Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 346 (2019) 413–423

www.elsevier.com/locate/entcs

Computing the Determinant of the Distance Matrix of a Bicyclic Graph

Ezequiel Dratman,[1]

CONICET - Instituto de Ciencias, Universidad Nacional de General Sarmiento, Los Polvorines, Argentina

Luciano N. Grippo,[2]

Instituto de Ciencias, Universidad Nacional de General Sarmiento, Los Polvorines, Argentina

Martín D. Safe,[3]

Departamento de Matemática, Universidad Nacional del Sur (UNS), Bahía Blanca, Argentina and INMABB, Universidad Nacional del Sur (UNS)-CONICET, Bahía Blanca, Argentina

Celso M. da Silva Jr.,[4]

DEMET and PPPRO, Centro Federal de Educação Tecnológica Celso Suckow da Fonseca, Rio de Janeiro, Brazil

Renata R. Del-Vecchio[5]

Departamento de Análise, Universidade Federal Fluminense, Niterói, Brazil

Abstract

Let G be a connected graph with vertex set $V = \{v_1, \ldots, v_n\}$. The distance $d(v_i, v_j)$ between two vertices v_i and v_j is the number of edges of a shortest path linking them. The distance matrix of G is the $n \times n$ matrix such that its (i, j)-entry is equal to $d(v_i, v_j)$. A formula to compute the determinant of this matrix in terms of the number of vertices was found when the graph either is a tree or is a unicyclic graph. For a byciclic graph, the determinant is known in the case where the cycles have no common edges. In this paper, we present some advances for the remaining cases; i.e., when the cycles share at least one edge. We also present a conjecture for the unsolved cases.

Keywords: bicyclic graphs, determinant, distance matrix.

https://doi.org/10.1016/j.entcs.2019.08.037

1571-0661/

1 Introduction

A *graph* $G = (V, E)$ consists of a set V of *vertices* and a set E of *edges*. We will consider graphs without multiple edges and without loops. Let G be a connected graph on n vertices with vertex set $V = \{v_1, \dots, v_n\}$. The *distance between vertices* v_i and v_j, denoted $d(v_i, v_j)$, is the number of edges of a shortest path from v_i to v_j. The *distance matrix* of G, denoted $D(G)$, is the $n \times n$ symmetric matrix having its (i, j)-entry equal to $d(v_i, v_j)$. Sometimes, we use $d_{i,j}$ to denote $d(v_i, v_j)$.

Distance matrices have been widely studied in the literature. The motivation to start investigating these matrices is due to the connection with a communication problem (see [4,6] for more details). In an early article, Graham and Pollack presented a remarkable result on the determinant of the distance matrix of a tree T on n vertices [4]. They proved that its determinant only depends on n, being equal to $(-1)^{n-1}(n-1)2^{n-2}$. Graham and Hoffman [5] found a formula for the determinant of the distance matrix of a graph in terms of its 2-connected components. Fourty years later, Bapat and Sivasubramanian took advantage from this result to present a formula for the determinant of the distance matrix of a block graph [2]. Graham and Lovász [6] obtained a formula for the inverse of the distance matrix of a tree. Bapat, Kirkland and Neumann [1] extended the result to the case of weighted trees. In the same article they also found a formula to compute the determinant of the distance matrix of a unicyclic graph. Specifically, they proved that the determinant is zero when its only cycle has an even number of edges, and if the graph has $2k + 1 + m$ vertices and a cycle with $2k + 1$ edges, the determinant is equal to $(-2)^m \left[k(k + 1) + \frac{2k+1}{2}m\right]$. In an attempt to generalize previous results in connections with trees, Gong, Zhang and Xu presented some advances in the direction of finding a formula for the determinant of a bicyclic graph [3], considering those bicyclic graphs with two edge-disjoint cycles. Nevertheless, the case of a bicyclic graph with two cycles sharing at least one edge remains open. In this article we present some advances in this direction. In addition, conjectures about the formulas to deal with the uncovered cases are presented.

This paper is organized as follows. In Section 2 we present some basic notations, preliminary results, and we briefly describe previous results in connection with the determinant of the distance matrix of a bicyclic graph. In Sections 3 and 4 we consider the determinant of the distance matrix of a θ-graph and a θ-graph plus a pendant vertex. In Section 5 we present formulas for the determinant of certain bicyclic graphs, where the cycles have at least one common edge. Finally, in section 6, we conjecture a formula of the determinant for the remaining cases.

[1] Email: edratman@ungs.edu.ar
[2] Email: lgrippo@ungs.edu.ar
[3] Email: msafe@uns.edu.ar
[4] Email: celso.silva@cefet-rj.br
[5] Email: rrdelvecchio@id.uff.br

2 Definitions and preliminary results

A *tree* is a connected acyclic graph. A *unicyclic graph* is a connected graph with as many edges as vertices. A *bicyclic graph* is a graph obtained by adding an edge to a unicyclic graph.

The path and the cycle on n vertices are denoted by C_n and P_n, respectively. We use $\mathcal{B}_n$ to denote the set of all bicyclic graphs on n vertices. We will define some special bicyclic graph, particularly important in order to determine the determinant of a graph in $\mathcal{B}_n$. Consider a copy of C_p and a copy of C_q having a vertex a_1 and a_l, respectively. We denote by $B(l,p,q)$ the graph obtained by joining a_1 and a_l by a path $a_1,\ldots,a_l$ of length $l-1$; if $l=1$ we identify vertex a_1 with vertex a_l. We call such a graph an ∞*-graph*. Let P_{l+1},P_{p+1},P_{q+1} be three vertex disjoint paths, $l\geq 1$ and $p,q\geq 2$, each of them having endpoints, $v_1^l,v_2^l,v_1^p,v_2^p,v_1^q,v_2^q$, respectively. We denote by $\theta(l,p,q)$-graph, or simply θ-graph, the graph obtained by identifying the vertices v_1^l,v_1^p,v_1^q as one vertex, and proceeding in the same way for v_2^l,v_2^p,v_2^q

Let G be a graph . We denote by $G+uv$ the graph that arises from G by adding an edge $uv\notin E(G)$. The neighborhood of a vertex v and the degree of a vertex v will be denoted by $N_G(v)$ and $d_G(v)$, respectively. When the context is clear we simply use $N(v)$ or $d(v)$. A *pendant vertex* of G is a vertex v of degree 1 and its incident edge is called a *pendant edge*. Two vertices u and v will be called *twins* if $N(u)\setminus\{v\}=N(v)\setminus\{u\}$. Let S be a subset of vertices of G, we denote by $G[S]$, the subgraph of G induced by S. Given a matrix A, $A(i|j)$ stands for the matrix obtained from A by deleting the row i and the column j. We use $\mathcal{S}(A)$ to denote the spectrum of A. By e_i we denote a vector having a 1 on the i-th coordinate and 0's in the remaining coordinates.

Clearly, the graph family $\mathcal{B}_n$ can be partitioned into two graph subfamilies. One, denoted $\mathcal{B}_n^\infty$, is the subfamily of those graphs having an ∞-graph, denoted by H_∞, as a subgraph; the other one, denoted $\mathcal{B}_n^\theta$, is the subfamily of those graphs having a θ-graph, denoted by H_θ, as a subgraph. Notice that, the graph obtained by deleting H_∞ (respectively H_θ) is an acyclic graph. It means that these graphs are those obtained from a graph $H_\infty\in\mathcal{B}_n^\infty$ (respectively a graph $H_\theta\in\mathcal{B}_n^\theta$) by adding pendant trees.

The following two lemmas essentially reduce the problem of computing the determinant of a graph $G\in\mathcal{B}_n$ to finding the determinant of a graph H either in $\mathcal{B}_n^\infty$ or in $\mathcal{B}_n^\theta$.

Lemma 2.1 *[3, Lemma 2.3] Let G be a graph. If H is the graph obtained from a graph G by adding a pendant vertex to any vertex of G, then* $\det(D(H))$ *is invariant, regardless of which vertex of G has been connected to the pendant vertex by an edge.*

Lemma 2.2 *[3, Lemma 2.4] Let G_1 and G_2 be two graphs with vertex sets $\{v_1,v_2,\ldots,v_k\}$ and $\{v_{k+1},v_{k+2},\ldots,v_n\}$, respectively. Let G be the graph obtained from G_1 and G_2 by adding an edge between vertices v_1 and v_n, and let H be graph obtained from G_1 and G_2 by identifying vertices v_1 and v_n and then adding a pendent vertex from v_1 (or v_n). Then,* $\det(D(G))=\det(D(H))$.

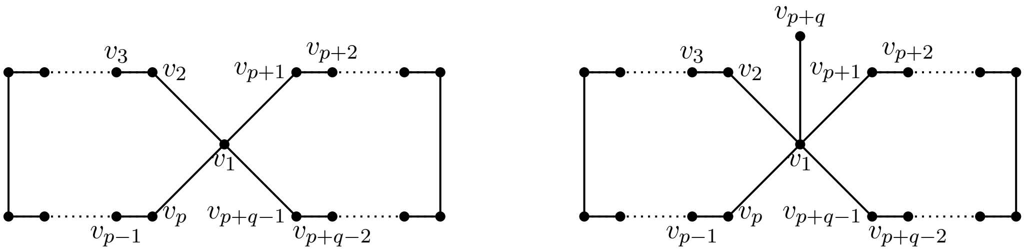

Fig. 1. Graph $B(1,p,q) = P(p,q,1)$ on the left and graph $P(p,q,2)$ on the right.

By Lemma 2.1, if a graph $G \in \mathcal{B}_n^\infty$ has $H = B(l,p,q)$ as induced subgraph, then $\det(D(G))$ is equal to the determinant of the graph H' obtained from H by identifying a vertex of degree 1 of a path on m vertices with the vertex of degree 4 of $B(1,p,q)$; where $m = n-p-q+1$. From now on we will denote such a graph H' by $P(p,q,m)$. Therefore, the problem of computing the determinant of the distance matrix of a bicyclic graph G having two edge-disjoint cycles as induced subgraph can be reduced to the problem a computing the determinant of $D(P(p,q,m))$. It can be also proved that computing such a determinant can be reduced to computing $f(0) = \det(D(P(p,q,1)))$ and $f(1) = \det(D(P(p,q,2)))$ (see Figure 1). This claim is deduced from the following result.

Lemma 2.3 *[3, Lemma 2.2] Suppose that the sequence $f(0), f(1), \cdots, f(n)$ satisfies the following linear recurrence relation*

$$\begin{cases} f(m) = -4f(m-1) - 4f(m-2) \\ f(0) = f_0 \\ f(1) = f_1. \end{cases}$$

Then

$$f(m) = \Big(2(m-1)f_0 + mf_1\Big)(-2)^{(m-1)}. \tag{1}$$

Having previously computed $f(0)$ and $f(1)$ and then proving that $f(m) = \det(D(P(p,q,m+1)))$ satisfies Equation 1 for each positive integer m, it is easy to find a formula to the determinant of any bicyclic graph having $B(p,q,l)$ as an induced subgraph.

Theorem 2.4 *[3, Theorem 3.4] Let G be an arbitrary bicyclic graph on $p+q+m-1$ vertices containing $B(p,q,l)$ as an induced subgraph with $m \geq l-1$. Then $\det(D(G)) = 0$ if one of the integers p or q is even, and otherwise*

$$\det(D(G)) = \left[\frac{(pq-1)(p+q)}{4} + \frac{m}{2}pq\right](-2)^m.$$

Next, we will state two technical results needed to deal with those cases of graphs in which there are at least two twin vertices.

Lemma 2.5 *Let G be a connected graph with vertex set $\{v_1, v_2, \dots, v_n\}$. If v_1 and v_2 are twins, then $\mathcal{S}(D(G)) = \{-d_{1,2}\} \cup \mathcal{S}(R)$, where*

$$R = \begin{pmatrix} d_{2,1} & \mathbf{0}_{1,n-2} \\ \vdots & \vdots \\ d_{n,1} & \mathbf{0}_{1,n-2} \end{pmatrix} + D(H),$$

and $H = G[\{v_2, \dots, v_n\}]$. In addition,

$$\det(D(G)) = -d_{1,2}(\det(D(H)(1|1))\ d_{2,1} + 2\det(D(H))).$$

Proof. Multiplying the matrix $D(G)$ by the elementary matrix

$$P = \begin{pmatrix} 1 & -1 & \mathbf{0}_{1,n-2} \\ 0 & 1 & \mathbf{0}_{1,n-2} \\ \mathbf{0}_{n-2,1} & \mathbf{0}_{n-2,1} & I_{n-2} \end{pmatrix}$$

on the left and by P^{-1} on the right, we obtain the first part of this result. The second part of this result can be easily proved expanding the determinant of this new matrix, which clearly has the same determinant as $D(G)$. □

Corollary 2.6 *Let G and H be graphs defined as in Lemma 2.5. If $\det(D(H)) \neq 0$, then*

$$\det(D(G)) = -d_{1,2}\det(D(H))(d_{2,1}((D(H))^{-1})_{1,1} + 2).$$

3 θ-graphs

In order to cover all the cases necessary to find a formula for computing the determinant of a distance matrix of a bicyclic graph, it remains to consider those bicyclic graphs with two cycles sharing at least one edge; i.e., those graphs having a θ-graph as induced subgraph. Below, we give details of those cases that need to be considered in order to completely solve the problem:

- $\theta(1, p, q)$, where p and q are even positive integers.
- $\theta(2, 2, q)$, where q is either equal to 2 or an odd positive integer greater than 1.
- $\theta(l, p, q)$ such that:
  · $l = 1$ and at least one of p and q is an odd integer greater than 1,
  · $l = p = 2$ and q is an even integer greater than 3, or
  · $l \geq 2$, $p \geq 3$, and $q \geq 3$.

The idea to find the desired formula in each of the cases is considering the determinant of a graph isomorphic to $\theta(l, p, q)$ and a graph isomorphic to $\theta(l, p, q)$ plus a pendant edge incident to one of its vertices of degree 3. Then, by applying Lemma 2.3, we will be able to compute the determinant of a graph obtained from

$\theta(l,p,q)$ by identifying one vertex of degree one of a path of length m with one of the vertices of degree three on $\theta(l,p,q)$. In the sequel, we will denote such a graph $F(l,p,q,m)$. Notice that Lemma 2.1 guarantees that the determinant of the distance matrix of this graph agrees with the determinant of the distance matrix of any bicyclic graph on $l+p+q+m-1$ vertices, having $\theta(l,p,q)$ as induced subgraph.

It is easy to verify by means of direct calculation that $\det(\theta(2,2,2)) = -16$.

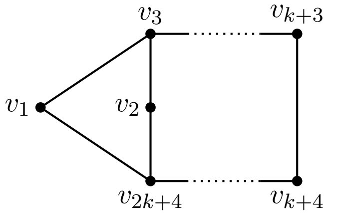

Fig. 2. $\theta(2,2,2k+1)$

Let us label the vertices of $\theta(2,2,2k+1)$ as in Figure 2. Clearly, v_1 and v_2 are twins. Besides, $\theta(2,2,2k+1) - v_1 = C_{2k+3}$, $\det(D(C_{2k+3})) = (k+1)(k+2)$, and $((D(C_{2k+3}))^{-1})_{1,1} = -2 + \frac{2k+3}{(k+1)(k+2)}$ (see [1]). By combining these observations with Corollary 2.6 the result below follows.

Lemma 3.1 *For every positive integer k,* $\det\left(D(\theta(2,2,2k+1))\right) = 4(k^2+k-1)$.

Lemma 3.2 *Let l, p and q be integers satisfying one of the following conditions:*

- *$l = 1$ and, at least, $p = 2k-1$ or $q = 2k-1$ for some $k \geq 2$,*
- *$l = p = 2$ and $q = 2k-2$ for some $k \geq 3$,*
- *$l \geq 2$, $p \geq 3$, and $q \geq 3$.*

Then,

$$det(D(\theta(l,p,q))) = 0.$$

Proof. (Sketch.) We will split the proof into three cases.

(i) We consider the graph $\theta(1,p,q)$, where $p = 2k-1$ or $q = 2k-1$ for some $k \geq 2$, and having its vertices labeled as in Figure 3.

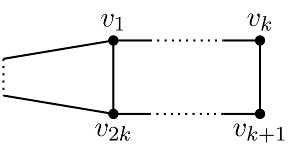

Fig. 3. $\theta(1,p,q)$

(ii) We consider the graph $\theta(2,2,q)$ with its vertices labeled as in Figure 4, where $q = 2k-2$ for some $k \geq 3$.

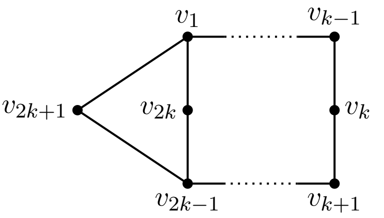

Fig. 4. $\theta(2, 2, q)$

(iii) We consider the graph $\theta(l, p, q)$, where $l \geq 2$, $p \geq 3$ and $q \geq 3$. Clearly, at least two of the three parameters (l, p and q) have the same parity. Hence, $l + p = 2k$, $l + q = 2k$, or $p + q = 2k$, for some $k \geq 3$. Accordingly, two cases should be considered: two of the parameters are even integers or two of the parameters are odd integers (see Figure 5).

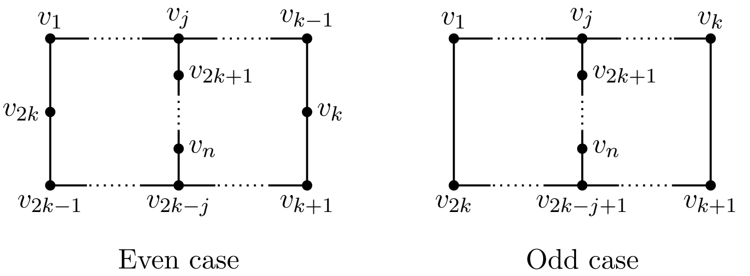

Fig. 5. $\theta(l, p, q)$ with $l \geq 2$, $p \geq 3$ and $q \geq 3$

For all those graphs considered in cases (i) and (iii) (where two paths connecting vertices of degree three have odd lengths) it can be proved that $d(v_1, v_i) - d(v_k, v_i) - d(v_{2k}, v_i) + d(v_{k+1}, v_i) = 0$, for every $1 \leq i \leq n$. Hence the vector $e_1 - e_k + e_{k+1} - e_{2k}$ is an eigenvector associated with the eigenvalue 0 for the distance matrix of these graphs. Analogously, it can be proved that, for all those graphs considered in cases (ii) and (iii) (where two paths connecting vertices of degree three have even lengths), $d(v_1, v_i) - d(v_{k-1}, v_i) - d(v_{2k-1}, v_i) + d(v_{k+1}, v_i) = 0$, for every $1 \leq i \leq n$. Hence the vector $e_1 - e_{k-1} + e_{k+1} - e_{2k-1}$ is an eigenvector associated with the eigenvalue 0 of the distance matrix for these graphs. □

4 θ-graphs plus a pendant vertex

Recall that $F(l, p, q, 1)$ stands for the graph obtained from $\theta(l, p, q)$ by adding a pendant edge incident to one of its vertices of degree 3. Below we give details of those cases to be considered in order to cover all the cases.

- $F(l, p, q, 1)$, where p and q are even positive integers.
- $F(2, 2, q, 1)$, where q is either equal to 2 or an odd positive integer greater than 1.
- $F(l, p, q, 1)$ such that:
  · $l = 1$ and at least one of p and q is an odd integer number greater than 1,

· $l = p = 2$ and q is an even integer greater than 3, or
· $p \geq 2$, $p \geq 3$, and $q \geq 3$.

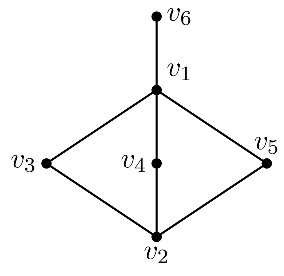

Fig. 6. $F(2, 2, 2, 1)$

Let us label the vertices of $F(2, 2, 2, 1)$ as in Figure 6. Clearly, vertices v_3, v_4 and v_5 satisfy conditions of Lemma 2.5. Hence, by applying this lemma repeatedly, we conclude that $\det(F(2, 2, 2, 1)) = 48$.

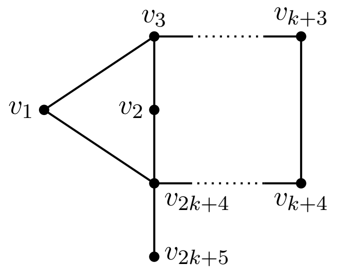

Fig. 7. $F(2, 2, 2k + 1, 1)$

Let us label the vertices of $F(2, 2, 2k + 1, 1)$ as in Figure 7. Clearly, v_1 and v_2 satisfy the hypothesis of Lemma 2.5. In addition, $H = G[\{v_2, \ldots, v_{2k+5}\}]$ is isomorphic to C_{2k+3} plus a pendant vertex, $\det(D(H)) = -2((k + 1)(k + 2) + \frac{2k+3}{2}) = -2k^2 - 8k - 7$ and $(D(H))^{-1})_{1,1} = \frac{-4k^2-12k-4}{2k^2+8k+7}$ (see [1] for more details). By combining these observations with Corollary 2.6 the result below follows.

Lemma 4.1 *Let k be a positive integer, then* $\det(D(F(2, 2, 2k+1, 1))) = 4(-2k^2 - 4k + 3)$.

Lemma 4.2 *Let l, p and q be integers satisfying one of the following conditions:*

- *$l = 1$ and either $p = 2k - 1$ or $q = 2k - 1$ for some $k \geq 2$,*
- *$l = p = 2$ and $q = 2k - 2$ for some $k \geq 3$,*
- *$l \geq 2$, $p \geq 3$, and $q \geq 3$.*

Then, $\det(D(F(l, p, q, 1))) = 0$.

Proof. (Sketch.)

We split the proof into the three cases described below.

(i) We consider the graph $F(1, p, q, 1)$, where $p = 2k - 1$ or $q = 2k - 1$ for some $k \geq 2$; with all its vertices labeled as in Figure 8.

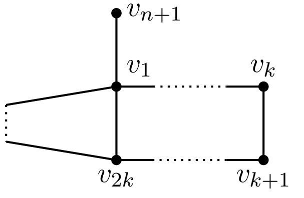

Fig. 8. $F(1, p, q, 1)$

(ii) We consider $F(2, 2, q, 1)$ with its vertices labeled as in Figure 9, where $q = 2k-2$ for some $k \geq 3$.

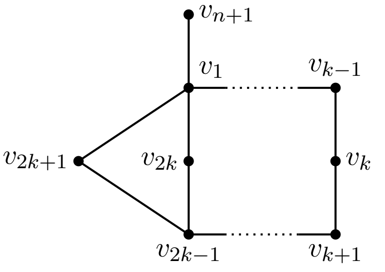

Fig. 9. $F(2, 2, q, 1)$

(iii) We consider $F(l, p, q, 1)$ where $l \geq 2$, $p \geq 3$, and $q \geq 3$. Hence at least two of the three parameter (l, p and q) have the same parity, meaning that at least one of the following conditions holds for some $k \geq 3$: $l+p = 2k$, $l+q = 2k$, or $p+q = 2k$. Accordingly, two cases should be considered: two of the parameters are even integers or two of the parameters are odd integers (see Figure 10).

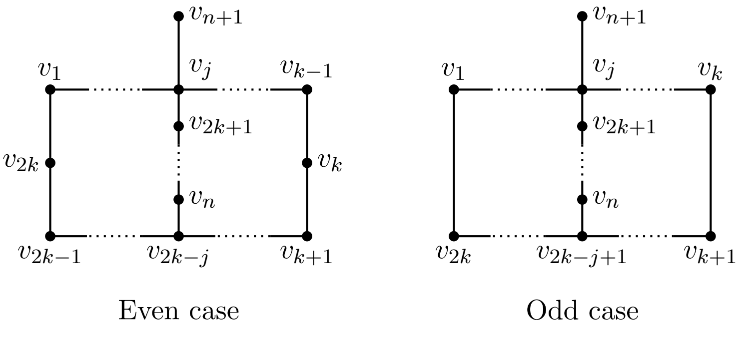

Fig. 10. $F(l, p, q, 1)$, where $l \geq 2$, $p \geq 3$, and $q \geq 3$.

In the cases (i) and (iii) (where the induced subgraph $\theta(l, p, q)$ has two paths connecting vertices of degree three with odd lengths), it can be proved that $d(v_1, v_i) - d(v_k, v_i) - d(v_{2k}, v_i) + d(v_{k+1}, v_i) = 0$ for every $1 \leq i \leq n+1$. Therefore the vector $e_1 - e_k + e_{k+1} - e_{2k}$ belongs to the kernel of the matrix $D(F(l, p, q, 1))$.

For cases (ii) and (iii) (where the induced subgraph $\theta(l, p, q)$ has two paths connecting vertices of degree three with even lengths), it can be proved that $d(v_1, v_i) - d(v_{k-1}, v_i) - d(v_{2k-1}, v_i) + d(v_{k+1}, v_i) = 0$ for all $1 \leq i \leq n+1$. Hence the vector $e_1 - e_{k-1} + e_{k+1} - e_{2k-1}$ belongs to the kernel of $D(F(l, p, q, 1))$. $\square$

5 Determinant of the distance matrix of graphs in $\mathcal{B}_n^\theta$

Recall that graphs in $\mathcal{B}_n^\theta$ have a θ-graph as induced subgraph. By Lemma 2.1, we obtain the following result.

Lemma 5.1 *Let G be a bicyclic graph on $p+q+l+m-1$ vertices having $\theta(p,q,l)$ as an induced subgraph for some integers p, q and l. Then* $\det(D(G)) = \det(D(F(p,q,l,m)))$.

By Lemma 2.3 we know that

$$\det\Big(D(F(p,q,l,m))\Big) = -4\det\Big(D(F(p,q,l,m-1))\Big) - 4\det\Big(D(F(p,q,l,m-2))\Big).$$

Therefore, combining lemmas 2.1 and 5.1 we obtain the following results.

Lemma 5.2 *Let G be a bicyclic graph on $m+p+q+l-1$ vertices having $\theta(p,q,l)$ as an induced subgraph for some integers p, q and l. Then,*

$$\det(D(G)) = \Big(2(m-1)\ \det(D_0) + m\ \det(D_1)\Big)(-2)^{(m-1)},$$

where $D_0 = D(F(p,q,l,0))$ and $D_1 = F(G(p,q,l,1))$.

Notice that $F(p,q,l{,}0) = \theta(l,p,q)$.

Summarizing, we were able to compute the determinant of those bicyclic graphs enumerated in the following result, whose proof is obtained by combining Lemma 5.2 with results and observations presented in Sections 3 and 4.

Theorem 5.3 *Let G be a bicyclic on $m+p+q+l-1$ vertices containing $\theta(p,q,l)$ as an induced subgraph for some integers p, q and l. Then:*

(i) *If $l=p=q=2$, then* $\det(D(G)) = -8(m+2)(-2)^m$.

(ii) *If $l=p=2$ and q is an odd positive integer greater than 1, then* $\det(D(G)) = (n^2-6n+4+2m(n-5))(-2)^m$, *where $n=p+q+l-1$.*

(iii) $\det(D(G)) = 0$ *whenever one of the following conditions holds:*
  - (a) *$l=1$ and at least one of p and q is an odd integer greater than 1;*
  - (b) *$l=p=2$ and q is an even integer greater than 3; or*
  - (c) *$l \geq 2$, $p \geq 3$, and $q \geq 3$.*

6 Conjectures

We leave unsolved the problem of computing the determinant of those bicyclic graphs on $l+p+q+m-1$ vertices having $\theta(l,p,q)$ as induced sugraph such that $l=1$ and p and q are even integers. Nevertheless, in those cases we have strong evidence, supported by numerical experiments performed in Sage, to conjecture that $\det(D(\theta(1,p,q)) = -\frac{(p+q)^2}{4}$ and $\det(D(F(1,p,q,1)) = \frac{(p+q+1)^2-1}{2}$, having considered all respective graphs up to 500 vertices. If our conjecture were true we could conclude the following.

Conjecture 6.1 *Let G be a bicyclic graph on $p+q+m$ vertices containing $\theta(1,p,q)$ as induced sugraph, such that p and q are even positive integers. Then,* $\det(D(G)) = -n(n+2m)(-2)^{m-2}$, *where $n = p+q$.*

If Conjecture 6.1 were true, then from [3, Theorem 3.4] and Theorem 5.3, it would follow that the determinant of a bicyclic graph only depends on the number of vertices and the length of its cycles.

Acknowledgement

E. Dratman, L.N. Grippo, and M.D. Safe acknowledge partial support from CONICET UNGS-144-20140100027-CO. L.N. Grippo and M.D. Safe acknowledge partial support from ANPCyT PICT 2017-1315 and UNS Grant PGI 24/L103. Safe acknowledges partial support from MATH-AmSud 18-MATH-01.Renata R. Del-Vecchio acknowledge partial support from CNPq Grants 408494/2016-6 and 305411/2016-0.

References

[1] Bapat, R., S.J. Kirkland, and M. Neumann, *On distance matrices and Laplacians*, Linear Algebra Appl. **401** (2005), 193–209.

[2] Bapat, R., and S. Sivasubramanian *Inverse of the distance matrix of a block graph*, Linear Multilinear Algebra **59** (2011), 1393–1397.

[3] Gong, S-C., J-L. Zhang, and G-H. Xu, *On the determinant of the distance matrix of a bicyclic graph*, arXiv:1308.2281v1 (2013).

[4] Graham, R.L., and H.O. Pollack, *On the addressing problem for loop switching*, Bell System Tech. J. **50** (1971), 2495–2519.

[5] Graham, R.L., A.J. Hoffman and H. Hosoya, *On the distance matrix of a directed graph*, J. Graph Theory, **1** (1977), 85–88.

[6] Graham, R.L., and L. Lovász, *Distance matrix polynomials of trees*, Adv. Math. **29** (1978), 60–88.