Asymptotic Behavior of the Steklov Eigenvalues For the $p-$Laplace Operator

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Abstract

In this paper we study the asymptotic behavior of the Steklov eigenvalues of the p-Laplacian. We show the existence of lower and upper bounds of a Weyl-type expansion of the function $N(\lambda)$ which counts the number of eigenvalues less than or equal to λ, and we derive from them asymptotic bounds for the eigenvalues.

1 Introduction

In this paper we study the following eigenvalue problem:

$$\begin{cases} \Delta_p u &= |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda|u|^{p-2}u & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in a bounded open set $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$ (at least, C^2), where $\Delta_p u$ is the p-laplacian operator $div(|\nabla u|^{p-2}\nabla u)$ for $1 < p < +\infty$, $\partial/\partial\nu$ is the outer unit normal

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derivative, and λ is a real parameter. The case $p = +\infty$ will be considered in a separate work (see [6]).

From [7], we know that the variational spectrum consists on a countable sequence of positive eigenvalues $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ (repeated according multiplicity) tending to $+\infty$, which are obtained with the aid of the Ljusternik-Schnirelmann theory. We recall that the variational characterization of the eigenvalues is as follows:

$$\lambda_k = \inf_{F \in C_k^\Omega} \sup_{u \in F} \int_\Omega |\nabla u|^p + |u|^p, \tag{1.2}$$

where

$$C_k^\Omega = \left\{ C \subset M^\Omega \ : \ C \text{ compact }, \ C = -C, \ \gamma(C) \geq k \right\},$$

$$M^\Omega = \left\{ u \in W^{1,p}(\Omega) \ : \ \int_{\partial\Omega} |u|^p = 1 \right\}$$

and $\gamma(C)$ is the Krasnoselskii genus (see [17] for the definition and properties of γ).

However, few facts are known about this sequence. It is not known if this variational sequence exhausts the spectrum, although it was proved that λ_1 is the only eigenvalue with a positive eigenfunction, it is simple, and there are no other eigenvalues between λ_1 and λ_2, see [15]. Moreover, it can be obtained by minimization of the Rayleigh quotient over all of $W^{1,p}(\Omega)$:

$$\lambda_1 = \min_{u \in W^{1,p}(\Omega)\backslash 0} \frac{\int_\Omega |\nabla u|^p + |u|^p}{\int_{\partial\Omega} |u|^p}.$$

Also, λ_2 is a variational eigenvalue which can be obtained by minimization in a different set of functions, and the corresponding eigenfunction changes signs (see [8] for details).

Our objective is to obtain the asymptotic growth of this sequence. To this end, we introduce the spectral counting function $N(\lambda)$ defined as the number of variational eigenvalues of problem (1.1) less than a given λ :

$$N(\lambda, \Omega) = \#\{k \ : \ \lambda_k \leq \lambda\}.$$

We obtain two side estimates of the asymptotic expansion of $N(\lambda)$, bounding it above and by below. We state here our main result:

Theorem 1.1 *Let $\{\lambda_k\}_k$ be the sequence of variational eigenvalues of Problem 1.1. Then,*

$$c|\partial\Omega|_{N-1}\lambda^{\frac{N-1}{p-1}} \leq N(\lambda) \leq C|\partial\Omega|_{N-1}\lambda^{\frac{N-1}{p-1}},$$

for certain positive constants c, C depending only on p and N.

The proof is based on variational arguments, by using an extension of the 'Dirichlet-Neumann bracketing' method of Courant, see [2]. As a corollary, we have:

Corollary 1.1 *Let $\{\lambda_k\}_k$ be the sequence of variational eigenvalues of Problem 1.1. Then, there exists two positive constants c_1 and c_2 depending only on p and N such that*

$$c_1 \left(\frac{k}{|\partial\Omega|_{N-1}} \right)^{\frac{p-1}{N-1}} \leq \lambda_k \leq c_2 \left(\frac{k}{|\partial\Omega|_{N-1}} \right)^{\frac{p-1}{N-1}}. \tag{1.3}$$

Let us emphasize the fact that c_1 and c_2 (and also c, C in the previous theorem) are independent of the set Ω.

The problem of estimating the spectral counting function of different elliptic operators has a long history in the linear case $p = 2$, see for instance [2, 13, 14] and the references therein for the Laplacian. The linear Steklov problem was studied by L. Sandgren in [18], and we will follow his method of proof. The main difference is due to our impossibility to compute the Steklov eigenvalues in a square explicitly.

The nonlinear case is by far more recent. Concerning the variational eigenvalues $\{\mu_k\}_k$ of the p-Laplacian with zero Dirichlet boundary conditions, the first result was obtained by Garcia Azorero and Peral Alonso in [12],

$$ ck^{p/N} \le \mu_k \le Ck^{p/N}, $$

which was improved by Friedlander in [11]. The one dimensional case was refined in [4, 16] where the following stronger result was obteined:

$$ \lim_{k\to\infty} \frac{\mu_k}{k^p} = \frac{\pi_p^p}{|\Omega|^p}, $$

where π_p is defined as $\pi_p = 2(p-1)^{1/p} \int_0^1 (1-s^p)^{-1/p} ds$ (see [3]).

The paper is organized as follows: In §2 we introduce several auxiliary results concerning the eigenvalue problem 1.1 together with different tools which will be used in the paper. In §3 we obtain an upper bound for $N(\lambda)$ in a cube Q with the eigenvalue in one face and zero Neumann boundary conditions in the others. §4 is devoted to the upper bound for $N(\lambda)$ in a rectangle R with the eigenvalue in one face and zero Dirichlet boundary conditions in the others. Finally, in §5 we prove Theorem 1.1.

2   Preliminaries

The main tool in the following sections is the Dirichlet Neumann Bracketing introduced by Courant in [2] and extended to the nonlinear setting in [11] and independently in [4]. For the Steklov problem it was extended in [18], and we will only sketch it for our case.

We introduce first certain spaces and eigenvalue problems. Given an open set $\Omega \subset \mathbb{R}^N$, let $\partial\Omega = \overline{\Gamma_1 \cup \Gamma_2}$, where Γ_1 and Γ_2 are relative open sets of $\partial\Omega$; and now we have the following problems:

Semi-Dirichlet Steklov eigenvalue problem: Find $\Lambda \in \mathbb{R}$ and $u \in W^{1,p}_{0,\Gamma_2}(\Omega)$ such that

$$ \begin{cases} \Delta_p u & = & |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} & = & \Lambda|u|^{p-2}u & \text{on } \Gamma_1 \\ u & = & 0 & \text{on } \Gamma_2. \end{cases} \tag{2.1} $$

Semi-Neumann Steklov eigenvalue problem: Find $\Lambda \in \mathbb{R}$ and $u \in W^{1,p}(\Omega)$ such that

$$ \begin{cases} \Delta_p u & = & |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} & = & \Lambda|u|^{p-2}u & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} & = & 0 & \text{on } \Gamma_2. \end{cases} \tag{2.2} $$

For both problems there exists a sequence of variational eigenvalues given by a characterization similar to 1.2, changing only the definition of M^Ω as

$$M^\Omega = \left\{ u \in W_{0,\Gamma_2}^{1,p}(\Omega) \;\; : \;\; \int_{\Gamma_1} |u|^p = 1 \right\}$$

for the semi-Dirichlet problem, and

$$M^\Omega = \left\{ u \in W^{1,p}(\Omega) \;\; : \;\; \int_{\Gamma_1} |u|^p = 1 \right\}$$

for the semi-Neumann problem.

Also, it is not difficult to show that for both problems the first eigenvalue is positive and the associate eigenfunction is nonnegative. We leave the details for the reader, although they can be grasped from [7] and [15].

The asymptotic number of Steklov eigenvalues of problem 1.1 will be obtained from the spectral asymptotic of both problems, and we will write $N_{sD}(\lambda, \Omega, \Gamma_1)$, $N_{sN}(\lambda, \Omega, \Gamma_1)$ to denote their counting functions. To this end, we introduce the *Dirichlet-Neumann bracketing*. We consider the Steklov problem 1.1 in Ω, and we divide it in two disjoint open sets $\Omega_1, \Omega_2 \in \mathbb{R}^N$ such that $(\overline{\Omega_1 \cup \Omega_2})^{int} = \Omega$ and $|\Omega \setminus \Omega_1 \cup \Omega_2|_N = 0$. Let Γ_2 be the common boundary $\overline{\Omega_1} \cap \overline{\Omega_2}$, and $\Gamma_{1,i} = \overline{\Omega_i} \cap \partial\Omega$ for $1 \le i \le 2$.

For such decomposition we have the following inequalities

$$N_{sD}(\lambda, \Omega_1, \Gamma_{1,1}) + N_{sD}(\lambda, \Omega_2, \Gamma_{1,2}) \le N(\lambda, \Omega) \tag{2.3}$$

and

$$N(\lambda, \Omega) \le N_{sN}(\lambda, \Omega_1, \Gamma_{1,1}) + N_{sN}(\lambda, \Omega_2, \Gamma_{1,2}), \tag{2.4}$$

which give the asymptotic behavior for problem 1.1 in terms of the other problems.

The proof of this fact is a consequence of the variational characterization of the eigenvalues 1.2 combined with the following inclusions:

$$W_{0,\Gamma_2}^{1,p}(\Omega_1 \cup \Omega_2) = W_{0,\Gamma_2}^{1,p}(\Omega_1) \oplus W_{0,\Gamma_2}^{1,p}(\Omega_2) \subset W^{1,p}(\Omega) \tag{2.5}$$

and

$$W^{1,p}(\Omega) \subset W^{1,p}(\Omega_1) \oplus W^{1,p}(\Omega_2) = W^{1,p}(\Omega_1 \cup \Omega_2). \tag{2.6}$$

Let us note that, since Ω_1 and Ω_2 are disjoint sets, the eigenvalues are the union of the eigenvalues of the problem in each set separately.

Also, we will need some auxiliary results for problems with the eigenvalue parameter in the equation.

Let us consider following mixed eigenvalue problem:

$$\begin{cases} -\Delta_p u &= \mu|u|^{p-2}u & \text{in } R_L \\ \frac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma \\ u &= 0 & \text{on } \partial R_L \setminus \Gamma \end{cases} \tag{2.7}$$

where $R_L = [0, L]^{N-1} \times [0, L/2]$, and Γ is the face $\{(x_1, \ldots, x_N) : 0 \le x_i \le L$ for $1 \le i \le N-1$; and $x_N = L/2\}$.

By a symmetry argument, the first eigenvalue μ_1 coincides with the first Dirichlet eigenvalue corresponding to the cube Q_L. Indeed, by extending the first eigenfunction u outside R_L as $u(x_1, \ldots, x_{N-1}, L/2 + x) = u(x_1, \ldots, x_{N-1}, L/2 - x)$ for $x \in (0, L/2)$, we obtain a Dirichlet eigenfunction, since it is a weak solution of problem 1.1, and being positive, it must be the first eigenfunction. Let us note also that the maximum of the function is attained at the center of Q_L.

We have the following result, which can be found also in [5]:

Proposition 2.1 *Let $\mu_1^{(R_L)}$ be the first eigenvalue of Problem 2.7. Then,*

$$\mu_1^{(R_L)} \leq \begin{cases} \frac{\pi_p^p N^{p/2}}{L^p} & \text{if } 2 < p, \\ \frac{\pi_p^p N}{L^p} & \text{if } p < 2. \end{cases}$$

Proof. In order to get a bound for $\mu_1^{(R_L)}$, we will study a different operator, the so-called pseudo p-Laplacian on Q_L:

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i}\right) = \nu|v|^{p-2}v & \text{in } Q_L \\ v = 0 & \text{in } \partial Q_L. \end{cases} \tag{2.8}$$

Due to the equivalence of norms in $\mathbb{R}^N$, we have $|\nabla u|_q \leq C_p |\nabla u|_p$, where $C_p = 1$ if $p \leq q$, and $C_p = N^{(p-q)/2q}$ if $p \geq q$ (see, for instance, [10]).

Since $\nu_1 = \inf_{u \in W_0^{1,p}} \frac{\int_{Q_L} \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^p}{\int_{Q_L} |u|^p}$, by fixing the set $B = \{u \in W_0^{1,p} : \int_{Q_L} |u|^p\}$, we have the following characterization of the first eigenvalues μ_1, ν_1 of the p-laplacian and the pseudo p-Laplacian in $Q_L \subset \mathbb{R}^N$ respectively:

$$\nu_1 = \inf_{u \in B} \||\nabla u|_p\|_p^p; \quad \mu_1 = \inf_{u \in B} \||\nabla u|_2\|_p^p,$$

and the norms inequality enable us to compare ν_1 and μ_1.

Now, we have an explicit expression for ν_1:

$$v_1 = \sin_p(\pi_p x_1/L) \cdots \sin_p(\pi_p x_N/L), \quad \nu_1 = \frac{\pi_p^p N}{L^p}$$

is the first eigenpair of the pseudo p-Laplacian on Q_L. This result follows directly by separation of variables, and v_1 is the first eigenfunction since there exists only one positive eigenfunction of the pseudo p-laplacian (see [1]). $\qquad \square$

3   An Upper Bound for $N(\lambda)$

We will study the asymptotic behavior of the eigenvalues of the following semi-Neumann problem:

$$\begin{cases} \Delta_p u & = & |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} & = & \Lambda|u|^{p-2}u & \text{on } \Gamma \\ \frac{\partial u}{\partial \nu} & = & 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

Let us consider first the particular case $\Omega = Q_L \subset \mathbb{R}^N$, the cube $[0, L]^N$, and let Γ be one of its faces, say $\{(x_1, \ldots, x_N) : 0 \leq x_i \leq L \text{ for } 1 \leq i \leq N-1; \text{ and } x_N = L\}$. We have the following result:

Proposition 3.1 *Let $\lambda_k^{(Q_L)}$ and $\Lambda_k^{(Q_L)}$ be the kth-eigenvalues of Problem 1.1 and Problem 2.2 in Q_L. Then,*

$$\Lambda_k^{(Q_L)} > \lambda_k^{(Q_L)}.$$

Moreover, when $k \geq 2$,

$$\lambda_k^{(Q_L)} \sim \lambda_k^{(Q_1)} L^{1-p}.$$

Proof. Let us recall the variational characterization of eigenvalues,

$$\lambda_k^{(Q_L)} = \inf_{F \in C_k^{Q_L}} \sup_{u \in F} \frac{\int_{Q_L} |\nabla u|^p + |u|^p}{\int_{\partial Q_L} |u|^p},$$

$$\Lambda_k^{(Q_L)} = \inf_{F \in C_k^{Q_L}} \sup_{u \in F} \frac{\int_{Q_L} |\nabla u|^p + |u|^p}{\int_{\Gamma} |u|^p},$$

where

$$C_k^{Q_L} = \left\{ C \subset W^{1,p}(Q_L) \; : \; C \text{ compact}, \; C = -C, \; \gamma(C) \geq k \right\}.$$

Since $\Gamma \subset \partial\Omega$, the inequality $\Lambda_k^{(Q_L)} > \lambda_k^{(Q_L)}$ holds.

Now, the proof that $\lambda_k^{(Q_L)} \sim \lambda_k^{(Q_1)} L^{1-p}$ can be found in [9], and follows by changing variables in the Rayleigh quotient, since $|\nabla u|^p$ contributes with the factor L^{-p}, and the differentials of volume and surface area differs by a factor L.

Remark 3.1 Let us note that for $k = 1$ this is not true, since $\lambda_k^{(Q_L)}$ goes to zero when L decreases. See [9] for further details on scaling.

Our next Theorem gives the upper bound for the eigenvalue counting function for certain domains.

Theorem 3.1 *Let us suppose that Γ_1 is an hyperplane. Then,*

$$N_{sN}(\lambda, \Omega, \Gamma_1) \leq \frac{|\Gamma_1|_{N-1} \lambda^{\frac{N-1}{p-1}}}{(\lambda_2^{(Q_1)})^{N-1}}.$$

Proof. Let λ be fixed, and let us divide Ω in several parts: cubes $\{Q_L^i\}_{1 \leq i \leq J}$ with one of its faces in Γ_1, and a complementary set $Q^c = \Omega \setminus \cup_{i=1}^J \overline{Q_L}^i$.

Now, applying the Dirichlet Neumann Bracketing by decomposing Ω as $Q^c \cup (\cup_{i=1}^J Q_L^i)$, with the eigenvalue parameter only on the faces of the cubes in Γ_1 and zero Neumann boundary conditions in the others and in ∂Q^c gives an upper bound for $N_{sN}(\lambda, \Omega, \Gamma_1)$ given by

$$\sum_{i=1}^J N(\lambda, Q_L^i).$$

We are free to choose the length L yet. By using Proposition 3.1, if $L < (\lambda_2^{(Q_1)}/\lambda)^{1/(p-1)}$, then $\Lambda_2^{(Q_L)} > \lambda$, and therefore $N(\lambda, Q_L^i) \leq 1$.

Hence, we choose $L = (\lambda_2^{(Q_1)}/\lambda)^{1/(p-1)}$ and we can bound $N(\lambda, \Omega, \Gamma_1)$ by the number of cubes J, and by equating surface areas we obtain $JL^{N-1} = |\Gamma_1|_{N-1}$, that is,

$$N_{sN}(\lambda, \Omega, \Gamma_1) \leq \frac{|\Gamma_1|_{N-1}\lambda^{\frac{N-1}{p-1}}}{(\lambda_2^{(Q_1)})^{N-1}}$$

and the proof is finished. $\qquad\square$

4   A Lower Bound for $N(\lambda)$

We will consider now the semi-Dirichlet eigenvalue problem:

$$\begin{cases} \Delta_p u &= |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \Lambda|u|^{p-2}u & \text{on } \Gamma \\ u &= 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

We consider the particular case $\Omega = R_L = [0, L]^{N-1} \times [0, L/2]$, and let Γ be the face $\{(x_1, \ldots, x_N) : 0 \leq x_i \leq L \text{ for } 1 \leq i \leq N-1; \text{ and } x_N = L/2\}$. We obtain upper bounds for $\Lambda_1^{(R_L)}$ from the previous bounds of $\mu_1^{(R_L)}$ in Proposition 2.1. We have the following result:

Proposition 4.1 *Let $\Lambda_1^{(R_L)}$ and $\mu_1^{(R_L)}$ be the first eigenvalues of Problems 2.1 and 2.7 in R_L. Then,*
$$\Lambda_1^{(R_L)} \leq (\mu_1^{(R_L)} + 1)\frac{L}{2}.$$

Proof. Let us recall the variational characterization of the first eigenvalue,

$$\Lambda_1^{(R_L)} = \inf_{u \in W^{1,p}(Q_L), u|_{\partial R_L \setminus \Gamma}=0} \frac{\int_{R_L} |\nabla u|^p + |u|^p}{\int_\Gamma |u|^p}.$$

Now, we use as a test function the first eigenfunction φ of problem 2.7, obtaining

$$\Lambda_1^{(R_L)} \leq \left(\frac{\int_{R_L} |\nabla\varphi|^p + |\varphi|^p}{\int_{R_L} |\varphi|^p}\right)\left(\frac{\int_{R_L} |\varphi|^p}{\int_\Gamma |\varphi|^p}\right) \leq (\mu_1^{(R_L)} + 1)\left(\frac{\int_{R_L} |\varphi|^p}{\int_\Gamma |\varphi|^p}\right).$$

Finally, by using Fubini's Theorem, the second factor could be bounded by $L/2$ and the proof is complete.

Remark 4.1 Combining the previous Proposition with Proposition 2.1, we have

$$\Lambda_1^{(R_L)} \leq \frac{C_p}{2}\left(\frac{\pi_p^p N}{L^{p-1}} + L\right) \sim \frac{\pi_p^p C_p N}{2L^{p-1}}$$

when $L \to 0$.

As in the previous section, the next Theorem gives the lower bound for the eigenvalue counting function in certain domains.

Theorem 4.1 *Let $N_{sD}(\lambda, \Omega, \Gamma_1)$ be the spectral counting function of Problem 2.1 where Γ_1 is an hyperplane. Then,*

$$N_{sD}(\lambda, \Omega, \Gamma_1) \geq c|\Gamma_1|_{N-1}\lambda^{\frac{N-1}{p-1}},$$

where c depends only on p and N.

Proof. Let λ be fixed, and let us divide Ω in several parts as before: rectangles $\{R_L^i\}_{1 \leq i \leq J}$ with one of this faces in Γ, and a complementary set $R^c = \Omega \setminus \cup_{i=1}^J \overline{R_L}^i$.

Now, applying the Dirichlet Neumann bracketing by decomposing Ω as $R^c \cup_{i=1}^J R_L^i$, with the eigenvalue parameter only on the faces of the rectangles in Γ_1 and zero Dirichlet boundary conditions in the others and in ∂R^c gives an upper bound for $N_{sD}(\lambda, \Omega, \Gamma_1)$ given by

$$\sum_{i=1}^J N(\lambda, R_L^i).$$

We choose now the length L. By using the remark below Proposition 2.1, if $L = (\pi_p^p C_p N/2\lambda)^{1/(p-1)}$, then each rectangle has at least one eigenvalue, and therefore $N(\lambda, R_L^i) \geq 1$.

Hence, $N_{sD}(\lambda, \Omega, \Gamma_1)$ is greater than or equal to J, and by equating surface areas we obtain $JL^{N-1} = |\Gamma_1|_{N-1}$, that is,

$$N_{sD}(\lambda, \Omega, \Gamma_1) \geq c|\Gamma_1|_{N-1}\lambda^{\frac{N-1}{p-1}},$$

$\square$

5   Proof of Theorem 1.1

The proof of Theorem 1.1 is rather lengthy and technical, and follows the same lines of sections 4.3 - 4.5 in [18]. We only sketch the main steps, since the critical estimations were given in Theorems 3.1 and 4.1.

Given an arbitrary open set Ω with smooth boundary, we divide its boundary in a finite number of connected parts $\{S_j\}_{1 \leq j \leq J}$, and there exist J coordinate neighborhoods $U_j \subset \mathbb{R}^{N-1}, \varphi_j$ such that any $x \in S_j$ is given by the equation $x = \varphi(y_1, \ldots, y_{N-1})$.

Now, given $x \in S_j$, let ν_x be the outer unit normal at x, and let us introduce the sets

$$D_j = \{z : z = x - y_N\nu_x, x \in S_j, 0 < y_N < \delta\}.$$

Clearly, δ could be chosen such that $D_j \subset \Omega$ for $1 \leq j \leq J$.

The first step in the proof of Theorem 1.1 is the following Lemma, which is a consequence of the Dirichlet Neumann bracketing:

Lemma 5.1 *Let D_j and S_j be defined as before. Then,*

$$\sum_{j=1}^{J} N_{sD}(\lambda, D_j, S_j) \leq N(\lambda, \Omega, \partial\Omega) \leq \sum_{j=1}^{J} N_{sN}(\lambda, D_j, S_j).$$

Let $\varepsilon > 0$ be fixed, and let us introduce the set $\varphi_j^{-1}(D_j) = V_j \times (0, \delta)$, where $V_j \subset U_j$. By a change of variables, the eigenvalue problem could be studied on the sets $V_j \subset U_j$. By subdividing S_j and after an orthogonal transformation if necessary, the jacobian of the transformation could be bounded as $1 \pm \varepsilon$ in the variational characterization of eigenvalues.

Hence, $|S_j|_{N-1} \sim |V_j|_{N-1}$, and we have (as in subsections 4.5.4 and 4.5.5 of [18]), the next step of the proof:

Lemma 5.2 *Let D_j, S_j and $V_j \times (0, \delta)$ be defined as before. Then,*

$$N_{sN}(\lambda, D_j, S_j) \sim N_{sN}(\lambda, V_j \times (0, \delta), V_j \times \{0\}),$$

$$N_{sD}(\lambda, D_j, S_j) \sim N_{sD}(\lambda, V_j \times (0, \delta), V_j \times \{0\}).$$

Let us note that the asymptotic expansion of $N_{sN}(\lambda, V_j \times (0, \delta), V_j \times \{0\})$ and $N_{sD}(\lambda, V_j \times (0, \delta), V_j \times \{0\})$ was obtained in Theorems 3.1 and 4.1 respectively, and therefore we have the last step of the proof:

Lemma 5.3 *Let S_j and $V_j \times (0, \delta)$ be defined as before. Then,*

$$N_{sN}(\lambda, V_j \times (0, \delta), V_j \times \{0\}) \leq C|S_j|_{N-1}\lambda^{\frac{N-1}{p-1}}$$

$$N_{sD}(\lambda, V_j \times (0, \delta), V_j \times \{0\}) \geq c|S_j|_{N-1}\lambda^{\frac{N-1}{p-1}}.$$

Combining this result with Lemma 5.1, we obtain the desired asymptotic bounds for $N(\lambda)$.

Remark 5.1 Let us note that Corollary 1.1 follows immediately from Theorem 1.1 since $N(\lambda_k) = k$ gives the inequalities

$$c|\partial\Omega|_{N-1}\lambda_k^{\frac{N-1}{p-1}} \leq k \leq C|\partial\Omega|_{N-1}\lambda_k^{\frac{N-1}{p-1}}.$$

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