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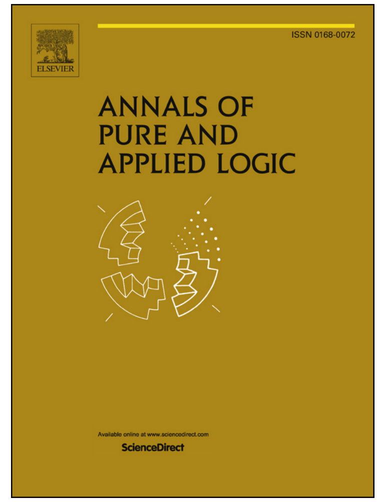

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Abstract

Consider any subvariety of BL-algebras generated by a single BL-chain which is the ordinal sum of the standard MV-algebra on [0,1] and a basic hoop $\mathbf{H}$. We present a geometrical characterization of elements in the finitely generated free algebra of each of these subvarieties. In this characterization there is a clear insight of the role of the regular and dense elements of the generating chain. As an application, we analyze maximal and prime filters in the free algebra.

Keywords: BL-algebras, Wajsberg hoops, Free algebras.
2008 MSC: 03G10, 03G27, 06B20, 03A99

1. Introduction

Basic Fuzzy Logic (BL for short) was introduced by Hájek in [19] to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on the real segment $[0, 1]$ and the implication by its corresponding adjoint. The equivalent algebraic semantics for BL, in the sense of Blok and Pigozzi, is the variety of BL-algebras $\mathcal{BL}$ ([19], [11]), that contains MV-algebras, Gödel algebras and Product algebras as proper subvarieties. Many algebraic properties of BL-algebras correspond to logical properties of BL. One of these properties, and what is our concern, is that the elements of free algebras in $\mathcal{BL}$ are in correspondence with equivalence classes of formulas in the logic. This is why many attempts to study free algebras in subvarieties of BL-algebras have been accomplished in the last decades. Some of these studies, as [14] and [7], describe free algebras in subvarieties of BL-algebras from an structural point of view, considering the representation of the algebra as weak boolean product of directly indecomposable BL-algebras over the Stone space corresponding to a free Boolean algebra. Some others provide a functional description of the elements in the free algebra. The most famous of such descriptions is the one of free algebras in the variety of MV-algebras presented by McNaughton in [22], which has been broadly used to investigate different aspects of Lukasiewicz's many-valued logic (see [13] and [25]). The case of Gödel functions is studied in [18] and functions in product logic are characterized in [15]. The functional description of the one-generated free BL-algebra is presented in [23] and it is generalized for the case of finitely many generators in [1] (see also [4]). The key point in these characterizations of free BL-algebras is that any n-generated BL-algebra is in the subvariety of BL-algebras generated by $n+1$-copies of the standard MV-chain on the real unit segment $[0,1]$, thus McNaughton functions can be used to describe the elements of the free algebra. But since the generating chain changes as the number of generators of the free algebra increases, the description of the functions on the algebra is recursive and it is hard to use it for further applications.

Our aim is to present a functional representation of the finitely generated free algebras in subvarieties of BL-algebras generated by a single BL-chain. The generating chain $\mathfrak{S}$ is the ordinal sum of the standard MV-algebra $[0,1]_{\mathbf{MV}}$ and an arbitrary totally ordered basic hoop $\mathbf{H}$, in symbols $\mathfrak{S} = [0,1]_{\mathbf{MV}} \oplus \mathbf{H}$. Therefore we are presenting a characterization of infinitely many free algebras in infinitely many subvarieties of BL-algebras. In particular, we are providing an alternative description of the free n-generated BL-algebra. The main advantage of this approach, is that unlike the work done in [1] and [4], when the number n of generators of the free algebra increase the generating chain remains fixed. This provides a clear insight of the role of the two main blocks of the generating chain in the description of the functions in the free algebra: the role of the regular elements and the role of the dense elements.

Once we fixed the chain $\mathbf{H}$ we denote $\mathcal{MS}$ the subvariety of BL-algebras that is our concern. To describe functions in the free algebra $Free_{\mathcal{MS}}(n)$ we first decompose the domain of the functions $\mathfrak{S}^n = ([0,1]_{\mathbf{MV}} \oplus \mathbf{H})^n$ in a finite number of pieces. In each piece a function in $Free_{\mathcal{MS}}(n)$ coincides either with a McNaughton function or a function in the free algebra in the subvariety of basic hoops generated by $\mathbf{H}$. Our description of functions in the free algebra in terms of the functions in the free algebras associated with the two blocks of the generating chain paves the way to understand some elements of the free BL-algebra. For example, we present a complete characterization of prime and maximal filters in the free BL-algebra.

The paper is organized as follows: In Section 2 we present all the background on hoops, BL-algebras and free algebras necessary to understand the main results of the paper. To have a geometrical intuition, in Section 3, we provide the characterization of the free algebras for the cases of one and two generators. These cases illustrate the behavior of term-functions in the different regions of the domain. Later, in Section 4 we present the general case, i.e., a characterization of free algebras in n generators. As an application, in the last section we analyze prime and maximal filters in these free algebras.

2. Preliminaries

2.1. Hoops and BL-algebras

A **basic hoop** is an algebra $\mathbf{A} = \langle A, \cdot, \rightarrow, \top\rangle$ of type $\langle 2, 2, 0\rangle$, such that $\langle A, \cdot, \top\rangle$ is a commutative monoid and for all $x, y, z \in A$:

1. $x \rightarrow x = \top$,
2. $x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)$,
3. $x \rightarrow (y \rightarrow z) = (x \cdot y) \rightarrow z$,
4. $(((x \rightarrow y) \rightarrow z) \cdot ((y \rightarrow x) \rightarrow z)) \rightarrow z = \top$.

A lattice order is defined in A by $x \leq y$ iff $x \rightarrow y = \top$ and the residuation condition that holds in A is

$$x \cdot y \leq z \text{ iff } x \leq y \rightarrow z.$$

A **BL-algebra** is a bounded basic hoop, that is, it is an algebra $\mathbf{A} = \langle A, \cdot, \rightarrow, \bot, \top\rangle$ of type $\langle 2, 2, 0, 0\rangle$ such that $\langle A, \cdot, \rightarrow, \top\rangle$ is a basic hoop and $\bot$ is the least element of $\mathbf{L}(\mathbf{A})$.

The varieties of BL-algebras and basic hoops will be denoted by $\mathcal{BL}$ and $\mathcal{BH}$, respectively. It is well known that both varieties are congruence distributive and congruence permutable.

As usual, three other important operations are defined in every BL-algebra $\mathbf{A}$. They are the negation and the lattice operations that are given by

$$\neg x = x \rightarrow \bot,$$

$$x \wedge y = x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x),$$
$$x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x).$$

Totally ordered BL-algebras, better known as **BL-chains** play a central role in the study of $\mathcal{BL}$ because they generate the whole variety and every proper subvariety ([2], [11]). Due to their importance, BL-chains have been deeply investigated ([2]) and different representation theorems for BL-chains can be found, most of them involving the decomposition into simpler structures which can be carried out considering the following ordinal sum construction ([2]): let $\mathbf{A}_1$ be a BL-chain and $\mathbf{A}_2$ a totally ordered basic hoop, and assume that $A_1 \cap A_2 = \{\top\}$. The **ordinal sum** is the BL-chain $\mathbf{A}_1 \oplus \mathbf{A}_2$ where the operations $\cdot, \rightarrow$ are given by:

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if } x, y \in A_i; \\ x & \text{if } x \in A_1 \setminus \{\top\},\ y \in A_2; \\ y & \text{if } y \in A_1 \setminus \{\top\},\ x \in A_2. \end{cases}$$

$$x \to y = \begin{cases} \top & \text{if } x \in A_1 \setminus \{\top\},\ y \in A_2; \\ x \to_i y & \text{if } x, y \in A_i; \\ y & \text{if } y \in A_1,\, x \in A_2. \end{cases}$$

Observe that in the ordinal sum $\mathbf{A}_1 \oplus \mathbf{A}_2$ all the elements in $A_1 \setminus \{\top\}$ are less than all the elements in the second summand A_2, as it is in the ordinal sum of posets.

We will recall one of the representations that we think is the most suitable to attack our problem, the one that decomposes each BL-chain into regular and dense elements. Given a BL-algebra $\mathbf{A}$, we can consider the set

$$MV(\mathbf{A}) = \{x \in A : \neg\neg x = x\}.$$

The algebra $\mathbf{MV}(\mathbf{A}) = \langle MV(\mathbf{A}), \cdot, \to, \bot, \top \rangle$ is an MV-algebra ([13]) which is a subalgebra of $\mathbf{A}$ whose elements are called **regular elements of A**.

If we also consider the set

$$D(\mathbf{A}) = \{x \in A : \neg x = \bot\},$$

the basic hoop $\mathbf{D}(\mathbf{A}) = \langle D(\mathbf{A}), \cdot, \to, \top \rangle$ contains all the **dense elements of A**.

Lemma 2.1. *([11, Theorem 3.3.1]) For each BL-chain $\mathbf{A}$, we have*

$$\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \oplus \mathbf{D}(\mathbf{A}).$$

Then for every element x in a BL-chain $\mathbf{A}$ we have that either:

$$x \in D(\mathbf{A}) \text{ so } \neg\neg x = \top \text{ and } \neg\neg x \to x = x \tag{1}$$

or

$$x \in MV(\mathbf{A}) \text{ so } \neg\neg x = x \text{ and } \neg\neg x \to x = \top \tag{2}$$

and $\top$ is the only element which is both: dense and regular. Because of this, every $x \in \mathbf{A}$ satisfies

$$x = (\neg\neg x) \cdot (\neg\neg x \to x),$$

where $\neg\neg x \in MV(\mathbf{A})$ and $\neg\neg x \to x \in D(\mathbf{A})$, then we can write any element of $\mathbf{A}$ as a product of two elements, one in each hoop of this decomposition.

2.2. Free algebras and term functions

For any $k \in \mathbb{N}$, a **BL-term** in the variables $x_1, \dots, x_k$ is a propositional formula in the language $\{\cdot, \to, \top, \bot\}$ whose variables are among $x_1, \dots, x_k$ (see [12, Chapter II]). A **hoop term** is a BL-term without the element $\bot$.

As usual, given a BL-term τ involving n variables $x_1, \dots, x_n$ and a BL-algebra $\mathbf{A}$ the corresponding term-function $\tau_{\mathbf{A}} : \mathbf{A}^n \to \mathbf{A}$ is defined inductively as follows:

- If $\tau = x_i$ for some $i = 1 \dots n$, then $\tau_{\mathbf{A}} = \pi_i$, i.e., the projection to the ith-coordinate. If $\tau = \perp$, $\tau_{\mathbf{A}}$ is the constant function $\perp$ and analogously if $\tau = \top$.
- if τ, υ are BL-terms, then $(\tau \cdot \upsilon)_{\mathbf{A}} = \tau_{\mathbf{A}} \cdot \upsilon_{\mathbf{A}}$ and $(\tau \to \upsilon)_{\mathbf{A}} = \tau_{\mathbf{A}} \to \upsilon_{\mathbf{A}}$.

Given a variety $\mathcal{V}$ of algebras which is generated by a single algebra $\mathbf{A}$, the free algebra of $\mathcal{V}$ on n generators $Free_{\mathcal{V}}(n)$ is isomorphic to the subalgebra of functions from A^n into A generated by the n-ary projections $\pi_1, \dots \pi_n$ over the variables $x_1, \dots, x_n$. In other words, $Free_{\mathcal{V}}(n)$ coincides with the algebra of equivalence classes of n variable term-functions in the language of the algebras of $\mathcal{V}$.

2.3. Functions of the free MV-algebra

We assume that the reader is familiar with the standard MV-algebra $[0,1]_{\mathbf{MV}}$ ([13]). This algebra is generic in the variety $\mathcal{MV}$ of MV-algebras and hence the free n-generated MV-algebra $Free_{\mathcal{MV}}(n)$ is the subalgebra of functions from $[0,1]^n_{\mathbf{MV}}$ to $[0,1]_{\mathbf{MV}}$ generated by the projections. These functions, known as McNaughton functions, can be described as follows:

Definition 2.2. ([22]) A continuous function $f : [0,1]^n \to [0,1]$ is a **McNaughton function** over $[0,1]^n$ if and only if there are finitely many linear polynomials $p_1, \dots, p_l$ with integer coefficients such that, for every $\bar{x} \in [0,1]^n$ there is $i \in \{1 \dots, l\}$ such that $f(\bar{x}) = p_i(\bar{x})$.

Theorem 2.3. *The free n-generated MV-algebra $Free_{\mathcal{MV}}(n)$ is isomorphic to the algebra of n-ary McNaughton functions.*

The description of $Free_{\mathcal{MV}}(n)$ as an algebra of continuous functions defined in the real unit interval allows the use of geometrical and topological techniques to study properties and characteristics of $\mathcal{MV}$ (see [25]).

We recall some of the definitions and results for the algebra $Free_{\mathcal{MV}}(n)$ given in [25] that will be needed in the next sections.

Definition 2.4. A point $\bar{x} \in \mathbb{R}^n$ is **rational** if all its coordinates are rational numbers, and a simplex T is called **rational** if every vertex of T is a rational point. Given a rational point $\bar{x} = (x_1, \dots, x_n) \in \mathbb{Q}^n$, we denote by $den(\bar{x})$ to the least common denominator of its coordinates, and by $\tilde{x}$ to the integer vector

$$\tilde{x} = \langle x_1 \cdot den(\bar{x}), \dots, x_n \cdot den(\bar{x}), den(\bar{x}) \rangle.$$

We say that a rational simplex $T \subseteq \mathbb{R}^n$ with vertices $v_1, \dots, v_m$, with $m \leq n$ is **unimodular** if the set $\{\tilde{v_1}, \dots, \tilde{v_m}\}$ can be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$.

A rational simplicial complex is said to be **unimodular** if every simplex in it is unimodular.

We also say that a triangulation is **unimodular** if its associated rational simplicial complex is **unimodular**.

Remark 2.5. Unimodular simplexes (complexes) are also known as regular simplexes (complexes), but we use the word unimodular in this work to avoid confusion with the regular elements in our generating chain.

Theorem 2.6. *(Corollary 2.10 of [25]) Let $\emptyset \neq P \subseteq [0,1]^n$. Then the following conditions are equivalent:*

1. *P is the support of some unimodular complex Δ.*
2. *$P = f^{-1}(1)$ for some $f \in Free_{\mathcal{MV}}(n)$.*
3. *P is a rational polyhedron.*

The next theorem is a consequence of Theorem 2.8 of [25] and the De Concini-Processi Lemma:

Theorem 2.7. *Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron and let Δ_1, Δ_2 be two unimodular triangulations of P. Then there is a unimodular triangulation Δ of P, called a refinement of Δ_1 and Δ_2, such that for each $T_1 \in \Delta_1$ and $T_2 \in \Delta_2$ there is a finite family of simplexes $S_i, i \in I$ in Δ such that $T_1 \cap T_2 = \bigcup_{i \in I} S_i$.*

Remark 2.8. The previous results imply that for any $f \in Free_{\mathcal{MV}}(n)$ there is a unimodular triangulation Δ of $[0,1]^n$ such that f is linear over each simplex of Δ.

Definition 2.9. Given a rational polyhedra $P \subseteq [0,1]^n$ and $\bar{x}_1, \ldots, \bar{x}_n$ rational points contained in P, we say that a rational triangulation Δ of P **respects** $\{\bar{x}_1, \ldots, \bar{x}_n\}$ if $\bar{x}_1, \ldots, \bar{x}_n$ are vertices of some simplexes of Δ.

Remark 2.10. As a consequence of Theorem 2.8 of [25] it can also be proved that for any simplicial complex $\mathcal{K} \subseteq [0,1]^n$ and $\bar{x}$ a rational point contained in $[0,1]^n$ there is a unimodular subdivision $\mathcal{K}_{\bar{x}}$ of $\mathcal{K}$ which respects $\bar{x}$.

Finally, we recall that if Δ is a unimodular triangulation of a rational polyhedron $P \in [0,1]^n$ and $T \in \Delta$ is a simplex, the set of faces of T is the set of elements $S \in \Delta$ such that $S \subseteq T$. The set of proper faces of T, that is $S \in \Delta$ such that $S \subsetneq T$ will be denoted by $\mathcal{F}_T$.

3. Characterization of free algebras: cases of one and two generators.

The subvariety of $\mathcal{BL}$ that we are planning to work with is going to be called $\mathcal{MS}$ and it is the variety generated by the BL-chain

$$\mathfrak{S} = [0,1]_{\mathbf{MV}} \oplus \mathbf{H}$$

where $\mathbf{H}$ is a fixed non-trivial totally ordered basic hoop. Then we have $\mathbf{MV}(\mathfrak{S}) \cong [0,1]_{\mathbf{MV}}$ and $\mathbf{D}(\mathfrak{S}) \cong \mathbf{H}$. We denote by $\mathcal{H}$ the subvariety of basic hoops generated by $\mathbf{H}$. Observe that the bottom element in $\mathfrak{S}$ is 0 and the top element is 1 which is in $[0,1]_{\mathbf{MV}} \cap \mathbf{H}$.

Our goal is to present a geometrical and functional description of the free algebra with n generators in $\mathcal{MS}$. That algebra, which we will be called $Free_{\mathcal{MS}}(n)$ is the subalgebra of functions from $\mathfrak{S}^n$ to $\mathfrak{S}$ generated by the n projection functions $\pi_1, \ldots \pi_n$. We will characterize term-functions evaluated in $\mathfrak{S}^n$ in terms of functions of $Free_{\mathcal{MV}}(n)$ and $Free_{\mathcal{H}}(m)$, with $m \leq n$.

It is important to recall that if $\mathbf{S}$ is a subalgebra of $\mathfrak{S}$ and f is a term-function in $Free_{\mathcal{MS}}(n)$, then for any $\bar{x} \in S^n$ then $f(\bar{x}) \in S$. In particular, $f(\bar{1}) \in \{0, 1\}$.

3.1. $Free_{\mathcal{MS}}(1)$

The case of the free algebra in one generator is an easy generalization of the description of the free algebra of one generator given in [23] (see also [1]), because that is a particular case when $\mathbf{H} = [0,1]_{\mathbf{MV}}$. Thus we give the explicit form of the functions in the algebra but we omit the proof.

Lemma 3.1. *Let $f \in Free_{\mathcal{MV}}(1)$ and $h \in Free_{\mathcal{H}}(1)$ be such that $f(1) = h(1) = 1$. Then the function*

$$\mathcal{F}(x) = \begin{cases} f(x) & \text{if} \quad x \in [0,1]_{\mathbf{MV}} \\ \\ h(x) & \text{if} \quad x \in \mathbf{H} \end{cases} \tag{3}$$

is in $Free_{\mathcal{MS}}(1)$. Conversely, for every function $\mathcal{F} \in Free_{\mathcal{MS}}(1)$ such that $\mathcal{F}(1) = 1$, there are two functions $f \in Free_{\mathcal{MV}}(1)$ and $h \in Free_{\mathcal{H}}(1)$ which satisfy (3).

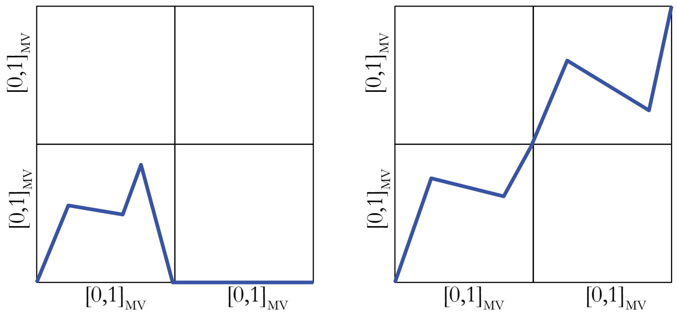
Lemma 3.2. *If $f \in Free_{\mathcal{MV}}(1)$ is a function such that $f(1) = 0$ then the function*

$$\mathcal{F}(x) = \begin{cases} f(x) & \text{if} \quad x \in [0,1]_{\mathbf{MV}} \\ \\ 0 & \text{if} \quad x \in \mathbf{H} \end{cases} \tag{4}$$

is in $Free_{\mathcal{MS}}(1)$. Conversely, if $\mathcal{F} \in Free_{\mathcal{MS}}(1)$ is such that $\mathcal{F}(1) = 0$, then there is a function $f \in Free_{\mathcal{MV}}(1)$ which satisfies (4).

So we have described all the functions in $Free_{\mathcal{MS}}(1)$.

Remark 3.3. For the case of $\mathbf{H} = [0,1]_{\mathbf{MV}}$, $Free_{\mathcal{MS}}(1)$ is the free BL-algebra with one generator ([23]). In the next figure we can see two examples of functions of this free BL-algebra. Note that in the first case, $\mathcal{F}(1) = 0$ and hence $\mathcal{F}(x) = 0$ for every x in the second summand of the generating chain, and in the second function, since $\mathcal{F}(1) = 1$ then the restriction of $\mathcal{F}$ to the second summand coincides with a function $g \in Free_{\mathcal{MV}}(n)$ with $g(1) = 1$.



3.2. $Free_{\mathcal{MS}}(2)$

To pave the way for the general case, we first present the details of the case of two generators. Recall that the chain that generates our variety is

$$\mathfrak{S} = [0,1]_{\mathbf{MV}} \oplus \mathbf{H}.$$

So we have to describe term-functions from $\mathfrak{S}^2$ to $\mathfrak{S}$. To achieve this aim, we will study the behavior of a term-function in each of the four regions of the domain:

$$R_1 = [0,1]^2_{\mathbf{MV}}, \quad R_2 = [0,1]_{\mathbf{MV}} \times \mathbf{H}, \quad R_3 = \mathbf{H} \times [0,1]_{\mathbf{MV}}, \quad \text{and} \quad R_4 = \mathbf{H}^2.$$

R_2	R_4
R_1	R_3

To succeed, we need to understand properly the division into regions: on one hand it is clear that $\mathfrak{S}^2 = \bigcup_{i=1}^4 R_i$. But from the definition of ordinal sum, since $[0,1]_{\mathbf{MV}} \cap \mathbf{H} = \{1\}$, these regions are not mutually disjoint. Indeed, if we define the **relative border** of the region R_1 as the set

$$\eth[0,1]^2_{\mathbf{MV}} = \{(x_1, x_2) \in [0,1]^2_{\mathbf{MV}} : x_i = 1 \text{ for some } 1 \le i \le 2\},$$

then for any $i = 2,3,4$ we have

$$R_1 \cap R_i \subseteq \eth[0,1]^2_{\mathbf{MV}}.$$

More precisely,

$$R_1 \cap R_2 = [0,1]_{\mathbf{MV}} \times \{1\}, \quad R_1 \cap R_3 = \{1\} \times [0,1]_{\mathbf{MV}}$$

and

$$R_2 \cap R_4 = \{1\} \times \mathbf{H}, \quad R_3 \cap R_4 = \mathbf{H} \times \{1\} \quad \text{and} \quad R_1 \cap R_4 = R_2 \cap R_3 = \{(1,1)\}.$$

These non-empty intersections will play a crucial role in the description of the functions. We will also need the following definition.

Definition 3.4. Given an interval $I \subseteq [0,1]_{\mathbf{MV}} \times \{1\}$ the **cylindrification** $\tilde{I}$ of I in R_2 will be the set

$$\tilde{I} = \{(x,y) \in R_2 : x \in I \text{ and } y \in \mathbf{H}\}.$$

Analogously one can define the cylindrification of $I \subseteq \{1\} \times [0,1]_{\mathbf{MV}}$ in R_3. In case I is just a point, i.e., $I = \bar{x}$ we write $\tilde{x}$ for the cylindrification of $\bar{x}$. This means that if $\bar{x} = (x,1)$ then $\tilde{x} = \{(x,y) : y \in \mathbf{H}\}$.

3.2.1. From term-functions to quadruples

Let's fix a BL-term α with two variables. If we denote by $\alpha_{\mathfrak{S}^2}$ the two-variables term-function in $Free_{\mathcal{MS}}(2)$ associated with α, our goal is to describe $\alpha_{\mathfrak{S}^2}$ as a quadruple of functions $\alpha_{\mathfrak{S}^2} = (f_1, f_2, f_3, f_4)$, each of them corresponding to the term function $\alpha_{R_i} = f_i$. Since α is a two-variables BL-term, it is also a term in the language of MV-algebras. From Theorem 2.3, there is a McNaughton function $f \in Free_{\mathcal{MV}}(2)$ such that

$$\alpha_{R_1} = \alpha_{[0,1]^2_{\mathbf{MV}}} = f. \tag{5}$$

The description of the term-functions in the other regions strongly depends on f. To prove this, we need some technical lemmas, whose proofs can be obtained by induction on the complexity of the formula α and the definition of the operations in the ordinal sum.

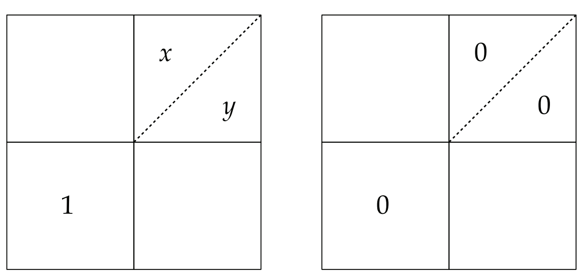
Lemma 3.5. *The following hold:*

- *If* $\alpha_{\mathfrak{S}^2}(1,1) = 1$ *then there is a function* $g \in Free_{\mathcal{H}}(2)$ *such that* $\alpha_{R_4} = g$.
- *If* $\alpha_{\mathfrak{S}^2}(1,1) = 0$ *then* $\alpha_{R_4} = 0$, *i.e.,* $\alpha_{\mathfrak{S}^2}$ *takes the value* 0 *all over* R_4.

As an example of the previous Lemma, one can consider the two-variable BL-term

$$\alpha = (\neg\neg x \to x) \wedge (\neg\neg y \to y).$$

Observe that if $(x,y) \in H^2$, from equation (1) we have that $\neg\neg x \to x = x$ and $\neg\neg y \to y = y$. Then $\alpha_{\mathfrak{S}^2}(1,1) = 1$ and then the function $g \in Free_{\mathcal{H}}(2)$, given by $g(x,y) = min\{x,y\}$ is such that $\alpha_{R_4} = g$. If $\beta = \neg\alpha$ then we have that $\beta_{\mathfrak{S}^2}(1,1) = 0$ and $\beta_{\mathfrak{S}^2}$ takes the value 0 all over R_4.



Since there is a symmetry between R_2 and R_3, to see what happens in these two remaining parts of the domain, we simply study α_{R_2}. As usual, the notation $[0,1)_{\mathbf{MV}}$ indicates the semiopen interval $[0,1]_{\mathbf{MV}} \setminus \{1\}$.

Lemma 3.6. *Let $x, z \in [0,1)_{\mathbf{MV}}$. Then we have:*

- *If $\alpha_{\mathfrak{S}^2}(x,1) = z$ then $\alpha_{\mathfrak{S}^2}(x,y) = z$, for every $y \in \mathbf{H}$.*
- *If $\alpha_{\mathfrak{S}^2}(x,1) = 1$ then there is a function $g \in Free_{\mathcal{H}}(1)$ such that $\alpha_{\mathfrak{S}^2}(x,y) = g(y)$, for every $y \in \mathbf{H}$.*

Lemma 3.6 asserts that if $x \in [0,1)_{\mathbf{MV}}$ is such that $\alpha_{\mathfrak{S}^2}(x,1) = 1$ then α_{R_2} coincides with a function $g \in Free_{\mathcal{H}}(1)$ in the cylindrification of $(x,1)$ in R_2, that is, for all the points of the form $(x,y) \in R_2$ with $y \in \mathbf{H}$. It also asserts that if $x, z \in [0,1)_{\mathbf{MV}}$ and $\alpha_{\mathfrak{S}^2}(x,1) = z$, then $\alpha_{R_2}(x,y) = z$ in the cylindrification of $(x,1)$ in R_2. Using this information, in what follows we show that we can partition the intersection $R_1 \cap R_2 = [0,1]_{\mathbf{MV}} \times \{1\}$ into a finite number of pieces to completely describe the behavior of α_{R_2} in the cylindrifications of the different parts.

First we see that it can be the case that $\alpha_{\mathfrak{S}^2}$ coincides with a function $g \in Free_{\mathcal{H}}(1)$ in the cylindrification of an interval of the form $I \times \{1\}$ for $I \subseteq [0,1)$, i.e., it coincides with g in $I \times \mathbf{H} \subseteq R_2$. To do so, we will write $\alpha_{\mathfrak{S}^2}(I,1) = 1$ whenever $\alpha_{\mathfrak{S}^2}(x,1) = 1$ for every $x \in I$.

Lemma 3.7. *Let $I \subseteq [0,1)$ be an interval and assume that $\alpha_{\mathfrak{S}^2}(I,1) = 1$. If for every subterm β of α we have that either $\beta_{\mathfrak{S}^2}(I,1) \in [0,1)_{\mathbf{MV}}$ or $\beta_{\mathfrak{S}^2}(I,1) = 1$, then there is a function $g \in Free_{\mathcal{H}}(1)$ such that $\alpha_{\mathfrak{S}^2}(I,y) = g(y)$, for every $y \in \mathbf{H}$.*

Proof. By Lemma 3.6, it is enough to prove that if α is such that $\alpha_{\mathfrak{S}^2}(I,1) = 1$ and for every subterm β of α we have that either $\beta_{\mathfrak{S}^2}(I,1) \in [0,1)_{\mathbf{MV}}$ or $\beta_{\mathfrak{S}^2}(I,1) = 1$, then for every pair $x_1, x_2 \in I$ with $x_1 \neq x_2$

$$\alpha_{\mathfrak{S}^2}(x_1,y) = \alpha_{\mathfrak{S}^2}(x_2,y) \tag{6}$$

for each $y \in \mathbf{H}$. We will show this by induction in the complexity of the term α.

If α is a term of complexity 0 such that $\alpha_{\mathfrak{S}^2}(I,1) = 1$ then we have two possibilites:

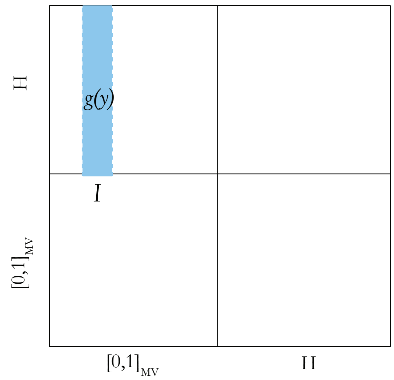
1. $\alpha = y$, then $\alpha_{\mathfrak{S}^2}(x_1,y) = y = \alpha_{\mathfrak{S}^2}(x_2,y)$, for every $x_1, x_2 \in I, y \in \mathbf{H}$.
2. $\alpha = 1$, then $\alpha_{\mathfrak{S}^2}(x_1,y) = 1 = \alpha_{\mathfrak{S}^2}(x_2,y)$, for every $x_1, x_2 \in I, y \in \mathbf{H}$.

Suppose that the statement holds for terms of complexity less than k and let α be a term of complexity k. Then we have two cases to consider:

1. $\alpha = \phi \cdot \psi$, with ϕ and ψ subterms of α of complexity less than k. Since $\alpha_{\mathfrak{S}^2}(I,1) = 1$ then necessarily $\phi_{\mathfrak{S}^2}(I,1) = 1$ and $\psi_{\mathfrak{S}^2}(I,1) = 1$. By inductive hypothesis, for every $x_1, x_2 \in I$ such that $x_1 \neq x_2$,

   $$\alpha_{\mathfrak{S}^2}(x_1,y) = \phi_{\mathfrak{S}^2}(x_1,y) \cdot \psi_{\mathfrak{S}^2}(x_1,y) = \phi_{\mathfrak{S}^2}(x_2,y) \cdot \psi_{\mathfrak{S}^2}(x_2,y) = \alpha_{\mathfrak{S}^2}(x_2,y),$$

   for every $y \in \mathbf{H}$, so the statement holds.



2. $\alpha = \phi \to \psi$, with ϕ and ψ terms of complexity less than k. By hypothesis we know that for every subterm β of α we have that either $\beta_{\mathfrak{G}^2}(I,1) \in [0,1)_{\mathbf{MV}}$ or $\beta_{\mathfrak{G}^2}(I,1) = 1$, then we have only three cases to consider for ϕ and ψ:

   (a) If $\phi_{\mathfrak{G}^2}(I,1) = 1$ and $\psi_{\mathfrak{G}^2}(I,1) = 1$: by inductive hypothesis we have that for $x_1, x_2 \in I$ such that $x_1 \neq x_2$,

$$\begin{aligned}\alpha_{\mathfrak{G}^2}(x_1,y) &= \phi_{\mathfrak{G}^2}(x_1,y) \to \psi_{\mathfrak{G}^2}(x_1,y)\\ &= \phi_{\mathfrak{G}^2}(x_2,y) \to \psi_{\mathfrak{G}^2}(x_2,y) = \alpha_{\mathfrak{G}^2}(x_2,y).\end{aligned}$$

   (b) If $\phi_{\mathfrak{G}^2}(I,1) \subseteq [0,1)_{\mathbf{MV}}$ and $\psi_{\mathfrak{G}^2}(I,1) \subseteq [0,1)_{\mathbf{MV}}$: by Lemma 3.6 we have $\phi_{\mathfrak{G}^2}(x_1,y) = \phi_{\mathfrak{G}^2}(x_1,1)$ for every $y \in \mathbf{H}$, and $\psi_{\mathfrak{G}^2}(x_1,y) = \psi_{\mathfrak{G}^2}(x_1,1)$ for every $y \in \mathbf{H}$. Analogously, $\phi_{\mathfrak{G}^2}(x_2,y) = \phi_{\mathfrak{G}^2}(x_2,1)$ for every $y \in \mathbf{H}$, and $\psi_{\mathfrak{G}^2}(x_2,y) = \psi_{\mathfrak{G}^2}(x_2,1)$ for every $y \in \mathbf{H}$. Since $\alpha_{\mathfrak{G}^2}(x_1,1) = \alpha_{\mathfrak{G}^2}(x_2,1) = 1$ then we have $\phi_{\mathfrak{G}^2}(x_1,1) \leq \psi_{\mathfrak{G}^2}(x_1,1)$ and $\phi_{\mathfrak{G}^2}(x_2,1) \leq \psi_{\mathfrak{G}^2}(x_2,1)$, and therefore

$$\alpha_{\mathfrak{G}^2}(x_1,y) = \phi_{\mathfrak{G}^2}(x_1,y) \to \psi_{\mathfrak{G}^2}(x_1,y) = \phi_{\mathfrak{G}^2}(x_1,1) \to \psi_{\mathfrak{G}^2}(x_1,1) = 1$$

   and

$$\alpha_{\mathfrak{G}^2}(x_2,y) = \phi_{\mathfrak{G}^2}(x_2,y) \to \psi_{\mathfrak{G}^2}(x_2,y) = \phi_{\mathfrak{G}^2}(x_2,1) \to \psi_{\mathfrak{G}^2}(x_2,1) = 1,$$

   so the statement holds for this case.

   (c) If $\phi_{\mathfrak{G}^2}(I,1) \subseteq [0,1)_{\mathbf{MV}}$ and $\psi_{\mathfrak{G}^2}(I,1) = 1$ we can prove the result using similar ideas to the ones used in the previous case.

□

Given a rational polyhedron P in $[0,1)_{\mathbf{MV}} \times \{1\}$ and a unimodular triangulation Δ of P, let S be a simplex in Δ. We denote by S° the relative interior of S when the dimension of S is one and $S^\circ = S$ if the dimension of S is zero. Then

S° is either a rational point or $S^\circ = I \times \{1\}$ for some open rational interval $I \subseteq [0,1)_{\mathbf{MV}}$. We shall work with the cylindrification of S in R_2, that is

$$\tilde{S}^\circ = \{(x,y) \in R_2 : (x,1) \in S^\circ\}.$$

Lemma 3.8. *Let P be a rational polyhedron on $[0,1)_{\mathbf{MV}} \times \{1\}$. If $\alpha_{\mathfrak{S}^2}(P) = 1$, then there is a unimodular triangulation Δ of P and a family of functions $\{g_S\}_{S\in\Delta}$ such that $g_S \in Free_{\mathcal{H}}(1)$ and*

$$\alpha_{\mathfrak{S}^2}(x,y) = g_S(y)$$

for every $(x,y) \in \tilde{S}^\circ$.

Proof. For each subterm β of α the term function β_P is the restriction to P of a McNaughton function. Let Δ be a unimodular triangulation of P that respects every function β_P for each subterm β of α, i.e., for each β subterm of α the function β_P is linear over each simplex S of Δ. If S is a rational point, from Lemma 3.6 there is $g_S \in Free_{\mathcal{H}}(1)$ such that

$$\alpha_{\mathfrak{S}^2}(x,y) = g_S(y)$$

for every $(x,y) \in \tilde{S}^\circ$. If S is one dimensional, Lemma 3.7 provides $g_S \in Free_{\mathcal{H}}(1)$

$$\alpha_{\mathfrak{S}^2}(x,y) = g_S(y)$$

for every $(x,y) \in \tilde{S}^\circ$ and we are done. □

We are now able to characterize the function α_{R_2}, whose domain is $R_2 = [0,1]_{\mathbf{MV}} \times \mathbf{H}$. According to Lemma 3.6 the behavior of the function $\alpha_{R_1} = f$ on the relative border $R_1 \cap R_2 \subseteq \eth[0,1]^2_{\mathbf{MV}}$ will determinate the value of the function in the rest of the domain. Let

$$1_{f,x} = \{(x,1) \in R_1 \cap R_2 : f(x,1) = 1\}.$$

The complement of $1_{f,x}$ relative to the relative border is the set

$$0_{f,x} = (R_1 \cap R_2) \setminus 1_{f,x} = \{(x,1) \in R_1 \cap R_2 : f(x,1) < 1\}.$$

If $\tilde{1}_{f,x}$ denotes the cylindrification of $1_{f,x}$ in R_2 and $\tilde{0}_{f,x}$ the cylindrification of $0_{f,x}$ in R_2 we observe that

$$R_2 = \tilde{1}_{f,x} \cup \tilde{0}_{f,x}.$$

With this notation define:

Definition 3.9. Given a function $f \in Free_{\mathcal{MV}}(2)$ we say that $g : R_2 \to \mathfrak{S}$ is an f-y-H-McNaughton function if the following conditions hold:

1. For each $(x,y) \in \tilde{0}_{f,x}$, $\;g(x,y) = f(x,1)$.
2. There is a unimodular triangulation Δ of $1_{f,x}$ which determines simplexes $S_1,\ldots,S_m$ and m functions in $Free_{\mathcal{H}}(1)$, $g_1,\ldots,g_m$, such that $g(x,y) = g_i(y)$, for every x in $\tilde{S}_i^\circ$.

Theorem 3.10. *Considering that $\alpha_{R_1} = f$ for the McNaughton function f, there is an f-y-H-McNaughton function h_y such that*

$$\alpha_{R_2} = h_y.$$

Proof. If $(x, y) \in \tilde{0}_{f,x}$ then $f(x, 1) \in [0, 1)_{\mathbf{MV}}$, and from Lemma 3.6 we get $\alpha_{R_2}(x, y) = f(x, 1)$. Otherwise, $(x, y) \in \tilde{1}_{f,x}$. But $1_{f,x}$ is the support of rational polyehdra thus from Lemma 3.8 there are a unimodular triangulation Δ of $1_{f,x}$ and a family $\{g_S\}_{S\in\Delta}$ of functions in $Free_{\mathcal{H}}(1)$ such that $\alpha_{R_2}(x, y) = g_S(y)$, for every x in $\tilde{S}^\circ$. Then α_{R_2} is an f-y-H-McNaughton function. $\square$

In a symmetric way we can define a *f-x-H-McNaughton function* and prove that:

Theorem 3.11. *Considering that $\alpha_{R_1} = f$ for the McNaughton function f, there is an f-x-H-McNaughton function h_x such that*

$$\alpha_{R_3} = h_x.$$

Definition 3.12. Given four functions $f \in Free_{\mathcal{MV}}(2)$, $g \in Free_{\mathcal{H}}(2) \cup \{0\}$ and h_x, h_y f-x-H-McNaughton and f-y-H-McNaughton functions respectively, we say that a function $\mathcal{F} : \mathfrak{S}^2 \to \mathfrak{S}$ is given by a MS-quadruple (f, h_x, h_y, g) if it satisfies:

$$\mathcal{F}(x, y) = \begin{cases} f(x, y) & \text{if} \quad (x, y) \in [0, 1]^2_{\mathbf{MV}} \\ h_x(x, y) & \text{if} \quad (x, y) \in [0, 1]_{\mathbf{MV}} \times \mathbf{H} \\ h_y(x, y) & \text{if} \quad (x, y) \in \mathbf{H} \times [0, 1]_{\mathbf{MV}} \\ g(x, y) & \text{if} \quad (x, y) \in \mathbf{H} \times \mathbf{H} \end{cases} \tag{7}$$

whenever $\mathcal{F}(1, 1) = 1$,

or

$$\mathcal{F}(x, y) = \begin{cases} f(x, y) & \text{if} \quad (x, y) \in [0, 1]^2_{\mathbf{MV}} \\ h_x(x, y) & \text{if} \quad (x, y) \in [0, 1]_{\mathbf{MV}} \times \mathbf{H} \\ h_y(x, y) & \text{if} \quad (x, y) \in \mathbf{H} \times [0, 1]_{\mathbf{MV}} \\ 0 & \text{if} \quad (x, y) \in \mathbf{H} \times \mathbf{H} \end{cases} \tag{8}$$

whenever $\mathcal{F}(1, 1) = 0$.

We conclude:

Theorem 3.13. *Given a two-variable BL-term α, the function $\alpha_{\mathfrak{S}^2} = \mathcal{F}$ is given by the MS-quadruple*

$$\mathcal{F} = (\alpha_{R_1}, \alpha_{R_2}, \alpha_{R_3}, \alpha_{R_4}).$$

3.2.2. From quadruples to term-functions

We will now prove that for every function $\mathcal{F}$ given by an MS-quadruple there is a two-variables BL-term whose evaluation on $\mathfrak{S}^2$ coincides with $\mathcal{F}$. Then we can conclude that the functions of $Free_{\mathcal{MS}}(2)$ are all given by quadruples. To that aim we fix an MS-quadruple

$$\mathcal{F} = (f_1, f_2, f_3, f_4).$$

$f_2(x, y)$	$f_4(x, y)$
$f_1(x, y)$	$f_3(x, y)$

To build the corresponding term we proceed as follows: we will find four two-variables BL-terms $\alpha^1, \alpha^2, \alpha^3$, and α^4, which are related to the four regions of the domain R_1, R_2, R_3 and R_4 and we will show that the BL-term

$$\alpha = \bigwedge_{i=1}^{4} \alpha^i \tag{9}$$

satisfies $\alpha_{\mathfrak{S}^2} = \mathcal{F}$.

Before reaching our main result we need to prove the existence of some auxiliary two-variables terms.

Lemma 3.14. *Given $g \in Free_{\mathcal{H}}(1)$ and a rational point $\bar{x}_0 = (x_0, 1) \in [0, 1)_{\mathbf{MV}} \times \{1\}$, there is a term $\mu_{\bar{x}_0}$ in two variables whose evaluation on $\mathfrak{S}^2$ satisfies:*

$$\mu_{\bar{x}_0 \mathfrak{S}^2}(x, y) = \begin{cases} g(y) & \text{if } (x, y) \in \tilde{x}_0 \\ \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

Proof. Since $x_0 \in [0, 1)_{\mathbf{MV}}$ is a rational number, there is a McNaughton function $f \in Free_{\mathcal{MV}}(2)$ such that $\bar{x}_0 = f^{-1}(\{1\})$. Let ϕ be a two-variables BL-term such that $\phi_{[0,1]^2_{\mathbf{MV}}} = f$, i.e., $\phi(\bar{x}_0) = 1$ and for every $\bar{x} \neq \bar{x}_0$ we have that $\phi(\bar{x}) \in [0, 1)_{\mathbf{MV}}$. From Lemma 3.6 we know that $\phi_{\mathfrak{S}^2}(x, y) \in [0, 1)_{\mathbf{MV}}$ for any $(x, y) \in \mathfrak{S}^2 \setminus \{\tilde{x}_0\}$. The same Lemma implies that $\phi_{\mathfrak{S}^2}(x, y) \in \mathbf{H}$ for each $(x, y) \in \tilde{x}_0$. Let $\tau = \neg\neg\phi$. Then from equation 2, if $(x, y) \in \mathfrak{S}^2 \setminus \{\tilde{x}_0\}$ we get that $\tau_{\mathfrak{S}^2}(x, y) = \phi_{\mathfrak{S}^2}(x, y)$ and from equation 1 we have that $\tau_{\mathfrak{S}^2}(x, y) = 1$ for each $(x, y) \in \tilde{x}_0$. Summing up

$$\tau_{\mathfrak{S}^2}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \tilde{x}_0 \\ \\ \phi(x, y) & \text{otherwise.} \end{cases} \tag{11}$$

Now we consider a one-variable hoop term ψ such that $\psi_H = g$ and the two-variables BL-term

$$\varphi(x,y) = \tau(x,y) \wedge \psi(y).$$

Since $\tau_{\mathfrak{S}^2}(x,y) \in [0,1)_{\mathbf{MV}}$ for each $(x,y) \notin \tilde{x}_0$ then $\varphi_{\mathfrak{S}^2}(x,y) = g(y)$ if $x = x_0$ and $y \in \mathbf{H}$ and $\phi_{\mathfrak{S}^2}(x,y) \in [0,1)_{\mathbf{MV}}$ otherwise. Another application of equations 1 and 2 guarantee that the term

$$\mu_{x_0}(x,y) = \neg\neg\varphi(x,y) \to \varphi(x,y)$$

satisfies

$$\mu_{x_0,\mathfrak{S}^2}(x,y) = \begin{cases} g(y) & \text{if } (x,y) \in \tilde{x}_0 \\ \\ 1 & \text{otherwise.} \end{cases} \tag{12}$$

□

Symmetrically we can prove the following result:

Lemma 3.15. *Given $g \in Free_{\mathcal{H}}(1)$ and a rational point $\bar{y}_0 = (1, y_0) \in \{1\} \times [0,1)_{\mathbf{MV}}$, there is a term $\nu_{\bar{y}_0}$ in two variables whose interpretation on $\mathfrak{S}^2$ satisfies:*

$$\nu_{\bar{y}_0\mathfrak{S}^2}(x,y) = \begin{cases} g(x) & \textit{if } (x,y) \in \tilde{y}_0 \\ \\ 1 & \textit{otherwise.} \end{cases} \tag{13}$$

Lemma 3.16. *Given $g \in Free_{\mathcal{H}}(1)$ and I a rational open interval contained in $[0,1]_{\mathbf{MV}}$, there is a two-variable BL-term γ_I that satisfies*

$$\gamma_{I\mathfrak{S}^2}(x,y) = \begin{cases} g(y) & \textit{if } (x,y) \in \tilde{I} \\ \\ 1 & \textit{otherwise.} \end{cases} \tag{14}$$

Proof. Let I be an open interval contained in $[0,1]$ with rational extrema. We know that the complement I^C of I in $[0,1]$ is a rational polyhedra. From Theorem 2.6, $I^C = h^{-1}(\{1\})$ for some function $h \in Free_{\mathcal{MV}}(1)$. Let ϕ be a one-variable BL-term such that $\phi_{[0,1]_{\mathbf{MV}}} = h$, that is, for each $x \in [0,1]_{\mathbf{MV}}$

$$\phi_{[0,1]_{\mathbf{MV}}}(x) = \begin{cases} h(x) \in [0,1)_{\mathbf{MV}} & \text{if} \quad x \in I \\ \\ 1 & \text{if} \quad x \in I^C \end{cases}$$

Consider the term $\varphi(x) = \phi(\neg\neg x)$ and its corresponding term-function on the algebra $\mathfrak{S}$. From equation 2, for $x \in [0,1]_{\mathbf{MV}}$ we have $\varphi_{\mathfrak{S}}(x) = \phi_{\mathfrak{S}}(x)$ and from equation 1 for $x \in \mathbf{H}$ we have that $\varphi_{\mathfrak{S}}(x) = \phi_{\mathfrak{S}}(1) = 1$. Thus for each $x \in \mathfrak{S}$ we have:

$$\varphi_{\mathfrak{S}}(x) = \begin{cases} h(x) \in [0,1]_{\mathbf{MV}} & \text{if } x \in I \\ \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand, since $g \in Free_{\mathcal{H}}(1)$, let δ be a one-variable hoop-term such that $\delta_H = g$. Recalling equation 1, we can consider $\neg\neg\delta \to \delta$, that satisfies for each $y \in \mathfrak{S}$

$$(\neg\neg\delta \to \delta)_{\mathfrak{S}}(y) = \begin{cases} g(y) & \text{if} \quad y \in \mathbf{H} \\ \\ 1 & \text{if} \quad y \in [0,1]_{\mathbf{MV}} \end{cases}$$

Thus the BL-term $\gamma_I(x,y) = [(\neg\neg\delta \to \delta)(y)) \vee \varphi(x)]$ has as interpretation on $\mathfrak{S}^2$ the function:

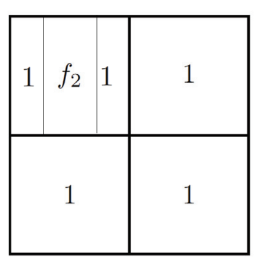
$$\gamma_{I\mathfrak{S}^2}(x,y) = \begin{cases} g(y) & \text{if } (x,y) \in \tilde{I} \\ \\ 1 & \text{otherwise.} \end{cases}$$

□

Recall the given quadruple $\mathcal{F} = (f_1, f_2, f_3, f_4)$. Since f_2 is an f_1-y-H-McNaughton function, then there is a unimodular triangulation Δ of $1_{f_1,x}$ which determines simplexes $S_1, \ldots, S_m$ and m functions in $Free_{\mathcal{H}}(1)$, $g_1, \ldots, g_m$, such that $f_2(x,y) = g_i(y)$, for every $(x,y) \in \tilde{S}_i^{\circ}$. From the previous result we can consider the terms $\gamma_{S_1}, \gamma_{S_2}, \ldots, \gamma_{S_m}$. Then $\alpha^2 = \bigwedge_{i=1}^{m} \gamma_{S_i}$ coincides with the function f_2 not in all R_2 but in $\tilde{1}_{f_1,x}$ the cylindrification of $1_{f_1,x}$. That is, it satisfies that

$$\alpha^2_{\mathfrak{S}^2}(x,y) = \begin{cases} f_2(x,y) & \text{if } (x,y) \in \tilde{1}_{f_1,x} \\ \\ 1 & \text{otherwise.} \end{cases}$$

graphically:



In a symmetrical way one can obtain a two-variable term α^3 such that

$$\alpha^3_{\mathfrak{S}^2}(x,y) = \begin{cases} f_3(x,y) & \text{if } (x,y) \in \tilde{1}_{f_1,y} \\ \\ 1 & \text{otherwise.} \end{cases}$$

graphically:

1	1
1	1 f_3 1

Lemma 3.17. *Given a function $g \in Free_{\mathcal{H}}(2)$ there is a two-variables BL-term η such that*

$$\eta_{\mathfrak{S}^2}(x,y) = \begin{cases} g(x,y) & \textit{if } (x,y) \in R_4 \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. Graphically, we need to find a BL-term η with two variables whose interpretation on $\mathfrak{S}^2$ is:

1	$g(x,y)$
1	1

Consider equations 1 and 2. Then $\beta_x = ((\neg\neg x \to x) \vee (\neg\neg y)) \to (\neg\neg x \to x)$ has as corresponding term-function on $\mathfrak{S}^2$

1	x
1	1

and the term $\beta_y = ((\neg\neg y \to y) \vee (\neg\neg x)) \to (\neg\neg y \to y)$ has as interpretation on $\mathfrak{S}^2$

1	y
1	1

Let $\bar{\eta}$ be a term such that $\bar{\eta}_H = g$, i.e., such that $g(x,y) = \bar{\eta}_{H^2}(x,y)$, for every $(x,y) \in H^2$. Since $\bar{\eta}(1,1) = 1$, then we take $\eta(x,y) = \bar{\eta}(\beta_x, \beta_y)$ and we conclude the proof. $\square$

As a consequence we obtain that if f_4 in the quadruple $\mathcal{F}$ is in $Free_{\mathcal{H}}(2)$ then there is a two-variables BL-term η such that $\eta_{\mathfrak{G}^2}$ satisfies

1	f_4
1	1

So we define

$$\alpha^4 = \begin{cases} \eta & \text{if } f_4 \in Free_{\mathcal{H}}(2)\ ; \\ \top & \text{otherwise.} \end{cases} \tag{15}$$

To complete the proof, let $\bar{\alpha}$ be a two-variables BL-term such that $\bar{\alpha}_{\mathfrak{G}^2} = f_1$. Then consider

$$\alpha^1 = \neg\neg\bar{\alpha}. \tag{16}$$

If $f_1(1,1) = 1$, by Lemmas 3.5 and 3.6 we have

$$\alpha^1_{\mathfrak{G}^2}(x,y) = \begin{cases} f_1(x,y) & \text{if} \quad (x,y) \in R_1 \\ 1 & \text{if} \quad (x,y) \in R_4 \\ 1 & \text{if} \quad (x,y) \in \tilde{1}_{f_1,x} \\ 1 & \text{if} \quad (x,y) \in \tilde{1}_{f_1,y} \\ f_1(x,1) & \text{if} \quad (x,y) \in \tilde{0}_{f_1,x} \\ f_1(1,y) & \text{if} \quad (x,y) \in \tilde{0}_{f_1,y} \end{cases}$$

If $f_1(1,1) = 0$, the same Lemmas 3.5 and 3.6 imply

$$\alpha^1_{\mathfrak{S}^2}(x,y) = \begin{cases} f_1(x,y) & \text{if} \quad (x,y) \in R_1 \\ 0 & \text{if} \quad (x,y) \in R_4 \\ 1 & \text{if} \quad (x,y) \in \tilde{1}_{f_1,x} \\ 1 & \text{if} \quad (x,y) \in \tilde{1}_{f_1,y} \\ f_1(x,1) & \text{if} \quad (x,y) \in \tilde{0}_{f_1,x} \\ f_1(1,y) & \text{if} \quad (x,y) \in \tilde{0}_{f_1,y} \end{cases}$$

Now let

$$\alpha = \bigwedge_{i=1}^{4} \alpha^i.$$

We have that $\alpha_{\mathfrak{S}^2} = \mathcal{F}$.

Example 3.18. *For the MS-quadruple* $\mathcal{F} = (f_1, f_2, f_3, f_4)$ *given by:*

$$f_1(x,y) = \begin{cases} 1 & \textit{if} \quad x \le \frac{1}{2} \\ 2-2x & \textit{if} \quad \frac{1}{2} < x \le \frac{3}{4} \\ -1+2x & \textit{if} \quad \frac{3}{4} < x \le 1 \end{cases}$$

$$f_2(x,y) = \begin{cases} y & \textit{if} \quad x < \frac{1}{2} \\ 1 & \textit{if} \quad x = \frac{1}{2} \\ 2-2x & \textit{if} \quad \frac{1}{2} < x \le \frac{3}{4} \\ -1+2x & \textit{if} \quad \frac{3}{4} < x \le 1 \end{cases}$$

$$f_3(x,y) = 1$$

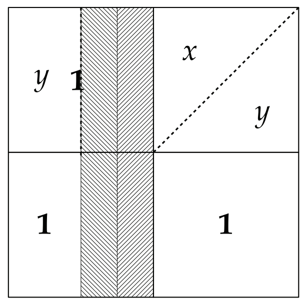
$$f_4(x,y) = min\{x,y\}$$

we can define the terms:

- $\alpha_1 = \neg(x \cdot x) \vee (x \cdot x)$
- $\alpha_2 = (\neg x) \cdot (\neg x) \vee (\neg\neg y \to y)$
- $\alpha_3 = \top$
- $\alpha_4 = (((\neg\neg x \to x) \vee (\neg\neg y)) \to (\neg\neg x \to x)) \wedge (((\neg\neg y \to y) \vee (\neg\neg x)) \to (\neg\neg y \to y))$.

If $\alpha = \bigwedge_{i=1}^{4} \alpha^i$ *then we have that* $\alpha_{\mathfrak{S}^2} = \mathcal{F}$.

Graphically,



Theorem 3.19. *$\mathcal{F} \in Free_{\mathcal{MS}}(2)$ if and only if $\mathcal{F}$ is given by a MS-quadruple (f_1, f_2, f_3, f_4).*

We can conclude with:

Corollary 3.20. *The algebra $Free_{\mathcal{MS}}(2)$ is the algebra whose elements are functions given by MS-quadruples and the operations $\cdot$ and $\to$ are defined pointwise.*

Remark 3.21. Observe that it can happen that different quadruples determine the same function. That can happen when two functions $h_{x,1}$ and $h_{x,2}$ (or $h_{y,1}$, $h_{y,2}$) are given by different triangulations, but coincide on every point. Therefore the correspondence between MS-quadruples and functions in $Free_{\mathcal{MS}}(2)$ is not bijective.

4. Characterization of free algebras: the general case

The aim of this section is to obtain a characterization of functions in $Free_{\mathcal{MS}}(n)$. Following the ideas in the case of two generators, we will describe each term-function as 2^n-tuples of functions in $Free_{\mathcal{MV}}(n)$ and in $Free_{\mathcal{H}}(m)$, $m \leq n$.

As we did in the case of two generators, we will separate the domain $\mathfrak{S}^n$ of the functions in $Free_{\mathcal{MS}}(n)$ into regions. For each subset $A = \{j_1, \ldots, j_m\} \subseteq \{1, \ldots, n\}$, we define the corresponding region

$$R_A = \prod_{i=1}^{n} E_i$$

where $E_i = H$ for each $i \in A$ and $E_i = [0,1]_{\mathbf{MV}}$ for each $i \notin A$. We denote by $\mathcal{R}$ the set of regions, whose cardinality is 2^n. For example, $A = \emptyset$ corresponds to $MV^n = [0,1]^n_{\mathbf{MV}}$. The nonempty intersections of the regions in $\mathcal{R}$ with the main region MV^n are going to be crucial in the characterization of the functions. Indeed, if we define the **relative border** of the region MV^n as the set

$$\eth[0,1]^n_{\mathbf{MV}} = \{\bar{x} \in [0,1]^n_{\mathbf{MV}} : x_i = 1 \text{ for some } 1 \leq i \leq n\},$$

we can see that for each $A \neq \emptyset$, the intersection $R_A \cap MV^n$ is included in $\eth[0,1]^n_{\mathbf{MV}}$.

Next we present the notation that we will use to achieve our aim. If $\bar{z} = (z_1, \ldots, z_n)$ is a point in a region R_A with $A = \{j_1, \ldots, j_m\}$ with an abuse of notation, we define

$$\pi_H(\bar{z}) = (z_{j_1}, \ldots, z_{j_m}) \in \mathbf{H}^m.$$

Definition 4.1. If $\bar{x} = (x_1, \ldots, x_n) \in MV^n$ we define:

- $1_{\bar{x}} = \{i \in \{1, \ldots, n\} : x_i = 1\}$ (one set of a point $\bar{x}$)
- $\|\bar{x}\| = |1_{\bar{x}}|$ (cardinality of the one set of $\bar{x}$)
- $\tilde{x} = \{\bar{z} \in \mathfrak{S}^n : z_j = x_j, \text{ for every } j \notin 1_{\bar{x}}, \text{ and } z_i \in H, \text{ for every } i \in 1_{\bar{x}}\}$ (cylindrification of the point $\bar{x}$)

For a fixed $\bar{x} = (x_1, \ldots, x_n) \in [0,1]^n_{\mathbf{MV}}$, if $A = 1_{\bar{x}}$ then $\tilde{x} \subseteq R_A$. Moreover, for every nonempty set $A \subseteq \{1, \ldots, n\}$ and each point $\bar{y} \in R_A$ there is a unique $\bar{x} \in R_A \cap MV^n$ such that $\bar{y} \in \tilde{x}$, i.e., $\bar{y}$ is in the cylindrification $\tilde{x}$.

We generalize the previous notation for arbitrary sets in the relative border.

Definition 4.2. If $T \subseteq \eth[0,1]^n_{\mathbf{MV}}$ is a simplex, we define:

- $1_T = \{i \in \{1, \ldots, n\} : x_i = 1 \text{ for all } \bar{x} \in T\}$ (one set of T)
- $\|T\| = |1_T|$ (cardinal of the one set of T)
- $\tilde{T} = \{\tilde{x} : \bar{x} \in T\}$ (cylindrification of the set T)
- $\mathcal{F}_T = \{F \subseteq T \cap \eth[0,1]^n_{\mathbf{MV}} : \forall \bar{x}, \bar{y} \in F, 1_{\bar{x}} = 1_{\bar{y}}\}$ (faces of the simplex T).

For a fixed $T \subseteq \eth[0,1]^n_{\mathbf{MV}}$, let $A = 1_T = \{j_1, \ldots, j_m\}$. Then $\tilde{T} \subseteq R_A$.

4.1. From term-functions to 2^n-tuples

Let's consider an n-variable BL-term

$$\alpha = \alpha(x_1, \ldots, x_n)$$

and let $\alpha_{\mathfrak{S}^n}$ be the corresponding term-function. This function will be described in terms of a 2^n-tuple of functions $\{\alpha_R\}_{R \in \mathcal{R}}$. As a first step, if we consider the region MV^n in $\mathcal{R}$, since α is a term in the language of MV-algebras as well, we denote by

$$\alpha_{MV^n} = f$$

the McNaughton function from $[0,1]^n_{\mathbf{MV}}$ into $[0,1]_{\mathbf{MV}}$ corresponding to α.

For every other region $R_A = \prod_{i=1}^n E_i$, with $A \neq \emptyset$, the value of the term function α_{R_A} will depend on the value of f in the intersection $R_A \cap MV^n$. We now state the analogous of Lemmas 3.5 and 3.6, that can be proved by induction on the complexity of α.

Lemma 4.3. *Let $\mathbf{H}^n$ be the region of $\mathfrak{S}^n$ given by $\mathbf{H}^n = \prod_{i=1}^n \mathbf{H}$. The following hold:*

- *If $\alpha_{\mathfrak{S}^n}(\bar{1}) = 1$ then there is a function $g \in Free_{\mathcal{H}}(n)$ such that $\alpha_{\mathbf{H}^n} = g$.*
- *If $\alpha_{\mathfrak{S}^n}(\bar{1}) = 0$ then $\alpha_{\mathbf{H}^n} = 0$, i.e., $\alpha_{\mathfrak{S}^n}$ takes the value 0 all over $\mathbf{H}^n$.*

As an example of the previous Lemma, we can consider the term

$$\alpha = \bigwedge_{i=1}^{n} (\neg\neg x_i \to x_i).$$

We get $\alpha_{\mathfrak{S}^n}(\bar{1}) = 1$ and from equation 1 for each $\bar{x} \in \mathbf{H}^n$ we have that $\alpha_{\mathfrak{S}^n}(\bar{x}) = min\{x_1, \ldots, x_n\}$. Thus the function $g \in Free_{\mathcal{H}}(n)$, given by $g(\bar{x}) = \bigwedge_{i=1}^n x_i$ is such that $\alpha_{\mathbf{H}^n} = g$.

If $\beta = \neg\alpha$ then we have that $\beta_{\mathfrak{S}^n}(\bar{1}) = 0$ and $\beta_{\mathfrak{S}^n}$ takes the value 0 all over $\mathbf{H}^n$.

The following result is analogous to Lemma 67 in [4].

Lemma 4.4. *Let $\bar{x} \in [0,1]^n_{\mathbf{MV}}$ be a point such that $1_{\bar{x}} \neq \emptyset$. The following hold:*

- *If $\alpha_{\mathfrak{S}^n}(\bar{x}) = c \in [0,1)_{\mathbf{MV}}$ then $\alpha_{\mathfrak{S}^n}(\bar{z}) = c$ for every $\bar{z} \in \tilde{x}$,*
- *If $\alpha_{\mathfrak{S}^n}(\bar{x}) = 1$ then there is $g \in Free_{\mathcal{H}}(m)$, $m = \|\bar{x}\|$ such that $\alpha_{\mathfrak{S}^n}(\bar{z}) = g(\pi_H(\bar{z}))$ for every $\bar{z} \in \tilde{x}$.*

We analyze now the behavior of $\alpha_{\mathfrak{S}^n}$ in the cylindrification of the relative interior of a simplex of dimension greater than 0. We recall that if a simplex $T \subseteq [0,1)^m \times \{1\}^{n-m}$, the relative interior of T is denoted by T° and the cylindrification of T° in $[0,1)^m \times \mathbf{H}^{n-m}$ is denoted by $\tilde{T}^\circ$

Lemma 4.5. *Let $m < n$ and let $T \in [0,1)^m \times \{1\}^{n-m}$, be a rational simplex of dimension greater or equal to 1. Assume that $\alpha_{\mathfrak{S}^n}(T) = 1$. If for every subterm β of α we have that either $\beta_{\mathfrak{S}^n}(T^\circ) \in [0,1)_{\mathbf{MV}}$ or $\beta_{\mathfrak{S}^n}(T^\circ) = 1$, then there is a function $g \in Free_{\mathcal{H}}(n-m)$ such that*

$$\alpha_{\mathfrak{S}^n}(\bar{x}) = g(\pi_H(\bar{x})),$$

for every $\bar{x} \in \tilde{T}^\circ$.

Proof. Using Lemma 4.4, for each $\bar{x} \in T$ with $\alpha_{\mathfrak{S}^n}(\bar{x}) = 1$ we denote by g_x the function in $Free_{\mathcal{H}}(n-m)$ that such that $\alpha_{\mathfrak{S}^n}(\bar{y}) = g_x(\pi_H(\bar{y}))$ for all $\bar{y}$ in the cylindrification $\tilde{x}$ of $\bar{x}$. To prove our result, it is enough to see that if $\alpha_{\mathfrak{S}^n}(T) = 1$ and for every subterm β of α we have that either $\beta_{\mathfrak{S}^n}(T^\circ) \in [0,1)_{\mathbf{MV}}$ or $\beta_{\mathfrak{S}^n}(T^\circ) = 1$, then for every $\bar{x}_1, \bar{x}_2 \in T^\circ$ with $\bar{x}_1 \neq \bar{x}_2$ we have that g_{x_1} and g_{x_2} coincide, which with an abuse of notation can be written as

$$\alpha_{\mathfrak{S}^n}(\tilde{x}_1) = \alpha_{\mathfrak{S}^n}(\tilde{x}_2). \tag{17}$$

Following the ideas in the proof of Lemma 3.7, we proceed by induction in the complexity of the term α.

If α is a term of complexity 0 such that $\alpha_{\mathfrak{S}^n}(T) = 1$ then we have two possibilities:

1. $\alpha = x_j$, with $j \in 1_T$ then $\alpha_{\mathfrak{S}^n}(\tilde{x}_1) = \pi_j = \alpha_{\mathfrak{S}^n}(\tilde{x}_2)$, for every $\bar{x}_1, \bar{x}_2 \in T^\circ$.
2. $\alpha = 1$, then $\alpha_{\mathfrak{S}^n}(\tilde{x}_1) = 1 = \alpha_{\mathfrak{S}^n}(\tilde{x}_2)$, for every $\bar{x}_1, \bar{x}_2 \in T^\circ$.

Suppose that the statement holds for terms of complexity less than k and let α be a term of complexity k. Then we have two cases to consider:

1. $\alpha = \phi \cdot \psi$, with ϕ and ψ subterms of α of complexity less than k. Since $\alpha_{\mathfrak{S}^n}(T) = 1$ then necessarily $\phi_{\mathfrak{S}^n}(T) = 1$ and $\psi_{\mathfrak{S}^n}(T) = 1$. By inductive hypothesis, for every $\bar{x}_1, \bar{x}_2 \in T^\circ$ such that $\bar{x}_1 \neq \bar{x}_2$,
$$\alpha_{\mathfrak{S}^n}(\tilde{x}_1) = \phi_{\mathfrak{S}^n}(\tilde{x}_1) \cdot \psi_{\mathfrak{S}^n}(\tilde{x}_1) = \phi_{\mathfrak{S}^n}(\tilde{x}_2) \cdot \psi_{\mathfrak{S}^n}(\tilde{x}_2) = \alpha_{\mathfrak{S}^n}(\tilde{x}_2),$$
so the statement holds.
2. $\alpha = \phi \to \psi$, with ϕ and ψ terms of complexity less than k. By hypothesis we know that for every subterm β of α we have that either $\beta_{\mathfrak{S}^n}(T^\circ) \in [0,1)_{\mathbf{MV}}$ or $\beta_{\mathfrak{S}^n}(T^\circ) = 1$, then we have only three cases to consider for ϕ and ψ:
   (a) If $\phi_{\mathfrak{S}^n}(T^\circ) = 1$ and $\psi_{\mathfrak{S}^n}(T^\circ) = 1$: by inductive hypothesis we have that for $\bar{x}_1, \bar{x}_2 \in T^\circ$ such that $\bar{x}_1 \neq \bar{x}_2$,
   $$\alpha_{\mathfrak{S}^n}(\tilde{x}_1) = \phi_{\mathfrak{S}^n}(\tilde{x}_1) \to \psi_{\mathfrak{S}^n}(\tilde{x}_1) = \phi_{\mathfrak{S}^n}(\tilde{x}_2) \to \psi_{\mathfrak{S}^n}(\tilde{x}_2) = \alpha_{\mathfrak{S}^n}(\tilde{x}_2).$$
   (b) If $\phi_{\mathfrak{S}^n}(T^\circ) \subseteq [0,1)_{\mathbf{MV}}$ and $\psi_{\mathfrak{S}^n}(T^\circ) \subseteq [0,1)_{\mathbf{MV}}$: by Lemma 4.4 we have $\phi_{\mathfrak{S}^n}(\tilde{x}_1) = \phi_{\mathfrak{S}^n}(\bar{x}_1)$, and $\psi_{\mathfrak{S}^n}(\tilde{x}_1) = \psi_{\mathfrak{S}^n}(\bar{x}_1)$. Analogously, $\phi_{\mathfrak{S}^n}(\tilde{x}_2) = \phi_{\mathfrak{S}^n}(\bar{x}_2)$, and $\psi_{\mathfrak{S}^n}(\tilde{x}_2) = \psi_{\mathfrak{S}^n}(\bar{x}_2)$.
   Since $\alpha_{\mathfrak{S}^n}(\bar{x}_1) = \alpha_{\mathfrak{S}^n}(\bar{x}_2) = 1$ then we have $\phi_{\mathfrak{S}^n}(\bar{x}_1) \leq \psi_{\mathfrak{S}^n}(\bar{x}_1)$ and $\phi_{\mathfrak{S}^n}(\bar{x}_2) \leq \psi_{\mathfrak{S}^n}(\bar{x}_2)$, and therefore
   $$\alpha_{\mathfrak{S}^n}(\tilde{x}_1) = \phi_{\mathfrak{S}^n}(\tilde{x}_1) \to \psi_{\mathfrak{S}^n}(\tilde{x}_1) = \phi_{\mathfrak{S}^n}(\bar{x}_1) \to \psi_{\mathfrak{S}^n}(\bar{x}_1) = 1$$
   and
   $$\alpha_{\mathfrak{S}^n}(\tilde{x}_2) = \phi_{\mathfrak{S}^n}(\tilde{x}_2) \to \psi_{\mathfrak{S}^n}(\tilde{x}_2) = \phi_{\mathfrak{S}^n}(\bar{x}_2) \to \psi_{\mathfrak{S}^n}(\bar{x}_2) = 1,$$
   so the statement holds for this case.
   (c) If $\phi_{\mathfrak{S}^n}(T^\circ) \subseteq [0,1)_{\mathbf{MV}}$ and $\psi_{\mathfrak{S}^n}(T^\circ) = 1$ we can prove the result using similar ideas of the previous cases.

□

Any rational polyhedra in $[0,1)^n_{\mathbf{MV}}$ can be triangulated in finitely many rational simplices $S_1, \ldots, S_k$ which verify the conditions of Lemma 4.5, so the previous result can be extended to cylindrifications of any polyhedra. We recall that if a simplex S is zero dimensional, then $S^\circ = S$, that is $\tilde{S}^\circ$ is the cylindrification of a point.

Lemma 4.6. *Let $P \subseteq [0,1)^m \times \{1\}^{n-m}$ be a rational polyhedral set. Assume that $\alpha_{\mathfrak{S}^n}(P) = 1$. Then there is a unimodular triangulation Δ of P such that for every $S \in \Delta$, there is a function $g_S \in Free_{\mathcal{H}}(n-m)$ that satisfies*

$$\alpha_{\mathfrak{S}^n}(\bar{y}) = g_S(\pi_H(\bar{y}))$$

for every $\bar{y} \in \tilde{S^\circ}$.

Proof. By Lemma 4.4, we know that for each point $\bar{x} \in P$, since $||\bar{x}|| = n - m$, there is a function $g_x \in Free_{\mathcal{H}}(n-m)$ such that

$$\alpha_{\mathfrak{S}^n}(\bar{y}) = g_x(\pi_H(\bar{y}))$$

for each $\bar{y} \in \tilde{x}$.

For each subterm β of α let f_β be the McNaugthon function in $Free_{\mathcal{MV}}(n)$ such that $\beta_{\mathfrak{S}^n}(\bar{x}) = f_\beta(\bar{x})$ for each $\bar{x} \in [0,1]^n_{\mathbf{MV}}$. Let Δ_β be a unimodular triangulation of $P \subseteq \eth[0,1]^n_{\mathbf{MV}}$, that respects f_β (see Theorem 2.7), that is, for each $S \in \Delta_\beta$, f_β is linear over Δ_β.

Following Theorem 2.7, let Δ be a unimodular triangulation of P which is a refinement of all Δ_β, for β subterm of α. This means that for each β subterm of α, f_β is linear over each $S \in \Delta$. Therefore, for every subterm β of α and each $S \in \Delta$ either $\beta_{\mathfrak{S}^n}(\bar{x}) \in [0,1)_{\mathbf{MV}}$ for every $\bar{x} \in S^\circ$ or $\beta_{\mathfrak{S}^n}(\bar{x}) = 1$. For each $S \in \Delta$ let $g_S \in Free_{\mathcal{H}}(n-m)$ be such that $g_S = g_z$ for some $\bar{z} \in S^\circ$. Lemma 4.5 guarantees that

$$\alpha_{\mathfrak{S}^n}(\bar{y}) = g_S(\pi_H(\bar{y}))$$

for every $\bar{y} \in \tilde{S^\circ}$.

□

Analogous results can be obtained for every rational polyhedral closed set P included in the intersection $R_A \cap MV^n$ for any region $R_A \in \mathcal{R}$ with $A \subsetneq \{1,\ldots,n\}$ and $A \neq \emptyset$. This leads to the following definition.

Definition 4.7. Let $f \in Free_{\mathcal{MV}}(n)$, $A = \{j_1,\ldots,j_m\} \subseteq \{1,\ldots,n\}$ and $U = R_A \cap f^{-1}(\{1\})$. We say that a function

$$g : R_A \to \mathfrak{S}$$

is an f-A-H-McNaughton function if there is a unimodular triangulation Δ of U and a family $\{g_S\}_{S\in\Delta}$ of functions in $Free_{\mathcal{H}}(m)$ such that for each $\bar{x} \in V$ it holds that:

- If $f(\bar{x}) = 1$ (i.e., $\bar{x} \in U$) and $\bar{x} \in S^\circ$ then $g(\bar{y}) = g_S(\pi_H(\bar{y}))$ for every $\bar{y} \in \tilde{x}$.
- If $f(\bar{x}) = c < 1$, then $g(\bar{y}) = c$ for every $\bar{y} \in \tilde{x}$.

Remark 4.8. Note that the function g in the previous definition is well defined, since for every point $\bar{y} \in R_A$ there is a unique $\bar{x} \in R_A \cap \eth[0,1]^n_{\mathbf{MV}}$ such that $\bar{y} \in \tilde{x}$, i.e., $\bar{y}$ is in the cylindrification $\tilde{x}$. Moreover, for each $\bar{x}$ such that $f(\bar{x}) = 1$ there is exactly one $S \in \Delta$ such that $\bar{x} \in S^\circ$.

Remark 4.9. Consider $n = 2$ and $A = \{2\}$, that is $R_A = [0,1]_{\mathbf{MV}} \times H$. The definition of f-A-H-McNaughton function coincides with that of f-y-H McNaughton function, so the developments for the general case are really a generalization of the two-variable case.

From Lemma 4.5 and Lemma 4.6, we can deduce that if $A \subsetneq \{1,\ldots,n\}$ and $A \neq \emptyset$, then the term function α_{R_A} is a f-A-H-McNaughton function. As a generalization of MS-quadruple we define:

Definition 4.10. A 2^n-tuple $(f, \{h_A : A \subsetneq \{1,\ldots,n\}, A \neq \emptyset\}, g)$ is said to be a MS_n-tuple if:

1. $f \in Free_{\mathcal{MV}}(n)$,
2. for each $A \subsetneq \{1,\ldots,n\}, A \neq \emptyset$, h_A is a f-A-H-McNaughton,
3. $g : H^n \to \mathfrak{S}$ is the zero function if $f(\bar{1}) = 0$ or $g \in Free_{\mathcal{H}}(n)$ otherwise.

A function $\mathcal{F} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be given by the MS_n-tuple $(f, \{h_A : \emptyset \neq A \subsetneq \{1,\ldots,n\}\}, g)$ if for each $\bar{x} \in \mathfrak{S}^n$ it satisfies:

$$\mathcal{F}(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if} \quad \bar{x} \in [0,1]^n_{\mathbf{MV}} \\ h_A(\bar{x}) & \text{if} \quad \bar{x} \in R_A \\ g(\bar{x}) & \text{if} \quad \bar{x} \in \mathbf{H}^n. \end{cases} \tag{18}$$

We write $\mathcal{F} = (f, \{h_A : \emptyset \neq A \subsetneq \{1,\ldots,n\}\}, g)$.

As a consequence of the results of this section, we have the following theorem:

Theorem 4.11. *Given an n-variable BL-term α there is a MS_n-tuple*

$$\mathcal{F} = (f, \{h_A : A \subsetneq \{1,\ldots,n\}, A \neq \emptyset\}, g)$$

such that the term function $\alpha_{\mathfrak{S}^n}$ coincides with $\mathcal{F}$.

4.2. From 2^n-tuples to term-functions

For a function $\mathcal{F} : \mathfrak{S}^n \to \mathfrak{S}$ given by an MS_n-tuple we aim to find an n-variable BL-term α such that the term function $\alpha_{\mathfrak{S}^n}$ coincides with $\mathcal{F}$.

To achieve our aim, as we did in the two-generator case, we build some terms that will help us to build any other.

Lemma 4.12. *Let $g \in Free_{\mathcal{H}}(n)$. Then there is a BL-term γ^g such that*

$$\gamma^g_{\mathfrak{S}^n}(\bar{x}) = \begin{cases} g(\bar{x}) & \text{if } \bar{x} \in \mathbf{H}^n \\ \\ 1 & \text{otherwise.} \end{cases} \tag{19}$$

Proof. From Theorem 2.6, there is a McNaughton function $f \in Free_{\mathcal{MV}}(n)$ such that f only takes the value 1 on the point $\bar{1} \in [0,1]^n_{\mathbf{MV}}$, in symbols $f^{-1}(\{1\}) = \bar{1}$. Let α be an n-variable BL-term such that $f = \alpha_{MV^n}$ and $\beta = \neg\neg\alpha$. Therefore $\beta_{\mathfrak{S}^n}(\bar{x}) = 1$ for each $\bar{x} \in H^n$ and acoording to Lemma 4.4, $\beta_{\mathfrak{S}^n}(\bar{x}) \in [0,1)_{\mathbf{MV}}$ for any other $\bar{x} \in \mathfrak{S}^n$. Let ϕ be an n-variables hoop term such that $\phi_{H^n} = g$. Then $\psi = \phi \wedge \beta$ is such that $\psi_{\mathfrak{S}^n}(\bar{x}) = g(\bar{x})$ for each $\bar{x} \in \mathbf{H}^n$ and $\psi_{\mathfrak{S}^n}(\bar{x}) \in [0,1)_{\mathbf{MV}}$ otherwise. Our desired term is then

$$\gamma^g = \neg\neg\psi \to \psi.$$

□

Lemma 4.13. *Let $n, m \in \mathbb{N}$ be such that $m < n$ and $T \subseteq [0,1)^m_{\mathbf{MV}} \times \{1\}^{n-m}$ a rational simplex. Then there is a BL-term σ^T such that $\sigma^T_{\mathfrak{S}^n}(\bar{x}) = 1$ for each $\bar{x} \in \tilde{T}$ and $\sigma^T_{\mathfrak{S}^n}(\bar{x}) \in [0,1)_{\mathbf{MV}}$ otherwise.*

Proof. Since T is a rational simplex from Theorem 2.6, there is a McNaughton function $h \in Free_{\mathcal{MV}}(n)$ such that $T = h^{-1}(\{1\})$. Let ϕ be a BL-term such that $\phi_{MV^n} = h$ and let $\sigma^T = \neg\neg\phi$. From Lemma 4.4 we get that $\sigma^T_{\mathfrak{S}^n}(\bar{x}) = 1$ for each $\bar{x} \in \tilde{T}$ and $\sigma^T_{\mathfrak{S}^n}(\bar{x}) \in [0,1)_{\mathbf{MV}}$ otherwise.

□

Lemma 4.14. *Let $n, m \in \mathbb{N}$ be such that $m < n$ and $T \subseteq [0,1)^m_{\mathbf{MV}} \times \{1\}^{n-m}$ a rational simplex. For each $g \in Free_{\mathcal{H}}(n-m)$ there exists an n-variable BL-term μ^T such that*

$$\mu^T_{\mathfrak{S}^n}(\bar{x}) = \begin{cases} g(\pi_H(\bar{x})) & \text{if } \bar{x} \in \tilde{T}^\circ \\ \\ 1 & \text{otherwise.} \end{cases} \tag{20}$$

Proof. It's worth to recall that if T is zero dimensional, then $T^\circ = T$. Otherwise, if the dimension of T is greater than 0, then T° is the relative interior of T, so it is in $[0,1)^m_{\mathbf{MV}} \times \{1\}^{n-m}$.

Let ϕ be an $n-m$ variable hoop term such that $\phi_{H^{n-m}} = g$. From Lemma 4.13 consider the term σ^T and let

$$\psi = \sigma^T \wedge \phi(\pi_H).$$

Then $\psi_{\mathfrak{S}^n}(\bar{x}) = g(\pi_H(\bar{x}))$ is $\bar{x} \in \tilde{T}$ and $\psi_{\mathfrak{S}^n}(\bar{x}) \in [0,1)_{\mathbf{MV}}$ otherwise. Hence the BL-term $\tau = \neg\neg\psi \to \psi$ satisfies

$$\tau_{\mathfrak{S}^n}(\bar{x}) = \begin{cases} g(\pi_H(\bar{x})) & \text{if } \bar{x} \in \tilde{T} \\ \\ 1 & \text{otherwise.} \end{cases} \tag{21}$$

Let $\mathcal{F}_T$ be the set of proper faces of T and consider for each $F \in \mathcal{F}_T$ the term σ^F of Lemma 4.13. Then

$$\mu^T = \tau \vee \bigvee_{F \in \mathcal{F}_T} \sigma^F$$

is such that

$$\mu^T_{\mathfrak{S}^n}(\bar{x}) = \begin{cases} g(\pi_H(\bar{x})) & \text{if } \bar{x} \in \tilde{T}^\circ \\ 1 & \text{otherwise.} \end{cases} \tag{22}$$

□

Lemma 4.15. *Let $f \in Free_{\mathcal{MV}}(n)$, $m < n$ and $A = \{m+1, m+2, \ldots n\}$. Consider $U = R_A \cap f^{-1}(\{1\})$. For any unimodular triangulation Δ of U and any family $\{g_S\}_{S \in \Delta}$ of functions in $Free_{\mathcal{H}}(n-m)$ there is a BL-term μ^Δ that satisfies:*

$$\mu^\Delta_{\mathfrak{S}^n}(\bar{x}) = \begin{cases} g_S(\pi_H(\bar{x})) & \textit{if } \bar{x} \in \tilde{S}^\circ \\ 1 & \textit{otherwise.} \end{cases} \tag{23}$$

Proof. The result is immediate from Lemma 4.14 by considering for each $S \in \Delta$ the term μ^S and then taking

$$\mu^\Delta = \bigwedge_{S \in \Delta} \mu^S.$$

□

We chose the notation μ^Δ for the BL-term in the previous lemma to make it simpler, but the reader should observe that the definition of μ^Δ depends not only on the triangulation Δ but also on the corresponding family of functions $\{g_S\}_{S\in\Delta}$. There will be no problems due to the ommision of $\{g_S\}_{S\in\Delta}$ in the notation in the following proofs.

Theorem 4.16. *Let $\mathfrak{F} : \mathfrak{S}^n \to \mathfrak{S}$ be a function given by the 2^n-tuple*

$$(f, \{h_A : \emptyset \neq A \subsetneq \{x_1, \ldots, x_n\}\}, g),$$

where $f \in Free_{\mathcal{MV}}(n)$, h_A is a function f-A-H-McNaughton and $g \in Free_{\mathcal{H}}(n) \cup \{0\}$, i.e., for every $\bar{x} \in \mathfrak{S}^n$ the function is given by:

$$\mathfrak{F}(\bar{x}) = \begin{cases} f(\bar{x}) & \textit{if} \quad \bar{x} \in [0,1]^n_{\mathbf{MV}}, \\ h_A(\bar{x}) & \textit{if} \quad \bar{x} \in R_A, \\ g(\bar{x}) & \textit{if} \quad \bar{x} \in \mathbf{H}^n. \end{cases} \tag{24}$$

Then $\mathfrak{F} \in Free_{\mathcal{MS}}(n)$.

Proof. We will build a BL-term γ such that $\gamma_{\mathfrak{S}^n} = \mathfrak{F}$. Consider a BL-term β such that $\beta_{MV^n} = f$. We define:

- $\gamma^{MV^n} = \neg\neg\beta$. From equation 2, for each $\bar{x} \in \mathfrak{S}^n$ if $\beta(\bar{x}) \in [0,1)_{\mathbf{MV}}$, then $\gamma^{MV^n}_{\mathfrak{S}^n}(\bar{x}) = \beta_{\mathfrak{S}^n}(\bar{x})$ and if $\beta(\bar{x}) \in \mathbf{H}$ then $\gamma^{MV^n}_{\mathfrak{S}^n}(\bar{x}) = 1$. From Lemma 4.4 we get

$$\gamma^{MV^n}_{\mathfrak{S}^n}(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if} \quad \bar{x} \in [0,1]^n_{\mathbf{MV}}, \\ f(\bar{x}) & \text{if} \quad \bar{x} \notin [0,1]^n_{\mathbf{MV}} \text{ and } \beta(\bar{x}) \in [0,1)_{\mathbf{MV}}, \\ 1 & \text{in any other case.} \end{cases} \tag{25}$$

- $\gamma^{\mathbf{H}^n} = \gamma^g$ as given in Lemma 4.12 if $f(\bar{1}) = 1$ and $\gamma^{\mathbf{H}^n} = 1$ if $g = 0$.
- $\gamma^A = \mu^{\Delta_A}$, where Δ_A is the unimodular triangulation of $R_A \cap f^{-1}(\{1\})$ corresponding to the f-A-H McNaughton function h_A, and μ^{Δ_A} is the term given in Lemma 4.15 corresponding to Δ_A and the family $\{g_S\}_{S\in\Delta_A}$ of the function h_A.

We define the BL-term γ in n variables by

$$\gamma = \bigwedge_{R\in\mathcal{R}} \gamma^R$$

Then we have that $\gamma_{\mathfrak{S}^n}(\bar{x}) = \mathcal{F}(\bar{x})$, for every $\bar{x} \in \mathfrak{S}^n$. □

Corollary 4.17. *The algebra $Free_{\mathcal{MS}}(n)$ is the algebra whose elements are functions given by MS-2^n-tuples and the operations $\cdot$ and $\to$ are defined pointwise, as it was in the case of two generators.*

As in the case with two variables, it can happen that two different 2^n-tuples $\mathcal{F}_1 = (f_1, \bar{h_1}, g_1)$ and $\mathcal{F}_2 = (f_2, \bar{h_2}, g_2)$ determine the same function (where we have $f_1, f_2 \in Free_{\mathcal{MV}}(n)$, $g_1, g_2 \in Free_{\mathcal{H}}(n)$ and $\bar{h_1}, \bar{h_2}$ are f_i-A-H-McNaughton functions for every subset A of $\{x_1, \ldots, x_n\}$ different from the total set of variables and the empty set. That happens in the case in which $f_1 = f_2$ and there are two different unimodular triangulations Δ_1 and Δ_2 of $\eth[0,1]^n_{\mathbf{MV}} \cap 1_{f_1}$ such that for every $\bar{x} \in \eth[0,1]^n_{\mathbf{MV}} \cap 1_{f_1}$, $h_{1_A}(\tilde{x}) = h_{2_A}(\tilde{x})$ for every h_{i_A} an f_i-A-H-McNaughton function with A a nonempty set properly contained in $\{x_1, \ldots, x_n\}$.

5. Filters in $\mathcal{MS}$-algebras

Using the characterization of the functions in $Free_{\mathcal{MS}}(n)$, in this section we study maximal and prime filters of this algebra. We will strongly use the fact that every prime filter is contained in a unique maximal filter.

An **implicative filter** (simply **filter** from now on) in a BL-algebra (or basic hoop) $\mathbf{A}$ is a subset $F \subseteq A$ satisfying that $1 \in F$ and if $x \in F$ and $x \to y \in F$ then $y \in F$. Filters can also be characterized as nonempty subsets of A upwards closed and closed under product.

A filter F is **proper** if $F \neq A$, **prime** if given two elements $a, b \in A$, if $a \vee b \in F$ then $a \in F$ or $b \in F$ and **maximal** if it is proper and none proper filter of $\mathbf{A}$ contains F.

To study filters we fix some notation. Given a point $\bar{x} = (x_1, \ldots, x_n) \in \mathfrak{S}^n$ let $\hat{x} \in \mathfrak{S}^n$ be given by:

$$\hat{x}_i = \begin{cases} x_i & \text{if} \quad x_i \in [0,1]_{\mathbf{MV}} \\ \\ 1 & \text{if} \qquad x_i \in \mathbf{H}. \end{cases}$$

The element $\hat{x}$ is in $[0,1]^n_{\mathbf{MV}}$ and $\bar{x}$ is in the cylindrification of $\hat{x}$.

Lemma 5.1. *Given a function $f \in Free_{\mathcal{MV}}(n)$, the function $f^\sharp$ given by*

$$f^\sharp(\bar{y}) = \begin{cases} f(\bar{y}) & \textit{if} \qquad \bar{y} \in [0,1]^n_{\mathbf{MV}} \\ \\ f(\hat{y}) & \textit{if} \quad \bar{y} \in \mathfrak{S}^n \setminus [0,1]^n_{\mathbf{MV}}, \end{cases}$$

is in $Free_{\mathcal{MS}}(n)$. Moreover, $f^\sharp$ is the greatest element in $Free_{\mathcal{MS}}(n)$ such that its restriction to $[0,1]^n_{\mathbf{MV}}$ coincides with f.

Proof. From the characterization of functions of $Free_{\mathcal{MS}}(n)$ given in the previous section, it is easy to see that $f^\sharp \in Free_{\mathcal{MS}}(n)$ corresponds to the 2^n-tuple

$$(f, \{h_A : \emptyset \neq A \subsetneq \{x_1, \ldots, x_n\}\}, g),$$

given as follows:

- for each $\emptyset \neq A \subsetneq \{x_1, \ldots, x_n\}$, there is a unimodular triangulation Δ of $U = R_A \cap f^{-1}(\{1\})$ and for each $\bar{x} \in R_A$ the function h_A is given by
  - If $f(\hat{x}) = 1$ (i.e., $\hat{x} \in U$) and $\hat{x} \in S^\circ$ then $h_A(\bar{x}) = 1$.
  - If $f(\hat{x}) = c < 1$, then $h_A(\bar{x}) = f(\hat{x})$.
- if $f(\bar{1}) = 1$ we take $g(\bar{x}) = 1$ for each $\bar{x} \in \mathbf{H}^n$. Otherwise g is identically 0 on $\mathbf{H}^n$.

Observe that for each point of the cylindrification of $\eth[0,1]^n_{\mathbf{MV}} \cap f^{-1}(1)$ the function $f^\sharp$ takes the value 1, and from Lemma 4.4 the rest of the values are totally determined by f. This makes $f^\sharp$ the greatest element in $Free_{\mathcal{MS}}(n)$ such that its restriction to $[0,1]^n_{\mathbf{MV}}$ coincides with f. $\square$

5.1. Relation with filters of $Free_{\mathcal{MV}}(n)$

Given a function $f \in Free_{\mathcal{MS}}(n)$ and a subset $S \subseteq \mathfrak{S}^n$ we let $f\restriction_S$ be the restriction of f to S. For each $F \subseteq Free_{\mathcal{MS}}(n)$ we define

$$F_{MV} = \{f\restriction_{[0,1]^n_{\mathbf{MV}}} : f \in F\},$$

and also we define the subset $G^{cyl} \subseteq Free_{\mathcal{MS}}(n)$ given by:

$$f \in G^{cyl} \text{ if and only if } f\restriction_{[0,1]^n_{\mathbf{MV}}} \in G.$$

With this notation we have:

Lemma 5.2. *If $F \subseteq Free_{\mathcal{MS}}(n)$ is a filter then F_{MV} is a filter in $Free_{\mathcal{MV}}(n)$. Moreover, if F is maximal (prime), then F_{MV} is maximal (prime).*

Proof. Clearly, the function which is identically 1 over $[0,1]^n_{\mathbf{MV}}$ is in F_{MV} (it is the restriction of the one which takes the value 1 over $\mathfrak{S}^n$, which is in F because it is a filter).

Suppose that $g \in F_{MV}$ and $f \in Free_{\mathcal{MV}}(n)$ are such that $f \geq g$ and let's see that $f \in F_{MV}$. Since $g \in F_{MV}$, we know that there is a function $\tilde{g} \in F$ such that $\tilde{g}\restriction_{[0,1]^n_{\mathbf{MV}}} = g$. We also have that $\tilde{g}^\sharp$ and $f^\sharp \in Free_{\mathcal{MS}}(n)$ and from their definition we have:

1. $f^\sharp\restriction_{[0,1]^n_{\mathbf{MV}}} = f$,

2. $\tilde{g}^\sharp\restriction_{[0,1]_{\mathbf{MV}^n}} = \tilde{g}\restriction_{[0,1]_{\mathbf{MV}^n}} = g$.

3. $f^\sharp \geq \tilde{g}^\sharp \geq \tilde{g}$.

Then $f^\sharp \in F$ and $f^\sharp\restriction_{[0,1]^n_{\mathbf{MV}}} = f \in F_{MV}$.

Finally let us see that F_{MV} is closed under product. Consider $f, g \in F_{MV}$, this means that there are functions $\tilde{f}, \tilde{g} \in F$ such that $\tilde{f}\restriction_{[0,1]^n_{\mathbf{MV}}} = f$ and $\tilde{g}\restriction_{[0,1]^n_{\mathbf{MV}}} = g$. Since F is closed under product, $\tilde{f} \cdot \tilde{g} \in F$, and we also have $\tilde{f} \cdot \tilde{g}\restriction_{[0,1]^n_{\mathbf{MV}}} = f \cdot g$, so $f \cdot g \in F_{MV}$. Therefore F_{MV} is a filter of $Free_{\mathcal{MV}}(n)$.

Assume now that F is maximal. Suppose that there is a filter $G \subsetneq Free_{\mathcal{MV}}(n)$ such that $F_{MV} \subsetneq G$.

From the definition it is easy to see that G^{cyl} is a filter and that $F \subsetneq G^{cyl}$. But $G^{cyl} \neq Free_{\mathcal{MS}}(n)$, because the function which is identically 0 is not in G^{cyl} and that contradicts the hypothesis that F is a maximal filter.

Finally, assume that F is prime. Let f, g be in $Free_{\mathcal{MV}}(n)$ and $f \vee g \in F_{MV}$. Then, there is $h \in F$ such that $h\restriction_{[0,1]^n_{\mathbf{MV}}} = f \vee g$. Besides $f^\sharp, g^\sharp \in Free_{\mathcal{MS}}(n)$ are such that $f^\sharp\restriction_{[0,1]^n_{\mathbf{MV}}} = f$, $g^\sharp\restriction_{[0,1]^n_{\mathbf{MV}}} = g$ and from the definition $f^\sharp \vee g^\sharp \geq h$. Since F is a prime filter, $f^\sharp \vee g^\sharp \in F$ and then $f^\sharp \in F$ or $g^\sharp \in F$, which implies that $f \in F_{MV}$ or $g \in F_{MV}$. □

5.2. Maximal filters

To characterize maximal filters in $Free_{\mathcal{MS}}(n)$ we recall a definition from the proof of Lemma 5.2. Given a filter $G \subseteq Free_{\mathcal{MV}}(n)$, we recall that G^{cyl} is the subset of $Free_{\mathcal{MS}}(n)$ given by:

$$f \in G^{cyl} \text{ if and only if } f\restriction_{[0,1]^n_{\mathbf{MV}}} \in G.$$

The reader can easily check that G^{cyl} is a filter of $Free_{\mathcal{MS}}(n)$ and that $(G^{cyl})_{MV} = G$.

Remark 5.3. From Lemma 5.2, if F is maximal in $Free_{\mathcal{MS}}(n)$, then F_{MV} is also maximal in $Free_{\mathcal{MV}}(n)$. But $(F_{MV})^{cyl}$ is a proper filter of $Free_{\mathcal{MS}}(n)$ such that $F \subseteq (F_{MV})^{cyl}$. Therefore if F is maximal, $(F_{MV})^{cyl}$ coincides with F, i.e, $F = (F_{MV})^{cyl}$.

Theorem 5.4. *The correspondence*

$$F \mapsto F_{MV}$$

is a bijection between the set of maximal filters of $Free_{\mathcal{MS}}(n)$ and the set of maximal filters of $Free_{\mathcal{MV}}(n)$.

Proof. From Lemma 5.2, if F is maximal in $Free_{\mathcal{MS}}(n)$, then F_{MV} is maximal in $Free_{\mathcal{MV}}(n)$.

To see that the map is onto, consider G maximal on $Free_{\mathcal{MV}}(n)$, thus $(G^{cyl})_{MV} = G$. We will prove that G^{cyl} is maximal in $Free_{\mathcal{MS}}(n)$ and that will imply the surjectivity of the application.

Assume that there is F a filter in $Free_{\mathcal{MS}}(n)$ such that $G^{cyl} \subsetneq F$. Then from the definition of G^{cyl}, we have $G = (G^{cyl})_{MV} \subsetneq F_{MV}$. Since G is maximal, $F_{MV} = Free_{\mathcal{MV}}(n)$, then the function identically 0 on $[0,1]^n_{\mathbf{MV}}$ is in F_{MV}. But, by Lemma 4.4, the only function $f \in Free_{\mathcal{MS}}(n)$ such that $f\restriction_{[0,1]^n_{\mathbf{MV}}}$ is 0 in every point is the function identically 0 on $\mathfrak{S}^n$. Thus the zero function is in F and we have $F = Free_{\mathcal{MS}}(n)$.

Finally let us check that the application is injective. Assume F, G maximal filters in $Free_{\mathcal{MS}}(n)$ such that $F_{MV} \neq G_{MV}$. Recall from Remark 5.3 that $F = (F_{MV})^{cyl}$ and $G = (G_{MV})^{cyl}$. Consider $f \in F_{MV} \setminus G_{MV}$. Then $f^\sharp \in (F_{MV})^{cyl} = F$. Clearly $f^\sharp \notin (G_{MV})^{cyl}$ and therefore, $f^\sharp \in F \setminus G$. □

Maximal filters of free finitely generated MV-algebras were studied in [13].

Theorem 5.5. *([13, Proposition 3.4.7]) There is a bijection between points of $[0,1]^n_{\mathbf{MV}}$ and maximal filters of $Free_{\mathcal{MV}}(n)$, that is given by*

$$\bar{x} \quad \mapsto \quad MV_{\bar{x}} = \{f \in Free_{\mathcal{MV}}(n) : f(\bar{x}) = 1\}.$$

For each $\bar{x} \in [0,1]^n_{\mathbf{MV}}$ we define the set

$$M_{\bar{x}} = \{f \in Free_{\mathcal{MS}}(n) : f(\bar{x}) = 1\}.$$

It is easy to check that $M_{\bar{x}}$ is a filter in $Free_{\mathcal{MS}}(n)$ and that $(M_{\bar{x}})_{MV} = MV_{\bar{x}}$. This fact together with the results of Theorems 5.4 and 5.5 yield:

Theorem 5.6. *The correspondence*

$$\bar{x} \mapsto M_{\bar{x}}$$

is a bijection between the points of $[0,1]^n$ and the maximal filters of $Free_{\mathcal{MS}}(n)$.

5.3. Prime filters

To study prime filters in our variety $Free_{\mathcal{MS}}(n)$, we will first recall some results about prime filters in $Free_{\mathcal{MV}}(n)$. A complete description about these filters can be found in [11], adapting the results of ideals given there.

Definition 5.7. Let $n \in \mathbb{N}$ and t be such that $0 \leq t \leq n$. We call **index** to the $(t+1)$-tuple of vectors $\mathbf{u} = (\bar{u}_0, \ldots, \bar{u}_t)$ in $\mathbb{R}^n$ such that $\bar{u}_1, \ldots, \bar{u}_t$ are linearly independent and for some values $\epsilon_1, \ldots, \epsilon_t > 0$ the convex hull

$$T = conv\{\bar{u}_0, \bar{u}_0 + \epsilon_1 \bar{u}_1, \ldots, \bar{u}_0 + \epsilon_1 \bar{u}_1 + \ldots + \epsilon_t \bar{u}_t\}$$

is a simplex contained in $[0,1]^n$. Any such T is called an **u-simplex**. We also define the set $F_{\mathbf{u}} \subseteq Free_{\mathcal{MV}}(n)$ as:

$$f \in F_{\mathbf{u}} \text{ if and only if the set } f^{-1}(\{1\}) \text{ contains some } \mathbf{u}\text{-simplex.}$$

Theorem 5.8. *$F_{\mathbf{u}}$ is a prime filter of $Free_{\mathcal{MV}}(n)$.*

Moreover, the converse also holds:

Theorem 5.9. *Every prime filter F in $Free_{\mathcal{MV}}(n)$ has the form $F = F_{\mathbf{u}}$ for some index $\mathbf{u}$.*

If $P \subseteq Free_{\mathcal{MS}}(n)$ is a prime filter, we know that there is a unique maximal filter $M_{\bar{x}}$ such that $P \subseteq M_{\bar{x}}$. We refer to this fact by saying that *the prime filter P is localized at $\bar{x}$*. We will now analyze separately what happens when a prime filter P is localized at a point $\bar{x}$ in the interior of the n-cube, i.e, $\bar{x} \in [0,1]^n_{\mathbf{MV}} \setminus \eth[0,1]^n_{\mathbf{MV}}$ and what happens when $\bar{x}$ is in the relative border, that is, $\bar{x} \in \eth[0,1]^n_{\mathbf{MV}}$.

Definition 5.10. For a given index $\mathbf{u}$, a unimodular triangulation τ is called a $\mathbf{u}$-triangulation if it contains a $\mathbf{u}$-simplex $S_{\mathbf{u}}$. Given a $\mathbf{u}$-triangulation τ, we define $ostar(\mathbf{u})$ as the interior of the set $\{T \in \tau : T \cap S_{\mathbf{u}} \neq \emptyset\}$.

Theorem 5.11. *Let $\bar{x} \in [0,1]^n_{\mathbf{MV}} \setminus \eth[0,1]^n_{\mathbf{MV}}$ and $P \subseteq M_{\bar{x}}$ be a prime filter in $Free_{\mathcal{MS}}(n)$. Then there is a prime filter $G \subseteq Free_{\mathcal{MV}}(n)$ such that*

$$P = G^{cyl} = \{f \in Free_{\mathcal{MS}}(n) : f\restriction_{[0,1]^n_{MV}} \in G\}.$$

Proof. By Lemma 5.2, we know that P_{MV} is a prime filter in $Free_{\mathcal{MV}}(n)$. Clearly, for every function $f \in P_{MV}$, $f(\bar{x}) = 1$ because $P \subseteq M_{\bar{x}}$. Then P_{MV} is a prime filter in $Free_{\mathcal{MV}}(n)$ contained in $MV_{\bar{x}}$. We call $G = P_{MV}$ and will see that $P = G^{cyl}$. The inclusion $P \subseteq G^{cyl}$ follows from definition. The proof of the inclusion $G^{cyl} \subseteq P$ is our task.

By Theorem 5.9 there is an index $\mathbf{u}$ such that for every function $f \in P_{MV}$, $f(S) = 1$, for some $\mathbf{u}$-simplex S.

Claim: For every $\mathbf{u}$-simplex $S_{\mathbf{u}}$ contained in $[0,1]^n_{\mathbf{MV}} \setminus \eth[0,1]^n_{\mathbf{MV}}$ and every $\mathbf{u}$-unimodular triangulation τ such that $S_{\mathbf{u}}$ is in τ and $ostar(S_{\mathbf{u}}) \subseteq [0,1]^n_{\mathbf{MV}} \setminus$

$\eth[0,1]^n_{\mathbf{MV}}$, there is a function $h_{S_{\mathbf{u}}} \in P$ such that

$$h_{S_{\mathbf{u}}}(\bar{y}) = \begin{cases} 0 & \text{if } \bar{y} \text{ is such that some coordinate } y_i \in \mathbf{H} \\ \\ 1 & \text{if } \bar{y} \in S_{\mathbf{u}} \\ \\ 0 & \text{if } \bar{y} \notin ostar(S_{\mathbf{u}}). \end{cases}$$

Proof of Claim: Let τ be a unimodular $\mathbf{u}$-triangulation such that $S_{\mathbf{u}}$ is in τ and $ostar(S_{\mathbf{u}}) \subseteq [0,1]^n_{\mathbf{MV}} \setminus \eth[0,1]^n_{\mathbf{MV}}$. The function $t_{S_{\mathbf{u}}}$ defined in the vertices of τ as:

$$t_{S_{\mathbf{u}}}(\bar{y}) = \begin{cases} 1 & \text{if } \bar{y} \text{ is a vertex of } S_{\mathbf{u}} \\ \\ 0 & \text{otherwise,} \end{cases}$$

and linearly extended to $[0,1]^n_{\mathbf{MV}}$ is a function in $Free_{\mathcal{MV}}(n)$. Also, $t_{S_{\mathbf{u}}}(S_{\mathbf{u}}) = 1$ so clearly we have that $t_{S_{\mathbf{u}}} \in G$. Then there is a function $h_{S_{\mathbf{u}}} \in Free_{\mathcal{MS}}(n)$ such that $h_{S_{\mathbf{u}}}\restriction_{[0,1]^n_{\mathbf{MV}}} = t_{S_{\mathbf{u}}}$.

Since $ostar(S_{\mathbf{u}}) \subseteq [0,1]^n_{\mathbf{MV}} \setminus \eth[0,1]^n_{\mathbf{MV}}$ for every $\bar{x} \in \eth[0,1]^n_{\mathbf{MV}}$ we have that $t_{S_{\mathbf{u}}}(\bar{x}) = 0$. By Lemma 4.4, the function $h_{S_{\mathbf{u}}} \in Free_{\mathcal{MS}}(n)$ such that $h_{S_{\mathbf{u}}}\restriction_{[0,1]^n_{\mathbf{MV}}} = t_{S_{\mathbf{u}}}$ is then unique, since it must be 0 in every $\bar{x} \notin [0,1)^n_{\mathbf{MV}}$. Hence, we proved the Claim.

Let f be a function in G^{cyl}. Since $f \in G^{cyl}$, $f\restriction_{[0,1]^n_{\mathbf{MV}}} \in G = P_{MV}$. Then there is a $\mathbf{u}$-simplex $S_{\mathbf{u}}$ such that $f\restriction_{[0,1]^n_{\mathbf{MV}}}$ takes the value 1 over $S_{\mathbf{u}}$. Let τ be a $\mathbf{u}$-unimodular triangulation such that f is linear in every simplex of τ and the $\mathbf{u}$-simplex $T_{\mathbf{u}}$ of τ satisfies that $T_{\mathbf{u}} \subseteq S_{\mathbf{u}} \cap [0,1)_{\mathbf{MV}}$. Then $f \geq h_{T_{\mathbf{u}}}$ and from the claim we get $f \in P$ as desired. $\square$

Hence if P is a prime filter in $Free_{\mathcal{MS}}(n)$ localized at $\bar{x}$ with $\bar{x} \in [0,1)^n_{\mathbf{MV}}$, then P is the cylindrification of a prime filter $G \in Free_{\mathcal{MV}}(n)$ which is localized at $MV_{\bar{x}}$. From Theorem 5.9 we can conclude:

Theorem 5.12. *Let $\bar{x} \in [0,1)^n_{\mathbf{MV}}$ and let P be a prime filter in $Free_{\mathcal{MS}}(n)$ localized at $\bar{x}$. Then there is an index $\mathbf{u}$ such that*

$$P = \{f \in Free_{\mathcal{MS}}(n) : f^{-1}(\{1\}) \textit{ contains some } \mathbf{u}\textit{-simplex in } [0,1)_{\mathbf{MV}}\}.$$

Finally we will characterize prime filters in $Free_{\mathcal{MS}}(n)$ localized at the relative border $\eth[0,1]^n_{\mathbf{MV}}$. To achieve that we will first recall some notation from Section 4 and introduce new one.

Remember that if $\bar{x} = (x_1, \ldots, x_n)$ is a point in a region R_A with $A = \{j_1, \ldots, j_m\}$ we write

$$\pi_H(\bar{x}) = (x_{j_1}, \ldots, x_{j_m}) \in \mathbf{H}^m.$$

We also recall that for each $\bar{x} \in \eth[0,1]^n_{\mathbf{MV}}$ then $1_{\bar{x}} = \{i \in \{1, \ldots, n\} : x_i = 1\}$ and the cylindrification of the point is the set

$$\tilde{x} = \{\bar{z} \in \mathfrak{S}^n : z_j = x_j, \text{ for every } j \notin 1_{\bar{x}}, \text{ and } z_i \in \mathbf{H}, \text{ for every } i \in 1_{\bar{x}}\}.$$

With this in mind, for each point $\bar{x} \in \eth[0,1]^n_{\mathbf{MV}}$ and each prime filter $P \in Free_{\mathcal{MS}}(n)$ localized at $\bar{x}$ we define the set $P_{\mathbf{H},\bar{x}}$ as

$$P_{\mathbf{H},\bar{x}} = \{f\restriction_{\tilde{x}} \;:\; f \in P\}.$$

From Lemma 4.4 the set $P_{\mathbf{H},\bar{x}}$ is contained in $Free_{\mathcal{H}}(m)$ with $m = \|\bar{x}\|$ and it is easy to verify that it is a filter. Similar to our previous notation, we can also define the set $(P_{\mathbf{H},\bar{x}})^{cyl} \subseteq Free_{\mathcal{MS}}(n)$ as

$$f \in (P_{\mathbf{H},\bar{x}})^{cyl} \text{ if and only if } f\restriction_{\tilde{x}} \in P_{\mathbf{H},\bar{x}}.$$

Thus $(P_{\mathbf{H},\bar{x}})^{cyl}$ is a filter in $Free_{\mathcal{MS}}(n)$ such that $P \subseteq (P_{\mathbf{H},\bar{x}})^{cyl}$.

Theorem 5.13. *Let $\bar{x} \in \eth[0,1]^n_{\mathbf{MV}}$ and $P \subseteq M_{\bar{x}}$ be a prime filter in $Free_{\mathcal{MS}}(n)$. Then at least one of the following properties hold:*

- $P_{\boldsymbol{H},\bar{x}} \cong Free_{\mathcal{H}}(m)$ *with* $m = \|\bar{x}\|$
- $P_{MV} = MV_{\bar{x}}$

Proof. Suppose, on the contrary, that $P_{\mathbf{H},\bar{x}} \subsetneq Free_{\mathcal{H}}(m)$ and P_{MV} is a proper filter in $MV_{\bar{x}}$.

Clearly, we have that $P \subseteq (P_{MV})^{cyl}$ and $P \subseteq (P_{\mathbf{H},\bar{x}})^{cyl}$. We will show that $(P_{MV})^{cyl} \nsubseteq (P_{\mathbf{H},\bar{x}})^{cyl}$ and $(P_{\mathbf{H},\bar{x}})^{cyl} \nsubseteq (P^{cyl}_{MV})$, and that will contradict the fact that P is prime, since the set of filters that contain a prime filter in a BL-algebra is totally ordered.

Since $P_{\mathbf{H},\bar{x}} \subsetneq Free_{\mathcal{H}}(m)$ there is a function $h \in Free_{\mathcal{H}}(m)$ such that $h \notin P_{\mathbf{H},\bar{x}}$. Let $h^{\ddagger} \in Free_{\mathcal{MS}}(n)$ be the function given by:

$$h^{\ddagger}(\bar{y}) = \begin{cases} h(\pi_{\mathbf{H}}(\bar{y})) & \text{if } \bar{y} \in \tilde{x} \\ \\ 1 & \text{otherwise.} \end{cases} \tag{26}$$

Consider $f = h^{\ddagger}\restriction_{[0,1]^n_{\mathbf{MV}}}$ in $Free_{\mathcal{MV}}(n)$. Clearly $f(\bar{y}) = 1$ for each $\bar{y} \in [0,1]_{\mathbf{MV}}$ then we have $h^{\ddagger} \in (P_{MV})^{cyl}$. But $h^{\ddagger} \notin (P_{\mathbf{H},\bar{x}})^{cyl}$, because $h^{\ddagger}\restriction_{\tilde{x}} = h \notin P_{\mathbf{H},\bar{x}}$. Hence, $(P_{\mathbf{H},\bar{x}})^{cyl} \nsubseteq (P_{MV})^{cyl}$.

Let $f \in Free_{\mathcal{MV}}(n)$ be a function such that $f \in MV_{\bar{x}} \setminus P_{MV}$. Then we can define the function $f^{\sharp} \in Free_{\mathcal{MS}}(n)$ as we did in the beggining of this section. By construction, $f^{\sharp} \notin (P_{MV})^{cyl}$. We also have that $f^{\sharp}\restriction_{\tilde{x}} = f^{\sharp}(\bar{x}) = 1$, so $f^{\sharp}\restriction_{\tilde{x}} \in P_{\mathbf{H},\bar{x}}$. Hence, $f^{\sharp} \in (P_{\mathbf{H},\bar{x}})^{cyl}$ and we can conclude that $(P_{MV})^{cyl} \nsubseteq (P_{\mathbf{H},\bar{x}})^{cyl}$, which completes the proof. □

We conclude the characterization of prime filters with the following two theorems:

Theorem 5.14. *Let $\bar{x} \in \eth[0,1]^n_{\boldsymbol{MV}}$, $P \subseteq M_{\bar{x}}$ be a prime filter in $Free_{\mathcal{MS}}(n)$ such that $P_{MV} = MV_{\bar{x}}$. Then $P_{\boldsymbol{H},\bar{x}}$ is a prime filter in $Free_{\mathcal{H}}(m)$ with $m = \|\bar{x}\|$ and*

$$P = \{f \in Free_{\mathcal{MS}}(n) : f\restriction_{\tilde{x}} \in P_{\boldsymbol{H},\bar{x}}\}.$$

Proof. To prove that $P_{\mathbf{H},\bar{x}}$ is a prime filter in $Free_{\mathcal{H}}(m)$ we can use an argument analogous to the one in the proof of Lemma 5.2 using the function $h^{\ddagger}$ instead of $h^{\sharp}$. Let F denote the set $F = \{f \in Free_{\mathcal{MS}}(n) : f\restriction_{\tilde{x}} \in P_{\mathbf{H},\bar{x}}\}$. The inclusion $P \subseteq F$ is straightforward from the definitions. To see the opposite inclusion, consider $f \in F$. Then $f\restriction_{\tilde{x}}$ is in $P_{\mathbf{H},\bar{x}}$, thus there is $h \in P$ such that $h\restriction_{\tilde{x}} = f\restriction_{\tilde{x}}$.

On the other hand, by Remark 2.8 there is a triangulation Δ_f of $[0,1]^n_{\mathbf{MV}}$ such that f is linear over each simplex of Δ. Since that triangulation is a simplicial complex, we can consider that complex and the point $\bar{x}$ and apply Remark 2.10 to obtain a unimodular triangulation Δ of $[0,1]^n_{\mathbf{MV}}$. We define a function g in the vertices of Δ as:

$$g(\bar{y}) = \begin{cases} 1 & \text{if } \bar{y} = \bar{x} \\ \\ 0 & \text{otherwise,} \end{cases}$$

and linearly extend it to $[0,1]^n_{\mathbf{MV}}$. Then g is a function in $Free_{\mathcal{MV}}(n)$, and $P_{MV} = MV_{\bar{x}}$, $g \in P$ and satisfies

- $g(\bar{x}) = 1$,
- $g(\bar{y}) < 1$ for each $\bar{y} \in [0,1]^n_{\mathbf{MV}} \setminus \{\bar{x}\}$,
- $g(\bar{y}) \leq f(\bar{y})$ for each $\bar{y} \in [0,1]^n_{\mathbf{MV}}$.

Therefore $f \geq h \wedge g$ and since $h \wedge g \in P$ we get $f \in P$ as desired. $\square$

Theorem 5.15. *Let $\bar{x} \in \eth[0,1]^n_{\mathbf{MV}}$, $P \subseteq M_{\bar{x}}$ be a prime filter in $Free_{\mathcal{MS}}(n)$ such that $P_{\mathbf{H},\bar{x}} = Free_{\mathcal{H}}(m)$ with $m = \|\bar{x}\|$. Then there is an index $\mathbf{u}$ such that*

$$P \subseteq \{f \in Free_{\mathcal{MS}}(n) : f^{-1}(\{1\}) \textit{ contains some } \mathbf{u}\textit{-simplex in } [0,1]_{\mathbf{MV}}\}.$$

The proof of this theorem is immediate, since, unlike the result in Theorem 5.12, Theorem 5.15 only states that P is included in the cylindrification of a prime filter in $Free_{\mathcal{MV}}(n)$. We present an example to show that the inclusion may be proper.

Example 5.16. *Let $\mathbf{H}$ be the three-elements Gödel chain, $\mathbf{H} = \{a < b < 1\}$. The set of elements of $\mathbf{H}$ that are greater than a form a proper prime filter of $\mathbf{H}$. Consider $n = 2$ and the index $\mathbf{u} = (\bar{x}, \bar{v})$ where $\bar{x} = (\frac{1}{2}, 1)$ and $\bar{v} = (1,0)$. Then an element $g \in Free_{\mathcal{MV}}(2)$ is in $F_{\mathbf{u}}$ if and only if there is $\delta > 0$ such that $[\bar{x}, \bar{x} + \delta\bar{v}) \subseteq g^{-1}(\{1\})$. Consider the set*

$$P = \{f \in Free_{\mathcal{MS}}(2) : f\restriction_{[0,1]_{\mathbf{MV}^2}} \in F_{\mathbf{u}} \textit{ and there is}$$

$$\epsilon > 0 \textit{ such that } \forall \bar{y} \in (\bar{x}, \bar{x} + \epsilon\bar{v}) \textit{ and } \forall \bar{z} \in \tilde{y},\ f(\bar{z}) > a\}.$$

Then P is a prime filter in $Free_{\mathcal{MS}}(2)$ that satisfies $P_{\mathbf{H},\bar{x}} = Free_{\mathcal{H}}(1)$ and $P_{MV} = F_{\mathbf{u}} \subsetneq M_{\bar{x}}$. However, P is properly included in the cylindrification of P_{MV}.

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