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Applied Acoustics 62 (2001) 1171–1182

applied acoustics

www.elsevier.com/locate/apacoust

Some observations on the application of the Rayleigh–Ritz method

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Received 8 November 1999; received in revised form 16 November 2000; accepted 20 November 2000

Abstract

This paper deals with the applicability of the Rayleigh–Ritz method for the determination of frequency coefficients of beams and plates with elastically restrained edges. Natural frequencies of beams elastically restrained against rotation and translation at both ends and of rectangular isotropic plates with elastic edge restraints are studied by using the Rayleigh–Ritz method along with orthogonal polynomials as co-ordinate functions. It is shown that the approximate satisfaction of boundary conditions introduces additional constraints into the formulation that bring unexpected values in the results. On the other hand, it is shown that there are defects in the approximations when the natural boundary conditions in beams are taken into account. The adequate procedure for constructing the co-ordinate functions to avoid numerical errors is presented. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Variational method; Plates; Beams; Vibrations

1. Introduction

The determination of natural frequencies in the transverse vibrations of rectangular plates with edges elastically restrained has been treated by several researchers [1–16]. In various of these works the Rayleigh–Ritz method has been applied.

When dealing with this method it is necessary to select a sequence of co-ordinate functions. The fact that the natural boundary conditions of a system need not be

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PII: S0003-682X(00)00097-9

satisfied by the chosen co-ordinate functions is a very important characteristic of the method, specially when dealing with problems for which such satisfaction is very difficult to achieve. For instance, this is the case of a rectangular isotropic plate with generally restrained edges. Consequently, it is possible to replace the original natural boundary conditions by more easily applied conditions or simply to ignore them. Laura and Grossi [7] showed the existence of a curious anomaly in the approximate analysis of a rectangular plate with edges elastically restrained against rotation and translation. The natural boundary conditions were satisfied approximately and the Rayleigh–Ritz method was used to derive and approximate the fundamental frequency equation. The anomaly was shown to exist in the frequency values for a certain range of the rotational and translational restraints parameters. In the present paper the Rayleigh–Ritz method with orthogonal polynomials as shape functions are used to explore this anomaly and to demonstrate that the origin of the problem is caused by the use of approximate natural boundary conditions. The eigenvalues have been calculated numerically and the effects of variation of rotational and translational restraint parameters, have been clarified quantitatively. A great number of problems were solved. Since these number of cases is prohibitively large, results are presented for only a few cases.

On the other hand, natural frequencies in the transverse vibration of beams with elastically restrained ends have been studied. Exact solutions for uniform beams with ends elastically restrained against rotation and translation have been obtained by the method of separation of variables. The approximate solutions have been obtained by using the Rayleigh–Ritz method. It is also the purpose of this paper to demonstrate the existence of defects in the approximate numerical values when using a set of orthogonal polynomials in the mentioned variational method, even when the natural boundary conditions are identically satisfied. It is shown that in certain cases choosing the co-ordinate functions which each satisfy identically the natural boundary conditions can lead to results with lower precision than those achievable by using co-ordinate functions which do not take into account these boundary conditions.

2. Analysis of the plate problem

Let us consider the rectangular isotropic plate with edges elastically restrained against rotation and translation shown in Fig. 1. Rayleigh–Ritz method requires the minimization of the Rayleigh quotient which for the fundamental frequency is given by:

$$\omega^2 = \frac{U_{\max}}{T_{\max}} \tag{1}$$

where $U_{\max} = U_{p,\max} + U_{r,\max} + U_{t,\max}$

$U_{p,\max}$ is the maximum strain energy of the plate and is given by

$$U_{p,\max} = \frac{1}{2}\int_0^b\int_0^a D\Big[(W_{xx})^2+\big(W_{yy}\big)^2+2\mu W_{xx}W_{yy}+2(1-\mu)\big(W_{xy}\big)^2\Big]\mathrm{d}x\mathrm{d}y \qquad (2)$$

The subscripts denote differentiation of W with respect to the subscripted variable, a and b are side lengths of the plate in the x and y directions, respectively.

The maximum strain energy associated to the rotational and traslational restraints in the edges are given by:

$$U_{r,\max} = \frac{1}{2} \times \left[r_1\int_0^b W_x^2(0,y)\mathrm{d}y + r_2\int_0^b W_x^2(a,y)\mathrm{d}y + r_3\int_0^a W_y^2(x,0)\mathrm{d}x + r_4\int_0^a W_y^2(x,b)\mathrm{d}x\right] \qquad (3)$$

$$U_{t,\max} = \frac{1}{2} \times \left[t_1\int_0^b W^2(0,y)\mathrm{d}y + t_2\int_0^b W^2(a,y)\mathrm{d}y + t_3\int_0^a W^2(x,0)\mathrm{d}x + t_4\int_0^a W^2(x,b)\mathrm{d}x\right] \qquad (4)$$

where r_i and t_i (i=1,4) are, respectively, the rotational and translational spring

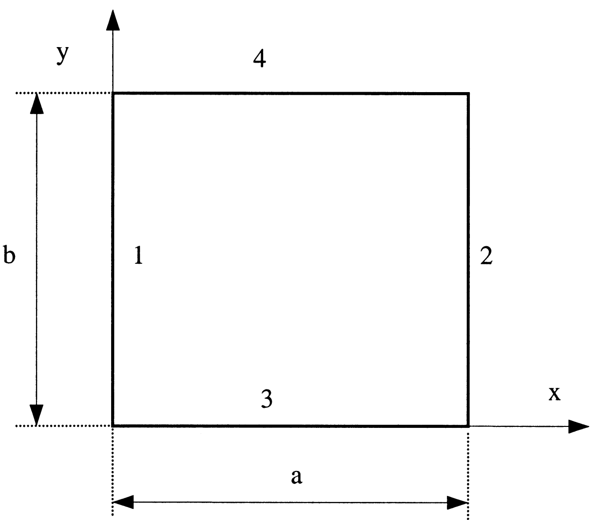

Fig. 1. Rectangular plate under study (numbers at the edges used as subscripts in defining edge restraint parameters).

constants along the corresponding edges. Finally, the maximum kinetic energy of the mechanical system is given by

$$T_{\max} = \frac{\rho\omega^2}{2}\int\int_{\mathrm{R}} h(x, y)W^2(x, y)\mathrm{d}x\mathrm{d}y \tag{5}$$

In the Rayleigh–Ritz method, one seeks an approximate solution in the form of a linear combinations of the so called co-ordinate functions $\varphi_i(x)\varphi_j(y)$ and undetermined parameters c_{ij}. These parameters are determined such that the solution $W(x, y) = \sum_{i=0}^{\mathrm{N}}\sum_{j=0}^{\mathrm{M}} c_{ij}\phi_i(x)\varphi_j(y)$ gives a minimum to the Rayleigh quotient (1).

In the case of normal modes of vibration, the boundary conditions are as follows

$$t_1 W(0, y) = -D\left(\frac{\partial^3 W(0, y)}{\partial x^3} + (2-\mu)\frac{\partial^3 W(0, y)}{\partial x \partial y^2}\right) \tag{6a}$$

$$r_1 \frac{\partial W(0, y)}{\partial x} = D\left(\frac{\partial^2 W(0, y)}{\partial x^2} + \mu\frac{\partial^2 W(0, y)}{\partial y^2}\right) \tag{6b}$$

$$t_2 W(a, y) = D\left(\frac{\partial^3 W(a, y)}{\partial x^3} + (2-\mu)\frac{\partial^3 W(a, y)}{\partial x \partial y^2}\right) \tag{6c}$$

$$r_2 \frac{\partial W(a, y)}{\partial x} = -D\left(\frac{\partial^2 W(a, y)}{\partial x^2} + \mu\frac{\partial^2 W(a, y)}{\partial y^2}\right) \tag{6d}$$

$$t_3 W(x, 0) = -D\left(\frac{\partial^3 W(x, 0)}{\partial y^3} + (2-\mu)\frac{\partial^3 W(x, 0)}{\partial y \partial x^2}\right) \tag{6e}$$

$$r_3 \frac{\partial W(x, 0)}{\partial y} = D\left(\frac{\partial^2 W(x, 0)}{\partial y^2} + \mu\frac{\partial^2 W(x, 0)}{\partial x^2}\right) \tag{6f}$$

$$t_4 W(x, b) = D\left(\frac{\partial^3 W(x, b)}{\partial y^3} + (2-\mu)\frac{\partial^3 W(x, b)}{\partial y \partial x^2}\right) \tag{6g}$$

$$r_4 \frac{\partial W(x, b)}{\partial y} = -D\left(\frac{\partial^2 W(x, b)}{\partial y^2} + \mu\frac{\partial^2 W(x, b)}{\partial x^2}\right) \tag{6h}$$

It is difficult to construct co-ordinate functions, which satisfy the boundary conditions (6). However, as it is not necessary to subject these functions to the natural boundary conditions, it is possible to replace the natural boundary conditions by more easily applied conditions. For instance, in this case it is possible to replace Eqs. (6) by the following approximate expressions

$$t_1 W(0, y) = -D\frac{\partial^3 W(0, y)}{\partial x^3}, \quad r_1 \frac{\partial W(0, y)}{\partial x} = D\frac{\partial^2 W(0, y)}{\partial x^2} \tag{7a, b}$$

$$t_2 W(a,y) = D\frac{\partial^3 W(a,y)}{\partial x^3}, \quad r_2 \frac{\partial W(a,y)}{\partial x} = -D\frac{\partial^2 W(a,y)}{\partial x^2} \tag{7c, d}$$

$$t_3 W(x,0) = -D\frac{\partial^3 W(x,0)}{\partial y^3}, \quad r_3 \frac{\partial W(x,0)}{\partial y} = D\frac{\partial^2 W(x,0)}{\partial y^2} \tag{7e, f}$$

$$t_4 W(x,b) = D\frac{\partial^3 W(x,b)}{\partial y^3}, \quad r_4 \frac{\partial W(x,b)}{\partial y} = -D\frac{\partial^2 W(x,b)}{\partial y^2} \tag{7g, h}$$

This procedure has been successfully used in several previous works [8–16].

Nevertheless, the assumption that the deflection function is a series of co-ordinate functions each of which satisfies conditions (7) represents an unnecessary restraint on the system. This can lead to poorer results than those achievable by using similar functions which are not constrained to satisfy these conditions. This topic is discussed by Bassily and Dickinson [17], where the use of beam vibration mode shapes in the Rayleigh-Ritz method to obtain solutions for the study of plates involving free edges is analysed.

In the present paper it is demonstrated that the use of approximate conditions (7) can yield numerical results which are against the physical nature of the problem, and it is shown how to remedy this situation.

3. Numerical results

The assumed shape function for using the Rayleigh–Ritz procedure is given by:

$$W(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_i(x) \ \varphi_j(y) \tag{8}$$

where c_{ij} are arbitrary coefficients and $\{\phi_i(x), \varphi_i(y)\}$ is the set of orthogonal polynomials. The procedure for constructing the orthogonal polynomials is well known and has been descripted in various articles included in the list of references [18–20]. The minimization of (1) leads to the following eigenvalue problem:

$$\sum_i \sum_j \left[K_{mnij} - \Omega^2 M_{mnij}\right] \ c_{ij} = 0 \quad \text{where} \quad \Omega^2 = \frac{\rho h \omega^2 a^4}{D}.$$

Table 1 depicts values of $\Omega_{00} = \sqrt{\frac{\rho h}{D}}\omega_{00} a^2$ for an isotropic rectangular plate with edges 1, 3 and 4 rigidly clamped while edge 2 is elastically restrained against rotation and translation. This values have been obtained with only one term in function (8)

Table 1
Values of $\Omega_{00} = \sqrt{\frac{\rho h}{D}}\omega_{00}a^2$ for an isotropic rectangular plate with edges 1, 3 and 4 rigidly clamped while edge 2 is elastically restrained against rotation and translation. These values have been obtained with only one term in Eq. (8). The corresponding polynomials were determined with the approximate natural boundary conditions (7)

$R_2 = \frac{r_2 b}{D}$	$T_2 = \frac{t_2 b^3}{D}$						
	20	30	40	50	80	130	140
0	25.44457	26.075093	26.633621	27.116369	28.193878	29.240413	29.384627
10	25.511582	25.985269	26.430480	26.844956	27.908709	29.194148	29.395874
40	25.525741	25.966520	26.387668	26.787038	27.850223	29.219787	29.443873
50	25.526901	25.964988	26.384174	26.782318	27.845618	29.223142	29.449385
100	25.529321	25.961793	26.376885	26.772475	27.836121	29.230905	29.461849
500	25.531363	25.959100	26.370745	26.764192	27.828247	29.238307	29.473437
1000	25.531625	25.958755	26.369957	26.763130	27.827246	29.239317	29.475000

where the polynomials $\phi_0(x)$ and $\varphi_0(y)$ have been constructed as the simplest polynomials with enough number of terms to satisfy the approximate conditions (7).

It is observed that the values of Ω_{00} do not increase when the values of R_2 increase, for the following ranges of values: $30 \leqslant T_2 = \frac{t_2 b^3}{D} < 130$ and $0 \leqslant R_2 = \frac{r_2 b}{D} \leqslant 1000$. Table 2 shows the effect of use of an increasing number M = N of polynomials in the assumed shape function. The values of Ω_{00} increase with the values of R_2 when $M = N \geqslant 6$. On the other hand Table 3 depicts values of the mentioned frequency coefficient obtained using function (8) with the first polynomials $\phi_0(x)$ and $\varphi_0(y)$ determined ignoring all natural boundary conditions and particularly the approximate conditions (7). In this case the variation of the values of Ω_{00} is correct and the convergence to the exact result is verified.

To sum up, the use of approximate conditions (7) with a small number of coordinate functions leads to incorrect variation of frequency values. This deficiency

Table 2
Values of $\Omega_{00} = \sqrt{\frac{\rho h}{D}}\omega_{00}a^2$ for an isotropic rectangular plate with edges 1, 3 and 4 rigidly clamped while edge 2 restrained against rotation and translation. $\left(R_2 = \frac{r_2 b}{D} = 0 \text{ and } 10; \quad T_2 = \frac{t_2 b^3}{D} = 80\right)$. These values have been obtained with different number of polynomials in Eq. (8). The first polynomials were determined with the approximate natural boundary conditions (7)

$R_2 = 0$	$R_2 = 10$	M = N (number of polynomials)
28.194	27.909	1
28.101	27.739	2
27.903	27.642	3
27.784	27.638	4
27.655	27.625	5
27.601	27.622	6
27.580	27.618	7

Table 3
Values of $\Omega_{00}=\sqrt{\frac{\rho h}{D}}\omega_{00}a^2$ for an isotropic rectangular plate with edges 1, 3 and 4 rigidly clamped while edge 2 restrained against rotation and translation. $\left(R_2=\frac{r_2 b}{D}=0 \text{ and } 10; \quad T_2=\frac{t_2 b^3}{D}=80\right)$. These values have been obtained with different number of polynomials in Eq. (8). The first polynomials were constructed without considering the natural boundary conditions

$R_2=0$	$R_2=10$	M = N (number of polynomials)
31.812	34.814	1
28.582	28.720	2
27.600	27.636	3
27.598	27.635	4
27.578	27.621	5

disappears when the number of co-ordinate functions is increased. However, if the natural boundary conditions are simply ignored, the convergence without defects to the correct solution is assured.

4. Analysis of the beam problem

Let us consider a uniform beam of length L whose ends are elastically restrained against rotation and translation as shown in Fig. 2. The differential equation for free flexural vibrations is given by:

$$EI\frac{\partial^4 u(x,t)}{\partial x^4}+\rho A\frac{\partial^2 u(x,t)}{\partial t^2}=0 \tag{9}$$

where u is the lateral deflection, EI the flexural rigidity, ρ the mass density and A the cross-sectional area of the beam. The boundary conditions are as follows

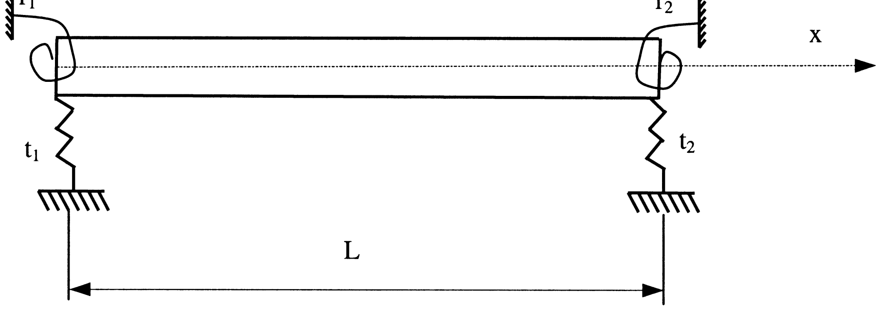

Fig. 2. Beams with elastically restrained ends under study.

$$r_1 \frac{\partial u(0,t)}{\partial x} = EI \frac{\partial^2 u(0,t)}{\partial x^2} \tag{10a}$$

$$t_1 u(0,t) = -EI \frac{\partial^3 u(0,t)}{\partial x^3} \tag{10b}$$

$$r_2 \frac{\partial u(L,t)}{\partial x} = -EI \frac{\partial^2 u(L,t)}{\partial x^2} \tag{10c}$$

$$t_2 u(L,t) = EI \frac{\partial^3 u(L,t)}{\partial x^3} \tag{10d}$$

Using the method of separation of variables one can assume as solution a series of the functions

$$u_n(x) = k_1 \sinh(\lambda x) + k_2 \cosh(\lambda x) + k_3 \sin(\lambda x) + k_4 \cos(\lambda x) \tag{11}$$

Substituting expression (11) in the boundary conditions (10) results in a homogeneous system of linear equations in the four constants k_1, k_2, k_3 and k_4. For a non-trivial solution the determinant of the matrix coefficient must be equal zero. This procedure yields the following frequency equation

$$\begin{aligned} &(A\sinh(\lambda) + B\cosh(\lambda))\sin(\lambda) + (C\cosh(\lambda) + D\sinh(\lambda))\cos(\lambda) + \lambda^8 \\ &\quad + \lambda^4(R_1T_1 + R_2T_2) + R_1R_2T_1T_2 = 0. \end{aligned} \tag{12}$$

where

$$\begin{aligned} A &= 2\lambda^2\left(T_1T_2 - \lambda^4 R_1R_2\right), \\ B &= -\lambda^7(R_1 + R_2) - \lambda^5(T_1 + T_2) + \lambda^3(R_1R_2T_2 + R_1R_2T_1) + \lambda(R_1T_1T_2 + R_2T_1T_2) \\ C &= -\lambda^8 + \lambda^4(R_1T_1 + R_2T_2 + 2R_2T_1 + 2R_1T_2) - R_1T_1R_2T_2, \\ D &= -\lambda^7(R_1 + R_2) + \lambda^5(T_1 + T_2) + \lambda^3(R_1R_2T_2 + R_1R_2T_1) - \lambda(R_1T_1T_2 + R_2T_1T_2) \end{aligned}$$

$$\lambda^4 = \frac{\rho A}{EI}\omega^2 L^4, \quad R_i = \frac{r_i L}{EI}, \quad T_i = \frac{t_i L^3}{EI}, \quad i = 1, 2.$$

Numerical results have also been obtained using orthogonal polynomials in the Rayleigh–Ritz method. In the case of normal modes the assumed shape function is given by:

$$u(x) = \sum_{i=0}^{N} c_i \phi_i(x) \tag{13}$$

Table 4 depicts values of the fundamental frequency coefficient λ_1, with

$$\lambda_1 = \sqrt{\sqrt{\frac{\rho A}{EI}}\omega_1 l^2}, \quad T_1 = \infty, \quad R_2 = 0, \quad R_1 = \frac{r_1 l}{EI} = 0, \quad T_2 = \frac{t_2 l^3}{EI} = 2000$$

The exact solution was obtained from Eq. (12). It can be observed that when the natural boundary conditions (10) are identically satisfied the convergence is affected numerically. On the other hand when these boundary conditions are not considered the convergence is not affected but obviously when only a few terms are used in (13) the results are not so accurate.

Table 5 shows values of the fundamental frequency coefficient λ_1, with

$$\lambda_1 = \sqrt{\sqrt{\frac{\rho A}{EI}}\omega_1 l^2}, \quad T_1 = \infty, \quad R_2 = 0, \quad R_1 = \frac{r_1 l}{EI} = 0, \quad T_2 = \frac{t_2 l^3}{EI} = 500$$

The same situation described for Table 4 is observed when the natural boundary conditions are ignored. Finally Fig. 3 shows the absolute error $E = \lambda_1^* - \lambda_1$ as a function of the rotational restraint parameter R_1. The values λ_1^* have been obtained using function (13) with N = 8 and the first polynomial determined by satisfying identically the corresponding natural boundary conditions (10) and the exact values λ_1 where obtained from Eq. (12). One immediately observes that, for certain values of the translational restraint parameter T_2, say $200 < T_2 < 750$ the error is greater.

If the corresponding natural boundary conditions are ignored in the application of Rayleigh–Ritz the method this error disappears.

Table 4
Values of λ_1 with $\lambda_1^4 = \frac{\rho A}{EI}\omega_1^2 l^4$, $T_1 = \infty$, $R_2 = 0$, $R_1 = \frac{r_1 l}{EI} = 0$, $T_2 = \frac{t_2 l^3}{EI} = 2000$. The exact solution λ_1 =3.1338386 was obtained from Eq. (12). (I) natural boundary conditions were identically satisfied; (II) natural boundary conditions were not considered

Number of polynomials	(I)	(II)
1	3.135304	8.801117
2	3.135304	3.299412
3	3.134464	3.299410
4	3.134463	3.134289
5	3.134170	3.134289
6	3.134170	3.133839
7	3.134168	
8	3.134168	
12	3.134168	

Table 5
Values of λ_1 with $\lambda_1^4 = \frac{\rho A}{EI}\omega_1^2 l^4$, $T_1 = \infty$, $R_2 = 0$, $R_1 = \frac{r_1 l}{EI} = 0$, $T_2 = \frac{t_2 l^3}{EI} = 500$. The exact solution $\lambda_1 = 3.1105681$ was obtained from Eq. (12). (I) natural boundary conditions were identically satisfied; (II) natural boundary conditions were not considered

Number of polynomials	(I)	(II)
1	3.112721	6.223330
2	3.112721	3.268483
3	3.111806	3.268457
4	3.111793	3.110982
5	3.111560	3.110976
6	3.111559	3.110568
7	3.111559	
8	3.111558	
15	3.111533	

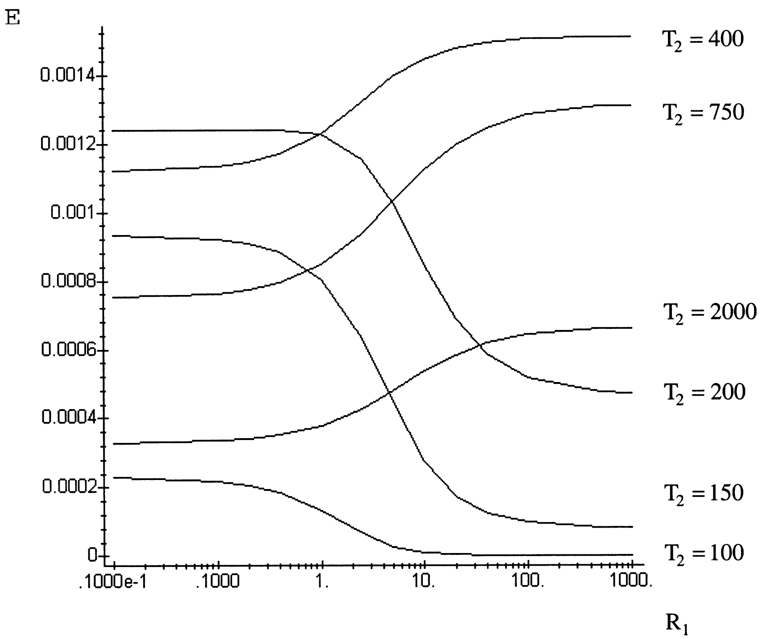

Fig. 3. Fundamental frequency error $E = \lambda_1^* - \lambda_1$ of beams with elastically restrained ends, where λ_1^* is the value obtained by the Rayleigh–Ritz method and λ_1 is the exact value obtained by Eq. (12).

5. Conclusions

When applying the Rayleigh–Ritz method it is not necessary to subject the coordinate functions to the natural boundary conditions, so it is possible to replace them by more easily applied conditions or to ignore them. In the present paper it was demonstrated:

(a) that the procedure of replacement of the natural boundary conditions by approximate expressions in the dynamical behaviour of rectangular plates with edges elastically restrained against rotation and translation, is adequate when only a small number of co-ordinate functions is used. In some cases, this procedure can lead to results counter to the physical nature of the problem under study.

When the approximate boundary conditions (7) are satisfied, this applies an artificial constraint on the deflection shape, thus increasing the natural frequency results. These artificial constraints do not vanish when the number of terms are increased. When only the geometrical conditions are satisfied, there are no such artificial constraints, and hence the convergence is very good. Only the approximate satisfaction of the boundary conditions, which introduces additional constraints into the formulation, brings defects in the results.

(b) That satisfaction of the natural boundary conditions in the dynamical behaviour of beams with ends elastically restrained against rotation and translation can lead to results affected by errors that disappear when those boundary conditions are simply ignored.

The results presented show that although individually each of the orthogonal polynomials used does not satisfy the natural boundary conditions, the series composed of these polynomials converges rapidly towards such satisfaction and very accurate numerical results are obtained.

Acknowledgements

The authors are grateful to Dr. Rama Bhat for his constructive criticism. The present study has been sponsored by Consejo de Investigación UNSa, Proyecto No. 744.

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