A GEOMETRIC DECOMPOSITION OF SPACES INTO CELLS OF DIFFERENT TYPES II: HOMOLOGY THEORY

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Abstract. We develop the homology theory of CW(A)-complexes, generalizing the classical cellular homology theory for CW-complexes. A CW(A)-complex is a topological space which is built up out of cells of a certain core A.

1. Introduction

This is the second article of a series of three papers in which we introduce and develop the theory of CW(A)-complexes. A CW(A)-complex is a topological space built up out of simple building blocks or cells of a certain type A (called the *core* of the complex). The theory of CW(A)-complexes generalizes the classical theory of CW-complexes, keeping the geometric intuition of J.H.C. Whitehead's original theory.

The main properties of standard CW-complexes arise from the following two basic facts: (1) the n-ball D^n is the topological (reduced) cone of the $(n-1)$-sphere S^{n-1} and (2) the n-sphere is the (reduced) n^{th}-suspension of the 0-sphere S^0. For example, the homotopy extension properties of CW-complexes are deduced from (1), since the inclusion $S^{n-1} \to D^n$ is a closed cofibration. Item (2) is closely related to the definition of the homotopy groups and is used to prove results such as Whitehead Theorem or Homotopy excision. These two basic facts suggest also that one might replace the original *core* S^0 by any other space A and construct spaces built up out of cells of different *shapes* or *types* using suspensions and cones of the base space A.

In the first article of this series [8] we developed the homotopy theory of such spaces. We proved various results, such as a generalized Whitehead Theorem, which allow a deeper insight into their homotopy properties. Some of these results, together with the basic definitions and ideas of this theory, are summarized in section 2. In this paper we investigate the homology theory of CW(A)-complexes. Our main goal is to develop tools and techniques which allow us to compute their singular homology out of the homology of the core A and their CW(A)-structure. These new tools appear as generalizations of the classical cellular homology theory. The general case is of course much more complicated to analyze than the standard case (i.e. when the core A is S^0), even if A itself is a CW-complex.

Note that the (reduced) homology of S^0 (with coefficients in $\mathbb{Z}$) has two key properties: it is concentrated in one degree (degree zero) and it is free (as an abelian group). Keeping this in mind, we begin our generalization of cellular homology by studying the case when the reduced homology of A is concentrated in a certain degree. In this case, the A-cellular chain complex (C_*, d) of a CW(A)-complex X is defined as follows. If the homology of A

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is concentrated in degree r, we take $C_n = \bigoplus_{A\text{-}(n-r)\text{-cells}} H_r(A)$ for every n, and we define $d_n : C_n \to C_{n-1}$ in a similar way to in the classical setting (see Section 3). The first significant result is the following.

Theorem. *Let A be a CW-complex with homology concentrated in degree r and let X be a CW(A)-complex. Then, the homology of the A-cellular chain complex defined as above coincides with the singular homology of X.*

The second case that we study is when the homology of A is free. We obtain the following result.

Theorem. *Let A be a CW-complex with free homology groups and let X be a finite dimensional CW(A)-complex. Then there exists a chain complex of $\mathbb{Z}$-modules (C_*, d) whose homology is the singular homology of X, with $C_n = \bigoplus_r H_{n-r}(A)^{\#r-cells}$.*

When the homology of the core A is neither concentrated nor free, the homology of X is more difficult to compute. Example 3.10 of Section 3 shows that, in that case, the homology of X cannot be computed from an A-cellular complex as in the theorems above. This case is analyzed in the third paper of this series [9]. As one might expect, there are spectral sequences which allow us to compute the singular homology of X in terms of the homology of the core and the CW(A)-structure.

In section 4 we investigate the A-Euler characteristic χ_A of CW(A)-complexes. We show that the A-Euler characteristic gives useful information on the space, although in some cases it might not be a topological invariant. More specifically, we prove below the following result.

Proposition. *Let A be a finite CW-complex and let X be a finite CW(A)-complex. Then $\chi(X) = \chi_A(X)\chi(A)$.*

We also define and investigate the *multiplicative Euler characteristic* when the core A has finite homology (see Theorem 4.7 below). The paper ends with some results which relate this theory with Moore spaces.

Throughout this paper, all spaces are assumed to be pointed spaces, all maps are pointed maps and homotopies are base-point preserving. Also homology will mean reduced homology with coefficients in $\mathbb{Z}$.

2. Preliminaries

In this section we recall briefly the main definitions and results on CW(A)-complexes. For a comprehensive exposition on the homotopy theory of CW(A)-complexes the interested reader might consult [8].

We denote by CX the reduced cone of X and by ΣX its reduced suspension. Also, S^n denotes the n-sphere and D^n the n-disk.

Let A be a fixed pointed topological space.

Definition 2.1. Let $n \in \mathbb{N}$. We say that a (pointed) space X *is obtained from a (pointed) space B by attaching an n-cell of type A* (or simply, an *A-n-cell*) if there exists a pushout diagram

$$\begin{array}{ccc} \Sigma^{n-1}A & \xrightarrow{g} & B \\ {\scriptstyle i}\downarrow & \textit{push} & \downarrow \\ \mathrm{C}\Sigma^{n-1}A & \xrightarrow[f]{} & X \end{array}$$

The A-*cell* is the image of f. The map g is the *attaching map* of the cell, and f is its *characteristic map.* We say that X *is obtained from* B *by attaching a* 0-*cell of type* A if $X = B \vee A$.

Note that attaching an S^0-n-cell is the same as attaching an n-cell in the usual sense, and that attaching an S^m-n-cell means attaching an $(m+n)$-cell in the usual sense.

For example, the reduced cone $\mathrm{C}A$ of A is obtained from A by attaching an A-1-cell. In particular, D^2 is obtained from D^1 by attaching a D^1-1-cell. Also, the reduced suspension ΣA can be obtained from the singleton $*$ by attaching an A-1-cell.

Definition 2.2. A *CW-structure with base* A *on a space* X, or simply a *CW(A)-structure on* X, is a sequence of spaces $* = X^{-1}, X^0, X^1, \ldots, X^n, \ldots$ such that, for $n \in \mathbb{N}_0$, X^n is obtained from X^{n-1} by attaching n-cells of type A, and X is the colimit of the diagram

$$* = X^{-1} \to X^0 \to X^1 \to \ldots \to X^n \to \ldots$$

The subspace X^n is called the n-*skeleton* of X. We say that the space X is a *CW(A)-complex* (or simply a *CW(A)*), if it admits some CW(A)-structure. In this case, the space A will be called the *core* or the *base space* of the structure.

Examples 2.3.

(1) A CW(S^0) is just a CW-complex and a CW(S^n) is a CW-complex with no cells of dimension less than n, apart from the base point.

(2) The space D^n admits several different CW(D^1)-structures. For instance, we can take $X^r = D^{r+1}$ for $0 \leq r \leq n-1$ since $\mathrm{C}D^r = D^{r+1}$. We may also define

$$X^0 = \ldots = X^{n-2} = * \qquad \text{and} \qquad X^{n-1} = D^n$$

since we have a pushout

$$\begin{array}{ccc} \Sigma^{n-2}D^1 = D^{n-1} & \longrightarrow & * \\ {\scriptstyle i}\downarrow & \textit{push} & \downarrow \\ \mathrm{C}\Sigma^{n-2}D^1 = \mathrm{C}D^{n-1} & \longrightarrow & \Sigma D^{n-1} = D^n \end{array}$$

It is clear that, in general, a topological space may admit many different decompositions into cells of different types. In [8] we studied the relationship between such different decompositions. In particular, we obtained results such as the following.

Theorem 2.4. *Let* A *be a* $CW(B)$*-complex of finite dimension and let* X *be a generalized* $CW(A)$*-complex. Then* X *is a generalized* $CW(B)$*-complex. In particular, if* A *is a standard finite dimensional CW-complex, then* X *is a generalized CW-complex and therefore it has the homotopy type of a CW-complex.*

By a *generalized complex* we mean a space which is obtained by attaching cells in countable many steps, allowing cells of any dimension to be attached in any step. The subspaces obtained at every step are called *layers*. They are the analogues of the n-skeletons.

We also analyzed the changing of the core A by a core B via a map $\alpha : A \to B$ and obtained the following result.

Theorem 2.5. *Let A and B be pointed topological spaces with closed base points, let X be a CW(A)-complex and let $\alpha : A \to B$ and $\beta : B \to A$ be continuous maps.*

i. *If $\beta\alpha = \mathrm{Id}_A$, then there exists a CW(B)-complex Y and maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi\varphi = \mathrm{Id}_X$.*
ii. *If β is a homotopy equivalence, then there is a CW(B)-complex Y and a homotopy equivalence $\varphi : X \to Y$.*
iii. *If $\beta\alpha = \mathrm{Id}_A$ and $\alpha\beta \simeq \mathrm{Id}_A$ then there exists a CW(B)-complex Y and maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi\varphi = \mathrm{Id}_X$ and $\varphi\psi = \mathrm{Id}_X$.*

In particular, when the core A is contractible, all CW(A)-complexes are also contractible.

Finally, we mention the following generalization of Whitehead Theorem.

Theorem 2.6. *Let X and Y be $CW(A)$-complexes and let $f : X \to Y$ be a continuous map. Then f is a homotopy equivalence if and only if it is an A-weak equivalence.*

We emphasize that our approach tries to keep the geometric intuition of Whitehead's original theory. There exist many generalizations of CW-complexes in the literature. We especially recommend Baues' generalization of complexes in Cofibration Categories [1]. There is also a categorical approach to cell complexes by the first named author of this paper [7]. The main advantage of the geometric point of view that we take in this article is that it allows the generalization of the most important classical results for CW-complexes and these new results can be applied in several concrete examples.

The theory of CW(A)-complexes that we develop in this series is also related to previous work of E. Dror Farjoun [4] and W. Chachólski [3] and to the underlying theory behind Kenzo, which is a set of computer programs for calculating homology and homotopy groups of spaces out of the homology and homotopy of simpler spaces [10]; although the approaches are quite different.

3. Homology of CW(A)-complexes

As we claimed in the introduction, our aim is to compute the singular homology groups of CW(A)-complexes out of the homology of A and the CW(A)-structure of the space.

Remark 3.1. Recall that if A and X are (pointed) CW-complexes and $g : A \to X$ is a continuous (cellular) map there is a long exact sequence

$$\cdots \longrightarrow H_n(A,*) \xrightarrow{g_*} H_n(X,*) \xrightarrow{i_*} H_n(\mathrm{C}g,*) \xrightarrow{q_*} H_{n-1}(A,*) \xrightarrow{g_*} \cdots$$

which induces short exact sequences

$$0 \longrightarrow \mathrm{Coker}\ g_* \xrightarrow{i_*} H_n(\mathrm{C}g,*) \xrightarrow{q_*} \mathrm{Ker}\ g_* \longrightarrow 0$$

Here, $\mathrm{C}g$ denotes the mapping cone of g. This has an evident analogy with the chain complex $\mathrm{C}g_*$, where g_* is the induced map in the singular chain complexes.

In case that all these short exact sequences split, the homology of Cg can be computed in the following way. The map g induces a morphism of chain complexes $g_* : H_*(A) \to H_*(X)$. The homology of the cone of this morphism

$$\cdots \longrightarrow H_{n+1}(X) \oplus H_n(A) \xrightarrow{\begin{pmatrix} 0 & g_* \\ 0 & 0 \end{pmatrix}} H_n(X) \oplus H_{n-1}(A) \xrightarrow{\begin{pmatrix} 0 & g_* \\ 0 & 0 \end{pmatrix}} H_{n-1}(X) \oplus H_{n-2}(A) \longrightarrow \cdots$$

is clearly the homology of Cg.

The well-known remark above will be our starting point to compute the singular homology of finite CW(A)-complexes. Consider the following example. Define $\mathbb{D}^2_4$ as the pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{g_4} & S^1 \\ \downarrow & push & \downarrow \\ D^2 & \longrightarrow & \mathbb{D}^2_4 \end{array}$$

where g_4 is a map of degree 4. Let the core A be $\mathbb{D}^2_4$ and let $g : D^2 \subseteq \mathbb{C} \to D^2$ be the map $g(z) = z^2$. The map g induces a well defined cellular map $g' : A \to A$. Let X be the CW(A)-complex of dimension one defined by the following pushout

$$\begin{array}{ccc} A & \xrightarrow{g'} & A \\ \downarrow & push & \downarrow \\ \mathrm{C}A & \longrightarrow & X \end{array}$$

Note that $H_1(A) = \mathbb{Z}_4$ and $H_r(A) = 0$ for $r \neq 1$. Also, the induced map $g'_* : H_1(A) \to H_1(A)$ is given by multiplication by 2. The cone of g' is in this case

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}_4 \xrightarrow{g'_*} \mathbb{Z}_4 \longrightarrow 0$$

where the group $\mathbb{Z}_4$ appears in degrees 1 and 2. Note that in the short exact sequences as above one gets $\ker g_* = 0$ or $\operatorname{coker} g_* = 0$. It follows that $H_r(X) = \mathbb{Z}_2$ for $r = 1, 2$ and $H_r(X) = 0$ for $r \neq 1, 2$.

The previous idea can also be applied to prove the following.

Proposition 3.2. *Let A be a CW-complex and let $n \in \mathbb{N}$. Let X be a CW(A)-complex with the property that, for every $r \in \mathbb{N}_0$, $H_{n-r}(A) = 0$ whenever X has at least one A-r-cell. Then $H_n(X) = 0$.*

Proof. Since A is a CW-complex, by cellular approximation we can suppose that X is also a standard CW-complex. Since all (standard) cells of dimension less than or equal to $n+1$ lie in the A-$(n+1)$-skeleton X^{n+1}, it suffices to prove that $H_n(X^{n+1}) = 0$.

We proceed by induction in the A-skeletons X^k. For $k = 0$ the result is clear. Now suppose the result holds for X^{k-1} and that X has A-k-cells. We denote by $g_\alpha : \Sigma^{k-1}A \to X^{k-1}$, $\alpha \in \Lambda$, their attaching maps. Consider the long exact sequence

$$\cdots \longrightarrow H_n(X^{k-1}) \xrightarrow{i_*} H_n(X^k, *) \xrightarrow{q_*} \bigoplus_{\alpha \in \Lambda} H_{n-1}(\Sigma^{k-1}A) = \bigoplus_{\alpha \in \Lambda} H_{n-k}(A) \xrightarrow{+(g_\alpha)_*} \cdots$$

By hypothesis, $H_n(X^{k-1}) = 0$ and since X^k has an A-k-cell, $H_{n-k}(A) = 0$. Hence, $H_n(X^k, *) = 0$. □

Corollary 3.3. *Let A be a CW-complex with homology concentrated in degree r and let X be a CW(A)-complex. If X does not have any A-n-cells, then $H_{n+r}(X) = 0$.* □

Given a CW(A)-complex X, our aim is to construct a suitable chain complex whose homology coincides with the homology of X. We investigate two particular cases: First we study the case when the homology of the core A is concentrated in one degree. The second case is when the homology of A is free. The constructions and results that we obtain in both cases generalize the standard results on cellular homology of CW-complexes.

We begin with the first case. Suppose $H_n(A) = 0$ for $n \neq r$, i.e. the (reduced) homology of A is concentrated in degree r.

In this case, given a CW(A)-complex X, we define the *A-cellular chain complex* (C_*, d) *of X* as follows. Take

$$C_n = \bigoplus_{A\text{-}(n-r)\text{-cells}} H_r(A)$$

and define $d_n : C_n \to C_{n-1}$ in the following way. Given e^n_α and e^{n-1}_β A-cells of dimensions n and $n-1$ respectively we consider $g_\alpha : \Sigma^{n-1}A \to X^{n-1}$ the attaching map of e^n_α (where X^{n-1} denotes the A-n-skeleton of X) and the quotient map

$$q_\beta : X^{n-1} \to X^{n-1}/(X^{n-1} - \overset{\circ}{e}{}^{n-1}_\beta) = \Sigma^{n-1}A.$$

The map $q_\beta g_\alpha : \Sigma^{n-1}A \to \Sigma^{n-1}A$ induces

$$(q_\beta g_\alpha)_* : H_{n+r-1}(\Sigma^{n-1}A) = H_r(A) \to H_{n+r-1}(\Sigma^{n-1}A) = H_r(A).$$

Finally, d_n is induced by the maps $d_n^{\alpha,\beta} = (q_\beta g_\alpha)_*$ from the α-th copy of $H_r(A)$ to the β-th copy of $H_r(A)$ (recall that $H_k(A) = 0$ if $k \neq r$).

Note that this chain complex is very similar to the standard (cellular) one. In fact, to prove that (C_*, d_*) is actually a chain complex one may proceed as in the classical case, but replacing S^{n-1} with $\Sigma^{n-1}A$ and D^n with $C\Sigma^{n-1}A$.

Theorem 3.4. *Let A be a CW-complex with homology concentrated in degree r and let X be a CW(A)-complex. Then, the homology of the A-cellular chain complex defined as above coincides with the singular homology of X.*

Proof. We proceed by induction in the A-n-skeleton X^n. For $n = 0$ the result is clear.

Suppose the result holds for X^{n-1}. For simplicity, we assume that X is obtained from X^{n-1} by attaching only one A-n-cell. The general case is similar.

Let (C'_*, d'_*) be the A-cellular chain complex of X^{n-1}. By hypothesis, the homology of (C'_*, d'_*) coincides with the singular homology of X^{n-1}. Hence, by 3.1, the singular homology of X^n can be computed as the homology of the chain complex

$$\cdots \longrightarrow H_{n+1}(C'_*) \oplus H_n(\Sigma^{n-1}A) \xrightarrow{\begin{pmatrix} 0 & g_* \\ 0 & 0 \end{pmatrix}} H_n(C'_*) \oplus H_{n-1}(\Sigma^{n-1}A) \longrightarrow \cdots$$

where $g : \Sigma^{n-1}A \to X^{n-1}$ is the attaching map of the A-n-cell.

We want to prove that this complex has the same homology as the A-cellular complex of X, namely

$$\cdots \longrightarrow 0 \longrightarrow H_{n+r-1}(\Sigma^{n-1}A) \xrightarrow{+(q_\beta g)_*} C'_{n+r-1} \xrightarrow{d'_{n+r-1}} \cdots$$

By the long exact sequence of the homology of the cone, it suffices to prove that $+q_\beta g_*$ induces the map g_* in homology. But this follows from the commutativity of the diagram

$$\begin{array}{ccc} H_{n+r-1}(\Sigma^{n-1}A) & \xrightarrow{g_*} & H_{n+r-1}(X^{n-1}) \\ & \swarrow{\simeq} & \downarrow{+(q_\beta)_*} \\ \ker d'_{n+r-1} & \xrightarrow[\text{inc}]{} & \bigoplus H_{n+r-1}(\Sigma^{n-1}A) \end{array}$$

where the isomorphism $H_{n+r-1}(X^{n-1}) \to \ker d'_{n+r-1}$ is induced by the map $+(q_\beta)_*$. □

Remark 3.5. The previous construction generalizes the classical one for cellular homology of CW-complexes. Note that the S^0-cellular chain complex of X is the standard cellular chain complex.

The following corollary is an example of one possible application of the theorem.

Corollary 3.6. *Let G and H be finite abelian groups with relatively prime orders. Let A and B be CW-complexes with homology concentrated in certain degrees n and m respectively, and with $H_n(A) = G$ and $H_m(B) = H$. Let X be a simply connected CW(A)-complex and let Y be a simply connected CW(B)-complex. Then X and Y have the same homotopy type if and only if both of them are contractible.*

Proof. By the hypothesis on the order of the elements, a quotient of $\bigoplus G$ different from 0 cannot be isomorphic to any quotient of $\bigoplus H$. It follows that if X and Y have the same homotopy type, then all their singular homology groups must vanish. □

We investigate now the case when the homology groups $H_n(A)$ are free for all n. The following lemma plays a key role in the proof of 3.8. Since its proof is standard, we only sketch the main ideas.

Lemma 3.7. *Let (C_*, d_*) and (D_*, d'_*) be chain complexes of $\mathbb{Z}$-modules, with C_n free for every n. Given morphisms $f_n : H_n(C_*) \to H_n(D_*)$, $n \in \mathbb{N}$, there exists a morphism of chain complexes $g : (C_*, d_*) \to (D_*, d'_*)$ which induces the maps f_n in homology.*

Proof. Since C_0 is projective, there exists a map $g_0 : C_0 \to D_0$ inducing f_0 in homology. Suppose that we have already defined $g_0, \ldots, g_{n-1}$ and they commute with the differentials and induce $f_0, \ldots, f_{n-1}$ in homology. Since $\ker d_n$ is projective there exists a map β in a commutative diagram

$$\begin{array}{ccc} \ker d_n & \xrightarrow{q_n^C} & \ker d_n/\mathrm{Im}\, d_{n+1} \\ \beta\downarrow & & \downarrow f_n \\ \ker d'_n & \xrightarrow{q_n^D} & \ker d'_n/\mathrm{Im}\, d'_{n+1} \end{array}$$

Note that $C_n \simeq \ker d_n \oplus \mathrm{Im}\, d_n$. We define $g_n = \beta$ in $\ker d_n$. Since $\mathrm{Im}\, d_n$ is projective, we can define g_n in $\mathrm{Im}\, d_n$ such that $g_{n-1}(y) = d'_n g_n(y)$ for all $y \in \mathrm{Im}\, d_n$. It is easy to check that $d'_n g_n = g_{n-1} d_n$ and that g_n induces the map f_n. □

Theorem 3.8. *Let A be a CW-complex with free homology groups and let X be a finite dimensional CW(A)-complex. Then there exists a chain complex of $\mathbb{Z}$-modules (C_*, d) whose homology is the singular homology of X, where*

$$C_n = \bigoplus_r H_{n-r}(A)^{\#r-cells}.$$

Proof. We proceed by induction in the dimension of X. If X has dimension zero, the result is trivial and if X has dimension one, the result follows from remark 3.1.

Suppose that the result is true for X' and that X is obtained from X' by attaching A-n-cells. For simplicity, we may suppose that only one A-n-cell is attached, and let g be its attaching map. We denote by $H_*(\Sigma^{n-1}A)$ and $H_*(X')$ the chain complexes of the homology of $\Sigma^{n-1}A$ and X' respectively with all differentials equal to zero, and by $C(X')$ the chain complex of X' of the inductive step. By remark 3.1, the homology of X can be computed as the homology of the chain complex $\mathrm{C}g_*$, where $g_* : H_*(\Sigma^{n-1}A) \to H_*(X')$ is the morphism of chain complexes (with zero differentials) induced by g in homology.

By lemma 3.7, there exists a morphism $\varphi : H_*(\Sigma^{n-1}A) \to C(X')$ inducing g_* in homology. It is easy to prove that the homology of $C\varphi$ coincides with the homology of $\mathrm{C}g_*$ which is the homology of X. □

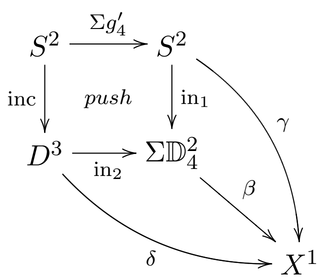
Example 3.9. Let A be a CW-complex such that $H_r(A) = \mathbb{Z}$ for $r = 1, 4$ and 0 otherwise. Let X be a CW(A)-complex having n A-0-cells and m A-2-cells. Note that all the maps in the chain complex of the previous theorem are 0 and hence

$$H_r(X) = \begin{cases} \mathbb{Z}^n & \text{for } r = 1, 4 \\ \mathbb{Z}^m & \text{for } r = 3, 6 \\ 0 & \text{otherwise} \end{cases}$$

Example 3.10. This example shows that theorem 3.8 may not hold if the hypothesis are not satisfied. Concretely, for the core $A = \mathbb{D}^2_4 \vee \Sigma\mathbb{D}^2_4$ (see page 5) we exhibit a CW(A)-complex X whose homology cannot be computed with a chain complex as in 3.8. Note that the homology of A is not concentrated in any degree and that its homology groups are not free. Moreover, the space X will constitute an example of a generalized CW($\mathbb{D}^4_2$)-complex which does not have the homotopy type of a CW($\mathbb{D}^4_2$)-complex.

The space X will consist of 3 A-cells, one of each dimension 0, 1 and 2. The attaching maps are defined as follows. For each $n \in \mathbb{Z}$ let $g'_n : D^2 \subseteq \mathbb{C} \to D^2$ be the map $g'_n(z) = z^n$. The map g'_n induces a well defined cellular map $g_n : \mathbb{D}^2_4 \to \mathbb{D}^2_4$. We will also denote $g'_n = g'_n|_{S^1} : S^1 \to S^1$.

Let X^1 be the CW(A)-complex of dimension one defined by attaching an A-1-cell to A by the map $* \vee \Sigma g_2$. We obtain X by attaching an A-2-cell to X^1 by the map $\beta \vee *$, where $\beta : \Sigma\mathbb{D}^2_4 \to X^1$ is the unique map induced by γ and δ in the following pushout



The map γ is defined as the composition

$$S^2 \xrightarrow{\Sigma g'_{-2}} S^2 \xrightarrow{\mathrm{in}_1} \Sigma\mathbb{D}^2_4 \xrightarrow{\mathrm{in}_3} X^1$$

(where in_3 is the canonical inclusion in the pushout) and $\delta = (\delta_1 \vee \delta_2) \circ q$, where δ_1, δ_2 and q are defined as follows. The map $q : D^2 \to D^2 \vee D^2$ is the quotient map that collapses the equator to a point. The map δ_1 is the composition

$$D^3 \xrightarrow{\Sigma g'_{-1}} D^3 \xrightarrow{\mathrm{in}_2} \Sigma\mathbb{D}^2_4 \xrightarrow{\mathrm{in}_3} X^1$$

and the map δ_2 is the composition

$$D^3 \xrightarrow{\Sigma g'_{-2}} D^3 \xrightarrow{\mathrm{Cin}_1} \mathrm{C}\Sigma\mathbb{D}^2_4 \xrightarrow{\mathrm{in}_4} X^1$$

The map in_4 is the canonical map induced in the pushout

$$\begin{array}{ccc} A = \mathbb{D}^2_4 \vee \Sigma\mathbb{D}^2_4 & \xrightarrow{* \vee \Sigma g_2} & A = \mathbb{D}^2_4 \vee \Sigma\mathbb{D}^2_4 \\ \downarrow & \textit{push} & \downarrow {\scriptstyle \mathrm{in}'_3 \vee \mathrm{in}_3} \\ \mathrm{C}A = \mathrm{C}\mathbb{D}^2_4 \vee \mathrm{C}\Sigma\mathbb{D}^2_4 & \xrightarrow[\mathrm{in}'_4 \vee \mathrm{in}_4]{} & X^1 \end{array}$$

It is easy to check that $\delta\mathrm{inc} = \gamma g_4$. Since the attaching maps are cellular, it follows that X is a CW-complex. We will show that $H_3(X) = \mathbb{Z}_8$. Hence, its homology cannot be computed with a chain complex as in 3.8 because $H_3(X)$ has an element of order 8. Note that X, as a standard CW-complex, has 1 0-cell, 1 1-cell, 3 2-cells, 4 3-cells, 3 4-cells and 1 5-cell. It is not difficult to prove that the rightmost part of its cellullar chain complex is the following.

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 4 & 0 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & -2 & 2 \\ 0 & 4 & 0 & 0 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 4 & 0 & 0 \end{pmatrix}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

It is easy to verify that $H_3(X) = \mathbb{Z}_8$, with generator being the class of $(0, 0, 1, 1)$.

Note that X is a generalized $\mathrm{CW}(\mathbb{D}^4_2)$-complex which does not have the homotopy type of a $\mathrm{CW}(\mathbb{D}^4_2)$-complex. Indeed, if there existed a $\mathrm{CW}(\mathbb{D}^4_2)$-complex Z, homotopy equivalent to X, then by theorem 3.4, $\mathbb{Z}_8 = H_3(X) = H_3(Z)$ would be a subquotient of $\bigoplus \mathbb{Z}_4$, which is impossible.

4. A-Euler characteristic and multiplicative characteristic

Let X be a pointed finite CW-complex. Recall that the reduced Euler characteristic of X is defined by

$$\chi(X) = \sum_{j \geq 0} (-1)^j \alpha_j$$

where α_j is the number of j-cells and where the base point does not count as a 0-cell. In this way the reduced Euler characteristic differs in 1 from the standard (unreduced) one.

Definition 4.1. Let A be a CW-complex and let X be a CW(A)-complex with a finite number of A-cells. We define the *A-Euler characteristic of X* by

$$\chi_A(X) = \sum_{j\geq 0}(-1)^j \alpha_j^A$$

where α_j^A is the number of A-j-cells of X.

Note that if $A = S^0$ then the A-Euler characteristic of X is the reduced Euler characteristic in the usual sense. Also, if $A = S^n$ then $\chi_A(X) = (-1)^n\chi(X)$. Recall that a CW($S^n$)-complex is a CW-complex with no cells of dimension less than n, apart from the base point.

The A-Euler characteristic gives us useful information about the space. For example, proposition 4.2 will show that if the core A is a finite CW-complex and X is a finite CW(A)-complex then $\chi(X)$ can be computed from $\chi(A)$ and $\chi_A(X)$. Note that $\chi(X)$ is well defined since X has the homotopy type of a finite CW-complex. When $\chi(A) \neq 0$, the A-Euler characteristic is a homotopical invariant. In case $\chi(A) = 0$, it might not be invariant by homotopy equivalences or even homeomorphisms, as the following example shows. Take the core A as D^1 (with 1 as base point). The disk D^2 is homeomorphic to $\mathrm{C}A$ and ΣA. We know that $\mathrm{C}A$ is obtained from A by attaching an A-1-cell, hence $\chi_A(\mathrm{C}A) = 0$. On the other hand, ΣA is obtained from $*$ by attaching a A-1-cell, so $\chi_A(\Sigma A) = -1$. Note that there are A-cellular approximations to the identity map of D^2 between these two different A-cellular structures, and that the homology of D^2 can be computed from the A-cellular complex by 3.4. But in this case the A-Euler characteristic cannot be computed from the A-cellular complex since, in contrast to the classical situation where the cellular complex has a copy of $\mathbb{Z}$ for each cell, the A-cellular complex has a trivial group for each A-cell of D^2.

Proposition 4.2. *Let A be a finite CW-complex and let X be a finite CW(A)-complex. Then $\chi(X) = \chi_A(X)\chi(A)$.*

Proof. The proposition follows from the fact that, for all $n \in \mathbb{N}_0$ the relative CW-complexes $(\mathrm{C}\Sigma^n A, \Sigma^n A)$ have exactly the same cells as A but shifted in dimension. Note also that X has the homotopy type of a CW-complex X' which is obtained by approximating the attaching maps of X by cellular maps. □

Corollary 4.3. *If $\chi(A) \neq 0$ and $\chi_A(X) \neq 0$ then X is not contractible.*

Note that in case $A = S^n$ the corollary does not say anything new. But, for example, if A is a torus ($\chi(A) = -1$) and X is a CW(A)-complex with an odd number of cells, then X is not contractible. Also, in this case, if X has any number of cells but only in even dimensions, it cannot be contractible.

We study now another interesting case: when the homology of A is a *finite graded group.* We say that graded group $(G_n)_{n\in\mathbb{N}_0}$ is *finite* if only a finite number of groups are non trivial and all of them are finite. In a similar way we say that a chain complex of abelian groups is *finite* if the underlying graded group is finite.

Definition 4.4. Let $\mathcal{G} = (G_n)_{n\in\mathbb{N}_0}$ be a finite graded group. We define the *multiplicative Euler characteristic of $\mathcal{G}$* as

$$\chi_m(\mathcal{G}) = \prod_{n\geq 0} \#(G_n)^{(-1)^n}$$

Let $C = (C_*, d_*)$ be a chain complex of abelian groups whose underlying graded group is finite. Since $H_n(C) = \ker d_n / \operatorname{Im} d_{n+1}$ and $C_n / \ker d_n = \operatorname{Im} d_n$, then

$$\#H_n(C) = \#\ker d_n / \#\operatorname{Im} d_{n+1} \quad \text{and} \quad \#C_n = \#\ker d_n . \#\operatorname{Im} d_n.$$

It follows that

$$\prod_{n\geq 0} \#(H_n(C))^{(-1)^n} = \prod_{n \text{ even}} \#\ker d_n . \#\operatorname{Im} d_n / \prod_{n \text{ odd}} \#\ker d_n . \#\operatorname{Im} d_n = \prod_{n\geq 0} \#(C_n)^{(-1)^n}$$

Therefore, the multiplicative Euler characteristic of C coincides with the multiplicative Euler characteristic of the graded group $H_*(C)$. In particular, the multiplicative Euler characteristic is invariant by quasi isomorphisms.

As a simple example, suppose (C_*, d_*) is a chain complex with $C_n = \bigoplus_{i\in I_n} \mathbb{Z}_4$ for all n (where I_n is any index set). Let (D_*, d'_*) be a chain complex with $H_k(D) = \mathbb{Z}_2$ for some k and $H_r(D) = 0$ for $r \neq k$. Then C and D are not quasi isomorphic, because $\chi_m(C) = 4^m$ for some $m \in \mathbb{Z}$, while $\chi_m(D) = 2$ or $\chi_m(D) = \frac{1}{2}$.

Remark 4.5. Let $C = (C_n)_{n\in\mathbb{N}_0}$, $D = (D_n)_{n\in\mathbb{N}_0}$ and $E = (E_n)_{n\in\mathbb{N}_0}$ be finite graded groups. It is easy to prove that if for each n there exists a short exact sequence

$$0 \to C_n \to D_n \to E_n \to 0$$

then $\chi_m(D) = \chi_m(C)\chi_m(E)$.The same holds in case there is an exact sequence

$$\ldots \to E_{n+1} \to C_n \to D_n \to E_n \to C_{n-1} \to \ldots$$

Definition 4.6. Let X be a topological space with finite homology. We define the *multiplicative Euler characteristic of* X as the multiplicative Euler characteristic of $H_*(X)$.

Theorem 4.7. *Let A be a CW-complex with finite homology and let X be a finite CW(A)-complex. Then*

$$\chi_m(X) = \prod_{n\geq 0} \chi_m(A)^{(-1)^n \#A\text{-}n\text{-}cells} = \chi_m(A)^{\chi_A(X)}$$

Proof. We proceed by induction in the number of cells of X. If X has only one cell the theorem trivially holds. Suppose the result is true for X' and that X is obtained from X' by attaching an A-r-cell. There exists a long exact sequence

$$\cdots \longrightarrow H_n(\Sigma^{r-1}A, *) \longrightarrow H_n(X', *) \longrightarrow H_n(X, *) \longrightarrow H_{n-1}(\Sigma^{r-1}A, *) \longrightarrow \cdots$$

Then, by 4.5,

$$\begin{aligned} \chi_m(X') &= \chi_m(H_*(X')) = \chi_m(H_*(\Sigma^{r-1}A))\chi_m(H_*(X)) = \\ &= \chi_m(H_*(A))^{(-1)^{r-1}}\chi_m(H_*(X)) = \chi_m(A)^{(-1)^{r-1}}\chi_m(X) \end{aligned}$$

Thus, $\chi_m(X) = \chi_m(X')\chi_m(A)^{(-1)^r}$. □

Example 4.8. Let A be a CW-complex with $H_1(A) = \mathbb{Z}_4$ and $H_r(A) = 0$ for $r \neq 1$. Let X be a topological space with $H_k(A) = \mathbb{Z}_2$ for some k and $H_r(A) = 0$ for $r \neq k$. Then X does not have the homotopy type of a CW(A)-complex.

The next result follows immediately from 4.7.

Proposition 4.9. *Let A and B be CW-complexes with finite homology. Let X be a topological space with finite homology such that $\chi_m(X) \neq 1$. Suppose, in addition, that X can be given both CW(A) and CW(B) structures. Then there exist $k, l \in \mathbb{Z} - \{0\}$ such that $\chi_m(A)^k = \chi_m(B)^l$.*

Example 4.10 (Moore spaces)**.** Fix a core A. Some questions that arise naturally are the following. For which abelian groups G and $n \in \mathbb{N}$ does there exist a CW(A)-complex X such that $H_n(X) = G$ and $H_r(X) = 0$ if $r \neq n$? Or more generally, for which sequences of abelian groups $(G_n)_{n\in\mathbb{N}_0}$ does there exist a CW(A)-complex X such that $H_n(X) = G_n$ for all n?

For example, if the core A is a simply connected CW-complex with $H_r(A) = \mathbb{Z}$ for $r = n$ and $H_r(A) = 0$ in other case, then A is homotopy equivalent to S^n. We know that for any abelian group G and for any $k \geq n$ there exists a CW-complex Z such that $H_k(Z) = G$ and $H_r(Z) = 0$ if $r \neq k$. Hence, by 2.5, there exists a CW(A)-complex X such that X has the same homology groups as Z. Therefore, in this particular case, for any sequence of abelian groups $(G_j)_{j\geq n}$ there exists a CW(A)-complex X such that $H_j(X) = G_j$ for all $j \geq n$.

If A is a CW-complex with finite homology, the results above provide necessary conditions for the required CW(A)-complex X to exist. For instance, 4.7 settles an easy-to-check necessary condition, as example 4.8 shows. In the case $A = \mathbb{D}^2_4$ (see page 5), we cannot construct a CW(A)-complex X such that $H_n(X) = \mathbb{Z}_5$ for some $n \in \mathbb{N}$ since, by 3.4, $H_n(X)$ must be a quotient of a subgroup of $\bigoplus \mathbb{Z}_4$.

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