us to take the limits  $N \rightarrow \infty$ ,  $M \rightarrow \infty$ , with  $N/M \rightarrow \beta$  constant, i.e., the *bulk limit*. In this case we get

$$\Phi_\beta^{els}(t) = \exp(-\beta S(t)), \quad (30)$$

where

$$S(t) = \sum_{\vec{s} \neq \vec{0}} (1 - \Phi_1(\vec{s}, t)), \quad (31)$$

is the average number of lattice points visited, at least once time, by one walker, until time  $t$ . We are assuming that the series in Eq. (31) converges.

In the continuous  $d$ -dimensional space, assuming that  $\mathcal{S}$  has a finite volume  $V$  and that initially each walker begin uniformly distributed in  $V$ , the generalization of Eq. (28) is immediate,

$$\Phi_N^{els}(t) = \left( \frac{1}{V} \int_{\mathcal{S}} \Phi_1(\vec{s}, t) d^d s \right)^N. \quad (32)$$

If the number of walkers per unit of volume,  $c = N/V$ , is constant, then in the limit  $N \rightarrow \infty$  and  $V \rightarrow \infty$  we also obtain

$$\Phi_c^{els}(t) = \exp \left( -c \int (1 - \Phi_1(\vec{s}, t)) d^d s \right), \quad (33)$$

where we assume that the integral over the whole space is bounded.

## 6. Illustrations

### 6.1. Finite and semi-infinite chain

As illustration, we now compute the MTL for a perfect trap at the origin of a chain, in presence of  $N$  walkers that jump from any site to its nearest neighbor with transition rate  $\lambda$ . For a finite chain of  $L$  sites with absorbing end at the origin and reflecting end at site  $L$ , there is an exact expression for the Laplace transform of the first-passage time density [27]

$$\hat{F}_1(j, u) = \frac{R(u)^j + R(u)^{2L+1-j}}{1 + R(u)^{2L+1}}, \quad (34)$$

( $j = 1, 2, \dots, L$ ) where  $R(u) = (r + 1 - \sqrt{r^2 + 2r})$  and  $r = u/\lambda$ . This expression can be Laplace antitransformed in exact way [28] but it is rather clumsy to display here. Moreover, for use Eqs. (25) or (28), we need previously make the integration involved in Eq. (2) and later make the integration in Eq. (7). We performed numerically these integrals [28]. In Fig. 1 we plot the values of  $T_N$  for a chain with  $L = 10$  and initial distribution of walkers given by Eqs. (24) and (27). Strikingly, we find in both cases a power-law behaviour (see quasi-linear relation in the log-log plot) for almost all values of  $N$ .

If the chain is semi-infinite, an explicit and simple expression for  $\Phi_1(j, t)$  ( $j = 1, 2, \dots$ ) can be derive from results in the literature [6, 29]

$$\Phi_1(j, t) = e^{-\lambda t} (I_0(\lambda t) - I_j(\lambda t)) + 2 e^{-\lambda t} \sum_{k=1}^j I_k(\lambda t), \quad (35)$$

where  $I_k(x)$  are the modified Bessel functions. Using this expression in Eqs. (25) or (30) and integrating numerically [28] in Eq. (7), we can evaluate the MTL for the initial distributions of Eqs. (24) and (27). Figure 2 plots the situation for walkers initially distributed with concentration  $\beta$  on the chain.

For comparison purposes, we also include in Fig. 1 the plots corresponding to a semi-infinite chain with all walkers initially concentrated, at site  $s = 1$  or site  $s = 10$ . In the later plots, the minimum number of walkers to obtain finite values of  $T_N$  is three, as has been quoted in Sec. 4.1. We want to stress the notorious resemblance obtained,



for not so large values of  $N$ , between the cases of finite chain and semi-infinite chain for all walkers initially concentrated. Fig. 2 also includes the plots corresponding to a finite chain of  $L = 10$  sites. In this graph the noticeable resemblance for large values of  $\beta$  is given between the cases of semi-infinite chain and finite chain with initial distribution of equally likely sites.

### 6.2. Bulk limit in $d$ -dimensions

For  $d$ -dimensional lattices with a perfect trap at the origin, the bulk limit is given by Eq. (30). Using Eq. (7), MTL can be written in this case as

$$T_\beta = \int_0^\infty \exp(-\beta S(t)) dt. \quad (36)$$

It can be shown that the laplace transform of  $S(t)$ , assuming a CTRW dynamics, is given by [30]

$$\hat{S}(u) = \frac{\hat{\psi}(u)}{u^2 P(\vec{0}, u | \vec{0}, t = 0)}, \quad (37)$$

where  $\hat{\psi}(u)$  is given by Eq. (19).

Using known expressions for  $P(\vec{0}, u | \vec{0}, t = 0)$  given in Ref. [19], we plot in Fig. (3)  $T_\beta$  as function of  $\beta$  for different lattices in  $d = 1$  (chain),  $d = 2$  (honeycomb, square, and triangular), and  $d = 3$  (SC, BCC, and FCC).  $T_\beta$  shows again a quasi-linear relation in the log-log plot and, as expected,  $T_\beta$  monotonously decrease as the coordination number of the lattice,  $\kappa$ , is increased.

### 6.3. Imperfect trapping in the continuous space

The semi-straight line with an imperfect trap at the origin is another interesting illustration of our concepts. The probability density (in presence of the trap),  $q(x, t | x_0, t = 0)$ , for finding a particle at the location  $x$  at time  $t$ , given that it departed from site  $x_0$  at the time  $t = 0$ , satisfies the classical diffusion equation

$$\frac{\partial q}{\partial t} = D \frac{\partial^2 q}{\partial x^2}, \quad (38)$$

where  $D$  is the diffusion coefficient. An imperfect trap is described by the radiation boundary condition [31]

$$\left. \frac{\partial q}{\partial x} \right|_{x=0} = \gamma q(x = 0, t | x_0, t = 0), \quad (39)$$

where  $\gamma$  measure the efficiency of the trap.  $\gamma = 0$  corresponds to a reflecting boundary and perfect trapping is reached in the limit  $\gamma \rightarrow \infty$ . The solution of Eqs. (38) and (39) can be read from Ref. [32]

$$q(x, t | x_0, t = 0) = q_a - \gamma q_b, \quad (40)$$

where

$$q_a = \frac{\exp\left(-\frac{(x-x_0)^2}{2Dt}\right) + \exp\left(-\frac{(x+x_0)^2}{2Dt}\right)}{2\sqrt{\pi Dt}} \quad (41)$$

and

$$q_b = \exp(D\gamma^2 t + \gamma(x + x_0)) \operatorname{erfc}\left(\frac{x + x_0}{2\sqrt{Dt}} + \gamma\sqrt{Dt}\right). \quad (42)$$

The relation between the survival probability  $\Phi_1(x_0, t)$  and  $q(x, t|x_0, t = 0)$  is given by Eq. (8). Therefore, for the imperfect trapping we get [33]

$$\Phi_1(x_0, t) = 1 - \operatorname{erfc}\left(\frac{x_0}{2\sqrt{Dt}}\right) + \exp(D\gamma^2 t + \gamma x_0) \operatorname{erfc}\left(\frac{x_0}{2\sqrt{Dt}} + \gamma\sqrt{Dt}\right). \quad (43)$$

Notice that the expected expression for perfect trapping ( $\gamma \rightarrow \infty$ ) is directly recovered

$$\Phi_1(x_0, t) = 1 - \operatorname{erfc}\left(\frac{x_0}{2\sqrt{Dt}}\right). \quad (44)$$

Figure 4 graphs  $T_N$  for several situations of trapping efficiency ( $\gamma$ ) and for the initial distribution of walkers concentrated at  $x_0 = 1$ . To make each plot we have used Eq. (43) and (25), and we have made numerically [28] the integration of Eq. (7). Also in the continuous space, we obtain quasi-linear relations in the log-log plots of  $T_N$  vs  $N$  for not so large values of  $N$ .

## 7. Target search with intermittent motion

Examples of intermittent processes may be found in many fields. For instance, the case of a reactant that freely diffuses in a solvent and intermittently binds to a cylinder [13]. Also, we found intermittent motion in the binding of a protein to specific sites on DNA for regulating transcription, as it is the case when the protein has the ability of diffuse in one dimension by sliding along the length of the DNA, in addition to their diffusion in bulk solution [14]. Moreover, intermitency could be associate with dynamical trapping problems [34]. We also found the search strategies like those implemented by animals in the pursuit of prey, or even in human activities such as victim localization, among the most representative examples of intermittent motion at macroscopic scales (see Ref. [15] and references therein). Recent works on intermittent search strategies were focused on the analysis of trapping of a single walker wandering in the presence of distributed traps. The magnitude that is usually estimated in this case is the named search time [17].

An interesting application of our present formalism is the related problem of a single static target among a set of initially uniformly distributed searchers, switching intermittently between two states of motion [15, 16]. In what follows, we calculate MTL for a target (perfect trap) at the origin of an infinite chain. We assume that each walker can be in either of two propagation states. In one of them, the displacement of the searchers is a random walk with symmetric jumps to nearest neighbors sites, with constant rate  $\lambda$ . In the other state, the searchers also perform a symmetrical random walk, but jumping to next-nearest neighbors, with the same constant rate  $\lambda$ . Thus, in the second state the diffusion is twice as big as in the other state, but in the second internal state, the walker can skip over the target without trapping. Transitions between the first and the second internal state, take place with rate constant  $\gamma_1$ , whereas the opposite transitions,

between the second and first state are at rate  $\gamma_2$ . The coupled master equations that describe this composite process of one walker are then

$$\frac{\partial P_1(j, t)}{\partial t} = \frac{\lambda}{2} (P_1(j+1, t) + P_1(j-1, t)) - \lambda P_1(j, t) + \gamma_2 P_2(j, t) - \gamma_1 P_1(j, t), \quad (45)$$

$$\frac{\partial P_2(j, t)}{\partial t} = \frac{\lambda}{2} (P_2(j+2, t) + P_2(j-2, t)) - \lambda P_2(j, t) + \gamma_1 P_1(j, t) - \gamma_2 P_2(j, t), \quad (46)$$

where  $P_1(j, t)$  ( $P_2(j, t)$ ) is the joint probability that the walker be at site  $j$  with internal state 1 (2) at time  $t$ . If we assume equally likely sites as initial distribution of walkers and the bulk limit defined in subsection 5.2, we can use Eqs.(36) and (37) with  $P(j=0, u) \equiv P_1(j=0, u) + P_2(j=0, u)$ . In this manner, we can compute  $T_\beta$  as a function of  $\gamma_1$  and  $\gamma_2$ .

In Fig. 5 we draw the behavior of MTL, in the bulk limit, as a function of the parameters of transition  $\gamma_1$  and  $\gamma_2$ . As can be seen from the figure, we obtain a region of optimal values in the parameter space  $(\gamma_1, \gamma_2)$  which can be appreciated by the grey scale (darker means a smaller value in  $T_\beta(\gamma_1, \gamma_2)$ ). The valley in the surface indicates that we can tune the parameters to optimize the search. Notice how the MTL adequately characterizes the improvement provided by intermittent search strategy [16]. Similar behavior has been exhibited in Ref. [17] despite the difference in the addressed problems.

## 8. Conclusions

This paper provides a simple, general, and unified formalism for the lifetime statistics of a fixed target in presence of a set of independent hunters or gatherers that diffuse in the space. Our framework compresses normal and anomalous diffusion on lattices as well as in the continuous space. Also, our scheme allows us to consider perfect and imperfect trapping and can be directly extended for dynamical traps. Our main quantity, the MTL was introduced and its connections with any other physical quantities, relevant for the temporal statistics of trapping, were established. Particularly, the role of the initial spatial distribution was discussed.

For trapping problems, where we deal with only one trap, MFPT approach is limited when the process is transient (see Eq. (5)) or the recurrence time in the lattice is infinite, because the MFPT diverge. Although MTL overcome this problem in one dimension, if the number of walkers is increased, it also diverges for  $d > 1$ . However, the bulk limit of MTL is finite in all situations. This robust property points out that MTL is the relevant physical quantity for situations where we consider the bulk density of diffusing particles and only one isolated capture center.

Additionally, a striking feature of MTL is it presents non-universal scaling laws when it is plotted as a function of the number or density of walkers. Moreover, MTL is an efficient global optimizer for search strategies using intermittent motion. The MTL surface has a valley, allowing us to tune up the parameters that regulate the intermittency, and thus to minimize the time of search.

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## Figure and Captions

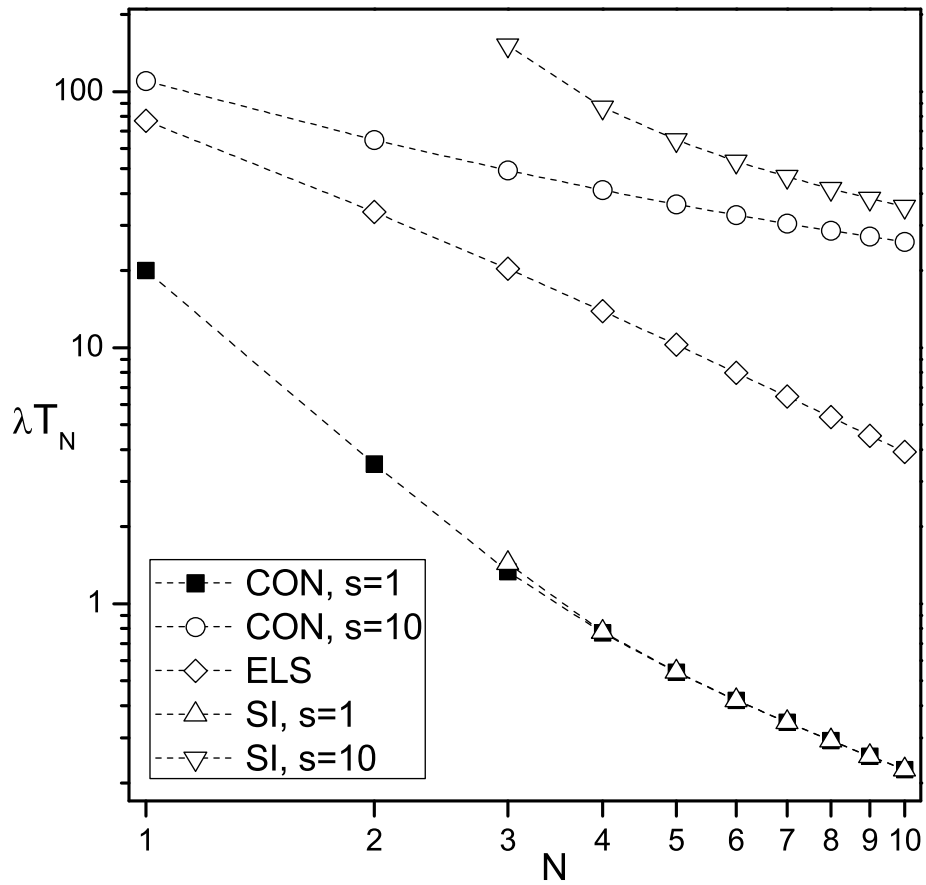


Figure 1:  $T_N$  as a function of the number  $N$  of walkers for a chain with the trap at the origin. The finite case corresponds to  $L = 10$  sites and two different initial distributions: Concentrated (CON), at site  $s = 1$  or site  $s=10$ ; and with all sites equally likely (ELS). The semi-infinite case (SI) corresponds to all walkers initially concentrated, at site  $s = 1$  or site  $s = 10$ . The dotted lines are only to guide the eye.

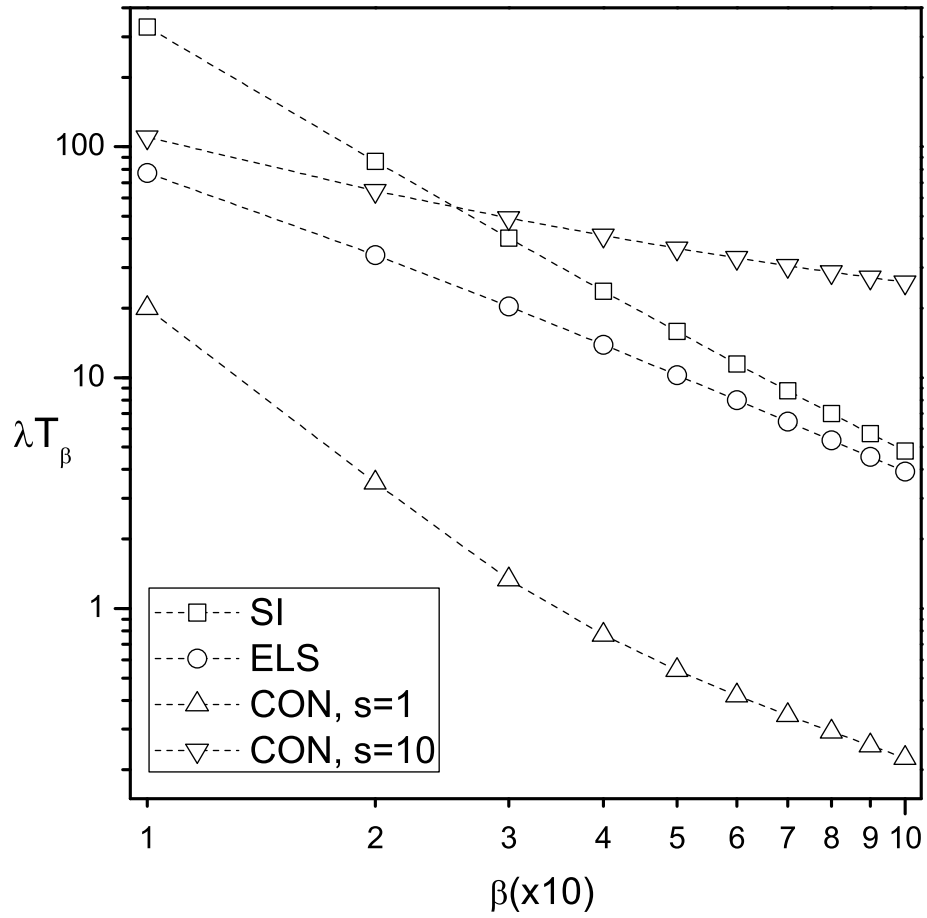


Figure 2:  $T_\beta$  as a function of the concentration of walkers  $\beta$ , for a chain with the target at the origin. The semi-infinite case (SI) corresponds to the bulk limit. The finite case corresponds to  $L = 10$  sites ( $\beta = N/L$ ) and all walkers initially concentrated (CON), at site  $s = 1$  or site  $s = 10$ ; or with all sites equally likely (ELS). The dotted lines are only to guide the eye.

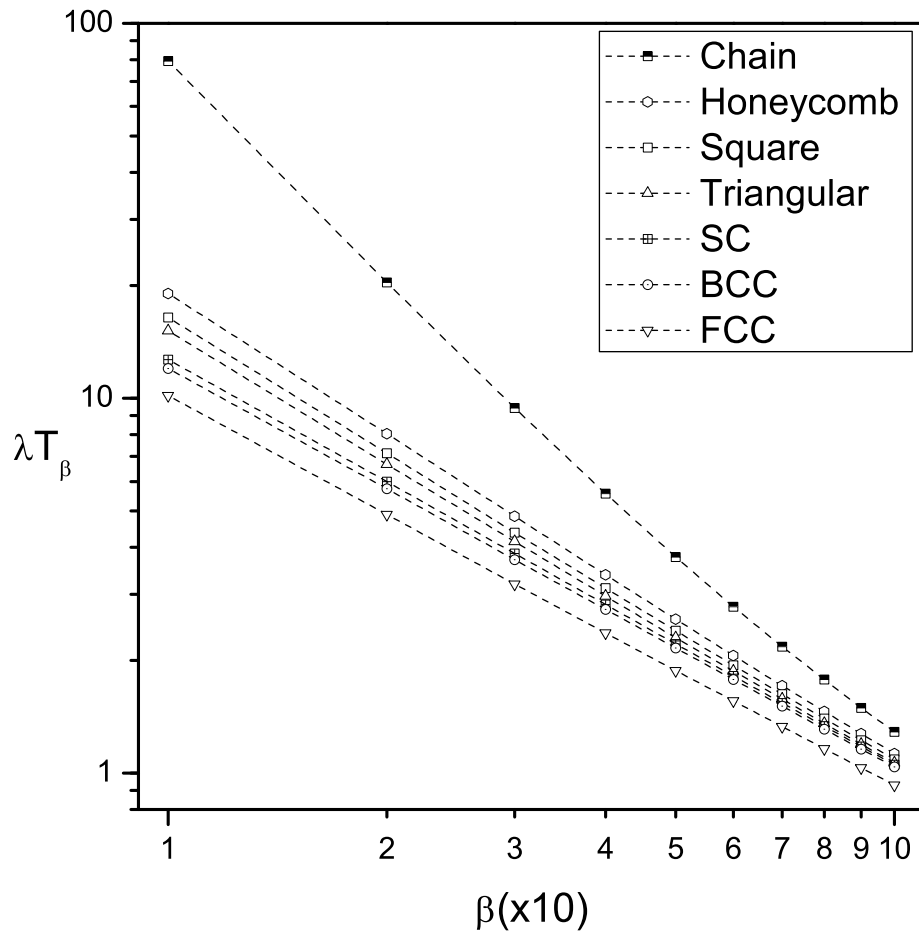


Figure 3:  $T_\beta$  as a function of the concentration of walkers,  $\beta$ , in the bulk limit, for different lattices. The coordination numbers of the lattices, from top to bottom, are  $\kappa = 2, 3, 4, 5, 6, 8, 12$ .



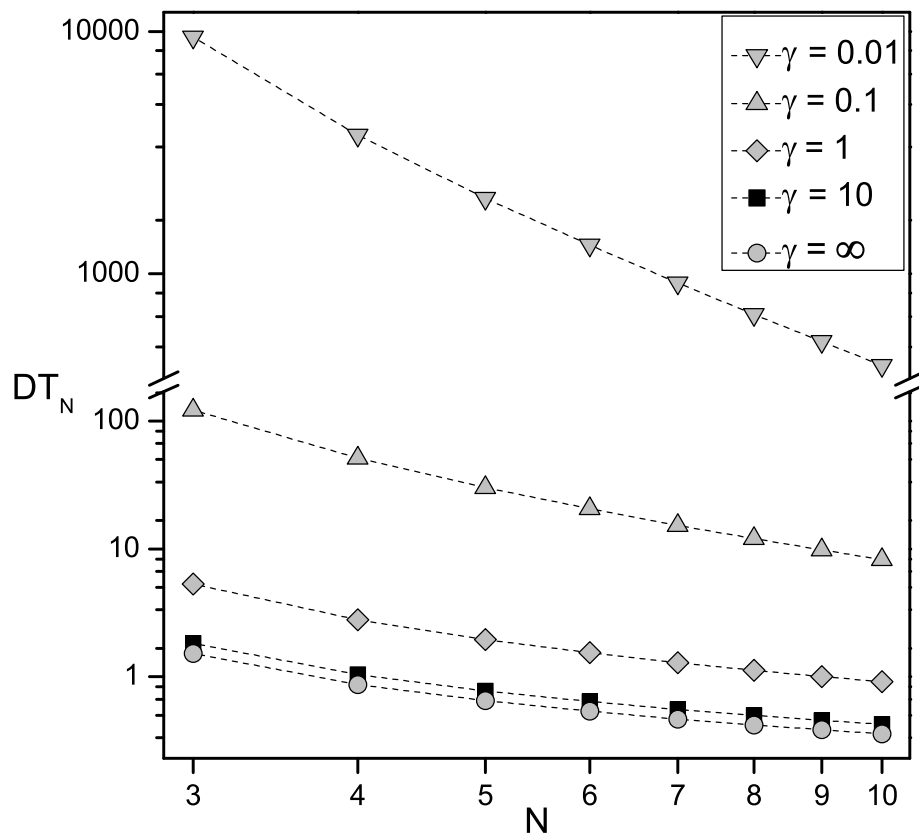


Figure 4:  $T_N$  as a function of the number  $N$  of walkers for semi-straight line with an imperfect trap at the origin. The dotted lines are only to guide the eye.

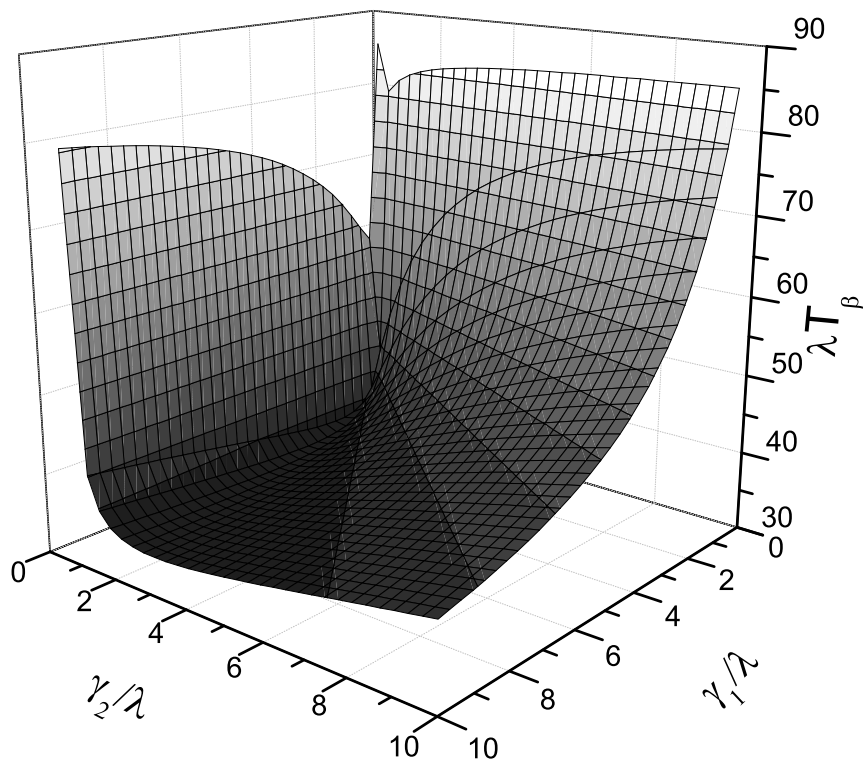


Figure 5:  $T_\beta$  in the bulk limit, for a concentration  $\beta = 0.1$  of walkers, as a function of parameters  $\gamma_1$  and  $\gamma_2$ .