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Ideals of Multilinear Forms – a Limit Order Approach

Daniel Carando, Verónica Dimant and Pablo Sevilla-Peris

Abstract. A general theory of limit orders for ideals of multilinear forms is developed. We relate the limit order of an ideal to those of its maximal hull and its adjoint ideal. We study the limit orders of the ideals of dominated and multiple summing multilinear forms. Finally, estimates of the diagonal of a (non-necessarily diagonal) multilinear form are presented, in terms of the limit order of the ideals to which it belongs.

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Introduction

In 1983, A. Pietsch [20] presented his "designs of a theory" for ideals of multilinear forms. His work provided a general framework from which different lines of investigation developed. Some ideals of multilinear forms appeared as the multilinear natural extension of ideals of linear operators (e.g., nuclear and integral multilinear forms). However, it is not always clear what the multilinear analogous of a linear operators ideal should be. For example, the ideal of absolutely r-summing operators lead to the development of many ideals of multilinear forms: absolutely r-summing, r-dominated, multiple r-summing, etc. Independently of their linear origin, many ideals of multilinear forms were studied by their own interest and also in relation to ideals of polynomials and holomorphy types. In any case, the theory of ideals of multilinear forms allows to deal with all the different situations in a unified way.

In the linear theory, the calculus of limit orders proved a useful tool, specially to compare different ideals of linear operators. The concept of limit order

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for multilinear forms was introduced in [4]. As an application, it could be shown that some properties of bilinear ideals were no longer valid for the n-linear case ($n \geq 3$).

The first aim of this article is to develop a bit further the general theory of limit orders for multilinear forms. Even though some of our proofs are adaptations of the linear analogous results, we chose to present them here. Our motivation was to give a self-contained treatment of the subject, since the linear versions of these results come from different sources and with different notations. The general theory developed, we focus on some particular ideals of multilinear forms.

In the first section, we recall the definitions of limit order and present some general properties. In section 2 we study the diagonal of a multilinear form. We show that the limit order of a multilinear ideal gives estimates of the diagonal of any multilinear form in the ideal. The third section deals with maximal and adjoint ideals of multilinear forms and their corresponding limit orders. In section 4, the ideal of r-dominated n-linear forms is shown to be dual to a tensor norm. This allows us to describe its adjoint ideal. In section 5 we estimate the limit order of the ideal of multiple 1-summing forms.

Given X, Y Banach spaces, we denote by $\mathcal{L}(X;Y)$ the space of continuous linear mappings $T : X \to Y$. If $X_1, \dots, X_n$ and Y are Banach spaces, $\mathcal{L}(X_1, \dots, X_n; Y)$ denotes the space of continuous n-linear mappings $T : X_1 \times \cdots \times X_n \to Y$. Whenever $X_1 = \cdots = X_n = X$ and $Y = \mathbb{C}$, the space of continuous n-linear mappings is simply denoted by $\mathcal{L}(^n X)$. We are going to deal with mappings $T \in \mathcal{L}(^n \ell_p)$. We denote by $x_1, \dots, x_n$ the elements in ℓ_p. If x is a sequence we write $x = (x(k))_{k=1}^{\infty}$, with $x(k) \in \mathbb{C}$. Given a sequence $(x(k))_k \subseteq \mathbb{C}$, its p-norm $(\sum |x(k)|^p)^{1/p}$ will be denoted by $\ell_p(x(i))$ or $\|x\|_p$.

If we have two sequences $(x(k))_k$ and $(y(k))_k$ we will denote $(x(k))_k \prec (y(k))_k$ if there exist a constant C such that $x(k) \leq C\, y(k)$ for every k. We will denote $(x(k))_k \asymp (y(k))_k$ if $(x(k))_k \prec (y(k))_k$ and $(y(k))_k \prec (x(k))_k$.
Although all the results in the article are proved for complex Banach spaces, standard modifications can be made to obtain the real version of most of them.

1. Limit order of ideals of multilinear forms

In [4], the limit order for ideals of multilinear forms is defined. We recall the definitions and some basic facts.

Given a sequence σ, the associated diagonal operator D_σ from ℓ_p to ℓ_q is defined by $D_\sigma(x) = (\sigma(k)x(k))_k$. With this, given an operator ideal $\mathfrak{A}$, the limit order $\lambda(\mathfrak{A}; p, q)$ (see [19, Section 14.4]) is the infimum over all $\lambda \geq 0$ such that every diagonal operator $D_\sigma : \ell_p \to \ell_q$ with $\sigma \in \ell_{1/\lambda}$ belongs to $\mathfrak{A}(\ell_p, \ell_q)$.

Ideals of multilinear forms were introduced in [20]. Let us recall the definition. An **ideal of multilinear forms** $\mathfrak{A}$ is a subclass of $\mathcal{L}$, the class of all multilinear forms such that, for any Banach spaces $X_1, \dots, X_n$ the set

$$\mathfrak{A}(X_1, \dots, X_n) = \mathfrak{A} \cap \mathcal{L}(X_1, \dots, X_n)$$

satisfies

1. For any $\gamma_1 \in X_1', \dots, \gamma_n \in X_n'$, the mapping $(x_1, \dots, x_n) \mapsto \gamma_1(x_1) \cdots \gamma_n(x_n)$ belongs to $\mathfrak{A}(X_1, \dots, X_n)$.
2. If $S, T \in \mathfrak{A}(X_1, \dots, X_n)$, then $S + T \in \mathfrak{A}(X_1, \dots, X_n)$.
3. If $T \in \mathfrak{A}(X_1, \dots, X_n)$ and $S_i \in \mathcal{L}(Y_i, X_i)$ for $i = 1, \dots, n$, then $T \circ (S_1, \dots S_n) \in \mathfrak{A}(Y_1, \dots, Y_n)$.

An ideal of multilinear forms is called **(quasi-) normed** if for each $X_1, \dots, X_n$ there is a (quasi-) norm $\|\cdot\|_{\mathfrak{A}(X_1,\dots,X_n)}$ in $\mathfrak{A}(X_1, \dots, X_n)$ such that

1. $\|(x_1, \dots, x_n) \mapsto \gamma_1(x_1) \cdots \gamma_n(x_n)\|_{\mathfrak{A}(X_1,\dots,X_n)} = \|\gamma_1\| \cdots \|\gamma_n\|$.
2. $\|T \circ (S_1, \dots S_n)\|_{\mathfrak{A}(Y_1,\dots,Y_n)} \leq \|T\|_{\mathfrak{A}(X_1,\dots,X_n)} \cdot \|S_1\| \cdots \|S_n\|$.

Following the spirit of the definition of limit order for operator ideals, the concept of limit order was defined in [4] for ideals of multilinear forms. First of all, if $T \in \mathcal{L}(^n\ell_p)$, we call it **diagonal** if there exists a sequence $\alpha = (\alpha(k))_k$ such that for all $x_1, \dots, x_n \in \ell_p$ we can write

$$T(x_1, \dots, x_n) = \sum_k \alpha(k) x_1(k) \cdots x_n(k).$$

We denote by T_α the diagonal multilinear mapping given by the sequence α. With this the limit order can be defined.

Definition 1.1. Let $\mathfrak{A}$ be an ideal of multilinear forms. For $1 \leq p \leq \infty$, the limit order $\lambda_n(\mathfrak{A}; p)$ is given by:

$$\lambda_n(\mathfrak{A}; p) = \inf\{\lambda : \text{ for each } \alpha \in \ell_{1/\lambda},\ T_\alpha \text{ belongs to } \mathfrak{A}(^n\ell_p)\}$$

From the definition we have that $0 \leq \lambda_n(\mathfrak{A}; p) \leq 1$ for every ideal $\mathfrak{A}$ and all p and n. Also, if $\mathfrak{A}, \mathfrak{B}$ are ideals such that $\mathfrak{A} \subseteq \mathfrak{B}$, then $\lambda_n(\mathfrak{A}; p) \geq \lambda_n(\mathfrak{B}; p)$.

With almost the same proof as in [19, Section 14.4], we obtain alternative expressions for $\lambda_n(\mathfrak{A}; p)$. First, we have:

$$\lambda_n(\mathfrak{A}; p) = \inf\{\lambda : \text{ if } \alpha = (k^{-\lambda})_k, \text{ then } T_\alpha \text{ belongs to } \mathfrak{A}(^n\ell_p)\}.$$

Given $N \in \mathbb{N}$, we define the n-linear form Φ_N on $\mathbb{C}^N$ by:

$$\Phi_N(x_1, \dots, x_n) = \sum_{k=1}^{N} x_1(k) \cdots x_n(k).$$

With this, if $\mathfrak{A}$ is quasi-normed and complete, then $\lambda_n(\mathfrak{A}; p)$ is the infimum of all $\lambda \geq 0$ such that

$$\|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \leq C N^\lambda \tag{1.1}$$

for all $N \in \mathbb{N}$, where $C > 0$ is a constant.

The following definition is the multilinear version of that introduced in [14] for linear operators. As in [14], we denote $\ell_n(\mathfrak{A}, p) := \{\alpha \in \ell_\infty : T_\alpha \in \mathfrak{A}(^n\ell_p)\}$. The sequence space $\ell_n(\mathfrak{A}, p)$ is a Banach space if $\mathfrak{A}$ is normed and we consider the norm

$$\|\alpha\|_{\ell_n(\mathfrak{A},p)} = \|T_\alpha\|_{\mathfrak{A}(^n\ell_p)}$$

Definition 1.2. Let $\mathfrak{A}$ be an ideal of n-linear forms and $1 \leq p \leq \infty$. We define the **defect** by

$$d_n(\mathfrak{A}, p) = \inf \left\{ \frac{1}{r} - \frac{1}{s} : \ell_r \subset \ell_n(\mathfrak{A}, p) \subset \ell_s \right\}.$$

Clearly, the definitions of limit order and defect can be generalized to n-linear forms on $\ell_{p_1} \times \cdots \times \ell_{p_n}$. All the results in this article can be extended to this situation. However, for the reader's convenience, we prefer to state them in the simple case $p_1 = \cdots = p_n = p$.

It is also possible to define the limit order of ideals of n-homogeneous polynomials. But this will not lead us to new horizons. Indeed, in [12] the authors showed that, given a λ-normed ideal of n-homogeneous polynomials $\mathcal{Q}$, there exists a λ-normed ideal of n-linear forms $\mathcal{Q}^\vee$ with the following property:

> A polynomial P is in $\mathcal{Q}$ if and only if its associated symmetric n-linear form $\check{P}$ is in $\mathcal{Q}^\vee$.

Moreover, $P \leftrightarrow \check{P}$ is a one-to-one correspondence between diagonal n-homogeneous polynomials and diagonal symmetric n-linear forms. Then, the limit orders of ideals of homogeneous polynomials can be seen as limit orders of ideals of multilinear forms.

Given a diagonal multilinear form $T_\alpha \in \mathcal{L}(^n\ell_p)$, we consider a sequence σ such that $\sigma(k)^n = \alpha(k)$ for all k. We take the diagonal operator $D_\sigma : \ell_p \to \ell_n$ associated to σ and define a mapping $\Phi : \ell_n \times \cdots \times \ell_n \to \mathbb{C}$ by $\Phi(x_1, \ldots, x_n) = \sum_k x_1(k) \cdots x_n(k)$. The fact that T_α is well defined on ℓ_p guarantees that $D_\sigma(\ell_p) \subset \ell_n$. Now, the diagonal n-linear mapping T_α can be rewritten as

$$T_\alpha(x_1, \ldots, x_n) = \Phi(D_\sigma(x_1), \ldots, D_\sigma(x_n)). \tag{1.2}$$

This decomposition allows to sometimes get some relations between limit orders of ideals of multilinear forms and of ideals of operators.

Examples: In [4] the limit orders of nuclear, integral and r-dominated multilinear forms (see [8, 17] for definitions) are computed:

$$\lambda_n(\mathcal{L}; p) = \begin{cases} 0 & \text{if } p \leq n \\ 1 - \frac{n}{p} & \text{if } p > n \end{cases}$$

$$\lambda_n(\mathcal{N}; p) = \lambda_n(\mathcal{I}; p) = \begin{cases} \frac{n}{p'} & \text{if } 1 \leq p < n' \\ 1 & \text{if } n' \leq p \end{cases}$$

$$\lambda_n(\mathcal{D}_r; p) = n\ \lambda(\Pi_r; p, n)$$

where $\lambda(\Pi_r; p, n)$ is the limit order of the ideal of absolutely r-summing linear operators (see [19, Section 22.4]). The relation of the limit order of r-dominated n-linear forms and absolutely r-summing operators follows from the following result:

Proposition 1.3. [4] *Let $T_\alpha \in \mathcal{L}(^n\ell_p)$ be diagonal and D_σ its associated diagonal operator. Then T_α is r-dominated if and only if D_σ is absolutely r-summing.*

Now we state some general properties of limit orders. As in the linear case, the following property holds for composition ideals:

Proposition 1.4. *Let $\mathfrak{A}$ be an ideal of n-linear forms and $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ linear operator ideals. Then:*

$$\lambda_n(\mathfrak{A} \circ (\mathfrak{B}_1, \ldots, \mathfrak{B}_n); p) \leq \lambda(\mathfrak{B}_1; p, s) + \cdots + \lambda(\mathfrak{B}_n; p, s) + \lambda_n(\mathfrak{A}; s)$$

As a consequence, since $\mathfrak{A} = \mathfrak{A} \circ (\mathcal{L}, \ldots, \mathcal{L})$, we have

$$|\lambda_n(\mathfrak{A}; p) - \lambda_n(\mathfrak{A}; p_0)| \leq n \left| \frac{1}{p} - \frac{1}{p_0} \right|.$$

Therefore, $\lambda_n(\mathfrak{A}; p)$ is a continuous function of $\frac{1}{p}$.

Let $\mathfrak{A}$ be a quasi-normed ideal of n-linear forms. The maximal hull $\mathfrak{A}^{max}$ of $\mathfrak{A}$ is defined as the class of all n-linear forms T such that

$$\|T\|_{\mathfrak{A}^{max}(E_1,\ldots,E_n)} := \sup\{\|T\,|_{M_1 \times \cdots \times M_n}\|_{\mathfrak{A}(M_1,\ldots,M_n)} : M_i \subset E_i,\ dim M_i < \infty\}$$

is finite.

$\mathfrak{A}^{max}$ is always complete and it is the largest ideal whose quasi-norm coincides with $\|\cdot\|_{\mathfrak{A}}$ in finite dimensional spaces. Hence, if $\mathfrak{A}$ is complete, equation (1.1) says that

$$\lambda_n(\mathfrak{A}; p) = \lambda_n(\mathfrak{A}^{max}; p)$$

for all p.

A quasi-normed ideal $\mathfrak{A}$ is called maximal if $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}}) = (\mathfrak{A}^{max}, \|\cdot\|_{\mathfrak{A}^{max}})$.

Following [5, 10, 13] we define now the adjoint of an ideal of multilinear mappings. Let $\mathfrak{A}$ be a normed ideal of n-linear mappings. If $M_1, \ldots, M_n$ are finite dimensional normed spaces, the multilinear norm $\|\cdot\|_{\mathfrak{A}}$ induces a tensor norm α in FIN (the class of all finite dimensional normed spaces) by means of the identification:

$$\left(\bigotimes_{i=1}^n M_i; \alpha\right) \stackrel{1}{=} \mathfrak{A}(M_1', \ldots, M_n')$$

This norm can be extended to a finitely generated tensor norm on the class of normed spaces by

$$\|s\|_{(\bigotimes_{i=1}^n E_i;\alpha)} := \inf\left\{\|s\|_{(\bigotimes_{i=1}^n M_i;\alpha)} : M_i \in \text{FIN}(E_i),\ s \in \bigotimes_{i=1}^n M_i\right\}$$

In this case, the tensor norm α and the ideal $\mathfrak{A}$ are said to be associated.

Given a normed ideal $\mathfrak{A}$ associated to the finitely generated tensor norm α, its adjoint ideal $\mathfrak{A}^*$ is defined by

$$\mathfrak{A}^*(E_1, \ldots, E_n) := \left(\bigotimes_{i=1}^n E_i; \alpha\right)'.$$

The adjoint ideal is called dual ideal in [10].

The tensor norm associated to $\mathfrak{A}^*$ is denoted by α^*. We also have the representation theorem [13, Section 3.2] (see also [10, Section 4]):

$$\mathfrak{A}^{max}(E_1, \dots, E_n) = \left(\bigotimes_{i=1}^{n} E_i; \alpha^* \right)'.$$

In particular, this shows that the adjoint ideal $\mathfrak{A}^*$ is maximal.

2. The diagonal of a multilinear form

Throughout this section, X_i will be Banach spaces with unconditional basis $\{e_j^i\}_j$ ($i = 1, \dots, n$). We define the mapping $D : \mathcal{L}(X_1, \dots, X_n) \to \mathcal{L}(X_1, \dots, X_n)$ given by

$$D(T)(x_1, \dots, x_n) = \sum_{j=1}^{\infty} T(e_j^1, \dots, e_j^n) x_1(j) \cdots x_n(j).$$

Note that $D(T)$ is the diagonal n-linear form given by the diagonal of T. The linear mapping D is well defined and continuous [7, Proposition 1.3]. Let us show now that it preserves some ideals of multilinear forms.

Proposition 2.1. *Let β be a tensor norm of order n. If $T \in \mathcal{L}(X_1, \dots, X_n)$ is β-continuous (i.e., $T \in (\bigotimes_{i=1}^n X_i; \beta)'$), then $D(T)$ is also β-continuous.*

Proof. For $0 \le t \le 1$ and $i = 1, \dots, n$, we define $\Lambda_t^i : X_i \to X_i$ by

$$\Lambda_t^i(x) = \sum_{j=1}^{\infty} x(j) r_j(t) e_j^i,$$

where $\{r_j\}_j$ are the generalized n-Rademacher functions [2, Section 1]. By the unconditionality of the basis, Λ_t^i is continuous and $\|\Lambda_t^i\| \le 2K_i$ (being K_i the unconditionality constant of the basis).

Let $s \in \bigotimes_{i=1}^n X_i$, $s = \sum_{k=1}^M x_k^1 \otimes \cdots \otimes x_k^n$. We have

$$\begin{aligned}
|D(T)(s)| &= \left| \sum_{k=1}^{M} D(T)(x_k^1, \dots, x_k^n) \right| \\
&= \left| \int_0^1 \sum_{k=1}^{M} T\left(\Lambda_t^1(x_k^1), \dots, \Lambda_t^n(x_k^n)\right) dt \right| \\
&= \left| \int_0^1 T\left((\Lambda_t^1 \otimes \cdots \otimes \Lambda_t^n)(s)\right) dt \right| \\
&\le \|T\|_{(\bigotimes_{i=1}^n X_i;\beta)'} \cdot \|\Lambda_t^1\| \cdots \|\Lambda_t^n\| \cdot \|s\|_\beta \\
&\le \|T\|_{(\bigotimes_{i=1}^n X_i;\beta)'} \cdot 2^n K_1 \cdots K_n \cdot \|s\|_\beta
\end{aligned}$$

□

Corollary 2.2. *If $\mathfrak{A}$ is a maximal ideal of n-linear forms, then $D : \mathfrak{A}(X_1, \ldots, X_n) \to \mathfrak{A}(X_1, \ldots, X_n)$ is well defined and continuous.*

In particular, the ideals of integral, extendible and r-dominated ($r \geq n$) n-linear forms are preserved by D. The same holds for the ideal of multiple r-summing multilinear forms (see [16, 3] for the definitions and Section 5 for the limit orders of the case $r = 1$). An example of an ideal which is not maximal but is preserved by D is given in the following:

Proposition 2.3. *If $T \in \mathcal{L}(X_1, \ldots, X_n)$ is weakly sequentially continuous, then so is $D(T)$*

Proof. Let $(x_k^i)_k \subseteq X_i$ be weakly convergent to $x^i \in X_i$. With the notation of the proof of Proposition 2.1, we have

$$D(T)(x_k^1, \ldots, x_k^n) = \int_0^1 T\left(\Lambda_t^1(x_k^1), \ldots, \Lambda_t^n(x_k^n)\right) dt.$$

Since each Λ_t^i is linear and T is weakly sequentially continuous, $T(\Lambda_t^1(x_k^1), \ldots, \Lambda_t^n(x_k^n))$ converges to $T\left(\Lambda_t^1(x^1), \ldots, \Lambda_t^n(x^n)\right)$ for every $t \in [0, 1]$. Now the result follows from the dominated convergence theorem. □

However, not every ideal is preserved by D:

Example. The ideals of nuclear, approximable and weakly continuous on bounded sets multilinear forms are not preserved by D:
Let $T \in \mathcal{L}(^n\ell_1)$ be given by

$$T(x^1, \ldots, x^n) = \left(\sum_{j=1}^{\infty} x^1(j)\right) \cdots \left(\sum_{j=1}^{\infty} x^n(j)\right).$$

Clearly, T is a finite type n-linear form. Now,

$$D(T)(x^1, \ldots, x^n) = \sum_{j=1}^{\infty} x^1(j) \cdots x^n(j)$$

which is not weakly continuous on bounded sets (see, for example, [8, Proposition 2.6]). Hence it is neither nuclear, nor approximable.

From the above results about the operator D and the definitions of limit order and defect, we can obtain some information about the diagonal of any n-linear form belonging to certain ideal.

Proposition 2.4. *If $\mathfrak{A}$ is a maximal ideal of n-linear forms (or any ideal of n-linear forms preserved by D) and $T \in \mathfrak{A}(^n\ell_p)$, then, for every $\varepsilon > 0$,*

$$\left(T(e_j, \ldots, e_j)\right)_j \in \ell_{r+\varepsilon}$$

where $r = \frac{1}{\lambda_n(\mathfrak{A};p) - d_n(\mathfrak{A};p)}$ and $(e_j)_j$ is the canonical basis of ℓ_p.

3. Maximal and adjoint ideals of multilinear mappings

The limit orders of an ideal and its adjoint ideal are related, as in the linear case, by the following equality (see [5, Sect. 17.19])

Lemma 3.1. $\|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)}\|\Phi_N\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} = N$

Proof. Let us note first that

$$\begin{aligned} N &= \sum_{k=1}^N \Phi_N(e_k,\dots,e_k) = \Phi_N\left(\sum_{k=1}^N e_k\otimes\cdots\otimes e_k\right) \\ &\leq \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)}\left\|\sum_{k=1}^N e_k\otimes\cdots\otimes e_k\right\|_{\left(\bigotimes^n \ell_p^N;\alpha^*\right)} = \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)}\cdot\|\Phi_N\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} \end{aligned}$$

For the reverse inequality, let us choose a norm one $s\in\left(\bigotimes^n\ell_p^N;\alpha^*\right)$ such that $\|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} = |\Phi_N(s)|$. Then

$$\|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)}\cdot\|\Phi_N\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} = \|\Phi_N(s)\cdot\Phi_N\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)}.$$

We write $s=\sum_{i_1,\dots,i_n=1}^N \alpha_{i_1,\dots,i_n}e_{i_1}\otimes\cdots\otimes e_{i_n}$ and define $s_0 := \sum_{i=1}^N \alpha_{i,\dots,i}e_i\otimes\cdots\otimes e_i$ the diagonal of s. Clearly, $\Phi_N(s)=\Phi_N(s_0)$.

Now we define $S:\ell_{p'}^N\to\ell_{p'}^N$ by $S(x_1,x_2,\dots,x_N)=(x_2\dots,x_N,x_1)$. It is easy to see that for each $1\leq j\leq n$, $\sum_{k=1}^N(e_j\otimes\cdots\otimes e_j)\circ(S^k,\cdots,S^k)=\Phi_N\in\mathfrak{A}^*({}^n\ell_{p'}^N)$. Therefore,

$$\begin{aligned} \|\Phi_N(s)\cdot\Phi_N\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} &= \|\Phi_N(s_0)\cdot\Phi_N\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} \\ &= \left\|\sum_{i=1}^N\alpha_{i,\dots,i}\cdot\sum_{k=1}^N(e_i\otimes\cdots\otimes e_i)\circ(S^k,\cdots,S^k)\right\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} \\ &= \left\|\sum_{k=1}^N s_0\circ(S^k,\dots,S^k)\right\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} \leq N\cdot\|s_0\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)}. \end{aligned}$$

Here we identify tensors with multilinear forms. Proceeding as in Proposition 2.1 and noting that in this situation the operator Λ_t has norm one, we can see that $\|s_0\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)}\leq\|s\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)}=\|s\|_{\left(\bigotimes^n\ell_p^N;\alpha^*\right)}=1$. This completes the proof. □

Inmmediatelly from this and (1.1) we have

Corollary 3.2. *Let $\mathfrak{A}$ be a normed ideal of multilinear forms. Then $\lambda_n(\mathfrak{A},p)+\lambda_n(\mathfrak{A}^*;p')\geq 1$.*

The remaining results in this section can be seen as multilinear versions of some results in [14].

Proposition 3.3. *Let $\mathfrak{A}$ be a Banach ideal of n-linear forms. Then $d_n(\mathfrak{A}, p)$ is the infimum of $\lambda - \mu$ where $\lambda, \mu \geq 0$ are such that*

$$CN^{\mu} \leq \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \leq DN^{\lambda} \tag{3.1}$$

for all $N \in \mathbb{N}$, for some constants $C, D > 0$.

Proof. Take r, s such that $\ell_r \subset \ell_n(\mathfrak{A}, p) \subset \ell_s$. From the closed graph theorem, the inclusions $\ell_r \hookrightarrow \ell_n(\mathfrak{A}, p)$ and $\ell_n(\mathfrak{A}, p) \hookrightarrow \ell_s$ are continuous. Thus, there exist constants C and D such that, for all $N \in \mathbb{N}$,

$$CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} = \|(1, \ldots, 1, 0, \ldots)\|_{\ell_n(\mathfrak{A},p)} \leq DN^{\frac{1}{r}}.$$

Conversely, suppose $CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \leq DN^{\frac{1}{r}}$ for all $N \in \mathbb{N}$. It is enough to show that, for all $\varepsilon > 0$, we have $\ell_{r-\varepsilon} \subset \ell_n(\mathfrak{A}, p) \subset \ell_{s+\varepsilon}$.

The first inclusion follows from the equivalence between the definitions of limit order. For the second inclusion, let $\sigma \in \ell_n(\mathfrak{A}, p)$. We first consider the case in which $(\sigma(k))_k$ converges to 0. We can assume that $(|\sigma(k)|)_k$ is non-increasing. We can factor Φ_N as

$$\begin{array}{ccccccc} \ell_p^N & \times & \cdots & \times & \ell_p^N & \xrightarrow{\Phi_N} & \mathbb{C} \\ D_\alpha \downarrow & & & & D_\alpha \downarrow & \nearrow_{T_\sigma} & \\ \ell_p^N & \times & \cdots & \times & \ell_p^N & & \end{array}$$

where $\alpha = \sigma^{-1/n}$. Therefore,

$$CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \leq \|T_\sigma\|_{\mathfrak{A}(^n\ell_p^N)} \cdot \|D_\alpha\|^n \leq \|T_\sigma\|_{\mathfrak{A}(^n\ell_p)} \cdot |\sigma(N)|^{-1}$$

Consequently, $|\sigma_N| \leq \frac{\|T_\sigma\|_{\mathfrak{A}(^n\ell_p)}}{DN^{1/s}}$ for all $N \in \mathbb{N}$. This implies that $\sigma \in \ell_{s+\varepsilon}$.

Now we assume that $(\sigma(k))_k$ does not converge to 0. Let us take constants a and b and a subsequence of $(\sigma(k))_k$ such that $a \leq \sigma(k_i) \leq b$ for all i. We will see that in this case, both $\ell_n(\mathfrak{A}, p)$ and ℓ_s coincide with ℓ_∞.

Let $\beta \in \ell_\infty$. We define $S : \ell_p \to \ell_p$ by

$$S(e_i) = \left(\frac{\beta(i)}{\sigma(k_i)}\right)^{\frac{1}{n}} e_{k_i}$$

Since $T_\beta = T_\sigma \circ (S, \ldots, S)$, we have that $T_\beta \in \mathfrak{A}(^n\ell_p)$ and then $\beta \in \ell_n(\mathfrak{A}, p)$.

Now, since $\ell_n(\mathfrak{A}, p) = \ell_\infty$, we have $\Phi = T_{(1,1,\ldots)} \in \mathfrak{A}(^n\ell_p)$. From the inequality $CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \leq \|\Phi\|_{\mathfrak{A}(^n\ell_p)}$ for all N, we obtain $s = \infty$. $\square$

Proposition 3.4. *Let $\mathfrak{A}$ be a Banach ideal of n-linear forms and $\mathfrak{A}^*$ its adjoint ideal. Then*

$$\lambda_n(\mathfrak{A}; p) + \lambda_n(\mathfrak{A}^*; p') = 1 + d_n(\mathfrak{A}; p).$$

Proof.

$$\begin{aligned}\lambda_n(\mathfrak{A};p)+\lambda_n(\mathfrak{A}^*;p') &= \inf\{\lambda>0:\ \exists D>0 \text{ with } \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)}\le DN^{\lambda},\ \forall N\}\\ &\quad+\inf\{\nu>0:\ \exists C>0 \text{ with } \|\Phi_N\|_{\mathfrak{A}^*(^n\ell_{p'}^N)}\le CN^{\nu},\ \forall N\}\\ &= \inf\{\lambda>0:\ \exists D>0 \text{ with } \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)}\le DN^{\lambda},\ \forall N\}\\ &\quad+\inf\{\nu>0:\ \exists \widetilde{C}>0 \text{ with } \widetilde{C}N^{1-\nu}\le\|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)},\ \forall N\}\\ &= 1+\inf\{\lambda-\mu:\ \widetilde{C}N^{\mu}\le\|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)}\le DN^{\lambda}\}\\ &= 1+d_n(\mathfrak{A};p).\end{aligned}$$

□

Note that Corollary 3.2 can also be obtained as a consequence of Proposition 3.4. We also get the following:

Corollary 3.5. *Let $\mathfrak{A}$ be a Banach ideal. The following are equivalent:*

(a) $\lambda_n(\mathfrak{A};p)+\lambda_n(\mathfrak{A}^*;p')=1$.
(b) *There exists $r>0$ such that for all $\varepsilon>0$, $\ell_{r-\varepsilon}\subset\ell_n(\mathfrak{A},p)\subset\ell_{r+\varepsilon}$.*
(c) *There exists $\lambda\ge 0$ such that for all $\varepsilon>0$ and all $N\in\mathbb{N}$, $CN^{\lambda-\varepsilon}\le\|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)}\le DN^{\lambda+\varepsilon}$ for some constants $C,D>0$.*
(d) $\left(\frac{\log\|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)}}{\log N}\right)_{N\in\mathbb{N}}$ *converges as $N\to\infty$.*

Moreover, if these equivalences hold, then $\frac{1}{r}=\lambda=\lambda_n(\mathfrak{A},p)$

Remark 3.6. The definition of limit order implies that, for all $\varepsilon>0$,

$$\ell_{1/\lambda_n(\mathfrak{A},p)-\varepsilon}\subset\ell_n(\mathfrak{A},p).$$

Therefore, the equality $\lambda_n(\mathfrak{A};p)+\lambda_n(\mathfrak{A}^*;p')=1$ is equivalent to the inclusion $\ell_n(\mathfrak{A},p)\subset\ell_{1/\lambda_n(\mathfrak{A},p)+\varepsilon}$ for all $\varepsilon>0$.

4. Dominated multilinear forms

Absolutely summing and p-summing operators (see [5, Ch. 11] or [19, Ch. 17]) admit different generalizations to the multilinear setting, depending on the different properties that we want to preserve. One of these posible generalizations is done attending to the Pietsch's Domination Theorem [19, Sect. 17.3.2]. With this we obtain the r-dominated multilinear mappings. In [4] the limit order of r-dominated n-linear forms was computed. Our aim in this section is to relate the adjoint ideal of r-dominated forms with the ideal of r-integral forms (see definitions below) and use the results in [4] and in the previous sections to compute the limit order of the latter. Let us recall first some definitions.

A sequence in a Banach space is **weakly p-summable** if $(\gamma(x_n))_n \in \ell_p$ for all $\gamma \in X'$. The space of weakly p-summable sequences endowed with the norm

$$w_p((x_n)_n) = \sup_{\gamma \in B_{X'}} \left(\sum_n |\gamma(x_n)|^p \right)^{1/p}$$

is a Banach space.

A map $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is said to be **absolutely** $(s; r_1, \ldots, r_n)$**-summing** (where $\frac{1}{s} \leq \frac{1}{r_1} + \cdots + \frac{1}{r_n}$) [1, 15] if there exists $C > 0$ such that for any finite choice of elements $x_j^i \in X_j$, $j = 1, \ldots, n$, $i = 1, \ldots, m$ we have

$$\left(\sum_{i=1}^m \|T(x_1^i, \ldots, x_n^i)\|^s \right)^{1/s} \leq C \cdot w_{r_1}(x_1^i) \cdots w_{r_n}(x_n^i).$$

A map $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is said to be r**-dominated** [21, 17] if it is absolutely $(r/n; r, \ldots, r)$-summing; that is, there exists $C > 0$ such that for every $x_j^i \in X_j$, $j = 1, \ldots, n$, $i = 1, \ldots, m$,

$$\left(\sum_{i=1}^m \|T(x_1^i, \ldots, x_n^i)\|^{r/n} \right)^{n/r} \leq C \cdot w_r(x_1^i) \cdots w_r(x_n^i).$$

We denote by $\mathcal{D}_r({}^n X)$ the space of r-dominated n-linear forms on X.

First we show that r-dominated n-linear forms are dual to a tensor norm whenever $r \geq n$. Next, this duality will be used to study the adjoint ideal $\mathcal{D}_r^*$.

For $r \geq n$, we define in $\bigotimes_{i=1}^n X_i$,

$$\alpha_{r'}^n(s) = \inf \left\{ \ell_u(\lambda(i)) \cdot w_r(x_i^1) \cdots w_r(x_i^n) : \; s = \sum_{i=1}^N \lambda(i) \cdot x_i^1 \otimes \cdots \otimes x_i^n \right\}$$

where $\frac{1}{u} + \frac{n}{r} = 1$.

A straightforward application of [9, §1.2] gives

Proposition 4.1. *$\alpha_{r'}^n$ is a finitely generated tensor norm of order n.*

Now we can show the desired duality:

Proposition 4.2. *If $r \geq n$, $\mathcal{D}_r({}^n X) = (\bigotimes^n X; \alpha_{r'}^n)'$.*

Proof. Let $T \in \mathcal{D}_r({}^n X)$. For $s = \sum_{i=1}^N \lambda(i) \cdot x_i^1 \otimes \cdots \otimes x_i^n \in \bigotimes^n X$ we have

$$\begin{aligned} |T(s)| = \left| \sum_{i=1}^N \lambda(i) \cdot T(x_i^1, \ldots, x_i^n) \right| &\leq \ell_{\frac{r}{n}}(T(x_i^1, \ldots, x_i^n)) \cdot \ell_u(\lambda(i)) \\ &\leq \|T\|_{\mathcal{D}_r({}^n X)} \cdot w_r(x_i^1) \cdots w_r(x_i^n) \cdot \ell_u(\lambda(i)). \end{aligned}$$

Since this is valid for any representation of s, we obtain that $T \in (\bigotimes^n X; \alpha_{r'}^n)'$ and $\|T\|_{(\bigotimes^n X; \alpha_{r'}^n)'} \leq \|T\|_{\mathcal{D}_r({}^n X)}$.

Conversely, let $T \in (\bigotimes^n X; \alpha^n_{r'})'$. For any sequences $(x_i^1)_{i=1}^N, \dots, (x_i^n)_{i=1}^N$ in X, there exist scalars $\lambda(1), \dots, \lambda(N)$ with $\ell_u(\lambda(i)) = 1$ such that

$$\begin{aligned}
\ell_{\frac{r}{n}}\left(T(x_i^1, \dots, x_i^n)\right) &= \sum_{i=1}^N \lambda(i) \cdot T(x_i^1, \dots, x_i^n) \\
&= T\left(\sum_{i=1}^N \lambda(i) \cdot x_i^1 \otimes \dots \otimes x_i^n\right) \\
&\le \|T\|_{(\bigotimes^n X; \alpha^n_{r'})'} \cdot \alpha^n_{r'}\left(\sum_{i=1}^N \lambda(i) \cdot x_i^1 \otimes \dots \otimes x_i^n\right) \\
&\le \|T\|_{(\bigotimes^n X; \alpha^n_{r'})'} \cdot w_r(x_i^1) \cdots w_r(x_i^n) \cdot \ell_u(\lambda(i)).
\end{aligned}$$

Thus, $T \in \mathcal{D}_r({}^n X)$ and $\|T\|_{\mathcal{D}_r({}^n X)} \le \|T\|_{(\bigotimes^n X; \alpha^n_{r'})'}$. □

Inspired by [5, Chapters 17 and 18], we study $\mathcal{D}_r^*$, the adjoint ideal to the ideal of r-dominated multilinear mappings.

Note that, since $\mathcal{D}_r$ is a maximal ideal, we have $\mathcal{D}_r = (\mathcal{D}_r^*)^*$. Therefore, for $M_1, \dots, M_n \in \mathrm{FIN}$,

$$\mathcal{D}_r^*(M_1, \dots, M_n) \overset{1}{=} \left(\bigotimes_{i=1}^n M_i; \alpha^n_{r'}\right).$$

Let $T \in \mathcal{D}_r^*(M_1, \dots, M_n)$ and fix $\varepsilon > 0$. T admits a representation in the following way:

$$T(x_1, \dots, x_n) = \sum_{k=1}^N \lambda(k) \cdot \gamma_k^1(x_1) \cdots \gamma_k^n(x_n), \tag{4.1}$$

where $(\lambda(k))_k \subset \mathbb{C}$, $(\gamma_k^i)_k \subset M_i'$ satisfy

$$\ell_u(\lambda(k)) \cdot w_r(\gamma_k^1) \cdots w_r(\gamma_k^n) = \|T\|_{\mathcal{D}_r^*(M_1, \dots, M_n)} \cdot (1 + \varepsilon)$$

with $\frac{1}{u} + \frac{n}{r} = 1$.

Then we can factor T as:

$$\begin{array}{cccccc}
M_1 & \times & \cdots & \times & M_n & \xrightarrow{T} \quad \mathbb{C} \\
R^1 \downarrow & & & & R^n \downarrow & \nearrow_{T_\lambda} \\
\ell_r^N & \times & \cdots & \times & \ell_r^N &
\end{array} \tag{4.2}$$

where $R^i(x) = (\gamma_k^i(x))_{k=1}^N$. Since $\|T_\lambda\| = \ell_u(\lambda(k))$ and $\|R^i\| = w_r(\gamma_k^i)$, we have

$$\|T_\lambda\| \cdot \|R^1\| \cdots \|R^n\| = \|T\|_{\mathcal{D}_r^*({}^n M)} \cdot (1 + \varepsilon).$$

Following these steps backwards, we obtain for each factorization of T as in (4.2), a representation of T as in equation (4.1). Therefore, we have

$$\|T\|_{\mathcal{D}_r^*({}^n M)} = \inf\left\{\|T_\lambda\| \cdot \|R^1\| \cdots \|R^n\| : T \text{ factors as in } (4.2)\right\}.$$

In [11, Section 4.3], the ideal of r-integral polynomials is defined. We define in an analogous way the ideal $\mathcal{I}_r$ of r-integral multilinear forms. If $r \geq n$, we say that $T \in \mathcal{L}(X_1, \dots, X_n)$ is **r-integral** if there exist a finite measure space (Ω, μ) and operators $S_i : X_i \to L_r(\mu)$ such that $T = Q^n_{\mu,r} \circ (S_1, \dots, S_n)$, where $Q^n_{\mu,r} \in \mathcal{L}(^n L_r(\mu))$ is the integrating n-linear form $Q^n_{\mu,r}(f_1, \dots, f_n) = \int_\Omega f_1 \cdots f_n \, d\mu$:

$$\begin{array}{ccccccc} X_1 & \times & \cdots & \times & X_n & \xrightarrow{T} & \mathbb{C} \\ S_1 \downarrow & & & & S_n \downarrow & \nearrow_{Q^n_{\mu,r}} & \\ L_r(\mu) & \times & \cdots & \times & L_r(\mu) & & \end{array}$$

$\mathcal{I}_r$ is a Banach ideal with the r-integral norm:

$$\|T\|_{\mathcal{I}_r(X_1,\dots,X_n)} = \inf \left\{ \|S_1\| \cdots \|S_n\| \cdot \|Q^n_{\mu,r}\| : T = Q^n_{\mu,r} \circ (S_1, \dots, S_n) \right\}.$$

Lemma 4.3. *The n-linear form $Q^n_{\mu,r}$ belongs to $\mathcal{D}^*_r$ and $\|Q^n_{\mu,r}\|_{\mathcal{D}^*_r(^n L_r(\mu))} = \|Q^n_{\mu,r}\| = \mu(\Omega)^{1/u}$, where $\frac{1}{u} + \frac{n}{r} = 1$.*

Proof. We have to show that $Q^n_{\mu,r}$ is a continuous linear form on $(\bigotimes^n L_r(\mu); (\alpha^n_{r'})^*)$ with norm $\mu(\Omega)^{1/u}$. This is shown in [5, Proposition 18.2] for bilinear forms. Their proof is also valid for $n \geq 3$. □

Corollary 4.4. *If $r \geq n$, $\mathcal{I}_r \subset \mathcal{D}^*_r$ and $\|T\|_{\mathcal{D}^*_r} \leq \|T\|_{\mathcal{I}_r}$ for each r-integral n-linear form T.*

Proof. If T is r-integral, it can be written as $T = Q^n_{\mu,r} \circ (S_1, \dots, S_n)$. Lemma 4.3 implies that $T \in \mathcal{D}^*_r$ and

$$\|T\|_{\mathcal{D}^*_r} \leq \|Q^n_{\mu,r}\|_{\mathcal{D}^*_r} \cdot \|S_1\| \cdots \|S_n\| = \|Q^n_{\mu,r}\| \cdot \|S_1\| \cdots \|S_n\|.$$

Taking the infimum over all representations of T we obtain the desired inequality. □

Theorem 4.5. *For $r \geq n$, we have $\mathcal{D}^*_r \overset{1}{=} \mathcal{I}^{max}_r$.*

Proof. It is enough to show, for $M_1, \dots, M_n \in \mathrm{FIN}$, that $\|T\|_{\mathcal{I}_r(M_1,\dots,M_n)} = \|T\|_{\mathcal{D}^*_r(M_1,\dots,M_n)}$. One inequality is given in Corollary 4.4. For the other one, we factor $T \in \mathcal{D}^*_r(M_1, \dots, M_n)$ as

$$\begin{array}{ccccccc} M_1 & \times & \cdots & \times & M_n & \xrightarrow{T} & \mathbb{C} \\ R^1 \downarrow & & & & R^n \downarrow & \nearrow_{T_\lambda} & \\ \ell^N_r & \times & \cdots & \times & \ell^N_r & & \end{array}$$

Let us show now that T_λ can be factored as

$$\begin{array}{ccccccc} \ell^N_r & \times & \cdots & \times & \ell^N_r & \xrightarrow{T_\lambda} & \mathbb{C} \\ J \downarrow & & & & J \downarrow & \nearrow_{Q^n_{\mu,r}} & \\ L_r(\mu) & \times & \cdots & \times & L_r(\mu) & & \end{array}$$

with $\|Q^n_{\mu,r}\| \cdot \|J\|^n \leq \|T_\lambda\|$. Since T_λ factors through $T_{|\lambda|}$, we can assume that $\lambda(k) \geq 0$ for each k. Let (Ω, μ) a measure space such that Ω can be splitted as a disjoint union of subsets $A_1, \dots, A_n$ with $\mu(A_k) = \lambda(k)^u$. Let $J : \ell_r^N \to L_r(\mu)$ be defined as

$$J(x) = \sum_{k=1}^{N} x(k) \cdot \lambda(k)^{\frac{1-u}{n}} \cdot \chi_{A_k}.$$

Simple computations show that $\|J\| = 1$ and $Q_{\mu,r}^n \circ (J, \dots, J) = T_\lambda$. Since $\|Q_{\mu,r}^n\| = \mu(\Omega)^{1/u} = \ell_u(\lambda(k))$ then $\|T_\lambda\| = \ell_u(\lambda(k))$. This completes the proof. □

In [19, Section 22.4], it is shown that absolutely r-summing operators satisfy $\lambda(\Pi_r, p, q) + \lambda(\Pi_r^*, q, p) = 1$. By [14, Corollary 1] (which is analogous to Corollary 3.5), this means that $\ell(\Pi_r, p, q) \subset \ell_{1/\lambda(\Pi_r,p,q)+\varepsilon}$ for all $\varepsilon > 0$. Now, by Proposition 1.3, $\ell_n(\mathcal{D}_r, p) \subset \ell_{1/\lambda_n(\mathcal{D}_r,p)+\varepsilon}$ for all $\varepsilon > 0$. Consequently, by Corollary 3.5 and the remark following it, we have that $\lambda_n(\mathcal{D}_r; p) + \lambda_n(\mathcal{D}_r^*; p') = 1$. So we have, for $r \geq n$,

$$\lambda_n(\mathcal{I}_r; p) = \lambda_n(\mathcal{D}_r^*; p) = 1 - \lambda_n(\mathcal{D}_r; p'). \tag{4.3}$$

We can use now the results of this sections to obtain some properties of the ideal of r-integral n-linear forms. For $n = 2$, the ideal of r-integral bilinear forms is isomorphic to its maximal hull [11, 4.4]. Then, by Theorem 4.5, it is isomorphic to the adjoint of the ideal of r-dominated bilinear forms. By [5] (see also [4]), r-dominated and 2-dominated bilinear forms coincide for all $r \geq 2$. Thus, the same holds for r-integral bilinear forms: $\mathcal{I}_r(^2X) = \mathcal{I}_2(^2X)$ for all Banach space X and all $r \geq 2$. This result is not longer true for $n \geq 3$. In fact, from [4, Proposition 2.6] and equality (4.3) we have:

Corollary 4.6. *Let $n \geq 3$. Given $r \geq n$, there exists p such that, for any $s > r$, there are diagonal s-integral n-linear forms on ℓ_p which are not r-integral.*

It is not known if $\mathcal{I}_r$ is a maximal ideal. This question is stated in [11, 4.4] as an open problem.

5. Multiple 1-summing forms

The r-dominated multilinear mappings generalize the absolutely r-summing linear mappings by preserving the Pietsch's Domination Theorem. This happened to be too restrictive a class, in the sense that many other interesting properties of summing operators were not preserved by r-dominated multilinear mappings. This led to the multiple summing multilinear operators. These have been introduced independently by M. Matos [16] and F. Bombal, D. Pérez-García and I. Villanueva [3]. A multilinear operator $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is **multiple r-summing** ($T \in \Pi_r(X_1, \dots, X_n; Y)$) if there exists $C > 0$ such that for every choice of finite sequences $(x_j^{i_j}) \subseteq X_j$ the following holds

$$\left(\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \|T(x_1^{i_1}, \dots, x_n^{i_n})\|^r \right)^{\frac{1}{r}} \leq C \cdot w_r((x_1^{i_1})_{i_1=1}^{m_1}) \cdots w_r((x_n^{i_n})_{i_n=1}^{m_n}).$$

The least of such constants C is called the **multiple p-summing norm** and denoted $\|T\|_{\Pi_r(X_1,\dots,X_n;Y)}$.

A. Defant and D. Pérez-García studied multiple 1-summing mappings in [6] and showed that its associated tensor norm preserves unconditionality. Some other properties of the norm were also proved and applied in [6, Section 6] to compute the limit order for bilinear multiple 1-summing operators. Their result can be written with our notation as

$$\lambda_2(\Pi_1;p)=\begin{cases}\dfrac{1}{p'} & \text{if } 2\le p\\ \dfrac{3}{2}-\dfrac{2}{p} & \text{if } \frac{4}{3}\le p<2\\ 0 & \text{if } 1\le p<\frac{4}{3}\end{cases}$$

Our aim is now to compute the limit order of Π_1 for higher n. In fact, what we do is to compute the Π_1-norm of $\Phi_N:\ell_p^N\times\cdots\times\ell_p^N\to\mathbb{C}$.
Let us begin by considering the case $p\le 2$. We follow the steps of [6, Section 6]. First of all, if $T\in\mathcal{L}(^n\ell_2^N)$ then

$$\|T\|_{\Pi_1(^n\ell_2^N)}\asymp\left(\sum_{i_1,\dots,i_n=1}^{N}|T(e_{i_1},\dots,e_{i_n})|^2\right)^{1/2} \tag{5.1}$$

(see [18, Theorem 4.2], also [6, Theorem 5.1]). On the other hand, by [6, Theorem 5.2], if X has 1-unconditional basis, has cotype 2 and $\dim X=N$, then for $S\in\mathcal{L}(^nX)$ we have

$$\|S\|_{\Pi_1(^nX)}\asymp\sup_{\sigma_j}\|S\circ(D_{\sigma_1},\dots,D_{\sigma_n})\|_{\Pi_1(^n\ell_2^N)}, \tag{5.2}$$

where $D_{\sigma_j}:\ell_2^N\to X$ are norm-one diagonal operators.

Applying (5.1) and (5.2) to Φ_N we obtain

$$\begin{aligned}\|\Phi_N\|_{\Pi_1(^n\ell_p^N)} &\asymp \sup_{\sigma_j}\left(\sum_{k_1,\dots,k_n=1}^{N}|\sigma_1(k_1)\cdots\sigma_n(k_n)\Phi_N(e_{k_1},\dots,e_{k_n})|^2\right)^{1/2}\\ &= \sup_{\sigma_j}\left(\sum_{k=1}^{N}|\sigma_1(k)\cdots\sigma_n(k)|^2\right)^{1/2}\end{aligned}$$

where the supremum is taken over all σ_j such that $D_{\sigma_j}:\ell_2^N\to\ell_p^N$, $j=1,\dots,n$ are norm-one operators. Note that $\|D_{\sigma_j}\|=\|\sigma_j\|_{\ell_r^N}$, where $\frac{1}{r}=\frac{1}{2}-\frac{1}{p'}$. If $r\ge n$, we have

$$\begin{aligned}\|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} &\asymp \sup\left\{\left(\sum_{k=1}^N |\sigma_1(k)\cdots\sigma_n(k)|^2\right)^{1/2} \ :\ \sigma_j \in B_{\ell_r^N}\right\} \\ &= \sup\left\{\left(\sum_{k=1}^N |\sigma(k)|^2\right)^{1/2} \ :\ \sigma \in B_{\ell_{r/n}^N}\right\} \\ &= \|id : \ell_{r/n}^N \to \ell_2^N\| = \begin{cases} 1 & \text{if } 1 \le \frac{r}{n} \le 2 \\ N^{\frac{1}{2}-\frac{n}{r}} & \text{if } 2 < \frac{r}{n} \end{cases}\end{aligned}$$

If $r < n$, with the same procedure we obtain $\|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} \preceq 1$. Since the reverse inequality is always true, we also have $\|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} \asymp 1$ for this case. This gives:

$$\|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} \asymp \begin{cases} 1 & \text{if } 1 \le p \le \frac{2n}{n+1} \\ N^{\frac{n+1}{2}-\frac{n}{p}} & \text{if } \frac{2n}{n+1} \le p \le 2. \end{cases}$$

We now consider $p > 2$. Let us see that in this case

$$\max\{\sqrt{N}, N^{1-\frac{n-1}{p}}(\log N)^{1/p'}\} \prec \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} \prec N^{\frac{n+1}{2}-\frac{n}{p}}.$$

To get the lower bound, first note that we can factor

$$\begin{array}{ccccccc} \ell_2^N & \times & \cdots & \times & \ell_2^N & \xrightarrow{\Phi_N} & \mathbb{C} \\ i\downarrow & & & & i\downarrow & \nearrow_{\Phi_N} & \\ \ell_p^N & \times & \cdots & \times & \ell_p^N & & \end{array}$$

This factorization together with (5.1) give $\sqrt{N} = \|\Phi_N\|_{\Pi_1({}^n\ell_2^N)} \le \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)}$.

We complete the lower bound by induction on n. By [6, Lemma 3.4] we have the isometry $\Pi_1({}^n\ell_p^N;\mathbb{C}) \overset{1}{=} \Pi_1({}^{n-1}\ell_p^N;\ell_{p'}^N)$, denoted by $T \leftrightarrow \widetilde{T}$. If $n = 2$, then $\widetilde{\Phi}_N = id : \ell_p^N \to \ell_{p'}^N$. Therefore, $\|\Phi_N\|_{\Pi_1({}^2\ell_p^N)} = \|id\|_{\Pi_1(\ell_p^N;\ell_{p'}^N)} \asymp (N\log N)^{1/p'}$, by [19, 22.4.11].

Let us now consider $\Sigma_N : \ell_{p'}^N \to \mathbb{C}$ given by $z \rightsquigarrow \sum_{k=1}^N z(k)$ and $\Psi_N = \Sigma_N \circ \widetilde{\Phi}_N : \overbrace{\ell_p^N \times \cdots \times \ell_p^N}^{(n-1)\text{ times}} \to \ell_{p'}^N \to \mathbb{C}$. By the induction hypothesis

$$N^{1-\frac{n-2}{p}}(\log N)^{1/p'} \prec \|\Psi_N\|_{\Pi_1({}^{n-1}\ell_p^N)} \le \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)}\|\Sigma_N\|.$$

Now, since $\|\Sigma_N\| = N^{\frac{1}{p}}$, we have the desired lower bound.
To get the upper bound, let us factor Φ_N in the following way

$$\begin{array}{ccccccc} \ell_p^N & \times & \cdots & \times & \ell_p^N & \xrightarrow{\Phi_N} & \mathbb{C} \\ \downarrow & & & & \downarrow & \nearrow & \\ \ell_2^N & \times & \cdots & \times & \ell_2^N & & \end{array}$$

Hence

$$\begin{aligned}\|\Phi_N\|_{\Pi_1(^n\ell_p^N)} &\leq \|id : \ell_p^N \to \ell_2^N\|^n \|\Phi_N\|_{\Pi_1(^n\ell_2^N)} \\ &\prec (N^{\frac{1}{2}-\frac{1}{p}})^n \sqrt{N} = N^{\frac{n+1}{2}-\frac{n}{p}}.\end{aligned}$$

This altogether gives the following situation

$$\begin{array}{rll} \|\Phi_N\|_{\Pi_1(^n\ell_p^N)} & \asymp 1 & \text{if } 1 \leq p \leq \frac{2n}{n+1} \\ \|\Phi_N\|_{\Pi_1(^n\ell_p^N)} & \asymp N^{\frac{n+1}{2}-\frac{n}{p}} & \text{if } \frac{2n}{n+1} \leq p \leq 2 \\ \max\{\sqrt{N}, N^{1-\frac{n-1}{p}}(\log N)^{1/p'}\} \prec \|\Phi_N\|_{\Pi_1(^n\ell_p^N)} & \prec N^{\frac{n+1}{2}-\frac{n}{p}} & \text{if } 2 \leq p. \end{array}$$

We can reformulate this results in terms of limit orders and defects:
For $p \leq 2$:

$$\lambda_n(\Pi_1; p) = \begin{cases} 0 & \text{if } 1 \leq p \leq \frac{2n}{n+1} \\ \dfrac{n+1}{2} - \dfrac{n}{p} & \text{if } \frac{2n}{n+1} \leq p \leq 2 \end{cases}$$

and $d_n(\Pi_1, p) = 0$.
For $p > 2$:

$$\max\left\{\frac{1}{2}, 1 - \frac{n-1}{p}\right\} \leq \lambda_n(\Pi_1; p) \leq \min\left\{\frac{n+1}{2} - \frac{n}{p}, 1\right\}$$

and $d_n(\Pi_1, p) \leq \frac{n-1}{2} - \frac{1}{p}$.

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References

[1] R. Alencar, M. Matos, *Some classes of multilinear mappings between banach spaces.* Pub. Dep. An. Mat. Univ. Complut. Madrid, 12, 1989.

[2] R. M. Aron, J. Globevnik, *Analytic functions on* c_0, Rev. Mat. Univ. Complut. Madrid **2**(suppl.) (1989), 27–33.

[3] F. Bombal, D. Pérez-García, I. Villanueva, *Multilinear extensions of Grothendieck's theorem.* Q. J. Math. **55**(4) (2004), 441–450.

[4] D. Carando, V. Dimant, P. Sevilla-Peris, *Limit orders and multilinear forms on* ℓ_p *spaces.* Publ. Res. Inst. Math. Sci. **42**(2) (2006), 507–522.

[5] A. Defant, K. Floret, *Tensor norms and operator ideals.* North Holland, Amsterdam (1993).

[6] A. Defant, D. Pérez-García, *A tensor norm that preserves unconditionality for $\mathcal{L}_p$ spaces.* Trans. Am. Math. Soc. (to appear).

[7] V. Dimant, R. Gonzalo, *Block diagonal polynomials.* Trans. Am. Math. Soc. **353**(2) (2001), 733–747.

[8] S. Dineen, *Complex analysis on infinite dimensional spaces.* Springer, London (1999).

[9] K. Floret, The extension theorem for norms on symmetric tensor products of normed spaces. In *Recent progress in functional analysis (Valencia, 2000)*, volume 189 of *North-Holland Math. Stud.*, pp. 225–237. North-Holland, Amsterdam (2001).

[10] K. Floret, *Minimal ideals of n-homogeneous polynomials on Banach spaces.* Results Math. **39**(3–4) (2001), 201–217.

[11] K. Floret, On ideals of n-homogeneous polynomials on Banach spaces. In *Topological algebras with applications to differential geometry and mathematical physics (Athens, 1999)*, pp. 19–38. Univ. Athens, Athens (2002).

[12] K. Floret, D. García, *On ideals of polynomials and multilinear mappings between Banach spaces.* Arch. Math. (Basel) **81**(3) (2003), 300–308.

[13] K. Floret, S. Hunfeld, *Ultrastability of ideals of homogeneous polynomials and multilinear mappings on Banach spaces.* Proc. Am. Math. Soc. **130**(5) (2002), 1425–1435.

[14] H. König, *Diagonal and convolution operators as elements of operator ideals.* Math. Ann. **218**(2) (1975), 97–106.

[15] M.C. Matos, *Absolutely summing holomorphic mappings.* An. Acad. Brasil. Ciênc. **68**(1) (1996), 1–13.

[16] M.C. Matos, *Fully absolutely summing and Hilbert-Schmidt multilinear mappings.* Collect. Math. **54**(2) (2003), 111–136.

[17] Y. Meléndez, A. Tonge, *Polynomials and the Piestsch domination theorem.* Math. Proc. Roy. Ir. Acad. **99**(2) (1999), 195–212.

[18] D. Pérez-García, *The inclusion theorem for multiple summing operators.* Studia Math. **165**(3) (2004), 275–290.

[19] A. Pietsch, *Operator ideals.* North Holland, Amsterdam (1980).

[20] A. Pietsch, Ideals of multilinear functionals (designs of a theory). In Teubner (ed.), *Proceedings of the second international conference on operator algebras, ideals, and their applications in theoretical physics (Leipzig 1983)*, volume Teubner-Texte Math. 67, pp. 185–199, Leipzig (1984).

[21] B. Schneider, *On absolutely p-summing and related multilinear mappings.* Wissenschaftliche Zeitschrift der Brandemburger Landeshochschule **35** (1991), 105–117.

Daniel Carando
Departamento de Matemática, FCEyN
Universidad de Buenos Aires
Pabellón I – Ciudad Universitaria 1428
Buenos Aires, Argentina
e-mail: dcarando@dm.uba.ar

Verónica Dimant
Departamento de Matemática
Universidad de San Andrés
Vito Dumas 284 (B1644BID) Victoria
Buenos Aires, Argentina
e-mail: vero@udesa.edu.ar

Pablo Sevilla-Peris
Departamento de Matemática Aplicada and IMPA
Universidad Politécnica de Valencia
ETSMRE, Av. Blasco Ibáñez 21 46010
Valencia, Spain
e-mail: Pablo.Sevilla@uv.es

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