ON MODULES OVER MATRIX QUANTUM PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. We classify all the quasifinite highest weight modules over the central extension of the Lie algebra of matrix quantum pseudo-differential operators, and obtain them in terms of representation theory of the Lie algebra $\widehat{gl}(\infty, R_m)$ of infinite matrices with only finitely many non-zero diagonals over the algebra $R_m = \mathbb{C}[t]/(t^{m+1})$. We also classify the unitary ones.

1. Introduction

The study of representation theory of the Lie algebra $\widehat{\mathcal{D}}$ (the universal central extension of the Lie algebra of differential operators on the circle, also denoted by $\mathcal{W}_{1+\infty}$), was initiated in [4]. In that paper, Kac and Radul classified the irreducible quasi-finite highest weight representations of $\widehat{\mathcal{D}}$, and it was shown that they can be realized in terms of irreducible highest weight representations of the Lie algebra of infinite matrices. At the end of that article, they did, very briefly, the same for the Lie algebra $\widehat{\mathcal{S}_q}$, the central extension of the Lie algebra of quantum pseudo-differential operators.

This study for $\widehat{\mathcal{D}}$ was continued in [2], [5] and [6] in the framework of vertex algebra theory. In [1] similar results were obtained for the matrix case, i.e., the universal central extension of the Lie algebra of matrix differential operators on the circle, denoted by $\widehat{\mathcal{D}^M}$. The main goal of the present paper is to get analogous results to those obtained in [4] for $\widehat{\mathcal{S}_q}$, but for the central extension of the Lie algebra of $M \times M$ matrix quantum pseudo-differential operators, denoted by $\widehat{\mathcal{S}_q^M}$.

The paper is organized as follows. In Sects. 2 and 3 we study the structure of $\widehat{\mathcal{S}_q^M}$, its parabolic subalgebras and the relation with $\widehat{gl}(\infty, R_m)$. In sects. 4 and 5 we classify and construct the quasifinite highest weight modules over $\widehat{\mathcal{S}_q^M}$ and the unitary ones.

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2. Lie algebras $\widehat{\mathcal{S}_q^M}$ and $\widehat{\mathcal{S}_q^{M\,\mathcal{O}}}$

Fix a positive integer M, and denote by T_q, $q \in \mathbb{C}^\times = \mathbb{C}/\{0\}$, the following operator on $\mathbb{C}[z, z^{-1}]$:

$$T_q f(z) = f(qz).$$

Denote by $\mathcal{S}_q^{M\,as}$ the associative algebra of all matrix pseudo-differential operators, i.e., the operators on $\mathbb{C}^M[z, z^{-1}]$ of the form

$$E = \sum_{k\in\mathbb{Z}} e_k(z) T_q^k, \text{ where } e_i(z) \in Mat_M\mathbb{C}[z, z^{-1}] \qquad \text{(sum is finite).}$$

Here and further we denote by $\mathrm{Mat}_M R$ the associative algebra of all $M \times M$ matrices over an algebra R. Any pseudo-differential operator can be written as linear combinations of elements of the form $z^k f(T_q)E_{ij}$, where $f \in \mathbb{C}[w, w^{-1}]$ and E_{ij} is the standard basis of $\mathrm{Mat}_M\mathbb{C}$. The product in $\mathcal{S}_q^{M\,as}$ is then given by

$$(z^r f(T_q)E_{ij})(z^s g(T_q)E_{kl}) = z^{r+s} f(q^s T_q) g(T_q) \delta_{jk} E_{il} \tag{2.1}$$

Denote by $\mathcal{S}_q^M$ the Lie algebra obtained from $\mathcal{S}_q^{M\,as}$ by taking th usual bracket. Let $\mathcal{S}_q^{M\prime} = [\mathcal{S}_q^M, \mathcal{S}_q^M]$. It is easy to check that we have:

$$\mathcal{S}_q^M = \mathcal{S}_q^{M\prime} \oplus \mathbb{C}\, T_q^0\, I_M \qquad \text{(direct sum of ideals)}$$

where I_M denotes the identity matrix in $\mathrm{Mat}_M\mathbb{C}$. Thus, the representation theory of $\mathcal{S}_q^M$ reduces to that of $\mathcal{S}_q^{M\prime}$. Taking the trace form in $\mathbb{C}[w, w^{-1}]$, namely $tr_0\left(\sum_j c_j w^j\right) = c_0$, and denoting by tr the usual trace in $\mathrm{Mat}_M\mathbb{C}$, we obtain, by a general construction (cf Sect 1.3 in [4]), the following 2-cocycle in $\mathcal{S}_q^M$:

$$\Psi(z^m f(T_q)A, z^k g(T_q)B) = m\ tr_0(f(q^{-m}w)g(w))\ tr(AB)\ \delta_{m,-k} \tag{2.2}$$

Let

$$\widehat{\mathcal{S}_q^M} = \mathcal{S}_q^{M\prime} + \mathbb{C}C$$

denote the central extension of $\mathcal{S}_q^{M\prime}$ corresponding to the cocycle (2.2). We will show that the representation theory of $\widehat{\mathcal{S}_q^M}$ with $|q| \neq 1$, is quite similar to that of $\widehat{\mathcal{D}^M}$. Some details and proofs will be omitted since they also are similar, (cf [1]).

The elements $z^k T_q^m E_{ij}$ $(k, m \in \mathbb{Z},\ i, j = 1, \cdots, M)$ form a basis of $\mathcal{S}_q^M$. Define the *weight* by

$$\mathrm{wt}\ z^k T_q^m E_{ij} = kM + i - j, \quad \mathrm{wt}\ C = 0 \tag{2.3.}$$

This gives us the principal $\mathbb{Z}$-gradation of $\widehat{\mathcal{S}_q^M}$:

$$\widehat{\mathcal{S}_q^M} = \bigoplus_{j\in\mathbb{Z}} (\widehat{\mathcal{S}_q^M})_j$$

Let $\mathcal{O}$ be the algebra of all holomorphic functions on $\mathbb{C}^\times$ with the topology of uniform convergence on compact sets. We consider the vector space $\mathcal{S}_q^{M\,\mathcal{O}^{as}}$ spanned by the quantum matrix pseudo-differential operators(of infinite order) of the form $z^k f(T_q)E_{ij}$, where $f \in \mathcal{O}$. The product in $\mathcal{S}_q^{M^{as}}$ extends to $\mathcal{S}_q^{M\,\mathcal{O}^{as}}$ by formula (2.1). The principal gradation extends as well by (2.3). Denote by $\mathcal{S}_q^{M\,\mathcal{O}}$ the corresponding Lie algebra. Then the cocycle Ψ extends by formula (2.2). Let $\widehat{\mathcal{S}_q^{M\,\mathcal{O}}} = \mathcal{S}_q^{M'\,\mathcal{O}} + \mathbb{C}C$ be the corresponding central extension.

Let us define a *parabolic* subalgebra $\mathfrak{p}$ of $\widehat{\mathcal{S}_q^M}$ as a subalgebra of the following form:

$$\mathfrak{p} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{p}_j, \quad \text{where } \mathfrak{p}_j = (\widehat{\mathcal{S}_q^M})_j \text{ for } j \geq 0 \text{ and } \mathfrak{p}_j \neq 0 \text{ for some } j < 0.$$

Observe that

$$(\widehat{\mathcal{S}_q^M})_j = \bigoplus_{k,l,m\,:\,j=kM+l-m} z^k\mathbb{C}[T_q, T_q^{-1}]E_{lm},$$

then, for each $j \in \mathbb{N}$ we have:

$$\mathfrak{p}_{-j} = \bigoplus_{k,l,m\,:\,kM+l-m=-j} z^k I^l_{-j} E_{lm}$$

where I^l_{-j} is a subspace of $\mathbb{C}[w, w^{-1}]$. Since

$$[f(T_q)E_{ll}, p(T_q)E_{lm}] = f(T_q)p(T_q)E_{lm} \qquad l \neq m$$

and

$$[f(T_q)I_M, z^k p(T_q)E_{lm}] = z^k(f(q^kT_q) - f(T_q))p(T_q)E_{lm}$$

where I_M is the identity matrix, then we have that I^l_{-j} is an ideal of $\mathbb{C}[w, w^{-1}]$.

Remark 2.2. There exist parabolic subalgebras $\mathfrak{p}$ of $\widehat{\mathcal{S}_q^M}$ $(M > 1)$ such that $\mathfrak{p}_j = 0$ for $j << 0$ (cf. [4], sect 2.4). For example:

$$\mathfrak{p} = \oplus_{j\in\mathbb{N}}(\widehat{\mathcal{S}_q^M})_j \oplus (\widehat{\mathcal{S}_q^M})_0 \oplus \mathbb{C}[T_q, T_q^{-1}]E_{12}$$

Given a parabolic subalgebra $\mathfrak{p}$, let $b^l_j(w)$ be the monic polynomial with $b^l_j(0) \neq 0$, which is a generator of the ideal $I^l_{-j} \subset \mathbb{C}[w, w^{-1}]$. We will assume that $b^l_j(w)$ is a monic polynomial as above if $I^l_j \neq 0$, and 0 otherwise. Thus, we have associated to $\mathfrak{p}$ a collection $\{b^l_j(w)\}_{j\in\mathbb{N},1\leq l\leq M}$ as above, called the *characteristic* polynomials of $\mathfrak{p}$.

Lemma 2.1. *Let $\{b_j^l\}$ be the set of characteristic polynomials of a parabolic subalgebra $\mathfrak{p}$ of the Lie algebra $\widehat{\mathcal{S}_q^M}$. Then*

1. *$b_j^{i+1}(w)$ divides $b_{j+1}^i(w)$ for all $j \in \mathbb{N}$ and $i = 1, \ldots, M$.*
2. *$b_j^i(w)$ divides $b_{j+1}^i(w)$ except in the special case: $-j = kM + i$, where $b_j^i(w)$ divides $b_{j+1}^i(qw)$.*
3. *$b_{(k+l)M+n-i}^i(w)$ divides $b_{kM+p-i}^i(q^{-l}w)b_{lM+n-p}^p(w)$ for all $k, l \in \mathbb{N}, 1 \leq i, p \leq M$, (where $b_j^{i+M} = b_j^i$).*

Proof. Since it is completely similar to the proof of Lemma 2.1 in [1], replacing D by T_q and keeping in mind formula (2.1), we omit the details.

It is easy to deduce from the proof of Lemma 2.1, the following Corollary.

Corollary 2.1. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\widehat{\mathcal{S}_q^M}$. The following statements are equivalent:*

1. *$\mathfrak{p}_{-j} \neq 0$ for all $j \in \mathbb{N}$.*
2. *$I_{-j}^l \neq 0$ for all $j \in \mathbb{N}$, $l = 1, \ldots, M$.*
3. *$I_{-1}^l \neq 0$ for all $l = 1, \ldots, M$.* $\square$

A parabolic subalgebra satifying any condition of Corollary 2.1 will be called *non-degenerate.*

Given M monic polynomials $b = (b_1(w), \ldots, b_M(w))$, with $b_i(0) \neq 0$, we define (cf. Lemma 2.1)

$$\begin{aligned}(b^{\min})_{kM+m-i}^i(w) = b_i(q^{-k}w)b_{i+1}(q^{-k}w)\ldots b_{M-1}(q^{-k})b_M(q^{-k+1}w)\\ b_1(q^{-k+1}w)\ldots b_{M-1}(q^{-k+1}w)b_M(q^{-k+2}w)b_1(q^{-k+2}w)\ldots\\ b_M(q^{-1}w)b_1(q^{-1}w)\ldots b_{M-1}(q^{-1}w)b_M(w)b_1(w)\ldots b_{m-1}(w)\end{aligned}$$

It follows that there exists (a unique non-degenerate) parabolic subalgebra, that we denote by $\mathfrak{p}_{\min}(b)$, for which the characteristic polynomials are $\{(b^{\min})_j^i\}$. We also have

$$\dim\left((\widehat{\mathcal{S}_q^M})_{-k}/\mathfrak{p}_{\min}(b)_{-k}\right) < \infty, \qquad k \in \mathbb{Z}. \tag{2.4}$$

Remark 2.3. Any parabolic subalgebra $\mathfrak{p}$ such that $b_1^i = b^i$, satisfies $\mathfrak{p}_{\min}(b) \subseteq \mathfrak{p}$. This follows from Lemma 2.1.

We shall need the following Proposition to study modules over $\widehat{\mathcal{S}_q^M}$ induced from its parabolic subalgebras. We omit its proof since it is completely similar to Proposition 2.2 in [1], with the same considerations we had for Lemma 2.1.

Proposition 2.2. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\widehat{\mathcal{S}_q^M}$ and let $b = (b_1(w), \dots, b_M(w))$ be its first characteristic polynomials. Then*

$$[\mathfrak{p}, \mathfrak{p}] = \left(\bigoplus_{k \neq 0} \mathfrak{p}_{-k} \right) \bigoplus (\widehat{\mathcal{S}_q^M})_0^b.$$

where

$$\begin{aligned}(\widehat{\mathcal{S}_q^M}_0)^b = \{ & b_i(T_q)g(T_q)E_{i+1,i+1} - b_i(T_q)g(T_q)E_{ii} \,|\, 1 \leq i \leq M-1, \\ & \text{and } g(w) \in \mathbb{C}[w, w^{-1}]\} \oplus \\ \{ & b_M(T_q)g(T_q)E_{1,1} - b_M(qT_q)g(qT_q)E_{MM} + tr_0(b_M(w)g(w))C | g(w) \in \mathbb{C}[w, w^-1]\}\end{aligned}$$

3. Embedding of $\widehat{\mathcal{S}_q^M}$ in $\widehat{gl}(\infty)[m]$

We shall construct an embedding of $\widehat{\mathcal{S}_q^M}$ into the Lie algebra of infinite matrices with only finitely many non zero diagonal, over the algebra of truncated polynomials.

Let R be an associative algebra over $\mathbb{C}$ and denote by R^∞ a free R-module with a fixed basis $\{v_j\}_{j \in \mathbb{Z}}$. Now, define the operators E_{ij} by $E_{ij} v_k = \delta_{jk} v_i$.

Let $\widetilde{M}(\infty, R)$ be the associative subalgebra of End R^∞ consisting of all operators $\sum_{i,j \in \mathbb{Z}} a_{ij} E_{ij}$ where $(a_{ij})_{i,j \in \mathbb{Z}}$ have a finite number of non-zero diagonals. Recall that $\widetilde{M}(\infty, R)$ is a $\mathbb{Z}$-graded algebra with the *principal* gradation defined by deg $E_{ij} = j - i$.

Let $\varphi : R^M[z, z^{-1}] \longrightarrow R[z, z^{-1}]$ be the isomorphism defined by

$$e_i z^j \mapsto z^{jM+i-1}$$

Now, let us fix $s \in \mathbb{C}^\times$ and a nilpotent element $t \in R$. Consider $R[z, z^{-1}]z^s$ as a free R-module with basis $v_j = z^{-j+s}$, $j \in \mathbb{Z}$. Using the isomorphism φ can construct (cf. [4] Sect 6 and [1] Sect.3) an embedding $\varphi_{s,t} : \mathcal{S}_q^{M\,as} \longrightarrow \widetilde{M}(\infty, R)$ of associative algebras over $\mathbb{C}$, which is compatible with the principal gradation. Explicitly:

$$\varphi_{s,t}(z^k f(T_q) E_{ij}) = \sum_{l \in \mathbb{Z}} f(sq^{-l+t}) E_{(l-k)M-i+1, lM-j+1} \tag{3.1}$$

and $\varphi_{s,t}$ extends to a homomorphism $\varphi_{s,t} : \mathcal{S}_q^{M\,\mathcal{O}^{as}} \longrightarrow \widetilde{M}(\infty, R)$.

Consider $R_m = \mathbb{C}[t]/(t^{m+1})$ where $m \in \mathbb{Z}_+$, and let $\widetilde{M}(\infty)[m] = \widetilde{M}(\infty, R_m)$. The homomorphism $\varphi_{s,t} : \mathcal{S}_q^{M\,\mathcal{O}^{as}} \longrightarrow \widetilde{M}(\infty)[m]$ given by (3.1) will be denoted by $\varphi_s^{[m]}$. Let

$$I_s^{[m]} = \{f \in \mathcal{O} \mid f^{(i)}(sq^j) = 0 \text{ for all } j \in \mathbb{Z}, i = 0, 1, \cdots, m\}$$

and

$$J_s^{[m]} = \bigoplus_{i,j=1}^{M} \bigoplus_{k\in\mathbb{Z}} z^k I_s^{[m]} E_{ij} \in \mathcal{S}_q^{M\,\mathcal{O}^{as}}$$

Therefore, It follows by the Taylor formula for $\varphi_s^{[m]}$ that:

$$\text{Ker } \varphi_s^{[m]} = J_s^{[m]} \tag{3.2}$$

Choose a branch of $\log q$. Let $\tau = \frac{\log q}{2\pi i}$. Then any $s \in \mathbb{C}$ is uniquely written as $s = q^a$, $a \in \mathbb{C}/\tau^{-1}\mathbb{Z}$. Now, let fix $\mathbf{s} = (s_1, \cdots, s_n) \in \mathbb{C}^n$ such that if we write each $s_i = q^{a_i}$, we have

$$a_i - a_j \notin \mathbb{Z} + \tau^{-1}\mathbb{Z} \text{ for } i \neq j, \tag{3.3}$$

and fix $\mathbf{m} = (m_1, \cdots, m_n) \in \mathbb{Z}_+^n$.

Let $\widetilde{M}(\infty)[\mathbf{m}] = \overset{n}{\underset{i=1}{\oplus}} \widetilde{M}(\infty)[m_i]$. Consider the homomorphism

$$\varphi_{\mathbf{s}}^{[\mathbf{m}]} = \bigoplus_{i=1}^{n} \varphi_{s_i}^{[m_i]} : \mathcal{S}_q^{M\,\mathcal{O}^{as}} \longrightarrow \widetilde{M}(\infty)[\mathbf{m}]$$

Proposition 3.1. *We have the exact sequence of $\mathbb{Z}$-graded associative algebras, provided that $|q| \neq 1$:*

$$0 \longrightarrow J_{\mathbf{s}}^{[\mathbf{m}]} \longrightarrow \mathcal{S}_q^{M\,\mathcal{O}^{as}} \overset{\varphi_{\mathbf{s}}^{[\mathbf{m}]}}{\longrightarrow} \widetilde{M}(\infty)[\mathbf{m}] \longrightarrow 0$$

where $J_{\mathbf{s}}^{[\mathbf{m}]} = \bigcap_{i=1}^{n} J_{s_i}^{[m_i]}$.

Proof. The first part is clear from (3.3). The surjectivity of $\varphi_{\mathbf{s}}^{[\mathbf{m}]}$ follows from the following well-known fact: for every discrete sequence of points in $\mathbb{C}$ and a non-negative integer m there exists $f(w) \in \mathcal{O}$ having prescribed values of its first m derivatives at these points. Note that, condition (3.3) and $|q| \neq 1$ are important in order to have a discrete sequence of points in $\mathbb{C}$. $\square$

Now, denote by $\widetilde{gl}(\infty)[m]$ the Lie algebra corresponding to $\widetilde{M}(\infty)[m]$. Let $\widehat{gl}(\infty)[m] = \widetilde{gl}(\infty)[m] + R_m$ be the central extension with respect to the 2-cocycle

$$\Phi(A, B) = tr([J, A]B)$$

where $J = \sum_{i\leq 0} E_{ii}$. The $\mathbb{Z}$-gradation of this Lie algebra extends from $\widetilde{gl}(\infty)[m]$ by letting $wt\ R_m = 0$.

The associative algebras homomorphism $\varphi_s^{[m]}$ defines a Lie algebras homomorphism, which we denote by the same letter.

The restriction of the cocycle Φ to $\varphi_s^{[m]}(\mathcal{S}_q^{M\,\mathcal{O}})$ gives us an R_m-valued 2-cocycle on $\mathcal{S}_q^{M\,\mathcal{O}}$ denoted by $\Psi_s^{[m]}$. Thus, as in [1], we have

Proposition 3.2. *The $\mathbb{C}$-linear map $\widehat{\varphi}_s^{[m]} : \widehat{\mathcal{S}_q^M} \longrightarrow \widehat{gl}(\infty)[m]$ defined by ($s = q^a$),*

$$\begin{aligned} \widehat{\varphi}_s^{[m]}\Big|_{\widehat{\mathcal{S}_q^M}_j} &= \varphi_s^{[m]}\Big|_{\mathcal{S}_q^M{}_j} && \text{if } j \neq 0, \\ \widehat{\varphi}_s^{[m]}(T_q^n E_{ii}) &= \varphi_s^{[m]}(T_q^n E_{ii}) + \frac{q^{an}}{1-q^n} \sum_{j=1}^{m} (n \log q)^j \frac{t^j}{j!} && (n \neq 0) \\ \widehat{\varphi}_s^{[m]}(C) &= 1 \in R_m \end{aligned} \tag{3.4}$$

is a homomorphism of Lie algebras over $\mathbb{C}$. □

4. Quasifinite Highest Weight Modules over $\widehat{\mathcal{S}_q^M}$

Recall that given $\lambda \in (\widehat{\mathcal{S}_q^M})_0^*$, the *Verma module* over $\widehat{\mathcal{S}_q^M}$ is defined as follows:

$$M(\lambda) = M(\widehat{\mathcal{S}_q^M}, \lambda) = \mathcal{U}(\widehat{\mathcal{S}_q^M}) \otimes_{\mathcal{U}\left(\oplus_{j\in\mathbb{Z}_+} (\widehat{\mathcal{S}_q^M})_j\right)} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one dimensional $\mathcal{U}\left(\oplus_{j\in\mathbb{Z}_+} (\widehat{\mathcal{S}_q^M})_j\right)$-module given by $h \mapsto \lambda(h)$ if $h \in (\widehat{\mathcal{S}_q^M})_0$, $(\widehat{\mathcal{S}_q^M})_j \mapsto 0$ for $j > 0$, and the action of $\widehat{\mathcal{S}_q^M}$ is induced by left multiplication in $\mathcal{U}(\widehat{\mathcal{S}_q^M})$. The vector $v_\lambda := 1 \otimes 1$ is called the *highest weight vector*.

Let $L(\lambda) = L(\widehat{\mathcal{S}_q^M}, \lambda)$ denote the unique irreducible quotient of $M(\widehat{\mathcal{S}_q^M}, \lambda)$. Modules $M(\lambda)$ and their quotients (including $L(\lambda)$) are called *highest weight modules* (with highest weight λ).

Note that the principal gradation on $\widehat{\mathcal{S}_q^M}$ induces a principal gradation on any highest weight module V:

$$V = \oplus_{j\in\mathbb{Z}_+} V_{-j}, \qquad \text{where } V_0 = \mathbb{C}v_\lambda \tag{4.1}$$

Take a parabolic subalgebra $\mathfrak{p}$ and let $\mathbf{b}$ be its first M characteristic polynomials. Let $\lambda \in (\widehat{\mathcal{S}_q^M})_0^*$ be such that $\lambda|_{(\widehat{\mathcal{S}_q^M})_0 \mathbf{b}} = 0$. Then the $\mathcal{U}\left(\oplus_{j\in\mathbb{Z}_+} (\widehat{\mathcal{S}_q^M})_j\right)$-module $\mathbb{C}_\lambda$ extends to $\mathfrak{p}$ by letting $[\mathfrak{p}, \mathfrak{p}] \mapsto 0$ (see Proposition 2.1). Denote by $M(\lambda; \mathbf{b})$ the generalized Verma module $M(\widehat{\mathcal{S}_q^M}, \mathfrak{p}, \lambda)$, i.e the $\widehat{\mathcal{S}_q^M}$-module,

$$M(\widehat{\mathcal{S}_q^M}, \mathfrak{p}, \lambda) = \mathcal{U}\left(\widehat{\mathcal{S}_q^M}\right) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}_\lambda.$$

Recall that a non-zero vector v in a highest weight module over $\widehat{\mathcal{S}_q^M}$ is called *singular* if $(\widehat{\mathcal{S}_q^M})_j v = 0$, for all $j > 0$.

Definition 4.1. A Verma module $M(\lambda)$ over $\widehat{\mathcal{S}_q^M}$ is called *highly degenerate* if there exists a singular vector $v_0 \in M(\lambda)_{-1}$ of the following form: $v_0 = Av_\lambda$, where $A \in (\widehat{\mathcal{S}_q^M})_{-1}$ and det $A \neq 0$.

The following key Proposition follows from Propositions 2.1 and formula (2.4).

Proposition 4.1. *The following conditions on $\lambda \in (\widehat{\mathcal{S}_q^M})_0^*$ are equivalent:*

1. *$M(\lambda)$ is highly degenerate;*
2. *$\lambda((\widehat{\mathcal{S}_q^M})_0^{\mathbf{b}}) = 0$ for some $\mathbf{b} = (b_1(w), \ldots, b_M(w))$, where all $b_i(w)$ are monic polynomials, with $b_i(0) \neq 0$.*
3. *$L(\lambda)$ is quasifinite;*
4. *$L(\lambda)$ is a quotient of a generalized Verma module $M(\lambda; \mathbf{b})$ for some monic polynomials $\mathbf{b}$.*

Let $L(\lambda)$ be a quasifinite highest weight module over $\widehat{\mathcal{S}_q^M}$. By Proposition 4.1 there exist some monic polynomials $b_1(w), \ldots, b_M(w)$ such that

$$(b_i(T_q)E_{i,i+1})v_\lambda = 0 \qquad (i = 1, \ldots, M-1), \text{ and } (z^{-1}b_M(T_q)E_{M,1})v_\lambda = 0$$

We shall call such monic polynomials of minimal degree the *characteristic* polynomials of $L(\lambda)$. Note that $L(\lambda)$ is the irreducible quotient of $M(\lambda; \mathbf{b})$, where $\mathbf{b} = (b_1(w), \ldots, b_M(w))$ are the characteristic polynomials of $L(\lambda)$.

A functional $\lambda \in (\widehat{\mathcal{S}_q^M})_0^*$ is described by its labels $\Delta_{i,m} = -\lambda(T_q^m E_{ii})$ and the central charge $c = \lambda(C)$. We shall consider the generating series

$$\Delta_i(x) = \sum_{m \in \mathbb{Z}} x^{-m} \Delta_{i,m}, \qquad i = 1, \cdots, M.$$

Recall that a *quasipolynomial* is a linear combination of functions of the form $p(x)e^{\alpha x}$, where $p(x)$ is a polynomial and $\alpha \in \mathbb{C}$. We also have the following well known

Proposition 4.2. *Given a quasipolinomial P, and a polynomial $B(x) = \prod_i (x - A_i)$, take $b(x) = \prod_i (x - a_i)$ where $a_i = e^{A_i}$, then $b(x)(\sum_{n \in \mathbb{Z}} P(n)x^{-n}) = 0$ if and only if $B(\frac{d}{dx})P(x) = 0$.*

Now we can state our main result.

Theorem 4.1. *(a) An irreducible highest weight module $L(\lambda)$ is quasifinite if and only if one of the following equivalent conditions holds:*

1. *There exist monic polynomials $b_1(w), \ldots, b_M(w)$ such that*

$$\begin{gathered} b_i(x)\left(\Delta_i(x) - \Delta_{i+1}(x)\right) = 0 \quad \text{for } i = 1, \cdots, M-1, \quad \text{and} \\ b_M(x)\left(\Delta_1(x) - \Delta_M(q^{-1}x) + c\right) = 0 \end{gathered} \tag{4.2}$$

(2) *There exist quasipolynomials P_i, $i = 1, \cdots, M$, such that ($n \in \mathbb{Z}$)*

$$\begin{aligned} \Delta_{i,n} - \Delta_{i+1,n} &= P_i(n) \qquad \textit{for } i = 1, \dots, M-1, \\ \Delta_{1,n} - q^n \Delta_{M,n} &= P_M(n) \end{aligned}$$

(b) The monic polynomials $b_1(w), \dots, b_M(w)$ of minimal degree satisfying equations (4.2), are the characteristic polynomials of a quasifinite module $L(\widehat{\mathcal{S}_q^M}, \lambda)$.

Proof. From Propositions 4.1 and 2.2, we have that $L(\lambda)$ is quasifinite if and only if there exist polynomials

$$b_i(w) = w^{m_i} + f_{i,m_i-1} w^{m_i-1} + \cdots + f_{i,0} \qquad \text{for } i = 1, \dots, M$$

such that for all $s = 0, 1, \dots$ we have

$$\lambda(b_i(T_q) T_q^s E_{i+1,i+1} - b_i(T_q) T_q^s E_{ii}) = 0 \qquad i = 1, \dots, M-1$$

$$\lambda(b_M(T_q) T_q^s E_{1,1} - b_M(qT_q)(qT_q)^s E_{MM} + tr_0\,(w^s b_M(w))C = 0$$

These conditions can be rewritten as follows:

$$\sum_{n=0}^{m_i} f_{i,n} F_{i,n+s} = 0 \qquad \text{for all } s = 0, 1, \dots \tag{4.1}$$

where

$$\begin{aligned} F_{i,n} &= \Delta_{i,n} - \Delta_{i+1,n} \qquad \text{for } i = 1, \dots, M-1 \\ F_{M,n} &= \Delta_{1,n} c - q^n \Delta_{M,n} + f_{m,-s} c \end{aligned}$$

Multiplying both sides of these equalities by x^{-s}, and summing over $s \in \mathbb{Z}$, we obtain (4.2). The equivalence of (1) and (2) follows from Proposition 4.2. Part (b) is also clear. $\square$

We shall need the following Proposition, which proof is completely similar to Proposition 4.3 in [1].

Proposition 4.3. *Let V be a quasifinite $\widehat{\mathcal{S}_q^M}$-module. Then the action of $\widehat{\mathcal{S}_q^M}$ on V naturally extends to the action of $(\widehat{\mathcal{S}_q^{M\mathcal{O}}})_k$ on V for any $k \neq 0$.*

We return now to the $\mathbb{Z}$-graded complex Lie algebra $\mathfrak{g}^{[m]} := \widehat{gl}(\infty)[m] = = \tilde{gl}(\infty, R_m) + R_m$ considered in section 3. Recall that it is a central extension of the Lie algebra $\tilde{gl}(\infty, R_m)$ over $\mathbb{C}$ by the $m+1$-dimensional space R_m.

An element $\lambda \in (\mathfrak{g}_0^{[m]})^*$ is characterized by its *labels*

$$\lambda_k^{(j)} = \lambda(t^j E_{kk}), \qquad k \in \mathbb{Z},\, j = 0, \dots, m\,,$$

and *central charges*

$$c_j = \lambda(t^j), \qquad j = 0, 1, \dots, m.$$

Let

$$h_k^{(j)} = \lambda_k^{(j)} - \lambda_{k+1}^{(j)} + \delta_{k,0} c_j, \qquad k \in \mathbb{Z},\ j = 0, \dots, m. \tag{4.3}$$

As usual, we have the irreducible highest weight $\mathfrak{g}^{[m]}$-module $L(\mathfrak{g}^{[m]}, \lambda)$ associated to λ. The proof of the following proposition is standard:

Proposition 4.4. *The $\mathfrak{g}^{[m]}$-module $L(\mathfrak{g}^{[m]}, \lambda)$ is quasifinite if and only if for each $j = 0, 1, \dots, m$ all but finitely many of the $h_k^{(j)}$ are zero.*

Given $\vec{m} = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$, we let $\mathfrak{g}^{[\mathbf{m}]} = \widehat{gl}(\infty)[\mathbf{m}] = \oplus_{i=1}^n \mathfrak{g}^{[m_i]}$. By Proposition 3.1, we have a surjective homomorphism of Lie algebras over $\mathbb{C}$:

$$\hat{\varphi}_{\mathbf{s}}^{[\mathbf{m}]} = \bigoplus_{i=1}^{n} \hat{\varphi}_{s_i}^{[m_i]} : \widehat{\mathcal{S}_q^{M\mathcal{O}}} \longrightarrow \mathfrak{g}^{[\mathbf{m}]} \tag{4.4}$$

Take $\lambda(i) \in (\mathfrak{g}_0^{[m_i]})^*$ such that $L(\mathfrak{g}^{[m_i]}, \lambda(i))$ is a quasifinite irreducible $\mathfrak{g}^{[m_i]}$-module. Then

$$L(\mathfrak{g}^{[\mathbf{m}]}, \vec{\lambda}) = \bigotimes_{i=1}^{n} L(\mathfrak{g}^{[m_i]}, \lambda(i))$$

is an irreducible $\mathfrak{g}^{[\mathbf{m}]}$-module. Now, by the homomorphism (4.4), we consider $L(\mathfrak{g}^{[\mathbf{m}]}, \vec{\lambda})$ as $\widehat{\mathcal{S}_q^M}$-module, and we denote it by $L_{\mathbf{s}}^{[\mathbf{m}]}(\vec{\lambda})$.

Now we have the following important Theorem.

Theorem 4.2. *Let $\mathbf{s} = (s_1, \cdots, s_n)$ satisfying (3.3). Consider the embedding $\hat{\varphi}_{\mathbf{s}}^{[\mathbf{m}]} : \widehat{\mathcal{S}_q^M} \longrightarrow \mathfrak{g}^{[\mathbf{m}]}$ and let V a quasifinite $\mathfrak{g}^{[\mathbf{m}]}$-module. Then any $\widehat{\mathcal{S}_q^M}$-submodule of V is a $\mathfrak{g}^{[\mathbf{m}]}$-submodule as well. In particular, the $\widehat{\mathcal{S}_q^M}$-modules $L_{\mathbf{s}}^{[\mathbf{m}]}(\vec{\lambda})$ are irreducible.*

Proof. The same proof as Theorem 4.5 in [4] $\square$

By Proposition 4.4 and Theorem 4.2 , we have that the $\widehat{\mathcal{S}_q^M}$-modules $L_{\mathbf{s}}^{[\mathbf{m}]}(\vec{\lambda})$ are irreducible quasifinite highest weight modules. Using the formulas (3.4), it is easy to calculate the generating series $\Delta_{\mathbf{m},\mathbf{s},\lambda,i}(x) = \sum_n \Delta_{i,n} x^{-n}$ of the highest weight and the central charge c of the $\widehat{\mathcal{S}_q^M}$-module $L_{\mathbf{s}}^{[\mathbf{m}]}(\vec{\lambda})$. We have ($s = q^a$):

$$\begin{aligned}\Delta_{m,s,\lambda,i,n} = &-\sum_{l\in\mathbb{Z}} q^{(a-l)n} \sum_{j=0}^{n} \frac{(n\log q)^j}{j!} \lambda_{lM-i+1}^{(j)} \\ &- \frac{q^{an}}{(1-q^n)} \sum_{j=0}^{m} c_j \frac{(n\log q)^j}{j!}\end{aligned} \tag{4.5}$$

$$c = c_0 \tag{4.6}$$

and

$$\Delta_{\mathbf{m},\mathbf{s},\vec{\lambda},i,n} = \sum_j \Delta_{m_j,s_j,\lambda(j),i,n}, \qquad c = \sum_j c_0(j) \tag{4.7}$$

Introduce

$$g_k(n) = \sum_{j=0}^{m} h_k^{(j)} \frac{(n\log q)^j}{j!} \qquad (k\in\mathbb{Z}) \tag{4.8}$$

Then, (4.5) can be rewritten as follows:

$$\Delta_{m,s,\lambda,i,n} = \sum_{l\in\mathbb{Z}} \frac{q^{(a-l)n}}{1-q^n} (g_{lM-i+1}(n) + g_{lM-i+2}(n) + \cdots + g_{lM+M-i}(n)) - c_0 \tag{4.9}$$

Furthermore, notice that

$$\Delta_{m,s,\lambda,i,n} - \Delta_{m,s,\lambda,i+1,n} = \sum_{l\in\mathbb{Z}} q^{(a-l)n} g_{lM-i}(n) \qquad \text{for } i = 1,\dots,M-1 \tag{4.10}$$

Note that these formulas, imply that any irreducible quasifinite highest weight module $L(\widehat{\mathcal{S}_q^M},\lambda)$ can be obtained in as above an essentially unique way. More precisely, we state the following result.

Theorem 4.3. *Let $L = L(\widehat{\mathcal{S}_q^M},\lambda)$ be an irreducible quasifinite highest weight module with central charge c and*

$$\Delta_{i,n} - \Delta_{i+1,n} = P_i(n) \qquad \textit{for } i = 1,2,\dots,M-1$$

$$\Delta_{M,n} = \frac{\sum_{i=1}^M P_i(n) - c}{q^n - 1}$$

where $P_i(x)$ are quasipolynomials and $\sum_{i=1}^M P_i(0) = c$ (see Theorem 4.1). We write $P_i(x) = \sum_{s\in\mathbb{C}} p_{i,a}(x) q^{ax}$ where $p_{a,i}(x)$ are polynomials. We decompose the set $\{a \in$

$\mathbb{C} \mid p_{i,a} \neq 0$ for some $i\}$ *in a disjoint union of congruence classes* $\mod \mathbb{Z}+\tau^{-1}\mathbb{Z}$. *Let* $S=\{a, a-k_1, a-k_2, \dots\}$ *be such a congruence class, let* $m = \max_{a\in S} \deg p_{a,i}$ *and let* $h^{(j)}_{k_rM-i} = (\frac{d}{dx})^j p_{i,a-k_r}(0)$, *for* $i = 1, 2, \dots, M$. *We associate to* S *the* $\mathfrak{g}^{[m]}$*-module* $L^{[m]}(\lambda_S)$ *with central charges*

$$c_j = \sum_{k_rM-i} h^{(j)}_{k_rM-i},$$

and labels

$$\lambda_l^{(j)} = \sum_{(k_r+\delta_{i,M})M-i\geq l} (h^{(j)}_{k_rM-i} - c_j\delta_{(k_r+\delta_{i,M})M-i,0})$$

Then the $\widehat{\mathcal{S}_q^M}$*-module* L *is isomorphic to the tensor product of all the modules* $L_s^{[m]}(\lambda_S)$.

Proof. The tensor product L' of all the modules $L_s^{[m]}(\lambda_S)$ is irreducible due to Theorem 4.2. It remains to show that L' has the same highest weight as L does. This is done by exploiting the formulas (4.5-10) $\square$

5. Unitary Quasifinite Highest Weight Modules over $\widehat{\mathcal{S}_q^M}$

In this section we shall classify all unitary (irreducible) quasifinite highest weight modules over $\widehat{\mathcal{S}_q^M}$, $q \in \mathbb{R}$, with respect to the anti-involution ω on ${\mathcal{S}_q^M}^{as}$ defined by

$$\omega(z^k f(T_q)E_{ij}) = z^{-k}\overline{f}(q^{-k}T_q)E_{ji}.$$

where for $f(T_q)=\sum_i f_iT_q^i$, we let $\overline{f}(T_q) = \sum_i \overline{f_i}T_q^i (f_i \in \mathbb{C})$. This anti-involution ω extends to the whole algebra ${\mathcal{S}_q^M}^{\mathcal{O}^{as}}$.

Observe that

$$\Psi(\omega(A), \omega(B)) = \overline{\Psi(B,A)}, \qquad A, B \in {\mathcal{S}_q^M}^{\mathcal{O}^{as}}.$$

Therefore, the anti-involution ω of the Lie algebra $\mathcal{S}_q^M$ and ${\mathcal{S}_q^M}^{\mathcal{O}}$ lifts to an anti-involution of their central extensions $\widehat{\mathcal{S}_q^M}$ and $\widehat{{\mathcal{S}_q^M}^{\mathcal{O}}}$, such that $\omega(C) = C$, which we again denote by ω.

Remark 5.1. (a) Under the homomorphism $\varphi_s = \varphi_{s,0} : {\mathcal{S}_q^M}^{\mathcal{O}^{as}} \longrightarrow \widetilde{M}(\infty, \mathbb{C})$ we have

$$(\varphi_s(z^k f(T_q)E_{ij}))^* = \varphi_{\bar{s}}(\omega(z^k f(T_q)E_{ij})).$$

Here A^* stands for the complex conjugate transpose of the matrix $A \in \widetilde{M}(\infty, \mathbb{C})$.

(b) (see e.g. [3]) For the involution ω of $\widehat{gl}(\infty, \mathbb{C})$ defined by $\omega(A) = {}^t\bar{A}$, $\omega(1) = 1$, a highest weight $\widehat{gl}(\infty, \mathbb{C})$-module with highest weight λ and central charge c is unitary if and only if the numbers $h_i^{(0)}$ are non-negative integers and $c = \sum_i h_i^{(0)}$.

We shall need the following Lemma

Lemma 5.1. *Let V be a unitary quasifinite highest weight module over $\widehat{\mathcal{S}_q^M}$ and let $\mathbf{b}$ be its characteristic polynomials. Then $b_i(w)$ has only simple real roots for $i = 1, 2, \ldots, M$.*

Proof. Let $S_i = T_q E_{ii}$ for $i = 1, 2, \ldots, M-1$, $S_M = \frac{1}{q^{-1}-1}(T_q + \Delta_{M,1})E_{MM}$ and $v_i = E_{i,i+1}v_\lambda, i = 1, 2, \ldots, M-1$, $v_M = z^{-1}E_{M,1}v_\lambda$. Then replace $(s, z^{-1}v_\lambda, b(w))$ by $(S_i, v_i, b_i(w))$ in the proof of Lemma 5.2 in [4]. $\square$

Now we can state,

Theorem 5.1. *(a) A non-trivial quasifinite irreducible highest weight module $L(\widehat{\mathcal{S}_q^M}, \lambda)$ is unitary if and only if the central charge c is a positive integer and there exists $r_1, \cdots, r_c \in \mathbb{R}$ and a partition of $\{1, \cdots, c\} = I_0 \cup \cdots \cup I_{M-1}$, such that*

$$\Delta_{i,n} = \sum_{j \in I_0 \cup \cdots \cup I_{i-1}} \frac{q^{nr_j} - 1}{q^n - 1} + \sum_{j \in I_i \cup \cdots \cup I_{M-1}} \frac{q^{n(r_j+1)} - 1}{q^n - 1} \tag{5.1}$$

(b) Any unitary quasifinite $\widehat{\mathcal{S}_q^M}$-module is obtained by taking tensor product of n unitary irreducible quasifinite highest weight modules over $\widehat{gl}(\infty, \mathbb{C})$ and restricting to $\widehat{\mathcal{S}_q^M}$ via an embedding $\hat{\varphi}_{\mathbf{s}}^{[0]}$, where $\mathbf{s} = (s_1, \ldots, s_n)$ is a real vector satisfying (3.3).

Proof. By Theorem 4.3, being a quasifinite irreducible highest weight $\widehat{\mathcal{S}_q^M}$- module, V is isomorphic to one of the modules $L_{\mathbf{s}}^{[\mathbf{m}]}(\vec{\lambda})$. It follows from Lemma 5.1 and Proposition 4.2 that $\mathbf{m} = 0$. Now the claim (b) follows from Remark 5.1 and the claim (a) follows from (b) and (4.9-10). $\square$

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