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LIFTINGS OF JORDAN AND SUPER JORDAN PLANES

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Abstract We classify pointed Hopf algebras with finite Gelfand–Kirillov dimension whose infinitesimal braiding has dimension 2 but is not of diagonal type, or equivalently is a block. These Hopf algebras are new and turn out to be liftings of either a Jordan or a super Jordan plane over a nilpotent-by-finite group.

Keywords: Hopf algebras; Gelfand–Kirillov dimension; Jordan plane

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1. Introduction

Let $\Bbbk = \overline{\Bbbk}$ be a field, $\operatorname{char}\Bbbk = 0$. Let H be a pointed Hopf algebra, $G = G(H)$, $\operatorname{gr} H$ the graded Hopf algebra associated with its coradical filtration, $R = \oplus_{n\geqslant 0} R^n$ the graded Hopf algebra in the category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of Yetter–Drinfeld modules such that $\operatorname{gr} H \simeq R\#\Bbbk G$ and $V = R^1$ the infinitesimal braiding of H. The classification of Hopf algebras with finite Gelfand–Kirillov dimension (GKdim for short) has attracted considerable interest recently (see [**9**]). Hopf algebras with trivial coradical and finite GKdim are quantum deformations of algebraic unipotent groups [**11**, Theorem 4.2]. Also, there are several results in low GKdim; see [**10**, **13**, **18**] and references therein. Further, the classification is known assuming that H is a domain, G is abelian and V is of diagonal type [**1**, **6**]. Here, we contribute to this question.

Let $\ell \in \mathbb{N}_{\geqslant 2}$ and $\mathbb{I}_\ell = \{1, 2, \dots, \ell\}$. Let $\epsilon \in \Bbbk^\times$. Let $\mathcal{V}(\epsilon, \ell)$ be the braided vector space with basis $(x_i)_{i\in\mathbb{I}_\ell}$ and braiding $c \in \operatorname{Aut}(V \otimes V)$ such that

$$c(x_i \otimes x_1) = \epsilon x_1 \otimes x_i, \quad c(x_i \otimes x_j) = (\epsilon x_j + x_{j-1}) \otimes x_i, \quad i, j \in \mathbb{I}_\ell. \tag{1.1}$$

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We say that a braided vector space is a *block* if it is isomorphic to $\mathcal{V}(\epsilon, \ell)$ for some $\epsilon \in \Bbbk^{\times}$, $\ell \in \mathbb{N}_{\geqslant 2}$.

Theorem 1.1 ([3, Theorem 1.2]). *The* GKdim *of the Nichols algebra* $\mathcal{B}(\mathcal{V}(\epsilon, \ell))$ *is finite if and only if* $\ell = 2$ *and* $\epsilon^2 = 1$.

Here is our main result.

Theorem 1.2. *Let* H *be a pointed Hopf algebra,* $G = G(H)$ *and* V *its infinitesimal braiding. Then the following are equivalent:*

(1) $\operatorname{GKdim} H < \infty$ *and* V *is a block.*

(2) $\operatorname{GKdim} H < \infty$, $\dim V = 2$ *and* V *is not of diagonal type.*

(3) G *is nilpotent-by-finite and there exists a Jordanian or super Jordanian YD-triple* $\mathcal{D} = (g, \chi, \eta)$ *and* $\lambda \in \Bbbk$, $\lambda = 0$ *when* $\chi^2 \neq \varepsilon$, *such that* $V = \mathcal{V}_g(\chi, \eta)$ *and* $H \simeq \mathfrak{U}(\mathcal{D}, \lambda)$, *cf.* §§ *4.1 and 4.2.*

We refer to Subsection 2.3 for the definition of Yetter-Drinfeld triple, YD-triple for short.

Proof. (1)⇒(2): by Theorem 1.1, $V \simeq \mathcal{V}(\epsilon, 2)$ with $\epsilon^2 = 1$, thus $\dim V = 2$ and V is not of diagonal type. (2)⇒(3): by Gromov's theorem, G is nilpotent-by-finite. By Lemma 2.3, V is a block, hence Propositions 4.2 and 4.4 apply; these Propositions also provide (1)⇐(3). □

The isomorphism classes of the Hopf algebras $\mathfrak{U}(\mathcal{D}, \lambda)$ are also determined in Propositions 4.2 and 4.4.

The paper is organized as follows. In § 2, we recall the definitions of the Nichols algebras called the Jordan and super Jordan planes. We then discuss indecomposable Yetter–Drinfeld modules of dimension 2 over groups. Section 3 is dedicated to a discussion of the problem of generation in degree 1, which is equivalent to the study of post-Nichols algebras with finite GKdim. We show how to reduce (in general) this problem to the study of pre-Nichols algebras with finite GKdim (see the relevant definitions below) and deduce from results in [3, § 4] that the only post-Nichols algebra of the Jordan, or super Jordan, plane with finite GKdim is the Nichols algebra itself. Finally, in § 4, we describe all possible liftings of the Jordan plane in Proposition 4.2, and those of the super Jordan plane in Proposition 4.4.

1.1. Notation

We refer to [5] for unexplained terminology and notation. If G is a group, then $\widehat{G}$ denotes its group of characters.

2. Yetter–Drinfeld modules of dimension 2

2.1. The Jordan and super Jordan planes

We assume from now on that $\epsilon^2 = 1$. Keep the notation above and set $x_{21} = \operatorname{ad}_c x_2\, x_1 = x_2x_1 - \epsilon x_1x_2$.

The Nichols algebra $\mathcal{B}(\mathcal{V}(1,2))$ is a well-known quadratic algebra, the so-called Jordan plane, related to the quantum Jordan $SL(2)$; it also appears in the classification of AS-regular graded algebras of global dimension 2 [7].

In turn, we call $\mathcal{B}(\mathcal{V}(-1,2))$ the *super Jordan plane*.

Proposition 2.1 ([3, Propositions 3.4 and 3.5]). *The algebras $\mathcal{B}(\mathcal{V}(\epsilon,2))$ have* GKdim *2 and are presented by generators x_1 and x_2 with defining relations*

$$x_2x_1 - x_1x_2 + \tfrac{1}{2}x_1^2, \quad \text{if } \epsilon = 1; \tag{2.1}$$

$$x_2x_{21} - x_{21}x_2 - x_1x_{21}, \quad x_1^2, \quad \text{if } \epsilon = -1. \tag{2.2}$$

Further, $\{x_1^a x_2^b : a, b \in \mathbb{N}_0\}$, respectively $\{x_1^a x_{21}^b x_2^c : a \in \{0,1\}, b, c \in \mathbb{N}_0\}$, is a basis of $\mathcal{B}(\mathcal{V}(1,2))$, respectively $\mathcal{B}(\mathcal{V}(-1,2))$.

2.2. Indecomposable modules over abelian groups

Let Γ be an abelian group. Let $g \in \Gamma$, $\chi \in \widehat{\Gamma}$ and $\eta : \Gamma \to \Bbbk$ a (χ,χ)-derivation, i.e.

$$\eta(ht) = \chi(h)\eta(t) + \eta(h)\chi(t), \quad h, t \in \Gamma.$$

Let $\mathcal{V}_g(\chi,\eta) \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ be a vector space of dimension 2, homogeneous of degree g and with action of Γ given in a basis $(x_i)_{i\in\mathbb{I}_2}$ by

$$h \cdot x_1 = \chi(h)x_1, \quad h \cdot x_2 = \chi(h)x_2 + \eta(h)x_1, \tag{2.3}$$

for all $h \in \Gamma$. Then $\mathcal{V}_g(\chi,\eta)$ is indecomposable in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} \iff \eta \neq 0$. As a braided vector space, $\mathcal{V}_g(\chi,\eta)$ is either of diagonal type, when $\eta(g) = 0$, or else isomorphic to $\mathcal{V}(\epsilon,2)$, $\epsilon = \chi(g)$ (note that indecomposability as Yetter–Drinfeld module is not the same as indecomposability as braided vector space).

Lemma 2.2. *Let $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, $\dim V = 2$. Then either V is of diagonal type or else $V \simeq \mathcal{V}_g(\chi,\eta)$ for unique g, χ and η with $\eta(g) = 1$.*

Proof. Assume that V is not of diagonal type; then V is indecomposable. Since $\Bbbk\Gamma$ is cosemisimple, there exists $g \in \Gamma$ such that V is homogeneous of degree g. Moreover, $\Bbbk = \overline{\Bbbk}$ implies that V is not simple. Hence there exist $\chi_1, \chi_2 \in \widehat{\Gamma}$ such that $\operatorname{soc} V \simeq \Bbbk_g^{\chi_1}$ and $V/\operatorname{soc} V \simeq \Bbbk_g^{\chi_2}$. Pick $x_1 \in \operatorname{soc} V - 0$ and $x_2 \in V_{g_2} - \operatorname{soc} V$; then $h \cdot x_2 = \chi_2(h)x_2 + \eta(h)x_1$ for all $h \in \Gamma$, where η is a (χ_1,χ_2)-derivation. Since V is not of diagonal type, $\chi_1(g) = \chi_2(g)$ and $\eta(g) \neq 0$. Now

$$\chi_1(h)\eta(g) + \eta(h)\chi_2(g) = \eta(hg) = \chi_1(g)\eta(h) + \eta(g)\chi_2(h) \Rightarrow \chi_1(h) = \chi_2(h)$$

for all $h \in \Gamma$. Finally, up to changing x_1, we may assume that $\eta(g) = 1$. □

2.3. Indecomposable modules over Hopf algebras

Let K be a Hopf algebra with bijective antipode. A *YD-pair* [2] for K is a pair $(g,\chi) \in G(K) \times \operatorname{Hom}_{\text{alg}}(K, \Bbbk)$ such that

$$\chi(h)\, g = \chi(h_{(2)}) h_{(1)}\, g\, \mathcal{S}(h_{(3)}), \quad h \in K. \tag{2.4}$$

If (g,χ) is a YD-pair, then the one-dimensional vector space $\Bbbk_g^\chi$, with action and coaction given by χ and g respectively, is in ${}^K_K\mathcal{YD}$. Conversely, any $V \in {}^K_K\mathcal{YD}$ with $\dim V = 1$ is like this, for unique g and χ. If (g,χ) is a YD-pair, then $g \in Z(G(K))$.

If $\chi_1, \chi_2 \in \operatorname{Hom}_{\text{alg}}(K, \Bbbk)$, then the space of (χ_1,χ_2)-derivations is

$$\operatorname{Der}_{\chi_1,\chi_2}(K, \Bbbk) = \{\eta \in K^* : \eta(ht) = \chi_1(h)\eta(t) + \chi_2(t)\eta(h)\, \forall h, t \in K\}.$$

A *YD-triple* for K is a collection (g,χ,η) where (g,χ), is a YD-pair for K, cf. (2.4), $\eta \in \operatorname{Der}_{\chi,\chi}(K,\Bbbk)$, $\eta(g) = 1$ and

$$\eta(h) g_1 = \eta(h_{(2)}) h_{(1)} g_2 \mathcal{S}(h_{(3)}), \quad h \in K. \tag{2.5}$$

If $K = \Bbbk G$ is a group algebra, then we can think of the collection (g,χ,η) as in G, $\widehat{G}$, $\operatorname{Der}_{\chi,\chi}(G,\Bbbk)$.

Let (g,χ,η) be a YD-triple for K. Let $\mathcal{V}_g(\chi,\eta)$ be a vector space with a basis $(x_i)_{i\in\mathbb{I}_2}$, where action and coaction are given by

$$h \cdot x_1 = \chi(h) x_1, \qquad h \cdot x_2 = \chi(h) x_2 + \eta(h) x_1, \qquad \delta(x_i) = g \otimes x_i,$$

$h \in K$, $i \in \mathbb{I}_2$. Then $\mathcal{V}_g(\chi,\eta) \in {}^K_K\mathcal{YD}$, the compatibility being granted by (2.4) and (2.5). Since $\eta(g) \neq 0$, then $\mathcal{V}_g(\chi,\eta)$ is indecomposable in ${}^K_K\mathcal{YD}$.

Lemma 2.3. *Let G be a group. Let $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, $\dim V = 2$. Then either V is of diagonal type as a braided vector space or else $V \simeq \mathcal{V}_g(\chi,\eta)$ for a unique YD-triple (g,χ,η).*

Proof. Assume first that $V = V_{g_1} + V_{g_2}$ as $\Bbbk G$-comodule, with $g_1 \neq g_2 \in G$. Now $g_2 \cdot V_{g_2} = V_{g_2}$, hence $g_2 \cdot V_{g_1} = V_{g_1}$ and similarly $g_1 \cdot V_{g_2} = V_{g_2}$. Thus V is of diagonal type, a contradiction. Thus, we may assume that $V = V_g$ for some $g \in G$, and Lemma 2.2 applies with $\Gamma = \langle g \rangle$, so that $V \simeq \mathcal{V}_g(\widetilde{\chi},\widetilde{\eta})$ for some $\widetilde{\chi} \in \widehat{\Gamma}$ and $\widetilde{\eta}$ a $(\widetilde{\chi},\widetilde{\chi})$-derivation. Then there is a basis $(x_i)_{i\in\mathbb{I}_2}$ where g acts by $A = \left(\begin{smallmatrix} \epsilon & 1 \\ 0 & \epsilon \end{smallmatrix}\right)$. However, $g \in Z(G)$, hence any $h \in G$ acts by a matrix in the centralizer of $A = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) : a \in \Bbbk^\times, b \in \Bbbk\}$. In other words, $V \simeq \mathcal{V}_g(\chi,\eta)$ for a unique YD-triple (g,χ,η). □

Let $\mathcal{D} = (g,\chi,\eta)$ be a YD-triple and $\epsilon := \chi(g)$. If $\epsilon = 1$, respectively -1, then we say that $\mathcal{D}$ is a *Jordanian*, respectively *super Jordanian*, YD-triple.

Remark 2.4. Let $f \in \operatorname{Aut}_{\text{Hopf}} H$, $g^f = f(g)$, $\chi^f = \chi \circ f^{-1}$, $\eta^f = \eta \circ f^{-1}$. Then $\mathcal{D}^f = (g^f, \chi^f, \eta^f)$ is a YD-triple and $\chi^f(g^f) = \chi(g)$, $\eta^f(g^f) = \eta(g)$. Thus, if $\mathcal{D}$ is Jordanian, respectively super Jordanian, then so is $\mathcal{D}^f$. Let $V^f = \mathcal{V}_{g^f}(\chi^f,\eta^f)$, with basis x_1', x_2'. Then f extends to a Hopf algebra isomorphism $\widetilde{f} : T(V)\#H \to T(V^f)\#H$ such that $f(x_i) = x_i'$. Let

$$\operatorname{Aut} \mathcal{D} := \{f \in \operatorname{Aut}_{\text{Hopf}} H : \mathcal{D}^f = \mathcal{D}\}.$$

Then we have a morphism of groups $\operatorname{Aut} \mathcal{D} \to \operatorname{Aut}_{\text{Hopf}}(T(V)\#H)$.

3. Generation in degree 1

3.1. A block plus a point

We shall need a result from [3] on some braided vector spaces of dimension 3. Let $\epsilon \in \{\pm 1\}$, $q_{12}, q_{21}, q_{22} \in \Bbbk^\times$ and $a \in \Bbbk$. Let V be the braided vector space with basis x_i, $i \in \mathbb{I}_3$, and braiding

$$(c(x_i \otimes x_j))_{i,j \in \mathbb{I}_3} = \begin{pmatrix} \epsilon x_1 \otimes x_1 & (\epsilon x_2 + x_1) \otimes x_1 & q_{12} x_3 \otimes x_1 \\ \epsilon x_1 \otimes x_2 & (\epsilon x_2 + x_1) \otimes x_2 & q_{12} x_3 \otimes x_2 \\ q_{21} x_1 \otimes x_3 & q_{21}(x_2 + a x_1) \otimes x_3 & q_{22} x_3 \otimes x_3 \end{pmatrix}. \tag{3.1}$$

The ghost is $\mathscr{G} = \begin{cases} -2a, & \epsilon = 1, \\ a, & \epsilon = -1. \end{cases}$ If $\mathscr{G} \in \mathbb{N}$, then the ghost is *discrete*.

Theorem 3.1 ([3, Theorem 4.1]). *If* $\operatorname{GKdim} \mathcal{B}(V) < \infty$, *then* V *is as in Table 1.*

Table 1. *Nichols algebras of a block and a point with finite GKdim.*

$q_{12}q_{21}$	ϵ	q_{22}	$\mathscr{G}$	GKdim
1	± 1	1 or $\notin \mathbb{G}_\infty$	0	3
		$\in \mathbb{G}_\infty - \{1\}$		2
	1	1	Discrete	$\mathscr{G} + 3$
		-1	Discrete	2
		$\in \mathbb{G}'_3$	1	2
	-1	1	Discrete	$\mathscr{G} + 3$
		-1	Discrete	$\mathscr{G} + 2$
-1	-1	-1	1	2

3.2. Pre-Nichols versus post-Nichols

Let $V \in {}^K_K\mathcal{YD}$ finite-dimensional. A *post-Nichols algebra* of V is a coradically graded connected Hopf algebra $\mathcal{E} = \oplus_{n \geqslant 0} \mathcal{E}^n$ in ${}^K_K\mathcal{YD}$ such that $\mathcal{E}^1 \simeq V$ [4]. A fundamental step in the classification of pointed Hopf algebras with finite GKdim is the following.

Question 3.2. Assume that $K = \Bbbk G$, with G nilpotent-by-finite. If $V \in {}^K_K\mathcal{YD}$ has $\operatorname{GKdim} \mathcal{B}(V) < \infty$, then determine all post-Nichols algebras $\mathcal{E}$ of V such that $\operatorname{GKdim} \mathcal{E} < \infty$.

A *pre-Nichols algebra* of V is a graded connected Hopf algebra $\mathcal{B} = \oplus_{n \geqslant 0} \mathcal{B}^n$ in ${}^K_K\mathcal{YD}$ such that $\mathcal{B}^1 \simeq V$ generates $\mathcal{B}$ as algebra [15]. If $\mathcal{E}$ is a post-Nichols algebra of V, then there is an inclusion $\mathcal{B}(V) \hookrightarrow \mathcal{E}$ of graded Hopf algebras in ${}^K_K\mathcal{YD}$ and the graded dual $\mathcal{E}^d$ is a pre-Nichols algebra of V^*, and vice versa.

Lemma 3.3. *Let $\mathcal{B}$ be a pre-Nichols algebra of V, $\mathcal{E} = \mathcal{B}^d$ (recall that $\dim V < \infty$). Then $\operatorname{GKdim} \mathcal{E} \leqslant \operatorname{GKdim} \mathcal{B}$. If $\mathcal{E}$ is finitely generated, then $\operatorname{GKdim} \mathcal{E} = \operatorname{GKdim} \mathcal{B}$.*

Proof. Let W be a finite-dimensional vector subspace of $\mathcal{E}$; without loss of generality, we may assume that W is graded. Let $\mathcal{E}_n = \sum_{0\leqslant j\leqslant n} W^j$. Now there exists $m \in \mathbb{N}$ such that $W \subseteq \oplus_{0\leqslant j\leqslant m}\mathcal{E}^j$; hence

$$\log_n \dim \mathcal{E}_n \leqslant \log_n \dim \oplus_{0\leqslant j\leqslant mn}\mathcal{E}^j = \log_n \dim \oplus_{0\leqslant j\leqslant mn}\mathcal{B}^j \overset{\clubsuit}{=} \log_n \dim(\oplus_{0\leqslant j\leqslant m}\mathcal{B}^j)^n \overset{\heartsuit}{\Rightarrow} \operatorname{GKdim} \mathcal{E} \leqslant \operatorname{GKdim} \mathcal{B}.$$

Here, $\oplus_{0\leqslant j\leqslant mn}\mathcal{B}^j = (\oplus_{0\leqslant j\leqslant m}\mathcal{B}^j)^n$ because V generates $\mathcal{B}$, hence ♣; while ♡ follows by the independence of the generators in the definition of GKdim.

Conversely, assume that W is a finite-dimensional graded vector subspace generating $\mathcal{E}$. We claim that $\mathcal{E}_n \supseteq \oplus_{0\leqslant j\leqslant n}\mathcal{E}^j$. Indeed, it suffices to show that $\mathcal{E}_n \supseteq \mathcal{E}^n$. For, take $x \in \mathcal{E}^n$; then $x = \sum w_{j_1} \dots w_{j_k}$ with $w_{j_h} \in W$, and

$$n = \deg w_{j_1} + \dots + \deg w_{j_k} \geqslant k \Rightarrow x \in \mathcal{E}_n.$$

Hence $\log_n \dim \mathcal{E}_n \geqslant \log_n \dim \oplus_{0\leqslant j\leqslant n}\mathcal{E}^j = \log_n \dim \oplus_{0\leqslant j\leqslant n}\mathcal{B}^j$, therefore $\operatorname{GKdim} \mathcal{E} \geqslant \operatorname{GKdim} \mathcal{B}$. □

Remark 3.4. The inequality in Lemma 3.3 might be strict: if $\mathcal{B} = \mathbf{k}[T]$ a polynomial ring with $\operatorname{char} \mathbf{k} > 0$, then $\mathcal{E}$ is the divided power algebra that has $\operatorname{GKdim} \mathcal{E} = 0 < 1 = \operatorname{GKdim} \mathbf{k}[T]$.

Question 3.5. If $V \in {}^K_K\mathcal{YD}$ has $\operatorname{GKdim} \mathcal{B}(V) < \infty$, then determine all pre-Nichols algebras $\mathcal{B}$ of V such that $\operatorname{GKdim} \mathcal{B} < \infty$.

To solve Question 3.5 for V is a first approximation to solve Question 3.2 for V^*, since it is open whether $\operatorname{GKdim} \mathcal{E} < \infty$ implies $\operatorname{GKdim} \mathcal{E}^d < \infty$ for a post-Nichols algebra $\mathcal{E}$. However, the next particular case is useful. Consider the partially ordered set of pre-Nichols algebras $\mathfrak{Pre}(V) = \{T(V)/I : I \in \mathfrak{S}\}$ with ordering given by the surjections. We say that V is *pre-bounded* if every chain

$$\dots < \mathcal{B}[3] < \mathcal{B}[2] < \mathcal{B}[1] < \mathcal{B}[0] = \mathcal{B}(V), \tag{3.2}$$

of pre-Nichols algebras over V with finite GKdim, is finite.

Lemma 3.6. *Let K be a Hopf algebra, $V \in {}^K_K\mathcal{YD}$ finite-dimensional and $\mathcal{E} \in {}^K_K\mathcal{YD}$ a post-Nichols algebra of V with $\operatorname{GKdim} \mathcal{E} < \infty$. If V^* is pre-bounded, then $\mathcal{E}$ is finitely generated and $\operatorname{GKdim} \mathcal{E} = \operatorname{GKdim} \mathcal{E}^d$. In particular, if the only pre-Nichols algebra of V^* with finite GKdim is $\mathcal{B}(V^*)$, then $\mathcal{E} = \mathcal{B}(V)$.*

Proof. First, we construct a chain $\mathcal{E}[0] = \mathcal{B}(V) \subsetneq \mathcal{E}[1] \dots \subseteq \mathcal{E}$ of finitely generated post-Nichols algebras of V. Suppose we have built $\mathcal{E}[n]$ and that $\mathcal{E} \supsetneq \mathcal{E}[n]$ (otherwise, we are done by Lemma 3.3). Pick $x \in \mathcal{E} - \mathcal{E}[n]$ homogeneous of minimal degree m. Let W be the Yetter–Drinfeld submodule of $\mathcal{E}^m$ generated by x and let $\mathcal{E}[n+1]$ be the subalgebra of $\mathcal{E}$ generated by $\mathcal{E}[n] + W$. Clearly $\mathcal{E}[n+1]$ is a Yetter–Drinfeld submodule of $\mathcal{E}$, hence $\mathcal{E}[n+1]\underline{\otimes}\mathcal{E}[n+1]$ is a subalgebra of $\mathcal{E}\underline{\otimes}\mathcal{E}$. By minimality of m, $\Delta(\mathcal{E}[n+1]) \subseteq \mathcal{E}[n+1]\underline{\otimes}\mathcal{E}[n+1]$. That is, $\mathcal{E}[n+1]$ is a finitely generated post-Nichols algebra of V with $\operatorname{GKdim}\mathcal{E}[n+1] < \infty$. Thus we have a chain (3.2) of pre-Nichols algebras with $\mathcal{B}[n] = \mathcal{E}[n]^d$, and $\operatorname{GKdim}\mathcal{B}[n] < \infty$ for all n by Lemma 3.3. By hypothesis, there is n such that $\mathcal{E}[n] = \mathcal{E}$ and we are done. Finally, if the only pre-Nichols algebra of V^* with finite GKdim is $\mathcal{B}(V^*)$, then $\mathcal{E} = \mathcal{B}(V)$, because there is only one chain (3.2) for V^*. □

3.3. Post-Nichols algebras of the Jordan and super Jordan planes

Lemma 3.7. *Assume that V is associated with either a Jordanian or a super Jordanian YD-triple $\mathcal{D} = (g, \chi, \eta)$ The only post-Nichols algebra of V in ${}^K_K\mathcal{YD}$ with finite* GKdim *is $\mathcal{B}(V)$.*

Proof. The dual V^* corresponds to the YD-triple $\mathcal{D}' = (g^{-1}, \chi^{-1}, \eta \circ \mathcal{S})$; by Lemma 3.6, it is enough to solve the analogous problem for pre-Nichols algebras. Let $\mathcal{B}$ be a pre-Nichols algebra of V such that $\operatorname{GKdim}\mathcal{B} < \infty$. We have canonical projections $T(V) \twoheadrightarrow \mathcal{B} \twoheadrightarrow \mathcal{B}(V)$. We shall prove that the defining relations of $\mathcal{B}(V)$ hold in $\mathcal{B}$.

Jordan case. Suppose that $y = x_2x_1 - x_1x_2 + (1/2)x_1^2 \neq 0$ in $\mathcal{B}$; note that y is primitive and $y \in \mathcal{B}_{g^2}$. Let $W \in {}^K_K\mathcal{YD}$ be spanned by the linearly independent primitive elements x_1, x_2 and y. Then $\mathcal{B}(W)$ is a quotient of $\mathcal{B}$, so $\operatorname{GKdim}\mathcal{B}(W) < \infty$. Notice that W satisfies (3.1) for $y = x_3$, $\epsilon = 1$, $q_{12} = q_{21} = q_{22} = 1$, $a = 2$. By Theorem 3.1, $\operatorname{GKdim}\mathcal{B}(W) = \infty$, a contradiction. Then $y = 0$ and $\mathcal{B} = \mathcal{B}(V)$.

Super Jordan case. Suppose first that $x_1^2 \neq 0$ in $\mathcal{B}$; note that x_1^2 is primitive and $x_1^2 \in \mathcal{B}_{g^2}$. Let $W \in {}^K_K\mathcal{YD}$ be spanned by the linearly independent primitives x_1, x_2 and x_1^2. Then $\mathcal{B}(W)$ is a quotient of $\mathcal{B}$, so $\operatorname{GKdim}\mathcal{B}(W) < \infty$. However, W satisfies (3.1) for $\epsilon = -1$, $x_3 = x_1^2$, $q_{12} = q_{21} = q_{22} = 1$, $a = -2$. By Theorem 3.1, $\operatorname{GKdim}\mathcal{B}(W) = \infty$, a contradiction, so $x_1^2 = 0$ in $\mathcal{B}$. Let $r = x_2x_{21} - x_{21}x_2 - x_1x_{21}$. In $T(V)\#K$ we have

$$\Delta(r) = r \otimes 1 + g^3 \otimes r + x_1g^2 \otimes x_1^2 - 2x_1^2g \otimes x_2. \tag{3.3}$$

Assume that $r \neq 0$ in $\mathcal{B}$. By the preceding and (3.3), r is primitive, and $r \in \mathcal{B}_{g^3}$. Let W' be the space spanned the linearly independent primitives x_1, x_2, r. Since $\mathcal{B}(W')$ is a quotient of $\mathcal{B}$, $\operatorname{GKdim}\mathcal{B}(W') < \infty$. However, W' satisfies (3.1) for $\epsilon = -1$, $x_3 = y$, $q_{12} = q_{21} = q_{22} = -1$, $a = -3$, so $\operatorname{GKdim}\mathcal{B}(W') = \infty$ by Theorem 3.1, a contradiction. Therefore, $\mathcal{B} = \mathcal{B}(V)$. □

4. Liftings

Let G be a nilpotent-by-finite group. If $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, then $T(V)$ is a Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ and we denote $\mathcal{T}(V) = T(V)\#\Bbbk G$. We compute all liftings of the Jordan and super Jordan planes $\mathcal{V}(\epsilon, 2)$ over $\Bbbk G$. We follow partly the strategy from [**2**], as the coradical is assumed to be finite-dimensional in [**2**]; instead we use [**14**, Theorem 8] being in the pointed context.

Remark 4.1. If $\mathcal{D} = (g, \chi, \eta)$ is a Jordanian or super Jordanian YD-triple, then $g^2 \neq 1$, since $g^2 \cdot x_2 = x_2 + 2\epsilon x_1$.

4.1. Liftings of Jordan planes

Let $\mathcal{D} = (g, \chi, \eta)$ be a Jordanian YD-triple for $\Bbbk G$ and $V = \mathcal{V}_g(\chi, \eta)$. Let $\lambda \in \Bbbk$ be such that

$$\lambda = 0, \qquad \text{if } \chi^2 \neq \varepsilon. \tag{4.1}$$

Let $\mathfrak{U} = \mathfrak{U}(\mathcal{D}, \lambda)$ be the quotient of $\mathcal{T}(V)$ by the relation

$$x_2x_1 - x_1x_2 + \tfrac{1}{2}x_1^2 = \lambda(1 - g^2). \tag{4.2}$$

Clearly, $\mathfrak{U}$ is a Hopf algebra quotient of $\mathcal{T}(V)$. We now show that any lifting of a Jordan plane over $\Bbbk G$ is $\mathfrak{U}(\mathcal{D}, \lambda)$ for some $\mathcal{D}$, λ.

Proposition 4.2. *$\mathfrak{U}$ is a pointed Hopf algebra, and a cocycle deformation of $\operatorname{gr}\mathfrak{U} \simeq \mathcal{B}(V)\#\Bbbk G$; it has $\operatorname{GKdim}\mathfrak{U} = \operatorname{GKdim}\Bbbk G + 2$.*

Conversely, let H be a pointed Hopf algebra with finite GKdim such that $G(H) \simeq G$ and the infinitesimal braiding of H is isomorphic to $\mathcal{V}(1, 2)$. Then $H \simeq \mathfrak{U}(\mathcal{D}, \lambda)$ for some YD-triple $\mathcal{D}$ and $\lambda \in \Bbbk$ satisfying (4.1).

Moreover, $\mathfrak{U}(\mathcal{D}, \lambda) \simeq \mathfrak{U}(\mathcal{D}', \lambda')$ if and only if there exist a Hopf algebra automorphism f of $\Bbbk G$ and $c \in \Bbbk^\times$ such that $\mathcal{D}' = f(D)$ and $\lambda = c\lambda'$.

Proof. First, we claim that there exists a $(\mathfrak{U}, \mathcal{B}(V)\#\Bbbk G)$-biGalois object $\mathcal{A}$, so that $\mathfrak{U}$ is a cocycle deformation of $\mathcal{B}(V)\#\Bbbk G$. Let $\mathcal{T} = \mathcal{T}(V)$.

Let X be the subalgebra of $\mathcal{T}$ generated by $t = (x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2)g^{-2}$, which is a polynomial algebra in t. Set $A = \mathcal{T}$. The algebra map $f : X \to A$ determined by $f(t) = t - \lambda g^{-2}$ is $\mathcal{T}$-colinear. Note that $x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2 - \lambda$ is stable by the action of $\Bbbk G$ because of (4.1). By [2, Remark 5.6 (b)],

$$\begin{aligned}\mathcal{A} := A/\langle f(X^+)\rangle &= \frac{\mathcal{T}}{\langle x_2x_1 - x_1x_2 + (1/2)x_1^2 - \lambda\rangle}\\ &\simeq \left(\frac{T(V)}{\langle x_2x_1 - x_1x_2 + (1/2)x_1^2 - \lambda\rangle}\right)\#\Bbbk G.\end{aligned}$$

We claim that $\mathcal{A} \neq 0$, which reduces to prove that

$$\mathcal{E} = \mathcal{E}(\mathcal{D}, \lambda) := \frac{T(V)}{\langle x_2x_1 - x_1x_2 + (1/2)x_1^2 - \lambda\rangle} \neq 0.$$

The algebra map $T(V) \to \Bbbk$, $x_1 \mapsto c$, $x_2 \mapsto 1$, where $c^2 = 2\lambda$, applies $x_2x_1 - x_1x_2 + (1/2)x_1^2 - \lambda$ to 0, so it factors through $\mathcal{E}$, and thus $\mathcal{E}$ is non-trivial. Now [14, Theorem 8] applies and $\mathcal{A}$ is a $\mathcal{B}(V)\#\Bbbk G$-Galois object. Now there exists a unique (up to isomorphism) Hopf algebra $L = L(\mathcal{A}, \mathcal{B}(V)\#\Bbbk G)$ such that $\mathcal{A}$ is a $(L, \mathcal{B}(V)\#\Bbbk G)$-biGalois object. However, $L \simeq \mathfrak{U}$ by a computation as in [2, Corollary 5.12], and the claim follows.

Moreover, $\mathrm{id}_{\mathcal{T}} : \mathcal{T} \to \mathcal{T} = A$ is a section that, restricted to $\Bbbk G$, is an algebra map. Arguing as in [2, Proposition 5.8 (b)] and applying [17, Theorem 4.2, Corollary 4.3], we conclude that there exists a section $\gamma : \mathcal{B}(V)\#\Bbbk G \to \mathcal{A}$ that, restricted to $\Bbbk G$, is an algebra map. Then [2, Proposition 4.14 (b), (c)] applies and $\operatorname{gr}\mathfrak{U} \simeq \mathcal{B}(V)\#\Bbbk G$, so its infinitesimal braiding is Jordanian. Also, $\operatorname{GKdim}\mathfrak{U} = \operatorname{GKdim}\operatorname{gr}\mathfrak{U} = \operatorname{GKdim}\Bbbk G + 2$ by [**19**, Theorem 5.4].

Conversely, let H be a pointed Hopf algebra with finite GKdim such that $G(H) \simeq G$ and the infinitesimal braiding of H is $V \simeq \mathcal{V}(1,2)$. By Lemma 2.3, there is a Jordanian YD-triple $\mathcal{D} = (g, \chi, \eta)$ such that $V = \mathcal{V}_g(\chi, \eta)$. Then $\operatorname{gr} H \simeq \mathcal{B}\#\Bbbk G$ for some post-Nichols algebra over V such that $\operatorname{GKdim}\mathcal{B} < \infty$. By Lemma 3.7, $\mathcal{B} = \mathcal{B}(V)$. In particular H is generated by $H_0 = \Bbbk G$ and H_1 as an algebra. Moreover, $H_1/H_0 \simeq V\#\Bbbk G$, so there exists a surjective Hopf algebra map $\pi : \mathcal{T} \twoheadrightarrow H$ that identifies $\Bbbk G$ and applies x_i to a $(g,1)$-primitive element $a_i \in H_1 \setminus H_0$. As $x_2x_1 - x_1x_2 + (1/2)x_1^2$ is a $(g^2,1)$-primitive element, $\pi(x_2x_1 - x_1x_2 + (1/2)x_1^2) \in \Bbbk(1-g^2)$, so there exists $\lambda \in \Bbbk$ satisfying (4.1) such that $a_2a_1 - a_1a_2 + (1/2)a_1^2 = \lambda(1-g^2)$. Then π factors through $\mathfrak{U}(\mathcal{D}, \lambda)$, and this map $\mathfrak{U}(\mathcal{D}, \lambda) \to H$ is an isomorphism since their associated coradically graded Hopf algebras coincide.

It remains to show the last statement. Let $F : \mathfrak{U}(\mathcal{D}, \lambda) \to \mathfrak{U}(\mathcal{D}', \lambda')$ be an isomorphism of Hopf algebras. Then $F|_{\Bbbk G}$ is an isomorphism of Hopf algebras since $\Bbbk G$ is the coradical. We may assume that $F_{|\Bbbk G} = \operatorname{id}_{\Bbbk G}$ by Remark 2.4. Now $g = g'$ since $\dim \mathcal{P}_{g,1}(\mathfrak{U}(\mathcal{D}, \lambda)) = 3$, $\dim \mathcal{P}_{h,1}(\mathfrak{U}(\mathcal{D}, \lambda)) = 1$ for all $h \in G(\Bbbk G)$, $h \neq g$, and the same for $\mathfrak{U}(\mathcal{D}', \lambda')$. As $F(x_i) \in \mathcal{P}_{g,1}(\mathfrak{U}(\mathcal{D}', \lambda'))$,

$$F(x_i) = a_i x_1' + b_i x_2' + c_i(1-g) \quad \text{for some} \quad a_i, b_i, c_i \in \Bbbk, i = 1, 2.$$

As $F(hx_ih^{-1}) = hF(x_i)h^{-1}$ for all $h \in G(\Bbbk G)$, we deduce that $b_1 = c_1 = 0$, $c_2 = 0$ if $\chi \neq \varepsilon$, $b_2 = a_1$, $\chi = \chi'$ and $\eta = \eta'$. Also,

$$0 = F\big(x_2x_1 - x_1x_2 + \tfrac{1}{2}x_1^2 - \lambda(1-g)\big) = (a_1^2\lambda' - \lambda)(1-g),$$

so $a_1^2\lambda' = \lambda$. The other implication is direct. □

Remark 4.3. Recall that G is nilpotent-by-finite. If G is torsion-free, then $H = \mathfrak{U}(\mathcal{D}, \lambda)$ is a domain. Indeed, $\Bbbk G$ is a domain by [**8**, **12**, **16**], hence $\operatorname{gr} H \simeq \mathcal{B}(V)\#\Bbbk G$ is a domain, and so is H. To see that $\mathcal{B}(V)\#\Bbbk G$ is a domain, filter by giving degree 0 to x_1 and G and degree 1 to x_2; the associated graded algebra is $S(V) \otimes \Bbbk G$, clearly a domain.

4.2. Liftings of super Jordan planes

Let $\mathcal{D} = (g, \chi, \eta)$ be a super Jordanian YD-triple for $\Bbbk G$ and $V = \mathcal{V}_g(\chi, \eta)$. Let $\lambda \in \Bbbk$ be such that

$$\lambda = 0, \quad \text{if } \chi^2 \neq \varepsilon. \tag{4.3}$$

Let $\mathfrak{U} = \mathfrak{U}(\mathcal{D}, \lambda)$ be the quotient of $\mathcal{T}(V)$ by the relations

$$x_1^2 = \lambda(1-g^2), \qquad x_2x_{21} - x_{21}x_2 - x_1x_{21} + 2\lambda x_2 + \lambda x_1 g^2 = 0. \tag{4.4}$$

We will prove that all liftings of super Jordan planes are the algebras $\mathfrak{U}(\mathcal{D}, \lambda)$. The proof follows the same steps as for Jordan planes.

Proposition 4.4. *$\mathfrak{U}$ is a pointed Hopf algebra, a cocycle deformation of* $\operatorname{gr}\mathfrak{U} \simeq \mathcal{B}(V)\#\Bbbk G$; $\operatorname{GKdim}\mathfrak{U} = \operatorname{GKdim}\Bbbk G + 2$. *Conversely, let H be a pointed Hopf algebra with finite* GKdim *such that $G(H) \simeq G$ and the infinitesimal braiding of H is $\simeq \mathcal{V}(-1, 2)$.*

Then $H \simeq \mathfrak{U}(\mathcal{D}, \lambda)$ for some YD-triple $\mathcal{D}$ and $\lambda \in \Bbbk$ satisfying (4.3).

Moreover, $\mathfrak{U}(\mathcal{D}, \lambda) \simeq \mathfrak{U}(\mathcal{D}', \lambda')$ if and only if there exist a Hopf algebra automorphism f of $\Bbbk G$ and $c \in \Bbbk^{\times}$ such that $\mathcal{D}' = f(D)$ and $\lambda = c\lambda'$.

Proof. First, we claim that there exists a $(\mathfrak{U}, \mathcal{B}(V)\#\Bbbk G)$-biGalois object $\mathcal{A}$, so $\mathfrak{U}$ is a cocycle deformation of $\mathcal{B}(V)\#\Bbbk G$. We proceed in two steps.

Let X_1 be the subalgebra of $\mathcal{T}$ generated by $t_1 = x_1^2 g^{-2}$, which is a polynomial algebra in t_1. Set $A = \mathcal{T}$. The algebra map $f : X_1 \to A$ determined by $f(t_1) = t_1 - \lambda g^{-2}$ is $\mathcal{T}$-colinear. Note that $x_1^2 - \lambda$ is stable by the action of $\Bbbk G$ because of (4.3). By [**2**, Remark 5.6],

$$\mathcal{A}_1 := A/\langle f(X_1^+)\rangle = \mathcal{T}/\langle x_1^2 - \lambda\rangle \simeq \big(T(V)/\langle x_1^2 - \lambda\rangle\big)\#\Bbbk G.$$

We claim that $\mathcal{A}_1 \neq 0$, which reduces to prove that $\mathcal{E}_1 = T(V)/\langle x_1^2 - \lambda\rangle \neq 0$. The algebra map $\psi : T(V) \to \Bbbk$, $x_1 \mapsto c$, $x_2 \mapsto 0$, where $c^2 = \lambda$, satisfies $\psi(x_1^2) = \lambda$, $\psi(x_2x_{21} - x_{21}x_2 - x_1x_{21} + 2\lambda x_2) = 0$. It induces an algebra map

$$\mathcal{E} = \mathcal{E}(\mathcal{D}, \lambda) = T(V)/\langle x_1^2 - \lambda, x_2x_{21} - x_{21}x_2 - x_1x_{21} + 2\lambda x_2\rangle \to \Bbbk.$$

Thus $\mathcal{E}$ is non-trivial, which implies that $\mathcal{E}_1$ is also non-trivial. Therefore [**14**, Theorem 8] applies and $\mathcal{A}_1$ is a $(T(V)/\langle x_1^2\rangle)\#\Bbbk G$-Galois object.

In the second step, we consider the subalgebra X_2 of $(T(V)/\langle x_1^2\rangle)\#\Bbbk G$ generated by $t_2 = (x_2x_{21} - x_{21}x_2 - x_1x_{21})g^{-3}$, a polynomial algebra in t_2. The algebra map $f : X_2 \to \mathcal{A}$ determined by

$$f(t_2) = (x_2x_{21} - x_{21}x_2 - x_1x_{21} + 2\lambda x_2)g^{-3}$$

is $\mathcal{T}$-colinear. By [**2**, Remark 5.6], $\mathcal{A} := \mathcal{A}_1/\langle f(X_2^+)\rangle = \mathcal{E}\#\Bbbk G$, so $\mathcal{A}$ is non-trivial. Then [**14**, Theorem 8] applies again and $\mathcal{A}$ is a $\mathcal{B}(V)\#\Bbbk G$-Galois object. Now there exists a unique (up to isomorphism) Hopf algebra $L = L(\mathcal{A}, \mathcal{B}(V)\#\Bbbk G)$ such that $\mathcal{A}$ is a $(L, \mathcal{B}(V)\#\Bbbk G)$-biGalois object. However, $L \simeq \mathfrak{U}$ by a computation as in [**2**, Cor. 5.12], and the claim follows. Moreover, $\operatorname{gr}\mathfrak{U} \simeq \mathcal{B}(V)\#\Bbbk G$ and $\operatorname{GKdim}\mathfrak{U} = \operatorname{GKdim}\Bbbk G + 2$ by [**19**, Theorem 5.4].

Conversely, let H be a Hopf algebra such that $H_0 \simeq \Bbbk G$, $\operatorname{GKdim} H < \infty$ and the infinitesimal braiding of H is $\simeq \mathcal{V}(-1, 2)$. By Lemma 2.3, there is a super Jordanian YD-triple $\mathcal{D} = (g, \chi, \eta)$ such that $V = \mathcal{V}_g(\chi, \eta)$. Then $\operatorname{gr} H \simeq \mathcal{B}\#\Bbbk G$ for some post-Nichols algebra over V with $\operatorname{GKdim}\mathcal{B} < \infty$; by Lemma 3.7, $\mathcal{B} = \mathcal{B}(V)$. Arguing as in Proposition 4.2, there exists a surjective Hopf algebra map $\pi : \mathcal{T} \twoheadrightarrow H$ that induces a surjective map $\mathfrak{U}(\mathcal{D}, \lambda) \to H$ for some λ as in (4.3), but π is an isomorphism since their associated coradically graded Hopf algebras coincide.

It remains to prove the last statement. Let $F : \mathfrak{U}(\mathcal{D}, \lambda) \to \mathfrak{U}(\mathcal{D}', \lambda')$ be a Hopf algebra isomorphism. As in the proof of Proposition 4.2, we may assume that $F_{|\Bbbk G} = \mathrm{id}_{\Bbbk G}$, and we have that $g = g'$. As $F(x_i) \in \mathcal{P}_{g,1}(\mathfrak{U}(\mathcal{D}', \lambda'))$,

$$F(x_i) = a_1 x_1' + b_i x_2' + c_i(1-g), \quad a_i, b_i, c_i \in \Bbbk, i = 1, 2.$$

Notice that $F(gx_ig^{-1}) = gF(x_i)g^{-1}$, so $b_1 = c_1 = c_2 = 0$, $b_2 = a_1$ since $\chi(g) = \chi'(g) = -1$. However, as $F(hx_ih^{-1}) = hF(x_i)h^{-1}$ for all $h \in G(\Bbbk G)$, we conclude that $\chi = \chi'$ and $\eta = \eta'$. Also,

$$0 = F\left(x_1^2 - \lambda(1-g)\right) = (a_1^2\lambda' - \lambda)(1-g),$$

so $a_1^2\lambda' = \lambda$. The other implication is direct. □

Remark 4.5. The algebra $H = \mathfrak{U}(\mathcal{D}, \lambda)$ is never a domain: $ab = 0$, where

$$a = \sqrt{\lambda}\,(g-1) + x, \quad b = \sqrt{\lambda}\,(g+1) + x.$$

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