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A general characterization for non-balanced games in terms of *U*-cycles

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Abstract

In a paper by Cesco [Cesco, J.C., 2003. Fundamental cycles of pre-imputations in non-balanced TU-games. International Journal of Game Theory 32, 211–222], it was proven that the existence of a certain type of cycles of pre-imputations, fundamental cycles, is equivalent to the non-balancedness of a *TU*-game, i.e., the emptiness of the core of the game. There are two characteristic sub-classes related to fundamental cycles: *U*-cycles and maximal *U*-cycles. In this note we show that it is enough to consider *U*-cycles in obtaining a similar characterization for non-balanced *TU*-games.

Keywords: Non-balanced games; Cycles; Characterization

1. Introduction

Balanced *TU*-games (games with transferable utilities) have been characterized in many different ways. The most famous result on this subject is the well-known Shapley–Bondareva Theorem (Bondareva, 1963; Shapley, 1967). Recently, in Cesco (2003), a characterization of non-balanced games was given in terms of the existence of certain cycles of pre-imputations (fundamental cycles). Later, for some class of *TU*-games, it was shown that the characterization theorem can still be obtained using narrow classes of cycles – *U*-cycles and maximal *U*-cycles (Cesco and Calí, 2006). The games studied are games where the only coalitions with non-zero value are the grand coalition and those having $n-1$ players, although these conditions could be substantially relaxed by asking only that the latter family is a minimal objecting family of coalitions (see Section 2). However, no general proof has been obtained for either *U*-cycles or maximal *U*-cycles. Here we do provide proof for the intermediate case of *U*-cycles.

It is already known (see for instance Cesco, 2003) that the existence of any kind of the above mentioned cycles implies the non-balancedness of the game. Here we show that every non-balanced game has a *U*-cycle. More precisely, we prove that, given any minimal balanced objecting family of coalitions, we are able to construct an associated *U*-cycle. A general existence of maximal *U*-cycles is still an open issue. We think that this is the most interesting case for several reasons. On one hand, an algorithm originally developed to reach points in the core of a balanced *TU*-game (Cesco, 1998) seems to have maximal *U*-cycles as limit cycles when it is run on non-balanced games. This fact has been observed numerically in all the examples tried, and formally proved in the framework of certain simple games (Cesco and Calí, 2004). On the other hand, maximal *U*-cycles are related to a well established solution concept which is the dynamic solution (Shenoy, 1980; Cesco and Calí, 2004). Moreover, since the notions involved in the dynamic solution concept studied in Cesco and Calí (2004) are much in the spirit of the ongoing literature of coalition formation (Sengupta and Sengupta, 1994, 1996; Koczy and Lauwers, 2004), maximal *U*-cycles could be useful in developing strategies to predict the formation of

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some coalition structures as the result of a well-defined dynamic bargaining process. The main result in this paper could be useful to later get an existence theorem about maximal U-cycles in non-balanced games which would support further studies along this line.

The note is organized as follows: Preliminaries and some notation are set forth in the next section. Cycles of pre-imputations are also defined in this section, as well as the most relevant result previously obtained. In Section 3 we prove the main existence result of this note. We close with some remarks about the possibility of obtaining a characterization result in terms of maximal cycles. We also include an Appendix exhibiting a non-balanced 5-person game which shows that a minimal balanced objecting family of coalitions is able to support both U-cycles and maximal U-cycles.

2. Preliminaries

A TU-game is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ represents the set of players and v the characteristic function. We assume that v is a real valued function defined on the family of subsets of $N, \mathscr{P}(N)$, satisfying $v(\Phi) = 0$. We will also assume that $v(N) = 1$ although this will not represent any restriction since the concepts we study here are invariant under strategic equivalence. The elements in $\mathscr{P}(N)$ are called coalitions.

The set of pre-imputations for a game (N, v) is

$$E = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \right\}$$

and the set of imputations is

$$A = \{x \in E : x_i \geqslant v(\{i\}) \text{ for all } i \in N\}.$$

Given a coalition $S \in \mathscr{P}(N)$ and a pre-imputation x, the excess of the coalition S with respect to x is $e(S, x|v) = v(S) - x(S)$, where $x(S) = \sum_{i \in S} x_i$ if $S \neq \Phi$ and 0 otherwise.

The core of a game (N, v) is defined by

$$C = \{x \in E : e(S, x|v) \leqslant 0 \text{ for all } S \in \mathscr{P}(N)\}$$

and it may be an empty set. Shapley–Bondareva's theorem characterizes the sub-class of TU-games with non-empty core where a central role is played by balanced families of coalitions. A family of non-empty coalitions $\mathscr{B} \subseteq \mathscr{P}(N)$ is called balanced if there exists a set of positive real numbers $(\lambda_S)_{S \in \mathscr{B}}$ satisfying $\sum_{\{S \in \mathscr{B} : i \in S\}} \lambda_S = 1$, for all $i \in N$. The numbers $\lambda_S, S \in \mathscr{B}$ are called the balancing weights for $\mathscr{B}$. $\mathscr{B}$ is minimal balanced if there is no proper balanced subfamily of it. In this case, the set of balanced weights is unique. Equivalently, if $\chi_S \in \mathbb{R}^n$ denotes the indicator vector of a coalition S defined by $(\chi_S)_i = 1$ if $i \in S$ and 0 if $i \in N \setminus S$, the family $\mathscr{B}$ is balanced if there exists a family of positive balancing weights $(\lambda_S)_{S \in \mathscr{B}}$, such that

$$\sum_{S \in \mathscr{B}} \lambda_S \cdot \chi_S = \chi_N. \tag{1}$$

A well-known result establishes that

$$\sum_{S \in \mathscr{B}} \lambda_S \cdot x(S) = x(N) \tag{2}$$

for any balanced family of coalitions $\mathscr{B}$ with balancing weights $(\lambda_S)_{S \in \mathscr{B}}$. At this point we mention that the set of characteristic vectors $(\chi_S)_{S \in \mathscr{B}}$ is linearly independent provided $\mathscr{B}$ is a minimal balanced family of coalitions.

A game (N, v) is balanced if

$$\sum_{S \in \mathscr{B}} \lambda_S \cdot v(S) \leqslant v(N) \tag{3}$$

for each balanced family $\mathscr{B}$ with balancing weights $(\lambda_S)_{S \in \mathscr{B}}$. The Shapley–Bondareva theorem states that the core of a game (N, v) is non-empty if and only if the game is balanced. An objectionable family is a balanced family not satisfying (3).

In what follows the notion of U-transfer in the framework of a TU-game (N, v) will play a central role. Given $x \in E$ and a proper coalition S, we say that y results from x by the U-transfer from $N \setminus S$ to S (shortly, y is a U-transfer from x) if

$$y = x + e(S, x|v) \cdot \beta_S, \tag{4}$$

with $e(S, x|v) > 0$. Here $\beta_S = \frac{\chi_S}{|S|} - \frac{\chi_{N \setminus S}}{|N \setminus S|}$ if S is a proper coalition and the zero vector of $\mathbb{R}^n$ otherwise. $|S|$ indicates the number of players in S. The vector β_S describes a transfer of one unit of utility from the members of $N \setminus S$ to the members of S. The U-transfer is called maximal if $e(S, x|v) \geqslant e(T, x|v)$ for all $T \in \mathscr{P}(N)$.

We now introduce some kinds of cycles of pre-imputations and state, without proof, several results proved in (Cesco and Aguirre, 2002; Cesco, 2003).

Definition 1. A *cycle* **c** in a TU-game (N, v) is a finite sequence of pre-imputations $(x^k)_{k=1}^m$, $m > 1$, such that there exist associated sequences of positive real numbers $(\mu_k)_{k=1}^m$ and $(S_k)_{k=1}^m$ of non-empty, proper coalitions of N (not necessarily all different) satisfying the neighboring transfer properties

$$x^{k+1} = x^k + \mu_k \cdot \beta_{S_k} \quad \text{for all } k = 1, \ldots, m \tag{5}$$

and

$$x^{m+1} = x^1 \tag{6}$$

as well.

A cycle is fundamental if $\mu_k \leqslant e(S_k, x^k|v)$ for all $k = 1, \ldots, m$.

A cycle is a U-cycle if $\mu_k = e(S_k, x^k|v)$ for all $k = 1, \ldots, m$.

A U-cycle is maximal if for all $k = 1, \ldots, m$, $e(S_k, x^k|v) \geqslant e(S, x^k|v)$ for all coalition S. A maximal U-cycle is regular if the strict inequality holds in the former conditions for all $S \neq S_k$. Otherwise, it will be called singular.

Given a cycle $\mathbf{c} = (x^k)_{k=1}^m$, we denote the vector of coalitions $(S_k)_{k=1}^m$ by supp($\mathbf{c}$). Let $\mathscr{B}(\mathbf{c}) = \{S : S = S_k$ for some entry of supp($\mathbf{c}$)$\}$. We will refer to $\mathscr{B}(\mathbf{c})$ as the family of coalitions supporting $\mathbf{c}$, and to the entries of the vector $(\mu_k)_{k=1}^m$ as the transfer amounts .$\mathscr{B}(\mathbf{c})$ is a balanced family of coalitions for every fundamental cycle $\mathbf{c}$ (Cesco, 2003, Theorem 1). In the latter paper it was also shown that the existence of a fundamental cycle implies the non-balancedness of the game and that every non-balanced TU-game has a fundamental cycle (Cesco, 2003, Theorem 3 and Theorem 9). These two results together provide a characterization of the non-balanced TU-games (i.e., games with non-empty core) in terms of fundamental cycles. During the proof of the latter theorem, the following claim which we state without proof, is included.

Proposition 1. *Let (N, v) be a TU-game and $\mathscr{B} = \{S_1, S_2, \ldots, S_m\}$ a minimal objectionable family of coalitions. Then there exists a fundamental cycle of pre-imputations $(x^k)_{k=1}^m$ whose support is $(S_k)_{k=1}^m$, i.e. there exist $(x^k)_{k=1}^m$ and $(\mu_k)_{k=1}^m, \mu_k > 0$ for all $k = 1, \ldots, m$ such that $x^{k+1} = x^k + \mu_k \cdot \beta_{S_k}$ for all $k = 1, \ldots, m$, and $x^{m+1} = x^1$.*

Remark 1. Proposition 1 implies that with every ordering of the coalitions in $\mathscr{B}$ there exists a fundamental cycle whose pre-imputations share the same cyclic ordering as the coalitions in $\mathscr{B}$. Its proof is based on a result about consistency of linear systems of inequalities due to *K*. (Fan, 1975) whose proof is not however, of a constructive type.

3. The existence of *U*-cycles

The existence of a U-cycle is proven in several steps. It rests on the observation that the U-cycles of a TU-game whose support is a family $\mathscr{B}$, are homothetic, except for a translation, to those of a canonical simple game. The key point in the proof is the explicit determination of the center and the expansion factor for the homothecy as well as the vector to carry out the translation. These observations will be formalized by proving successive results.

Let (N, v)be a TU-game and $\mathscr{B} = \{S_1, S_2, \ldots, S_m\}$ a minimal objectionable balanced family of coalitions in (N, v) having $(\lambda_k)_{k=1}^m$ as its set of balancing weights. Both the game (N, v) and the family $\mathscr{B}$ will remain the same in the rest of the note. For further references, given a real number $\mu \geqslant 0$, we will denote with (N, v_μ) the related game having characteristic function given by $v_\mu(S_k) = \mu$ for all $S_k \in \mathscr{B}$ and $v_\mu(S) = v(S)$ otherwise. Usually we will call (N, v_μ) a μ-uniform $\mathscr{B}$-game related to (N, v) or simply, a μ-uniform game. Furthermore, we will denote $v(S_k)$ by μ_k for all $S_k \in \mathscr{B}$. $E_\mathscr{B}$ will stand for the $m \times n$ matrix having χ_{S_k} as its kth row, $k = 1, \ldots, m$. We note that $m \leqslant n$ and that the rank of $E_\mathscr{B}$ is m. Thus, the linear system $E_\mathscr{B} \cdot t' = b'$ is always consistent for any vector b. Now, let

$$\mu = \frac{\sum_{k=1}^m \lambda_k \cdot \mu_k}{\sum_{k=1}^m \lambda_k} \tag{7}$$

and $t = (t_1, t_2, \ldots, t_n)$ be any solution of the linear system

$$E_\mathscr{B} \cdot t' = b', \tag{8}$$

where $b = (\mu_1 - \mu, \ldots, \mu_m - \mu)$. In the next result we show that the homeomorphism on $\mathbb{R}^n$ assigning

$$x = \bar{x} + t, \tag{9}$$

to any vector $\bar{x} \in \mathbb{R}^n$ relates U-cycles in the μ-uniform game (N, v_μ) to U-cycles in (N, v), while the inverse homeomorphisms induces an inverse relationship between U-cycles in (N, v) to those in (N, v_μ).

Proposition 2. *Let μ be given by (7) and let t be any solution of system (8). If $\bar{\mathbf{c}} = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m)$ is a U-cycle in (N, v_μ) whose support is $(S_k)_{k=1}^m$, then $\mathbf{c} = (\bar{x}^1 + t, \bar{x}^2 + t, \ldots, \bar{x}^m + t)$ is a U-cycle in (N, v) with the same support as $\bar{c}$. Conversely, if $\mathbf{c} = (x^1, x^2, \ldots, x^m)$ is a U-cycle in (N, v) whose support is $(S_k)_{k=1}^m$, then, $\bar{\mathbf{c}} = (x^1 - t, x^2 - t, \ldots, x^m - t)$ is a U-cycle in (N, v_μ) with the same support as* $\mathbf{c}$.

Proof. We first point out that any solution $t = (t_1, \ldots, t_n)$ of (8) satisfies $\sum_{i=1}^n t_i = 0$. In fact, by associating the following matrix product in two different ways, where $\boldsymbol{\lambda}$ is the m-vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, and taking into account (1), we get that

$$\boldsymbol{\lambda} \cdot E_\mathscr{B} \cdot t' = \sum_{i=1}^n t_i = \sum_{k=1}^m \lambda_k \cdot \mu_k - \sum_{k=1}^m \lambda_k \cdot \mu = 0,$$

which proves our claim. Now, let $\bar{\mathbf{c}} = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m)$ be a U-cycle in (N, v_μ) whose support is $(S_k)_{k=1}^m$ and let

$$x^k = \bar{x}^k + t$$

for all $k = 1, 2, \ldots, m$. We are going to show that $\mathbf{c} = (x^k)_{k=1}^n$ is a U-cycle of pre-imputations in (N, v). To prove this, we first note that

$$\begin{aligned} e(S_k, x^1|v) &= \mu_k - x^1(S_k) = \mu_k - \bar{x}^1(S_k) - t(S_k) \\ &= \mu - \bar{x}^1(S_k) + (\mu_k - \mu - t(S_k)) = \mu - \bar{x}^1(S_k) \\ &= e(S_k, \bar{x}^1|v_\mu) \end{aligned}$$

for all $S_k \in \mathscr{B}$. Here we have used the fact that t is a solution of the linear system (8) to justify that $\mu_k - \mu - t(S_k) = 0$. Thus

$$x^1 + e(S_1, x^1|v) \cdot \beta_{S_1} = \bar{x}^1 + e(S_1, \bar{x}^1|v_\mu) \cdot \beta_{S_1} + t = \bar{x}^2 + t = x^2.$$

With a similar argument, we are able to prove that $e(S_i, x^k|v) = e(S_i, \bar{x}^k|v_\mu)$ for all $k = 1, \ldots, m$, and $S_i \in \mathscr{B}$ which allow us to get that

$$x^k + e(S_k, x^k|v) \cdot \beta_{S_k} = x^{k+1}$$

for all $k = 2, \ldots, m$. Since $x^{n+1} = x^1$, we conclude that $\mathbf{c}$ is a U-cycle in (N, v).

The converse statement is proven similarly by using the inverse transformation $\bar{x}^i = x^i - t$ of (9). $\square$

Remark 2. The family $\mathscr{B}$ has the same worth $w(\mathscr{B},v)=\sum_{k=1}^{m}\lambda_k\cdot v(S_k)$ with respect to both of the above games (N,v) and (N,v_μ).

Lemma 3. *Let (N,v_1) be the 1-uniform game related to (N,v) and let $\alpha=1-\frac{1}{\sum_{k=1}^{m}\lambda_k}$. Then, there exists a pre-imputation x such that $e(S,x|v_1)=\alpha$ for all $S_k\in\mathscr{B}$.*

Proof. Let $\mathbf{1}-\boldsymbol{\alpha}=(1-\alpha,\dots,1-\alpha)$ be the m-vector having all its components equal to $1-\alpha$. Let x be a solution of the linear system $E_{\mathscr{B}}\cdot x'=(\mathbf{1}-\boldsymbol{\alpha})'$ which is, as we mentioned above, always consistent. Once more, since by (2)

$$\boldsymbol{\lambda}\cdot E_{\mathscr{B}}\cdot x'=x(N)$$

and $\boldsymbol{\lambda}\cdot(\mathbf{1}-\boldsymbol{\alpha})'=1$, we get that x is also a pre-imputation which satisfies $e(S,x|v_1)=\alpha$ for all $S_k\in\mathscr{B}$. □

Remark 3. As a referee pointed us, any solution of the latter system is a $\mathscr{B}$-pre-nucleolus of (N,v_1), namely, a pre-nucleolus obtained in the agreement that the only coalitions considered to define the vector of excesses associated to a pre-imputation, are those in $\mathscr{B}$. Thus, there will be a unique $\mathscr{B}$-pre-nucleolus if and only if the matrix $E_{\mathscr{B}}$ associated to $\mathscr{B}$ is non-singular, and this happens if and only the minimal balanced family $\mathscr{B}$ contains n coalitions.

Now let $\hat{x}$ be a pre-imputation satisfying $e(S,\hat{x}|v_1)=\alpha$ for all $S\in\mathscr{B}$, which exists because of Lemma 3 and let

$$x=\frac{1-\mu}{\alpha}\cdot\hat{x}+\left(1-\frac{1-\mu}{\alpha}\right)\cdot\bar{x}, \tag{10}$$

define an homeomorphism on $\mathbb{R}^n$ whose inverse is given by

$$\bar{x}=\frac{\alpha}{\alpha-(1-\mu)}\cdot x+\frac{\mu-1}{\alpha-(1-\mu)}\cdot\hat{x}.$$

Like in Proposition 2, this transformation allows us to relate U-cycles in the 1-uniform game related to (N,v) to U-cycles in any non-balanced μ-uniform game (N,v_μ) related to (N,v).

Proposition 4. *Let μ be a number satisfying $1-\alpha=\frac{1}{\sum_{k=1}^{m}\lambda_k}<\mu$. If $\bar{\mathbf{c}}=(\bar{x}^j)_{j=1}^m$ is a U-cycle of pre-imputations with $\text{supp}(\bar{\mathbf{c}})=(S_k)_{k=1}^m$ in the game (N,v_1), then $\mathbf{c}=(x^j)_{j=1}^m$, where*

$$x^j=\frac{1-\mu}{\alpha}.\hat{x}+\left(1-\frac{1-\mu}{\alpha}\right).\bar{x}^j,$$

$j=1,\dots,m$, is a U-cycle of pre-imputations in the game (N,v_μ) with the same support $(S_k)_{k=1}^m$. Conversely, if $\mathbf{c}=(x^j)_{j=1}^m$ is a U-cycle of pre-imputations with $\text{supp}(\mathbf{c})=(S_k)_{k=1}^m$ in the game (N,v_μ), then $\bar{\mathbf{c}}=(\bar{x}^j)_{j=1}^m$ where

$$\bar{x}^j=\frac{\alpha}{\alpha-(1-\mu)}.x^j+\frac{\mu-1}{\alpha-(1-\mu)}.\hat{x},$$

$j=1,\dots,m$, is a U-cycle of pre-imputations in the game (N,v_1) with the same support $(S_k)_{k=1}^m$.

Proof. The condition $1-\alpha<\mu$ is to limit ourselves to stay in the class of non-balanced games. Let $\bar{\mathbf{c}}=(\bar{x}^j)_{j=1}^m$ be a U-cycle of pre-imputations in the game (N,v_1) with $\text{supp}(\bar{\mathbf{c}})=(S_k)_{k=1}^m$. We stress on the fact that we are considering a labeling of the coalitions in $\mathscr{B}$ such that $\bar{x}^{j+1}=\bar{x}^j+e(S_j,\bar{x}^j|v_1)\cdot\beta_j, j=1,\dots,m$.

To prove that $\mathbf{c}=(x^j)_{j=1}^m$ is a U-cycle in (N,v_μ), let $i,j=1,\dots,m$ be given. Since $x^j(S_i)=1-x^j(N\setminus S_i)$, we have that

$$\begin{aligned}
e(S_i,x^j|v_\mu)&=\mu-1+x^j(N\setminus S_i)\\
&=\mu-1+\frac{1-\mu}{\alpha}.\hat{x}(N\setminus S_i)\\
&\quad+\left(1-\frac{1-\mu}{\alpha}\right)\cdot\bar{x}^j(N\setminus S_i)\\
&=\mu-1+\frac{1-\mu}{\alpha}\cdot\alpha+\left(1-\frac{1-\mu}{\alpha}\right)\cdot\bar{x}^j(N\setminus S_i)\\
&=\left(1-\frac{1-\mu}{\alpha}\right)\cdot\bar{x}^j(N\setminus S_i)\\
&=\left(1-\frac{1-\mu}{\alpha}\right)\cdot(1-\bar{x}^j(S_i))\\
&=\left(1-\frac{1-\mu}{\alpha}\right)\cdot e(S_i,\bar{x}^j|v_1).
\end{aligned} \tag{11}$$

The condition $1-\alpha<\mu$ guarantees that the coefficient $(1-\frac{1-\mu}{\alpha})$ of $e(S_i,\bar{x}^j|v_1)$ in (11) is positive. This implies that the excesses $e(S_i,x^j|v_\mu)$ related to x^j are ordered in the same way as those related to $\bar{x}^j$, for all $S_i\in\mathscr{B}$. Then, because of (10) and (11) we get that

$$\begin{aligned}
x^1+e(S_1,x^1|v_\mu)\cdot\beta_{S_1}&=\frac{1-\mu}{\alpha}\cdot\hat{x}+\left(1-\frac{1-\mu}{\alpha}\right)\cdot\bar{x}^1\\
&\quad+\left(1-\frac{1-\mu}{\alpha}\right)\cdot e(S_i,\bar{x}^1|v_1)\cdot\beta_{S_1}\\
&=\frac{1-\mu}{\alpha}\cdot\hat{x}+\left(1-\frac{1-\mu}{\alpha}\right)\cdot\bar{x}^2=x^2.
\end{aligned}$$

An inductive argument completes the first part of the proof. The converse is proven in a similar way by using the inverse transformation of (10). □

Remark 4. The relationships between U-cycles in (N,v_1) and U-cycles in (N,v_μ) showed in Proposition 4 can be easily extended to relationships between U-cycles in any pair of non-balanced games (N,v_μ) and $(N,v_{\bar{\mu}})$ related to (N,v).

Remark 5. Transformations (9) and (10) are not unique unless the rank of matrix $E_{\mathscr{B}}$ is n (see also Remark 3). The games studied in Cesco and Calí (2006) satisfy this condition. The cycles studied there are isolated, in the sense that each pre-imputation in cycle has a neighborhood which does not contain a pre-imputation belonging to another cycle. However, when the rank of $E_{\mathscr{B}}$ is less than n, each vector t belonging to the kernel of $E_{\mathscr{B}}$, which is a subspace of dimension $r\geqslant 1$ included in the orthogonal subspace to the vector $(1,\dots,1)$, induces a U-cycle in the game (N,v) related to a cycle $(x^k)_{k=1}^m$ in the game (N,v_μ). The family of cycles obtained in this way, are not isolated.

Two cycles in such a class translate to one another with the translation vector belonging to the kernel of $E_{\mathscr{B}}$.

We now prove the existence result.

Theorem 5. *If $\mathscr{B} = \{S_1, S_2, \ldots, S_m\}$ is an objectionable family of coalitions in (N, v), then there always exists a U-cycle of pre-imputations whose support is $(S_k)_{k=1}^m$.*

Proof. We will develop the proof in several steps. First, according to Proposition 1, there exists a fundamental cycle of pre-imputations $(x^k)_{k=1}^m$ such that $x^{k+1} = x^k + \mu_k \cdot \beta_{S_k}, k = 1, \ldots, m$, and $x^{m+1} = x^1$. We claim that this fundamental cycle is also a U-cycle in the modified game $(N, \bar{v})$ where $\bar{v}(S_k) = x^k(S_k) + \mu_k$ for all $S_k \in \mathscr{B}$, and $\bar{v}(S) = v(S)$ otherwise. In fact

$$e(S_1, x^1|\bar{v}) = \bar{v}(S_1) - x^1(S_1) = x^1(S_1) + \mu_1 - x^1(S_1) = \mu_1 > 0,$$

and

$$x^1 + e(S_1, x^1|\bar{v}) \cdot \beta_{S_1} = x^1 + \mu_1 \cdot \beta_{S_1} = x^2,$$

which shows that x^2 is U-transfer from x^1. An inductive argument proves that $x^{k+1} = x^k + e(S_k, x^k|\bar{v}) \cdot \beta_k, e(S_k, x^k|\bar{v}) > 0$ for all $k = 1, \ldots, m$, and that $x^{m+1} = x^1$. Thus, $(x^k)_{k=1}^m$ is a U-cycle in $(N, \bar{v})$.

In a second step and from the U-cycle $(x^k)_{k=1}^m$ in $(N, \bar{v})$,we construct, by the procedure described in Proposition 2, a U-cycle in the game $(N, \bar{v}_{\bar{\mu}})$ associated to $(N, \bar{v})$ with $\bar{\mu} = \frac{\sum_{k=1}^m \lambda_k \cdot \bar{v}(S_k)}{\sum_{k=1}^m \lambda_k}$. In the third step we construct from the latter, a new U-cycle in the game $(N, \bar{v}_\mu)$ also associated to $(N, \bar{v})$, but with $\mu = \frac{\sum_{k=1}^m \lambda_k \cdot v(S_k)}{\sum_{k=1}^m \lambda_k}$. This is carried out by the procedure described in Proposition 4. Finally, the last step is to get from this U-cycle, a U-cycle in the game (N, v) once more by the procedure indicated in Proposition 2. This last cycle is the cycle we are looking for. $\square$

Remark 6. In the first part of the proof of the Theorem 5, $\bar{v}(S) \leqslant v(S)$ for all $S \subseteq N$. If all of the elements in the cycle are imputations, then $v(S_k) > 0$ for all $S_k \in \mathscr{B}$.

Remark 7. From Proposition 1 (see Remark 1) it follows that with each permutation of the coalitions in $\mathscr{B}$ there is an associated fundamental cycle. Therefore, by the procedure described in the former result, we can construct up to $m!$ U-cycles whose support is $\mathscr{B}$, and have no repetition in the coalitions appearing in the support. However, several of these cycles define the same sequence of pre-imputations except for the starting point. They are in fact, only $(m-1)!$ essentially different classes of U-cycles with the two properties previously mentioned.

4. Final remarks

We close this note with a few remarks regarding maximal U-cycles, which are related to the U-cycles but are, in our opinion, the ones we consider more attractive. The method described in Section 3 does not provide, in general, maximal U-cycles as it is shown by the example given in the Appendix. However, for some games with a distinguished family of coalitions $\mathscr{C}$, like those studied in (Cesco and Calí, 2006), the U-cycles obtained for the associated canonical game, i.e. the simple game with $v(S) = 1$ if $S \in \mathscr{C}$ or $S = N$, and zero otherwise, are also maximal U-cycles. Further studies to characterize games with this property should be useful, although we have to mention that, since maximal U-cycles are not preserved under the transformations defined by (9) and (10), some extra limitations should be imposed on the games in order to apply a similar technique to guarantee the existence of maximal U-cycles.

Another way to obtain the existence of maximal U-cycles is by proving a general 'convergence' theorem showing that the algorithm described in Cesco (1998) always has a limit cycle which is a maximal cycle. Up to now, only partial results have been obtained. This point of view has been successful in dealing with games having strong symmetry properties.

Regarding the computational complexity of the algorithm, we have to mention that it is associated to a problem which is NP-hard. The heavy computational component of the algorithm is that devoted to the computation of the maximum excess over all the set of coalitions, which grows like 2^n, n being the number of players in the game. However, for some special, although very interesting games, like assignment games, the set of relevant coalition reduces substantially, making the algorithm work in polynomial time. Furthermore, the maximum excess problem has the advantage of being highly parallelizable, which opens a gateway for implementing efficient algorithms for problems of moderate size. The reader is referred to Deng and Papadimitriou (1994) for some observations about the complexity of the solutions in cooperative game theory and to Faigle et al. (2001) for the specific maximum excess problem.

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Appendix

The example we develop below is useful in showing that, given an objectionable family $\mathscr{B}$, the method presented in Section 3 can provide a U-cycle **c** which is not maximal and moreover, there can be another maximal U-cycle sharing the same supporting family as **c**.

Let $S_1 = \{1, 2\}, S_2 = \{2, 3\}, S_3 = \{1, 3\}$, and $S_4 = \{4, 5\}$. Then, the family $\mathscr{B} = \{S_1, S_2, S_3, S_4\}$ is a minimal balanced family in the framework of 5-person games. Let us now consider the 5-person game with characteristic function given by

$$v(N) = 1, v(S) = 1 \quad \text{for all } S \in \mathscr{B},$$
$$v(S) = 0 \quad \text{otherwise.}$$

This game is a non-balanced game and $\mathscr{B}$ is an objectionable family of coalitions. The pre-imputations

$$x^1 = \begin{pmatrix} 0.0000 \\ 0.1429 \\ -0.1429 \\ 0.5000 \\ 0.5000 \end{pmatrix}, \quad x^2 = \begin{pmatrix} 0.4286 \\ 0.5714 \\ -0.4286 \\ 0.2143 \\ 0.2143 \end{pmatrix},$$

$$x^3 = \begin{pmatrix} 0.1429 \\ 1.000 \\ 0.000 \\ -0.07145 \\ -0.07145 \end{pmatrix}, \quad x^4 = \begin{pmatrix} 0.5714 \\ 0.7143 \\ 0.4286 \\ -0.35715 \\ -0.35715 \end{pmatrix},$$

form a U-cycle $\mathbf{c} = (x^1, x^2, x^3, x^4)$ with $\text{supp}(\mathbf{c}) = (S_1, S_2, S_3, S_4)$, which is not, however, a maximal U-cycle. Indeed,

$$e(\{1,3\}, x^1) = 1 + 0.1429 > 1 - 0.1429 = e(\{1,2\}, x^1).$$

On the other side, a maximal U-cycle $\tilde{\mathbf{c}}$ can be found having

$$\text{supp}(\tilde{\mathbf{c}}) = (S_2, S_3, S_4, S_1, S_3, S_4, S_2, S_4, S_3, S_1, S_4, S_2,$$
$$S_4, S_1, S_3, S_4, S_2, S_3, S_4, S_1, S_4, S_3, S_2, S_4, S_1, S_4),$$

with 26 elements exhibiting a rather chaotic pattern. An initial pre-imputation for the cycle $\mathbf{c}$(exact up to the fourth decimal place) is

$$x = \begin{pmatrix} 0.5779 \\ 0.4221 \\ -0.2236 \\ 0.1118 \\ 0.1118 \end{pmatrix}.$$

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