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Characterizing paths graphs on bounded degree trees by minimal forbidden induced subgraphs

L. Alcón [a], M. Gutierrez [a,b], M.P. Mazzoleni [a,b,*]

[a] Departamento de Matemática, Universidad Nacional de La Plata, CC 172, (1900) La Plata, Argentina

[b] CONICET, Argentina

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ABSTRACT

An undirected graph G is called a VPT graph if it is the vertex intersection graph of a family of paths in a tree. The class of graphs which admit a VPT representation in a host tree with maximum degree at most h is denoted by $[h, 2, 1]$. The classes $[h, 2, 1]$ are closed under taking induced subgraphs, therefore each one can be characterized by a family of minimal forbidden induced subgraphs. In this paper we associate the minimal forbidden induced subgraphs for $[h, 2, 1]$ which are VPT with (color) h-critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of VPT minimal forbidden induced subgraphs for $[h, 2, 1]$. The members of this family together with the minimal forbidden induced subgraphs for VPT (Lévêque et al., 2009; Tondato, 2009), are the minimal forbidden induced subgraphs for $[h, 2, 1]$, with $h \geq 3$. By taking $h = 3$ we obtain a characterization by minimal forbidden induced subgraphs of the class VPT $\cap$ EPT $=$ EPT $\cap$ Chordal $= [3, 2, 2] = [3, 2, 1]$ (see Golumbic and Jamison, 1985).

1. Introduction

The **intersection graph** of a family is a graph whose vertices are the members of the family, and the adjacency between vertices is defined by a non-empty intersection of the corresponding sets. Classic examples are interval graphs and chordal graphs.

An **interval graph** is the intersection graph of a family of intervals of the real line, or, equivalently, the vertex intersection graph of a family of subpaths of a path. A **chordal graph** is a graph without chordless cycles of length at least four. Gavril [5] proved that a graph is chordal if and only if it is the vertex intersection graph of a family of subtrees of a tree. Both classes have been widely studied [3].

In order to allow larger families of graphs to be represented by subtrees, several graph classes are defined imposing conditions on trees, subtrees and intersection sizes [9,10]. Let h, s and t be positive integers; an (h, s, t)-representation of a graph G consists in a host tree T and a collection $(T_v)_{v \in V(G)}$ of subtrees of T, such that (i) the maximum degree of T is at most h, (ii) every subtree T_v has maximum degree at most s, and (iii) two vertices v and v' are adjacent in G if and only if the corresponding subtrees T_v and $T_{v'}$ have at least t vertices in common in T. The class of graphs that have an (h, s, t)-representation is denoted by $[h, s, t]$. When there is no restriction on the maximum degree of T or on the maximum degree of the subtrees, we use $h = \infty$ and $s = \infty$ respectively. Therefore, $[\infty, \infty, 1]$ is the class of chordal graphs and $[2, 2, 1]$ is the class of interval graphs. The classes $[\infty, 2, 1]$ and $[\infty, 2, 2]$ are called VPT and EPT respectively in [7]; and UV and UE, respectively in [13].

* Corresponding author at: Departamento de Matemática, Universidad Nacional de La Plata, CC 172, (1900) La Plata, Argentina.
E-mail addresses: liliana@mate.unlp.edu.ar (L. Alcón), marisa@mate.unlp.edu.ar (M. Gutierrez), pia@mate.unlp.edu.ar (M.P. Mazzoleni).

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In [6,14], it is shown that the problem of recognizing VPT graphs is polynomial time solvable. Recently, in [1], generalizing a result given in [7], we have proved that the problem of deciding whether a given VPT graph belongs to $[h, 2, 1]$ is NP-complete even when restricted to the class VPT ∩ Split without dominated stable vertices. The classes $[h, 2, 1]$, $h \geq 2$, are closed under taking induced subgraphs; therefore each one can be characterized by a family of minimal forbidden induced subgraphs. Such a family is known only for $h = 2$ [11] and there are some partial results for $h = 3$ [4]. In this paper we associate the VPT minimal forbidden induced subgraphs for $[h, 2, 1]$ with (color) h-critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of VPT minimal forbidden induced subgraphs for $[h, 2, 1]$. The members of this family together with the minimal forbidden induced subgraphs for VPT (see Fig. 2) which were determined in [12,15], are the minimal forbidden induced subgraphs for $[h, 2, 1]$, with $h \geq 3$. Notice that by taking $h = 3$ we obtain a characterization by minimal forbidden induced subgraphs of the class VPT ∩ EPT = EPT ∩ Chordal = $[3, 2, 2]$ = $[3, 2, 1]$ [7].

The paper is organized as follows: in Section 2, we provide basic definitions and basic results. In Section 3, we give necessary conditions for VPT minimal non-$[h, 2, 1]$ graphs. In Section 4, we show a procedure to construct minimal non-$[h, 2, 1]$ graphs. In Section 5, we describe the family of all minimal non-$[h, 2, 1]$ graphs.

2. Preliminaries

Throughout this paper, graphs are connected, finite and simple. The **vertex set** and the **edge set** of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The **open neighborhood** of a vertex v, represented by $N_G(v)$, is the set of vertices adjacent to v. The **closed neighborhood** $N_G[v]$ is $N_G(v) \cup \{v\}$. The **degree** of v, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. For simplicity, when no confusion can arise, we omit the subindex G and write $N(v)$, $N[v]$ or $d(v)$. Two vertices $x, y \in V(G)$ are called **true twins** if $N[x] = N[y]$.

A **complete set** is a subset of mutually adjacent vertices. A **clique** is a maximal complete set. The family of cliques of G is denoted by $\mathcal{C}(G)$. A **stable set**, also called an independent set, is a subset of pairwise non-adjacent vertices.

A graph G is **k-colorable** if its vertices can be colored with at most k colors in such a way that no two adjacent vertices share the same color. The **chromatic number** of G, denoted by $\chi(G)$, is the smallest k such that G is k-colorable. A vertex $v \in V(G)$ or an edge $e \in E(G)$ is a **critical element** of G if $\chi(G - v) < \chi(G)$ or $\chi(G - e) < \chi(G)$ respectively. A graph G with chromatic number h is **h-vertex critical** (resp. **h-critical**) if each of its vertices (resp. edges) is a critical element.

A **VPT representation** of G is a pair $\langle \mathcal{P}, T\rangle$ where $\mathcal{P}$ is a family $(P_v)_{v\in V(G)}$ of subpaths of a host tree T satisfying that two vertices v and v' of G are adjacent if and only if P_v and $P_{v'}$ have at least one vertex in common; in such case we say that P_v intersects $P_{v'}$. When the maximum degree of the host tree is at most h the VPT representation of G is called an $(h, 2, 1)$-representation of G. The class of graphs which admit an $(h, 2, 1)$-representation is denoted by **[h, 2, 1]**.

Since a family of vertex paths in a tree satisfies the Helly property [2], if C is a clique of G then there exists a vertex q of T such that $C = \{v \in V(G) \mid q \in V(P_v)\}$. On the other hand, if q is any vertex of the host tree T, the set $\{v \in V(G) \mid q \in V(P_v)\}$, denoted by $\mathbf{C_q}$, is a complete set of G, but not necessarily a clique. In order to avoid this drawback, we introduce the notion of full representation at q.

Let $\langle \mathcal{P}, T\rangle$ be a VPT representation of G and let q be a vertex of T with degree h. The connected components of $T - q$ are called the **branches of T at q**. A path is **contained** in a branch if all its vertices are vertices of the branch. Notice that if $N_T(q) = \{q_1, q_2, \dots, q_h\}$, then T has exactly h branches at q. The branch containing q_i is denoted by $\mathbf{T_i}$. Two branches T_i and T_j are **linked** by a path $P_v \in \mathcal{P}$ if both vertices q_i and q_j belong to $V(P_v)$.

Definition 1. A VPT representation $\langle \mathcal{P}, T\rangle$ is **full at a vertex q** of T if, for every two branches T_i and T_j of T at q, there exist paths $P_v, P_w, P_u \in \mathcal{P}$ such that: (i) the branches T_i and T_j are linked by P_v; (ii) P_w is contained in T_i and intersects P_v in at least one vertex; and (iii) P_u is contained in T_j and intersects P_v in at least one vertex.

A clear consequence of the previous definition is that if $\langle \mathcal{P}, T\rangle$ is full at a vertex q of T, with $d_T(q) = h \geq 3$, then C_q is a clique of G.

The following theorem from [1] shows that any VPT representation which is not full at some vertex q of T with $d_T(q) \geq 4$ can be modified to obtain a new VPT representation without increasing the maximum degree of the host tree; while decreasing the degree of the vertex q.

Theorem 2 *([1]). Let $\langle \mathcal{P}, T\rangle$ be a VPT representation of G. Assume there exists a vertex $q \in V(T)$ with $d_T(q) = h \geq 4$ and two branches of T at q which are linked by no path of $\mathcal{P}$. Then there exists a VPT representation $\langle \mathcal{P}', T'\rangle$ of G with $V(T') = V(T) \cup \{q'\}$, $q' \notin V(T)$, and*

$$d_{T'}(x) = \begin{cases} 3 & \text{if } x = q'; \\ h - 1 & \text{if } x = q; \\ d_T(x) & \text{if } x \in V(T') \setminus \{q, q'\}. \end{cases}$$

Branch graphs defined below are used in the following results to describe intrinsic properties of VPT representations.

Definition 3 ([7]). Let $C \in \mathcal{C}(G)$. The **branch graph** of G for the clique C, denoted by $\mathbf{B(G/C)}$, is defined as follows: its vertices are the vertices of $V(G) \setminus C$ which are adjacent to at least one vertex of C. Two vertices u and v are adjacent in $B(G/C)$ if and only if

(1) $uv \notin E(G)$;
(2) there exists a vertex $x \in C$ such that $xu \in E(G)$ and $xv \in E(G)$;
(3) there exists a vertex $y \in C$ such that $yu \in E(G)$ and $yv \notin E(G)$;
(4) there exists a vertex $z \in C$ such that $zu \notin E(G)$ and $zv \in E(G)$.

It is clear that if $C \in \mathcal{C}(G)$ and $v \in V(G) - C$, then $C \in \mathcal{C}(G - v)$. The following claim, whose proof is trivial, describes the branch graph of $G - v$ for the clique C in terms of $B(G/C)$.

Claim 4. *Let $C \in \mathcal{C}(G)$ and $v \in V(G) - C$:* (i) *If $v \notin V(B(G/C))$ then $B((G - v)/C) = B(G/C)$;* (ii) *if $v \in V(B(G/C))$ then $B((G - v)/C) = B(G/C) - v$.*

Lemma 5 ([1]). *Let C be a clique of a VPT graph G, $\langle \mathcal{P}, T\rangle$ be a VPT representation of G and q be a vertex of T such that $C = C_q$. If v is a vertex of $B(G/C)$, then P_v is contained in some branch of T at q. If two vertices u and v are adjacent in $B(G/C)$, then P_u and P_v are not contained in the same branch of T at q.*

In [1] we proved the following two results which show that there is a relationship between the VPT graphs that can be represented in a tree with maximum degree at most h and the chromatic number of their branch graphs.

Lemma 6 ([1]). *Let C be a clique of a VPT graph G, $\langle \mathcal{P}, T\rangle$ be a VPT representation of G and q be a vertex of T such that $C = C_q$. If $d_T(q) = h$, then $B(G/C)$ is h-colorable.*

Theorem 7 ([1]). *Let $G \in VPT$ and $h \geq 4$. The graph G belongs to $[h, 2, 1] - [h-1, 2, 1]$ if and only if $\max_{C\in\mathcal{C}(G)}(\chi(B(G/C))) = h$. The reciprocal implication is also true for $h = 3$.*

Definition 8. A clique K of a graph G is called **principal** if

$$\max_{C\in\mathcal{C}(G)}(\chi(B(G/C))) = \chi(B(G/K)).$$

A graph G is **split** if $V(G)$ can be partitioned into a stable set S and a clique K. The pair $(\mathbf{S}, \mathbf{K})$ is the **split partition** of G and this partition is unique up to isomorphisms. The vertices in S are called **stable vertices**, and K is called the **central clique** of G. We say that a vertex s is **a dominated stable vertex** if $s \in S$ and there exists $s' \in S$ such that $N(s) \subseteq N(s')$. Notice that if G is split then $\mathcal{C}(G) = \{K\} \cup \{N[s] \mid s \in S\}$. We will write **Split** for the class of split graphs.

Lemma 9. *If (S, K) is the split partition of $G \in VPT \cap Split$, then K is a principal clique of G.*

Proof. Let $s \in S$, we know that $N[s] \in \mathcal{C}(G)$. Observe that $V(B(G/N[s])) = (K - N(s)) \cup S'$, with $S' = \{x \in S \mid N(x) \cap N(s) \neq \emptyset\}$. We claim that the vertices of $K - N(s)$ are isolated in $B(G/N[s])$. Indeed, let $x \in K - N(s)$; if $y \in K - N(s)$, then $xy \notin E(B(G/N[s]))$ because $xy \in E(G)$ and, if $y \in S'$ then $xy \notin E(B(G/N[s]))$ because $N(y) \subseteq N[x]$. Then, the chromatic number of $B(G/N[s])$ is equal to the chromatic number of the subgraph of $B(G/N[s])$ induced by S', which clearly is a subgraph of $B(G/K)$. Thus $\chi(B(G/N[s])) \leq \chi(B(G/K))$. Hence $\max_{C\in\mathcal{C}(G)}(\chi(B(G/C))) = \chi(B(G/K))$, that is, K is a principal clique of G. □

3. Necessary conditions for VPT minimal non-[h, 2, 1] graphs

In this section we give necessary conditions for being a VPT minimal non-$[h, 2, 1]$ graph; recall that:

Definition 10. A **minimal non-[h,2,1] graph** is a minimal forbidden induced subgraph for the class $[h, 2, 1]$, this means any graph G such that $G \notin [h, 2, 1]$ and $G - v \in [h, 2, 1]$ for every vertex $v \in V(G)$.

Theorem 11. *Let $G \in VPT$ and let $h \geq 3$. If G is a minimal non-$[h, 2, 1]$ graph, then $G \in [h + 1, 2, 1]$.*

Proof. Let $C \in \mathcal{C}(G)$ and let $v \notin C$. We know that $G - v \in [h, 2, 1]$ then, by Theorem 7, $\chi(B((G - v)/C)) \leq h$. By Claim 4, $\chi(B((G - v)/C)) \geq \chi(B(G/C) - v) \geq \chi(B(G/C)) - 1$. Thus $\chi(B(G/C)) - 1 \leq h$ and hence $\chi(B(G/C)) \leq h + 1$. Then, by Theorem 7, $G \in [h + 1, 2, 1]$. □

Theorem 12. *Let K be a principal clique of a VPT minimal non-$[h, 2, 1]$ graph G, with $h \geq 3$. Then:* (i) *$V(B(G/K)) = V(G) - K$;* (ii) *if $v \in V(G) - K$ then $|N(v) \cap K| > 1$;* (iii) *$B(G/K)$ is $(h + 1)$-vertex critical;* (iv) *if $s_1, s_2 \in V(G) - K$ then $N(s_1) \cap K \neq N(s_2) \cap K$.*

Proof. By Theorem 11, $G \in [h+1, 2, 1]$. Then, by Theorem 7, since K is a principal clique of G, we have that

$$\chi(B(G/K)) = h + 1. \tag{1}$$

(i) It is clear that $V(B(G/K)) \subseteq V(G) - K$. Suppose there exists $v \in V(G) - K$ such that $v \notin V(B(G/K))$. Thus, by Claim 4, $B((G-v)/K) = B(G/K)$. Since G is a minimal non-$[h, 2, 1]$ graph, $G - v \in [h, 2, 1]$ and, by Theorem 7, $B((G-v)/K)$ is h-colorable. Thus $B(G/K)$ is h-colorable which contradicts the condition (1).

(ii) By item (i), we know that $v \in V(B(G/K))$; then $|N(v) \cap K| \geq 1$. If $|N(v) \cap K| = 1$, v would be an isolated vertex of $B(G/K)$ and $\chi(B(G/K)) = \chi(B(G/K) - v)$. But, by Claim 4 and Theorem 7, $\chi(B(G/K)) = \chi(B(G/K) - v) = \chi(B((G-v)/K)) = h$, which also contradicts the condition (1).

(iii) By the condition (1), $\chi(B(G/K)) = h + 1$; assume, in order to obtain a contradiction, that $B(G/K)$ is not $(h+1)$-vertex critical; then there exists $v \in V(B(G/K))$ such that $\chi(B(G/K) - v) = h + 1$. Then, since $v \in V(B(G/K))$, by Claim 4, $\chi(B((G-v)/K)) = \chi(B(G/K) - v) = h + 1$, which contradicts the fact that G is a minimal non-$[h, 2, 1]$ graph.

(iv) We will see that if $N(s_1) \cap K = N(s_2) \cap K$ then $s_1 s_2 \notin E(B(G/K))$ and $N_{B(G/K)}(s_1) = N_{B(G/K)}(s_2)$, which will contradict the fact that $B(G/K)$ is $(h+1)$-vertex critical. Indeed, if $N(s_1) \cap K = N(s_2) \cap K$ then $s_1 s_2 \notin E(B(G/K))$ by the definition of branch graph. Moreover, if $s_3 \in N_{B(G/K)}(s_1)$ then there exist $k_1, k_2, k_3 \in K$ such that: $k_1 s_1 \in E(G)$, $k_1 s_3 \in E(G)$; $k_2 s_1 \in E(G)$, $k_2 s_3 \notin E(G)$; $k_3 s_1 \notin E(G)$, $k_3 s_3 \in E(G)$. And, since $N(s_1) \cap K = N(s_2) \cap K$, it follows that $k_1 s_2 \in E(G)$, $k_2 s_2 \in E(G)$, $k_3 s_2 \notin E(G)$. In addition, $s_3 s_2 \notin E(G)$ because in other case there would be an induced 4-cycle $\{s_2, k_2, k_3, s_3\}$ in G, contradicting the fact that $G \in$ VPT (see Fig. 2). Hence $s_3 \in N_{B(G/K)}(s_2)$; we have proved that $N_{B(G/K)}(s_1) \subseteq N_{B(G/K)}(s_2)$. By symmetry, it is easy to see that $N_{B(G/K)}(s_2) \subseteq N_{B(G/K)}(s_1)$. □

The following lemma and definition are used in the proof of Theorem 15 which states that any VPT minimal non-$[h, 2, 1]$ graph is split and has no dominated stable vertices.

Lemma 13. *Let K be a principal clique of a VPT minimal non-$[h, 2, 1]$ graph G, with $h \geq 3$. Then, $K - \{k\} \in \mathcal{C}(G - k)$ for all $k \in K$.*

Proof. Let $\langle \mathcal{P}, T \rangle$ be an $(h+1, 2, 1)$-representation of G and let $q \in V(T)$ such that $K = C_q$. We claim that $\langle \mathcal{P}, T \rangle$ is full at q. Indeed, suppose, for a contradiction, that $\langle \mathcal{P}, T \rangle$ is not full at q. We can assume, without loss of generality, that if x is an end vertex of a path $P_v \in \mathcal{P}$ then there exists a path $P_u \in \mathcal{P}$ intersecting P_v only in x, in other case the vertex x can be removed from P_v. This implies that any path of $\mathcal{P}$ linking two branches intersects paths contained in those branches. Hence since $\langle \mathcal{P}, T \rangle$ is not full at q, there exist branches T_i and T_j of T at q which are linked by no path of $\mathcal{P}$. Then, by Theorem 2, we can obtain a new VPT representation $\langle \mathcal{P}', T' \rangle$ of G with $d_{T'}(q) \leq h$. Thus, by Lemma 6, $B(G/C_q)$ is h-colorable which contradicts the fact that C_q is a principal clique of G.

Hence since $\langle \mathcal{P}, T \rangle$ is full at q, every pair of branches of T at q are linked by a path of $\mathcal{P}$. If there exists $k \in C_q$ such that $C_q - \{k\}$ is not a clique of $G - k$, there must exists $v \in V(G) - C_q$ such that v is adjacent to all the vertices of $C_q - \{k\}$. Let $T_1, T_2, \ldots, T_{h+1}$ be the branches of T at q. Assume, without loss of generality, that P_k is contained in $\{q\} \cup T_1 \cup T_2$. Since $v \in V(G) - C_q$, there exists i such that P_v is contained in T_i. And, since $h \geq 3$, there exists a branch T_s, with $s \neq 1, 2, i$. Let P_u be the path of $\mathcal{P}$ linking T_s and T_r, with $r \neq i$. It is clear that $u \in C_q$ and v is not adjacent to u, which contradicts the fact that v is adjacent to all the vertices of $C_q - \{k\}$. Thus $C_q - \{k\} \in \mathcal{C}(G - k)$. □

Definition 14. A **canonical VPT representation** of G is a pair $\langle \mathcal{P}, T \rangle$ where T is a tree whose vertices are the members of $\mathcal{C}(G)$, $\mathcal{P}$ is the family $(P_v)_{v \in V(G)}$ with $P_v = \{C \in \mathcal{C}(G) \mid v \in C\}$ and P_v is a subpath of T for all $v \in V(G)$.

In [13] it was proved that every VPT graph admits a canonical VPT representation.

Theorem 15. *Let G be a VPT graph and let $h \geq 3$. If G is a minimal non-$[h, 2, 1]$ graph, then $G \in$ Split without dominated stable vertices.*

Proof. Case (1): Suppose that $G \in$ Split with split partition (S, K), and G has dominated stable vertices. Let $\langle \mathcal{P}, T \rangle$ be a canonical VPT representation of G, and let $q \in V(T)$ such that $K = C_q$. Assume that $N_T(q) = \{q_1, q_2, \ldots, q_k\}$, with $k > h$, and call $T_1, T_2, \ldots, T_k$ to the branches of T at q containing the vertices $q_1, q_2, \ldots, q_k$ respectively. It is clear that for each q_i, with $1 \leq i \leq k$, there exists $P_{w_i} \in \mathcal{P}$ such that $q_i \in V(P_{w_i})$ and $q \notin V(P_{w_i})$. Notice that every $w_i \in S$.

Suppose that $S = \{w_1, w_2, \ldots, w_k\}$. Since G has dominated stable vertices, by item (iv) of Theorem 12 we can assume, without loss of generality, that $N(w_1) \subsetneqq N(w_2)$. This means that w_1 and w_2 are not adjacent in $B(G/C_q)$; thus, by item (iii) of Theorem 12, $N_{B(G/C_q)}(w_1) \not\subseteq N_{B(G/C_q)}(w_2)$. Hence there exists $l \in V(B(G/C_q)) - \{w_1, w_2\}$, such that $l \in N_{B(G/C_q)}(w_1) - N_{B(G/C_q)}(w_2)$. Since $V(B(G/C_q)) = S$ we can assume that $l = w_3$. Then, by the definition of branch graph, there exists $z \in C_q$ such that $z w_1 \in E(G)$, $z w_3 \in E(G)$ and, since $N(w_1) \subsetneqq N(w_2)$, $z w_2 \in E(G)$, which implies that P_z contains the vertices q_1, q_2 and q_3. Then P_z is not a path. This contradicts the fact that $\langle \mathcal{P}, T \rangle$ is a VPT representation of G.

We conclude that $S' = S - \{w_1, w_2, \ldots, w_k\} \neq \emptyset$. Let $G' = G - S'$. Notice that $C_q \in \mathcal{C}(G')$ and $V(B(G'/C_q)) = \{w_1, w_2, \ldots, w_k\}$. Since G is a minimal non-$[h, 2, 1]$ graph, then $G' \in [h, 2, 1]$ and $\chi(B(G'/C_q)) \leq h$.

We claim that there exists an h-coloring of $B(G'/C_q)$ such that if there exists $x \in C_q$ and $w_i, w_j \in \{w_1, w_2, \ldots, w_k\}$ with $x w_i \in E(G)$, $x w_j \in E(G)$ then

$$w_i \text{ and } w_j \text{ have different colors in } B(G'/C_q). \tag{2}$$

Indeed, if w_i and w_j have the same color in $B(G'/C_q)$ then $w_iw_j \notin E(B(G'/C_q))$. Then we can assume that $N(w_i) \subseteq N(w_j)$, since, by hypothesis, there exists $x \in C_q$ such that $xw_i \in E(G)$ and $xw_j \in E(G)$; thus, for any $s \neq j$, no vertex of C_q is adjacent to w_i and w_s. This implies that w_i is an isolated vertex of $B(G'/C_q)$. Therefore, we can change the color of w_i to either of the $h-1$ remaining colors. This process can be repeated until we have the desired h-coloring of $B(G'/C_q)$.

Hence we consider an h-coloring, say c', of $B(G'/C_q)$ satisfying the condition (2).

Now, we give an h-coloring, denoted c, of $B(G/C_q)$ as follows: given $w \in V(B(G/C_q))$, by Lemma 5, there exists $1 \leq i \leq k$ such that P_w is contained in T_i, we define $c(w) = c'(w_i)$. Notice that, in particular, $c(w_i) = c'(w_i)$.

We will see that c is a proper coloring of $B(G/C_q)$. That is, we have to see that if $uv \in E(B(G/C_q))$ then $c(u) \neq c(v)$. Since $uv \in E(B(G/C_q))$, by Lemma 5, P_u and P_v are in different branches of T at q say T_i and T_j. Moreover, there exists $x \in C_q$ such that $xu \in E(G)$ and $xv \in E(G)$, but this implies that $xw_i \in E(G)$ and $xw_j \in E(G)$. Hence since our coloring satisfies the condition (2), $c'(w_i) \neq c'(w_j)$. Thus $c(u) \neq c(v)$. Therefore, our coloring is proper.

Thus we have an h-coloring of $B(G/C_q)$ which contradicts the fact that C_q is a principal clique of G. We conclude that, if $G \in$ Split then G has no dominated stable vertices.

Case (2): Suppose that $G \notin$ Split. Since G is a minimal non-$[h, 2, 1]$ graph, by Theorem 11, $G \in [h+1, 2, 1]$. Let $\langle \mathcal{P}, T\rangle$ be an $(h+1, 2, 1)$-representation of G and let $q \in V(T)$ such that C_q is a principal clique of G. We obtain from G a new graph $\tilde{G}$ with $V(\tilde{G}) = V(G)$, as follows.

$\tilde{G}$ has the $(h+1, 2, 1)$-representation $\langle \mathcal{P}', T\rangle$, $\mathcal{P}' = (P'_v)_{v\in V(G)}$, given by:

$$P'_v = \begin{cases} P_v & \text{if } v \in C_q; \\ \{q_v\} & \text{if } v \in V(G) - C_q, \text{ where } q_v \text{ is the vertex of } P_v \text{ closest to } q\,. \end{cases}$$

We will prove that $\tilde{G}$ is a split graph, $\tilde{G} \notin [h, 2, 1]$ and $\tilde{G}$ has dominated stable vertices.

It is clear that C_q is a clique of $\tilde{G}$. We claim that $(V(G) - C_q, C_q)$ is a split partition of $\tilde{G}$. Indeed, if $x, y \in V(G) - C_q$ and $xy \in E(\tilde{G})$ then $q_x = q_y$. Thus $N_G(x) \cap C_q = N_G(y) \cap C_q$ which contradicts item (iv) of Theorem 12.

By Lemma 9, C_q is a principal clique of $\tilde{G}$. Then, to see that $\tilde{G} \notin [h, 2, 1]$ it is enough to see that $B(\tilde{G}/C_q) = B(G/C_q)$ because in such case $\chi(B(\tilde{G}/C_q)) = \chi(B(G/C_q)) = h+1$, and it follows by Theorem 7.

It is straightforward to show that $V(B(\tilde{G}/C_q)) = V(B(G/C_q))$ and that $E(B(G/C_q)) \subseteq E(B(\tilde{G}/C_q))$. On the other hand, notice that if $xy \in E(B(\tilde{G}/C_q))$ and $xy \notin E(B(G/C_q))$ then $xy \in E(G)$. Thus, without loss of generality, we can assume that $N_G(x)\cap C_q \subseteq N_G(y)\cap C_q$ which contradicts the fact that x and y are adjacent in $B(\tilde{G}/C_q)$. We conclude that $B(\tilde{G}/C_q) = B(G/C_q)$.

Now, to see that $\tilde{G}$ has dominated stable vertices, let x and y be vertices of $V(\tilde{G}) - C_q$ adjacent in G, they exist since $G \notin$ Split. Since $xy \in E(G)$, we can assume, without loss of generality, that q_x lies on the path of T between q and q_y, which implies that x dominates y in $\tilde{G}$.

Then, by Case (1), $\tilde{G}$ is not a minimal non-$[h, 2, 1]$ graph. Thus there exists $v \in V(\tilde{G})$ such that $(\tilde{G} - v) \in [h+1, 2, 1]$.

If $v \in V(B(\tilde{G}/C_q))$, then $\chi(B((\tilde{G} - v)/C_q)) = h+1$. Moreover, by Claim 4 and since $B(\tilde{G}/C_q) = B(G/C_q)$, we have that $B((\tilde{G} - v)/C_q) = B(\tilde{G}/C_q) - v = B(G/C_q) - v = B((G - v)/C_q)$. Hence $\chi(B((G - v)/C_q)) = h+1$ which contradicts the fact that G is a minimal non-$[h, 2, 1]$ graph.

If $v \in C_q$, then, by Lemma 13, $C_q - v \in \mathcal{C}(G - v)$; therefore $C_q - v \in \mathcal{C}(\tilde{G} - v)$. Thus $\tilde{G} - v \in$ Split with split partition $(V(G)-C_q, C_q-v)$. Then, by Lemma 9, C_q-v is a principal clique of $\tilde{G}-v$. Hence $\chi(B((\tilde{G}-v)/(C_q-v))) = h+1$. Moreover, it is easy to see that $B((\tilde{G}-v)/(C_q-v)) = B((G-v)/(C_q-v))$; thus $\chi(B((\tilde{G}-v)/(C_q-v))) = \chi(B((G-v)/(C_q-v))) = h+1$ which contradicts the fact that G is a minimal non-$[h, 2, 1]$ graph.

We conclude that $G \in$ Split. □

In Theorem 12 we gave some necessary conditions on the branch graph with respect to a principal clique of a minimal non-$[h, 2, 1]$ graph. In what follows, in Theorem 16, using the fact that all minimal non-$[h, 2, 1]$ graphs are split without dominated stable vertices and the fact that the central clique of a split graph is principal, we will give more necessary conditions for minimal non-$[h, 2, 1]$ graphs.

Theorem 16. *Let G be a VPT graph and let $h \geq 3$. If G is a minimal non-$[h, 2, 1]$ graph with split partition (S, K) then:* (i) *for all $k \in K$, $|N(k) \cap S| = 2$;* (ii) *$|E(B(G/K))| = |K|$;* (iii) *$B(G/K)$ is $(h+1)$-critical.*

Proof. By Theorem 15, $G \in$ Split without dominated stable vertices. Let (S, K) be the split partition of G. By Lemma 9 K is a principal clique of G, by Theorem 11 $\chi(B(G/K)) = h+1$ and by item (i) of Theorem 12 $V(B(G/K)) = S$.

(i) Since $G \in$ VPT $\cap$ Split without dominated stable vertices, $|N(k) \cap S| \leq 2$ for all $k \in K$. Suppose there exists $k' \in K$ such that $|N(k') \cap S| < 2$.

By Theorem 11, $G \in [h+1, 2, 1]$. Let $\langle \mathcal{P}, T\rangle$ be an $(h+1, 2, 1)$-representation of G and let $q \in V(T)$ such that $K = C_q$. By Lemma 13, $C_q - \{k'\} \in \mathcal{C}(G - k')$.

1. If $|N(k') \cap S| = 0$: Then $B((G - k')/(C_q - \{k'\})) = B(G/C_q)$. Thus $\chi(B((G - k')/(C_q - \{k'\}))) = \chi(B(G/C_q)) = h+1$, which contradicts the fact that G is a minimal non-$[h, 2, 1]$ graph.

2. If $|N(k') \cap S| = 1$: We will see that $B((G - k')/(C_q - \{k'\})) = B(G/C_q)$. It is clear, by item (ii) of Theorem 12, that $V(B((G - k')/(C_q - \{k'\}))) = V(B(G/C_q))$ and $E(B((G - k')/(C_q - \{k'\}))) \subseteq E(B(G/C_q))$. Let $uv \in E(B(G/C_q))$ such that $uv \notin E(B((G - k')/(C_q - \{k'\})))$. Since $|N(k') \cap S| = 1$ we can assume, without loss of generality, that $(N(v) \cap C_q) - (N(u) \cap C_q) = \{k'\}$. Therefore, we have that $N_{B(G/C_q)}(v) = \{u\}$, because if there is $w \neq u$ such that $w \in N_{B(G/C_q)}(v)$ then there is $\tilde{k} \in C_q$ such that $\tilde{k}w \in E(G)$, $\tilde{k}v \in E(G)$. And, so $\tilde{k}$ is not k' since $|N(k') \cap S| = 1$; thus $\tilde{k}u \in E(G)$, because $(N(v) \cap C_q) - (N(u) \cap C_q) = \{k'\}$. But then $\{u, v, w\} \subseteq N(\tilde{k}) \cap S$, which contradicts the fact that $|N(k) \cap S| \leq 2$ for all $k \in K$.

Hence $d_{B(G/C_q)}(v) = 1$, which contradicts the fact that H is $(h + 1)$-vertex critical.

(ii) First we will prove that $|E(B(G/K))| \leq |K|$. Let $e = uv \in E(B(G/K))$. By the definition of branch graph, there exists $k \in K$ such that $ku \in E(G)$, $kv \in E(G)$. Thus for each $e \in E(B(G/K))$ there exists $k \in K$. Hence, by item (i), $|E(B(G/K))| \leq |K|$. Now we will see that $|K| \leq |E(B(G/K))|$. Let $k \in K$. By item (i), $|N(k) \cap S| = 2$. Suppose that $N(k) \cap S = \{u, v\}$, hence $N(u) \cap N(v) \neq \emptyset$. Since there are no dominated stable vertices, $N(u) \nsubseteq N(v)$, $N(v) \nsubseteq N(u)$. Thus $uv \in E(B(G/K))$. Hence for each $k \in K$ there exist $u,v \in S$ such that $uv \in E(B(G/K))$. Observe that if $\tilde{k} \in K$ such that $\tilde{k} \neq k$, then $N(\tilde{k}) \cap S \neq N(k) \cap S$. Because if $N(\tilde{k}) \cap S = N(k) \cap S$, then $\tilde{k}$ and k are true twins in G which contradicts the fact that G is a minimal non-$[h, 2, 1]$ graph. Therefore, $|K| \leq |E(B(G/K))|$.

(iii) Let $e = uv$ by any edge of $B(G/K)$, we will prove that $\chi(B(G/K) - e) < \chi(B(G/K))$. Notice it is enough to show that there exists $k \in K$ such that $B(G/K) - e = B((G - k)/(K - \{k\}))$; in fact, since G is a minimal non-$[h, 2, 1]$ graph, we have that $G - k \in [h, 2, 1]$, and so $\chi(B((G - k)/(K - \{k\}))) \leq h < h + 1 = \chi(B(G/K))$.

Let k be a vertex of K adjacent to both u and v. Observe that, by Lemma 13, $K - \{k\}$ is a clique of $G - k$.

Since every vertex of S is adjacent to at least two vertices of K, it follows that $V(B((G - k)/(K - \{k\}))) = V(B(G/K)) = V(B(G/K) - e)$. On the other hand, it is clear that any two vertices adjacent in $B((G - k)/(K - \{k\}))$ are adjacent in $B(G/K)$. Then, it remains to see that $E(B(G/K) - e) \subseteq E(B((G - k)/(K - \{k\})))$.

Let xy be an edge of $B(G/K) - e$, and let k_1, k_2, k_3 be vertices of K such that $k_1x \in E(G)$, $k_1y \in E(G)$, $k_2x \in E(G)$, $k_2y \notin E(G)$, $k_3x \notin E(G)$, $k_3y \in E(G)$. Notice that $k_1 \neq k$ because every vertex of K has exactly two neighbors in S and $e \neq xy$. Therefore, if $k_2 \neq k$ and $k_3 \neq k$, we have that $xy \in E(B((G - k)/(K - \{k\})))$ and the proof is completed. We can assume without lost of generality that $k_2 = k$ and, since the only neighbors of k in S are u and v, also, without lost of generality, we can assume that $x = u$.

Since, by item (iii) of Theorem 12, $B(G/K)$ is $(h + 1)$-vertex critical, then any vertex of $B(G/K)$ has degree at least h. Thus there exists $k_4 \in K$ such that $k_4x \in E(G)$ and $k_4 \neq k$, $k_4 \neq k_1$. If $k_4y \in E(G)$, then $N[k_1] = N[k_4] = \{x, y\} \cup K$ which means that k_1 and k_4 are true twins contradicting the minimality of G. Hence $k_4y \notin E(G)$. The existence of vertices k_1, k_3, k_4 implies that $xy \in E(B((G - k)/(K - \{k\})))$. $\square$

4. Building minimal non-[h,2,1] graphs

The construction presented here is similar to that done in [1], and a generalization of that used in [4]. Given a graph H with $V(H) = \{v_1, \ldots, v_n\}$, let G_H be the graph with vertices:

$$V(G_H) = \begin{cases} v_i & \text{for each } 1 \leq i \leq n; \\ v_{ij} & \text{for each } 1 \leq i < j \leq n \text{ such that } v_iv_j \in E(H); \\ \tilde{v}_i & \text{for each } 1 \leq i \leq n \text{ with } d_H(v_i) = 1; \end{cases}$$

and the cliques of G_H are K_H and C_{v_i} for $1 \leq i \leq n$, where:

$$K_H = \{v_{ij} \mid 1 \leq i < j \leq n\} \cup \{\tilde{v}_i \mid 1 \leq i \leq n \text{ and } d_H(v_i) = 1\},$$
$$C_{v_i} = \{v_i\} \cup \{v_{ij} \mid v_j \in N_H(v_i)\} \cup \{\tilde{v}_i \mid d_H(v_i) = 1\}.$$

In Fig. 1 we offer an example.

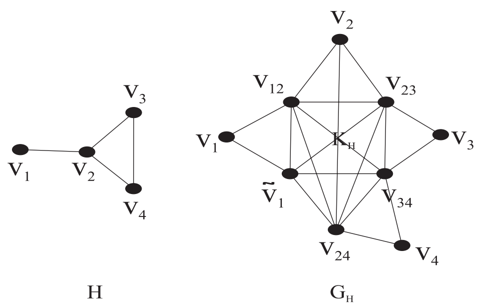

Fig. 1. A graph H and the graph G_H.

Notice that the vertices of G_H are partitioned into a stable set S_H of size $n = |V(H)|$ corresponding to the vertices v_i, and a central clique K_H of size $|E(H)| + |\{v \in V(H) \mid d_H(v) = 1\}|$ corresponding to the remaining vertices. The usefulness of G_H relies on the properties described in the following lemma used in the proof of Theorem 18.

Lemma 17 (*[1]*). (i) *G_H is a VPT $\cap$ Split graph without dominated stable vertices;* (ii) $B(G_H/K_H) = H$.

Theorem 18. *Let $h \geq 3$. The graph G_H is a minimal non-$[h, 2, 1]$ graph if and only if H is $(h+1)$-critical.*

Proof. Assume that G_H is a minimal non-$[h, 2, 1]$ graph. By item (ii) of Lemma 17, $B(G_H/K_H) = H$. Hence, by item (iii) of Theorem 16, H is $(h+1)$-critical.

Now, let H be an $(h+1)$-critical graph with $V(H) = \{v_1, v_2, \ldots, v_n\}$. By Lemmas 17 and 9, $\max_{C\in\mathcal{C}(G_H)}(\chi(B(G_H/C))) = \chi(B(G_H/K_H)) = \chi(H) = h+1$. Hence, by Theorem 7, $G_H \in [h+1, 2, 1] - [h, 2, 1]$. Let us see that $G_H - v \in [h, 2, 1]$ for all $v \in V(G_H)$. First, if $v = v_i \in V(H)$, using Claim 4 and item (ii) of Lemma 17, we have that $B((G_H - v_i)/K_H) = B(G_H/K_H) - v_i = H - v_i$. Thus since H is $(h+1)$-vertex critical, $\chi(B((G_H - v_i)/K_H)) = h$. Since $G_H - v_i$ is a split graph with central clique K_H, by Lemma 9 and Theorem 7, we have that $G_H - v_i \in [h, 2, 1]$. Secondly, if $v = v_{ij}$ where $e = v_iv_j \in E(H)$, as a direct consequence of the way in which the graph G_H was obtained from H, we have that $B((G_H - v_{ij})/(K_H - \{v_{ij}\})) = H - e$. Then $\chi(B((G_H - v_{ij})/(K_H - \{v_{ij}\}))) = \chi(H - e)$. And, $\chi(H - e) = h$ because H is $(h+1)$-critical. Hence $G_H - v_{ij} \in [h, 2, 1]$. Since H is a critical graph it has no degree 1 vertices, and so G_H has no more vertices. $\square$

5. Characterization of minimal non-[h,2,1] graphs

In this section, we give a characterization of VPT minimal non-$[h, 2, 1]$ graphs, with $h \geq 3$. The main result of this section is Theorem 19 which states that the only VPT minimal non-$[h, 2, 1]$ graphs are the graphs G_H constructed from $(h+1)$-critical graphs H.

Moreover, in Theorem 20, we show that the family of graphs constructed from $(h+1)$-critical graphs together with the family of minimal forbidden induced subgraphs for VPT [12,15], is the family of minimal forbidden induced subgraphs for $[h, 2, 1]$, with $h \geq 3$.

Theorem 19. *Let $h \geq 3$ and let G be a VPT graph. G is a minimal non-$[h, 2, 1]$ graph if and only if there exists an $(h+1)$-critical graph H such that $G \simeq G_H$.*

Proof. The reciprocal implication follows directly applying Theorem 18.

Let G be a minimal non-$[h, 2, 1]$ graph. By Theorem 15, we know that $G \in$ Split without dominated stable vertices. Let (S, K) be a split partition of G. By Theorem 11, $G \in [h+1, 2, 1]$. Let $H = B(G/K)$. By item (iii) of Theorem 16, H is an $(h+1)$-critical graph. Let us see that $G \simeq G_H$. Let $G_H = (S_H, K_H)$. By item (ii) of Lemma 17, $B(G_H/K_H) = H$; then $B(G_H/K_H) = B(G/K)$. So, since $V(B(G_H/K_H)) = V(B(G/K))$, $S_H = S$. Moreover, since $E(B(G_H/K_H)) = E(B(G/K))$, by item (ii) of Theorem 16, $|K_H| = |K|$ and, by item (i) of Theorem 16, $|N(k) \cap S| = 2$ for all $k \in K$. Suppose that $N(k) \cap S = \{v_i, v_j\}$; we will see that $v_iv_j \in E(H)$. It is clear that $v_ik \in E(G)$ and $v_jk \in E(G)$. Moreover, by item (ii) of Theorem 12, there exist $k', k'' \in K$ such that $k'v_i \in E(G)$, $k''v_j \in E(G)$. If $k' = k''$ then, since $|N(k) \cap S| = 2$ for all $k \in K$, we have that k' and k are true twins in G, which contradicts the fact that G is minimal non-$[h, 2, 1]$ graph. Hence $k' \neq k''$. Thus $k'v_j \notin E(G)$ and $k''v_i \notin E(G)$. Therefore, $v_iv_j \in E(H)$.

Hence we can define a function f that assigns to each vertex $k \in K$ an edge $v_iv_j \in E(H)$, that is, an element of K_H. Note that in G_H the vertex $v_{ij} \in K_H$ is adjacent exactly to v_i and v_j. Hence the function f can be extended to a new function $\tilde{f}$ from $K \cup S$ to $K_H \cup S_H$, being the identity function from S to S_H. Moreover, $\tilde{f}$ is an isomorphism between G and G_H. $\square$

Theorem 20. *Let $h \geq 3$. A graph G is a minimal non-$[h, 2, 1]$ if and only if G is one of the members of $F_0, F_1, \ldots, F_{16}$ or $G \simeq G_H$, where H is an $(h+1)$-critical graph.*

Proof. By Theorem 19, if $G \simeq G_H$ where H is an $(h+1)$-critical graph, then G is a minimal non-$[h, 2, 1]$ graph.

If G is any of the members of $F_0, \ldots, F_{16}$, then $G \notin$ VPT and $G - v \in$ VPT for all $v \in V(G)$. Moreover, in [4] it was proved that $G - v \in$ EPT for all $v \in V(G)$. Thus $G - v \in$ VPT $\cap$ EPT$= [3, 2, 1]$ [8], which implies that $G - v \in [h, 2, 1]$. Hence G is a minimal non-$[h, 2, 1]$ graph.

Let $h \geq 3$ and let G be a minimal non-$[h, 2, 1]$ graph.

Case (1): $G \notin$ VPT. Since G is a minimal non-$[h, 2, 1]$ graph, then $G - v \in [h, 2, 1]$ for all $v \in V(G)$. Thus $G - v \in$ VPT for all $v \in V(G)$. Then, G is a minimal forbidden induced subgraph for VPT. Hence G is one of the members of $F_0, F_1, \ldots, F_{16}$.

Case (2): $G \in$ VPT. Then, by Theorem 19, $G \simeq G_H$, where H is an $(h+1)$-critical graph.

Notice that, since every G_H is VPT, no member of $F_0, F_1, \ldots, F_{16}$ is an induced subgraph of G_H. $\square$

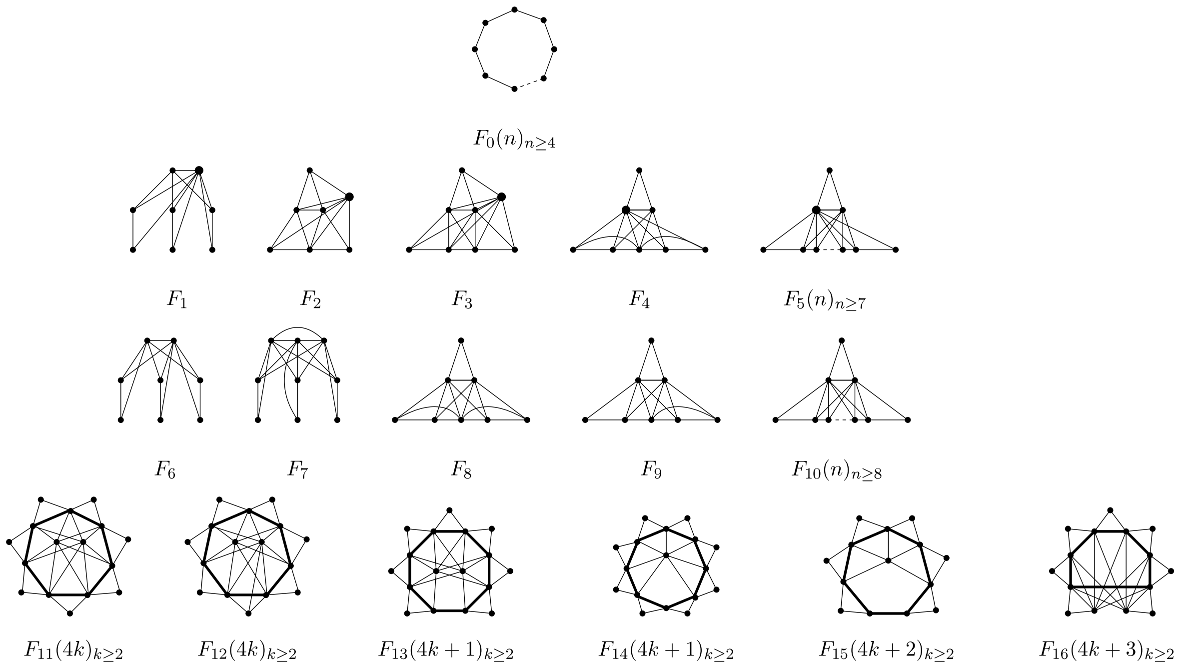

Fig. 2. Minimal forbidden induced subgraphs for VPT graphs (the vertices in the cycle marked by bold edges form a clique). See [12,15] for more details.

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