HYPERCOMPLEX NILPOTENT LIE GROUPS

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1. Introduction

An invariant hypercomplex structure on a Lie group G is a triple of integrable complex structures $\{J_i\}_{i=1,2,3}$ on the corresponding Lie algebra $\mathfrak{g}$ satisfying the quaternion identities $J_i^2 = -1, i = 1, 2, 3$ and $J_1J_2 = J_3 = -J_2J_1$. Invariant hypercomplex structures on compact Lie groups were studied in [15] in the context of $N = 4$ supersymmetry. D. Joyce (see [8]) constructed, for any compact Lie group G, an invariant hypercomplex structure on $T^{2n-r} \times G$, r the rank of G. This extends a result of Samelson who showed that every compact Lie group G of even dimension has a complex structure such that left translations are holomorphic mappings (see [12]).

There have been a number of recent results on invariant hypercomplex structures on non-compact Lie groups. The classification, for 4-dimensional real Lie groups was given in [2]. In [5] it was shown that every solvable Lie group associated to a rank one symmetric space of non compact type with dimension divisible by 4, admits such a structure. On the other hand, examples were given of solvable Lie groups G such that no product with $\mathbf{R}^s$ admits a hypercomplex structure, in contrast to the compact case discussed in [8]. The case of nilpotent Lie groups had already been considered in [9], [4] and [3]. The construction given in [9] allows to obtain as a byproduct, many non homogeneous hypercomplex manifolds.

In [7] a classification of hypercomplex structures on 8-dimensional nilpotent Lie algebras is given. In the present note, using the description in the main result in [7], we prove that every hypercomplex 8-dimensional nilmanifold is quaternionic, i.e the Obata conection is flat (see Section 2). In Section 3 we concentrate on *abelian* hypercomplex structures, which have the special property that for one (hence for all) of the complex structures in the family, the associated $(1, 0)$-vector fields commute. We will prove that such a structure can only occur on solvable Lie groups (Proposition 3.1). For such groups these structures are very common. For instance, in 2-step nilpotent Lie groups many examples have been constructed (see [6], [3]). In a nilpotent Lie group of dimension 8, the existence of a hypercomplex structure forces $\mathfrak{g}$ to be 2-step (see [7]). However, in Section 3 we provide an

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example of an abelian hypercomplex structure on a 3-step, 12-dimensional nilpotent Lie group that is not Obata flat.

2. 8-dimensional hypercomplex nilmanifolds

We note that the only nilpotent Lie group of dimension 4 which admits an hypercomplex structure is abelian. On the other hand, in the case of nilpotent Lie groups of dimension 8, the condition of admitting an invariant hypercomplex structure, imposes strong restrictions on G. In [7] the following result is proved

Theorem 2.1. *Let G be an 8-dimensional nilpotent Lie group endowed with a hypercomplex structure. Then G is at most 2-step nilpotent and the first Betti number of the corresponding Lie algebra $\mathfrak{g}$ satisfies $b_1(\mathfrak{g}) \geq 4$.*

It was also proved (in [7]) that there exists a basis of left invariant 1-forms

$$\{e^1, e^2 = J_1e^1, e^3 = J_2e^1, e^4 = J_3e^1, e^5, e^6 = J_1e^5, e^7 = J_2e^5, e^8 = J_3e^5\}$$

such that, if $b_1(\mathfrak{g}) = 4$ and e^{kl} stands for $e^k \wedge e^l$ then

$$\begin{array}{ll} de^i & = 0, i = 1, \dots, 4, \\ de^5 & = a_1e^{12} + a_2e^{13} + a_3e^{14} + (-a_3 + t_3)e^{23} + (a_2 - t_2)e^{24} + (-a_1 + t_1)e^{34}, \\ de^6 & = b_1e^{12} + b_2e^{13} + b_3e^{14} + (-b_3 + t_2)e^{23} + (b_2 + t_3)e^{24} + (-b_1 + t_4)e^{34}, \\ de^7 & = c_1e^{12} + c_2e^{13} + c_3e^{14} + (-c_3 - t_1)e^{23} + (c_2 - t_4)e^{24} + (-c_1 + t_3)e^{34}, \\ de^8 & = d_1e^{12} + d_2e^{13} + d_3e^{14} + (-d_3 + t_4)e^{23} + (d_2 - t_1)e^{24} + (-d_1 - t_2)e^{34}, \end{array}$$

with $t_1^2 + t_2^2 + t_3^2 + t_4^2 \neq 0$. If $b_1(\mathfrak{g}) \geq 5$, then

$$\begin{array}{ll} de^i & = 0, i = 1, \dots, 5, \\ de^6 & = b_1\,(e^{12} - e^{34}) + b_2(e^{13} + e^{24}) + b_3\,(e^{14} - e^{23}) + te^{34}, \\ de^7 & = c_1\,(e^{12} - e^{34}) + c_2\,(e^{13} + e^{24}) + c_3(e^{14} - e^{23}) - te^{24}, \\ de^8 & = d_1\,(e^{12} - e^{34}) + d_2(e^{13} + e^{24}) + d_3\,(e^{14} - e^{23}) + te^{23}. \end{array}$$

Remark 1. Note that the center $\mathfrak{z}$ of $\mathfrak{g}$ contains the 4-dimensional subspace $W = span\{e_5, e_6, e_7, e_8\}$, which is invariant with respect to J_i, $i = 1, 2, 3$.

It is known (see [10]) that every hypercomplex structure $\{J_i\}_{i=1,2}$ on a $4k$-dimensional differentiable manifold M uniquely determines an affine, torsion free connection (called the Obata connection), with respect to which the complex structures $J_i, i = 1, 2, 3$ are parallel. In other words, the $GL(k, \mathbb{H})$-structure they determine admits a torsion free connection. The above description of the hypercomplex structures on 8-dimensional nilpotent Lie groups allows us to show that its Obata connection is flat. Indeed one has

Proposition 2.1. *Every hypercomplex structure on a 8-dimensional nilpotent Lie group is flat.*

Proof. The Obata connection associated to an invariant hypercomplex structure $J_1, J_2, J_3 = J_1J_2$ on a nilpotent Lie group G is given by

$$\nabla_X Y = \frac{1}{2}[X,Y] + \frac{1}{12}\sigma([J_1X, J_3Y] + [J_2X, J_3Y]) + \frac{1}{6}\sum_{i=1}^{i=3}[J_iX, Y] + [J_iY, X],$$

where X, Y lie in $\mathfrak{g}$ and σ denotes the cyclic sum with respect to the indices $1, 2, 3$.

Using Remark 1 one can prove that

$$\nabla_X Y = 0, \quad \text{for } X \in W \quad \text{or} \quad Y \in W,$$

since W is J_i-invariant, for $i = 1, 2, 3$. Moreover, for any $X, Y \in \mathfrak{g}$, we have that $\nabla_X Y \in W$. Hence, the curvature tensor R associated to ∇ vanishes.

If the connection is flat, then M is quaternionic in the sense of Sommese ([14]), that is, M can be covered by coordinate neighborhoods such that the transition functions are quaternionic. In particular, the $GL(k, \mathbb{H})$-structure is integrable, hence a flat affine structure exists on M. Thus, as a consequence of the previous proposition it will follow that any 8-dimensional hypercomplex nilmanifold M is quaternionic in the sense of Sommese.

Furthermore using the previous Proposition it is easy to prove that (G, J_i, ∇) is a special complex manifold in the sense of [1], since the Obata connection ∇ is a flat torsion-free connection such that $\nabla J_i = 0$. In this case any affine function f (i.e. a function satisfying $\nabla df = 0$) can be extended to a holomorphic function F such that $Re\, F = f$.

3. Abelian hypercomplex structures

Among the left invariant almost complex structures J on a Lie group G one may consider as a special class, those $J's$ satisfying

(1) $[JX, JY] = [X, Y]$ for all $X, Y \in \mathfrak{g}$.

One verifies easily that these almost complex structures are integrable, that is,

$$0 = N_J(X, Y) = J[X, Y] - [JX, Y] - [X, JY] - J[JX, JY].$$

Complex structure satisfying condition (1) are called *abelian* since the associated $(1, 0)$ vector fields commute. They have been studied previously in [5], [6], [4], [11], [3]. Abelian complex structures are common on solvable Lie algebras. See [13] for families of abelian complex structures on solvable, non nilpotent, 4-dimensional Lie algebras. The following proposition shows that only solvable Lie groups can carry this class of structures.

Proposition 3.1. *If a Lie group G carries an invariant abelian complex structure then G is solvable.*

Proof. We note first that semisimple Lie groups do not admit abelian complex structures of the above type. To prove the assertion, let $\langle , \rangle$ be any

Ad(G)-invariant non degenerate bilinear form on $\mathfrak{g}$. Then, if J is abelian one has $[JX, Y] = -[X, JY]$ giving

$$\langle J^*[X,Y], Z\rangle = \langle X, [Y, JZ]\rangle = -\langle X, [JY, Z]\rangle = -\langle [X, JY], Z\rangle = \langle [JX, Y], Z\rangle,$$

where J^* is the adjoint with respect to $\langle , \rangle$. Hence $J^*[X,Y] = [JX,Y] = -[X,JY] = [JY,X] = J^*[Y,X]$ thus $J^* = 0$ on $\mathfrak{g}^1 = \mathfrak{g}$, which is not possible.

Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$, the Lie algebra of G. Then J abelian implies $\mathfrak{r} + J\mathfrak{r}$ is a solvable ideal and by maximality it must coincide with $\mathfrak{r}$. We claim $\mathfrak{r} = \mathfrak{g}$. Otherwise, J induces on the quotient semisimple Lie algebra $\mathfrak{g}/\mathfrak{r}$ an abelian complex structure contradicting what we have proved above.

Every hypercomplex structure gives rise to a sphere of complex structures. We show next that if one of the (invariant) complex structures on the sphere is abelian then all of the complex structures in the sphere are abelian.

Lemma 3.1. *If J, K are anticommuting complex structures on $\mathfrak{g}$ and J satisfies (1) then K also satisfies (1).*

Proof. Since K is integrable

$$[KX, Y] + [X, KY] + K[KX, KY] = K[X, Y] = K[JX, JY] =$$

$$[KJX, JY]+[JX, KJY]+K[KJX, KJY] = -[KX, Y]-[X, KY]+K[KX, KY],$$

showing that K is abelian.

The hypercomplex structure is called *abelian* if $[J_iX, J_iY] = [X, Y]$, for any $i = 1, 2, 3$ and $X, Y \in \mathfrak{n}$. For this, it is sufficient by the above lemma, that one of the complex structures in the associated sphere is abelian.

In [6] it is proved that an 8-dimensional nilpotent Lie group with an abelian hypercomplex structure is either abelian or a trivial extension of a group of Heisenberg type. In [3], the 2-step nilpotent Lie algebras carrying abelian hypercomplex structures are characterized and it is shown in particular that the result quoted above is not longer true in higher dimensions.

A natural question that comes is whether there exist examples of s-step nilpotent Lie groups (with $s > 2$) in dimension greater than eight endowed with a hypercomplex structure. The following example gives a positive answer to this question.

Example 1. Consider the 3-step nilpotent Lie algebra $\mathfrak{g}$ of dimension 12 defined by

$$\begin{array}{l} de^i = 0, i = 1, \dots, 9, \\ de^{10} = e^1 \wedge e^2 - e^5 \wedge e^6, \\ de^{11} = e^2 \wedge e^5 - e^1 \wedge e^6, \\ de^{12} = e^1 \wedge e^4 + e^2 \wedge e^{10} + e^5 \wedge e^8 + e^6 \wedge e^{11} \end{array}.$$

It is possible to check that $\mathfrak{g}$ admits an abelian hypercomplex structure $\{J_1, J_2\}$, with corresponding basis of $(1,0)$-forms,

$$\left\{ \begin{array}{lll} \omega_1^1 = e^1 - ie^2, & \omega_2^1 = e^3 - ie^{12}, & \omega_3^1 = e^4 - ie^{10}, \\ \omega_4^1 = e^5 - ie^6, & \omega_5^1 = e^7 - ie^9, & \omega_6^1 = e^8 - ie^{11}, \end{array} \right.$$

$$\begin{cases} \omega_1^2 = e^1 - ie^6, & \omega_2^2 = e^2 - ie^5, & \omega_3^2 = e^3 - ie^9, \\ \omega_4^2 = e^4 - ie^{11}, & \omega_5^2 = e^{10} - ie^8, & \omega_6^2 = e^{12} - ie^7. \end{cases}$$

Furthermore we note that the assertion in Proposition 2.1 does not hold in this case. Indeed, the curvature tensor R associated to Obata connection is not zero; for example one can check that $R_{e_1 e_2} e_6 = \frac{1}{3} e_{12}$, where $\{e_i\}$ is the dual basis of $\{e^i\}$.

References

1. D. V. Alekseevsky, V. Cortes, C. Devchand: Special complex manifolds, preprint math. DG/9910091.
2. M. L. Barberis: Hypercomplex structures on four-dimensional Lie groups, *Proc. Amer. Math. Soc.* **125** (1997), no. 4, 1043-1054.
3. M. L. Barberis: Abelian hypercomplex structures on central extensions of H-type algebras, *Journal of Pure and Applied Algebra*, to appear.
4. M. L. Barberis, I. Dotti Miatello, R. Miatello: On certain locally homogeneous Clifford manifolds, *Ann. Glob. Anal. and Geom.* **13** (1995), 289-301.
5. M. L. Barberis, I. Dotti Miatello: Hypercomplex structures on a class of solvable Lie groups, *Quart. J. Math. Oxford (2)* **47** (1996), 389-404.
6. I. Dotti, A. Fino: Abelian hypercomplex 8-dimensional nilmanifolds, *Ann. Glob. Anal. and Geom.* **18** (2000), 47-59.
7. I. Dotti, A. Fino: Hypercomplex 8-dimensional nilpotent Lie groups, preprint (2000).
8. D. Joyce: Compact hypercomplex and quaternionic manifolds, *J. Diff. Geom.* **35** (1992), 743-761.
9. D. Joyce: Manifolds with many complex structures, *Quart. J. Math. Oxford (2)* **35** (1992), 743-761.
10. M. Obata: Affine connections on manifolds with almost complex, quaternionic or Hermitian structures, *Japan. Jour. Math.* **26** (1956), 43-79.
11. S. Salamon: Complex structures on nilpotent Lie algebras, *J. Pure Appl. Algebra*, to appear.
12. H. Samelson: A class of complex analytic manifolds, *Portugal. Math.* **12** (1953) 129-132.
13. J. E.Snow: Invariant complex structures on four dimensional solvable real Lie groups, *Manuscripta Math.* **66** (1990), 397-412.
14. A. Sommese: Quaternionic manifolds, *Math. Ann.* **212** (1975), 191-214.
15. A. Sevrin, W. Troost, A. Van Proeyen, Ph. Spindel: Extended supersymmetric σ-models on group manifolds. II. Current algebras, *Nuclear Phys. B* **311** (1988/89), no. 2, 465-492.

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