Extended Hamilton-Jacobi theory, contact manifolds and integrability by quadratures

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Abstract

A Hamilton-Jacobi theory for general dynamical systems, defined on fibered phase spaces, has been recently developed. In this paper we shall apply such a theory to contact Hamiltonian systems, as those appearing in thermodynamics and on geodesic flows in fluid mechanics. We first study the partial and complete solutions of the Hamilton-Jacobi Equation (HJE) related to these systems. Then we show that, for a given contact system, the knowledge of what we have called a complete *pseudo-isotropic solution* ensures the integrability by quadratures of its equations of motion. This extends to contact manifolds a recent result obtained in the context of general symplectic and Poisson manifolds.

1 Introduction

In the last few years, different versions of the Hamilton-Jacobi Theory have been developed beyond the scenario of Hamiltonian systems in a cotangent bundle. See for instance [3, 6, 7, 16, 17]. In a recent paper [10], an extension of the theory to general dynamical systems (on fibered phase spaces), which contains as particular cases the previously mentioned versions, has been presented. Let us briefly describe it. Consider a finite-dimensional smooth manifold M and a vector field $X \in \mathfrak{X}(M)$. Suppose that M is a fibered bundle with base manifold N and surjective submersion $\Pi : M \to N$ (*ipso facto* an open map). Related to these data (see [10]) we have the **Π-Hamilton-Jacobi equation** (**Π-HJE**) for X:

$$\sigma_* \circ \Pi_* \circ X \circ \sigma = X \circ \sigma, \tag{1}$$

whose unknown is a section $\sigma : N \to M$ of Π (*ipso facto* a closed map). If σ solves the equation above we shall say that σ is a **(global) solution of the Π-HJE for X**. On the other hand, given an open subset $U \subseteq M$, we shall call **local solution of the Π-HJE for X along U** to any solution of the $\Pi_{|U}$-HJE for $X_{|U}$. (Here, we are seeing $\Pi_{|U}$ as a fibration onto $\Pi(U)$ and $X_{|U}$ as a vector field on U). Note that σ is a solution of the Π-HJE for

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X if and only if

$$\sigma_* \circ X^\sigma = X \circ \sigma, \tag{2}$$

where $X^\sigma := \Pi_* \circ X \circ \sigma$, i.e. the vector fields $X \in \mathfrak{X}(M)$ and $X^\sigma \in \mathfrak{X}(N)$ are σ-related. (Moreover, it can be shown that σ is a solution of above equation if and only if its image is an X-invariant submanifold). This means that, in order to find the trajectories of X along the image of σ, we can look for the integral curves of X^σ.

The standard Hamilton-Jacobi Theory is obtained when M is a co-tangent bundle T^*Q, the fibration Π is the canonical projection $\pi_Q : T^*Q \to Q$ and $\sigma : Q \to T^*Q$ is an exact 1-form. A Hamilton-Jacobi Theory for closed 1-forms may be found, for instance, in Ref. [6].

Given another manifold Λ such that $\dim \Lambda + \dim N = \dim M$, a **complete solution of the Π-HJE for** X is a surjective local diffeomorphism $\Sigma : N \times \Lambda \to M$ such that, for all $\lambda \in \Lambda$,

$$\sigma_\lambda := \Sigma(\cdot, \lambda) : p \in N \longmapsto \Sigma(p, \lambda) \in M \tag{3}$$

is a solution of the Π-HJE for X. The local version is obtained by replacing M, X, Π and N by U, $X_{|U}$, $\Pi_{|U}$ and $\Pi(U)$, respectively, being U an open subset of M. Each section σ_λ is called a **partial solution**. We showed in [10] that a (local) complete solution Σ exists around every point $p \in M$ for which $X(p) \notin \ker \Pi_{*,p}$. Denoting by $\mathfrak{p}_N$ and $\mathfrak{p}_\Lambda$ the projections of $N \times \Lambda$ onto N and Λ, respectively, it is easy to prove that a surjective local diffeomorphism Σ is a complete solution if and only if

$$\Pi \circ \Sigma = \mathfrak{p}_N \quad \text{and} \quad \Sigma_* \circ X^\Sigma = X \circ \Sigma, \tag{4}$$

being $X^\Sigma \in \mathfrak{X}(N \times \Lambda)$ the unique vector field on $N \times \Lambda$ satisfying

$$(\mathfrak{p}_N)_* \circ X^\Sigma = \Pi_* \circ X \circ \Sigma \quad \text{and} \quad (\mathfrak{p}_\Lambda)_* \circ X^\Sigma = 0. \tag{5}$$

Note that $X^\Sigma(p, \lambda) = (X^{\sigma_\lambda}(p), 0)$, with $X^{\sigma_\lambda} := \Pi_* \circ X \circ \sigma_\lambda \in \mathfrak{X}(N)$, so, in particular,

$$\mathsf{Im} X^\Sigma \subseteq TN \times \{0\}. \tag{6}$$

Also, the fields X and X^Σ are Σ-related. This implies that all the trajectories of X can be obtained from those of X^Σ. More precisely, since each trajectory of X^Σ is clearly of the form $t \mapsto (\gamma(t), \lambda) \in N \times \Lambda$, for some fixed point $\lambda \in \Lambda$ [see Eq. (6)], those of X are given by $t \mapsto \Sigma(\gamma(t), \lambda)$. So, for each λ, it rests to find the curves γ, which are the integral curves of the vector field $X^{\sigma_\lambda} \in \mathfrak{X}(N)$. In the standard case, it is well-known that such curves can be found up to quadratures. This gives rise to a natural problem: in the general case, when can such curves be constructed up to quadratures? Or, which conditions must a complete solution satisfy, in the general case, in order to ensure integrability by quadratures of the vector fields X^{σ_λ}?

When M is a general symplectic or a Poisson manifold and X is a Hamiltonian vector field, we have shown in [10] that, under a rather simple additional condition, the integral curves of each X^{σ_λ} can be found up to quadratures. Let us concisely see how this works. Suppose for instance that M is a symplectic manifold with symplectic form ω and X is a Hamiltonian vector field defined by a function H. Let us write $X = X_H$. In such a case (see Ref. [10]), the second part of (4) is equivalent to

$$i_{X_H^\Sigma} \Sigma^* \omega = \Sigma^* \mathsf{d} H. \tag{7}$$

In other words, $X_H^\Sigma \in \mathfrak{X}(N \times \Lambda)$ is a Hamiltonian vector field w.r.t. the symplectic form $\Sigma^* \omega$ and with Hamiltonian function $H \circ \Sigma$. Now, the annunciated condition: Σ is said to be **isotropic** if

$$\sigma_\lambda^* \omega = 0, \quad \forall \lambda \in \Lambda.$$

It was shown in [10] that, given an isotropic complete solution, Eq. (7) implies that (unless locally)

$$\mathsf{L}_{X_H^\Sigma} (\mathsf{d}W - \theta) = \mathfrak{p}_\Lambda^* \mathsf{d}h,$$

where W and h are functions related to Σ, H and ω, which can be found up to quadratures, and $-\theta$ is a (local) primitive of ω (which, by the Poincaré lemma, can also be found up to quadratures). So, defining for each $\lambda \in \Lambda$ the function $\varphi_\lambda : N \to T_\lambda^* \Lambda$ such that

$$\langle \varphi_\lambda (p), z \rangle = \langle (\mathsf{d}W - \theta)(p, \lambda), (0, z) \rangle, \quad \forall z \in T_\lambda^* \Lambda, \tag{8}$$

we have that each curve γ is given by

$$\frac{d}{dt} \varphi_\lambda (\gamma(t)) = \mathsf{d}h(\lambda),$$

or equivalently by

$$\varphi_\lambda (\gamma(t)) = \varphi_\lambda (\gamma(0)) + t \, \mathsf{d}h(\lambda). \tag{9}$$

Since each φ_λ is an immersion, we can solve the last algebraic equation for $\gamma(t)$, obtaining in this way the integrals curves of $X_H^{\sigma_\lambda}$.

In this paper we shall study the above mentioned problem in the case in which M is a contact manifold and X is a contact vector field. To do that, we could consider the symplectification of the contact manifold and then try to apply the procedure described above. But we wanted to develop our study completely inside the contact manifold scenario, and this is what we shall do.

The organization of the paper is as follows. In Section 2 we make a brief review on contact manifolds and contact (Hamiltonian) systems. In Section 3 we study the partial and complete solutions of the HJE in the context of contact systems. In Section 4 we introduce the concept of pseudo-isotropic complete solutions and show that the knowledge of one of such solutions ensures that the trajectories of the system can be found up to quadratures. At the end of the section, some comments are made relating the non-commutative integrability of contact systems, or *contact non-commutative integrability* (CNCI), and our pseudo-isotropic complete solutions. In particular, we show that some of the conditions imposed by the CNCI are not needed to ensure integrability by quadratures. A similar result was obtained in [10], but in the context of symplectic and Poisson manifolds. The integration procedure which we use in Section 4, in some sense, works better when the Hamiltonian function H defining the contact system is a non-vanishing function. For vanishing Hamiltonian functions, we develop in Section 5 an alternative procedure. This allowed us to deal with a case that is not explored (as far as we know) in the literature of integrable contact systems: the one for which $\xi(H) \neq 0$, where ξ is the Reeb vector field of the contact manifold.

We assume that the reader is familiar with the basic concepts of Differential Geometry (see [4, 15, 22]), and in particular Symplectic Geometry, and with the basic ideas related to Hamiltonian systems in the context of Geometric Mechanics (see [1, 2, 20]). We shall work in the smooth (i.e. C^∞) category, focusing exclusively on finite-dimensional smooth manifolds.

2   Contact Hamiltonian systems

In this section we recall the definition of contact manifolds, Legendrian submanifolds, Darboux coordinates for contact manifolds, Reeb vector fields, contact vector fields, and we say what we shall mean by a contact Hamiltonian system. For more details, see for instance Refs. [2, 19, 23].

2.1 Contact manifolds and contact Hamiltonian vector fields

By a **contact manifold** we shall understand a pair (M, η), where M is a $(2n + 1)$-dimensional manifold and η is a 1-form on M such that $(d\eta)^n \wedge \eta$ is a volume form, i.e.

$$\overbrace{(d\eta)_p \wedge \cdots \wedge (d\eta)_p}^{\text{n-times}} \wedge \eta_p \neq 0, \quad \forall p \in M. \tag{10}$$

In particular, since $d\eta \wedge \eta \neq 0$, the distribution $\mathsf{Ker}\, \eta$ is not completely integrable. Moreover, it can be shown that its maximal integral submanifolds are n-dimensional. Let us say that all the results which we will describe in this section are proved in Ref. [2].

For later convenience, consider the next definition.

Definition 1. A submanifold $S \subset M$ is said to be **isotropic** (resp. **pseudo-isotropic**) if $\mathfrak{i}^* \eta = 0$ (resp. $\mathfrak{i}^* d\eta = 0$), being $\mathfrak{i} : S \to M$ the inclusion map. S is said to be a **Legendrian submanifold** if is isotropic and n-dimensional.

Note that, if S is isotropic, then $\mathsf{Im}\, \mathfrak{i}_* = TS \subset \mathsf{Ker}\, \eta$, i.e. S is an integral submanifold of $\mathsf{Ker}\, \eta$. So, the Legendrian submanifolds are the maximal integral submanifolds of $\mathsf{Ker}\, \eta$.

Around every $p \in M$ of a contact manifold M, there exists a coordinate chart $\left(U, \left(x^1, ..., x^n, y_1, ..., y_n, z\right)\right)$, called **Darboux** or **(contact) canonical coordinate chart**, such that the local expression of η is (sum over repeated index convention is assumed from now on)

$$\eta_{|U} = y_i\, dx^i + dz.$$

In these coordinates, it can be shown that the Legendrian submanifolds are locally given by equations of the form

$$y_i = -\frac{\partial \Phi}{\partial x^i}, \quad x^j = \frac{\partial \Phi}{\partial y_j} \quad \text{and} \quad z = \Phi - y_j \frac{\partial \Phi}{\partial y_j}, \quad i \in I, \quad j \in J, \tag{11}$$

where I and J give a partition of $\{1, ..., n\}$ and Φ is any function of the variables x^i and y_j, with $i \in I$ and $j \in J$.

Remark 1. Given a cotangent bundle T^*Q, its canonical 1-form θ induces on $T^*Q \times \mathbb{R}$ a contact form given by $\eta_Q = p_1^* \theta + dz$, being p_1 the projection of $T^*Q \times \mathbb{R}$ onto T^*Q and z the global coordinate of the factor $\mathbb{R}$. It is clear that, if $\left(x^1, ..., x^n, y_1, ..., y_n\right) = (\mathbf{x}, \mathbf{y})$ is a system of canonical coordinates for the symplectic manifold $(T^*Q, -d\theta)$, then $(\mathbf{x}, \mathbf{y}, z)$ is a system of canonical coordinates for the contact manifold $(T^*Q \times \mathbb{R}, \eta_Q)$. Locally, every contact manifold looks like the last one.

Eq. (10) is the same as saying that $\mathsf{Ker}\, \eta$ and $\mathsf{Ker}\, d\eta$ are regular distributions satisfying $TM = \mathsf{Ker}\, \eta \oplus \mathsf{Ker}\, d\eta$. This implies that $\dim \mathsf{Ker}\, (d\eta)_p = 1$ and $d\eta_p$ is (symplectic) non-degenerated when restricted to $\mathsf{Ker}\, \eta_p$, for all $p \in M$. As a consequence, there exists a unique global vector field $\xi \in \mathfrak{X}(M)$, called **Reeb vector field**, characterized by

$$i_\xi d\eta = 0 \quad \text{and} \quad i_\xi \eta = 1. \tag{12}$$

Note that ξ is a global generator of $\mathsf{Ker}\, d\eta$. Therefore, $TM = \mathsf{Ker}\, \eta \oplus \langle \xi \rangle$. Also, in a Darboux chart U, we have that the local expression of the Reeb vector field of (M, η) is

$$\xi_{|U} = \frac{\partial}{\partial z}. \tag{13}$$

Moreover, given $F \in C^\infty(M)$ and $\alpha \in \Omega^1(M)$, there exists a unique $X \in \mathfrak{X}(M)$ such that

$$i_X d\eta = \alpha - i_\xi \alpha\, \eta \quad \text{and} \quad i_X \eta = F. \tag{14}$$

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In particular, if for some function $H \in C^\infty(M)$ we take $F = H$ and $\alpha = -\mathsf{d}H$, Eq. (14) defines the so-called **contact field** X_H related to H. Thus, X_H is characterized by

$$i_{X_H}\mathsf{d}\eta = -\mathsf{d}H + \xi(H)\,\eta \quad \text{and} \quad i_{X_H}\eta = H. \tag{15}$$

In Darboux coordinates, the local expression of X_H is, around an open neighborhood U,

$$(X_H)_{|U} = \left[\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x^i} + \left(y_i\frac{\partial H}{\partial z} - \frac{\partial H}{\partial x^i}\right)\frac{\partial}{\partial y_i} + \left(H - y_i\frac{\partial H}{\partial y_i}\right)\frac{\partial}{\partial z}\right]. \tag{16}$$

Note that the Reeb vector field is the contact field related to the constant function equal to 1, i.e. $\xi = X_1$.

Finally, given a contact manifold (M, η) and a function $H \in C^\infty(M)$, we shall call **contact Hamiltonian system**, or simply **contact system**, to the pair (M, X_H).

2.2 Some examples

In this section we present some physical systems that can be described in terms of contact systems. For more examples, see Ref. [5].

2.2.1 Thermodynamic processes

Let us consider the contact manifold of Remark 1 for $Q = \mathbb{R}^n$, i.e. take $M = T^*\mathbb{R}^n \times \mathbb{R}$. Let us identify M with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$. If $\left(x^1, ..., x^n, y_1, ..., y_n, z\right) = (\mathbf{x}, \mathbf{y}, z)$ are the global coordinates for M, then $\eta_{\mathbb{R}^n} = y_i\,\mathsf{d}x^i + \mathsf{d}z$. Let us define $H : \mathbb{R}^{2n+1} \to \mathbb{R}$ as

$$H(\mathbf{x}, \mathbf{y}, z) = a^0(\mathbf{x})\,(z - \Phi(\mathbf{x})) + a^j(\mathbf{x})\left(y_j + \frac{\partial\Phi}{\partial x^j}(\mathbf{x})\right), \tag{17}$$

for certain functions $a^j, \Phi : \mathbb{R}^n \to \mathbb{R}$ with some non-null a^j. According to (16), its related contact field is

$$\begin{aligned}
X_H = &\; a^i\frac{\partial}{\partial x^i} + \left[a^0\left(y_i + \frac{\partial\Phi}{\partial x^i}\right) - \frac{\partial a^0}{\partial x^i}(z - \Phi) - \left(\frac{\partial a^j}{\partial x^i}\left(y_j + \frac{\partial\Phi}{\partial x^j}\right) + a^j\frac{\partial^2\Phi}{\partial x^i\partial x^j}\right)\right]\frac{\partial}{\partial y_i} \\
&+ \left(a^0(z - \Phi) + a^j\frac{\partial\Phi}{\partial x^j}\right)\frac{\partial}{\partial z}.
\end{aligned} \tag{18}$$

It is clear that Φ defines a Legendrian submanifold $\mathcal{L} \subseteq H^{-1}(0)$ by the equations [recall Eq. (11)]

$$y_i = -\frac{\partial\Phi}{\partial x^i}, \quad i = 1, ..., n, \quad \text{and} \quad z = \Phi. \tag{19}$$

It can be shown that, in general, given a Legendrian submanifold $\mathcal{L}$, a contact vector field X_H is tangent to $\mathcal{L}$ if and only if $\mathcal{L} \subseteq H^{-1}(0)$ (see [23], Theorem 3). Then, we have in our case that $\mathsf{Im}\left(X_{H|\mathcal{L}}\right) \subseteq T\mathcal{L}$, for $\mathcal{L}$ given by Eq. (19). In addition, taking $\left(x^1, ..., x^n\right)$ as coordinates for $\mathcal{L}$, it can be shown that the restriction of X_H to $\mathcal{L}$ is

$$\hat{X}(\mathbf{x}) := X_{H|\mathcal{L}}(\mathbf{x}) = a^i(\mathbf{x})\left.\frac{\partial}{\partial x^i}\right|_{\mathbf{x}}. \tag{20}$$

The physical meaning of X_H is as follows (see Ref. [23]). The quasi-static processes of a thermodynamic system are curves on a Legendrian submanifold $\mathcal{L}$ of some contact manifold. In many cases, only certain processes are taking into account, given by the integral curves of some vector field along $\mathcal{L}$. So, if we have a thermodynamic system with processes contained in the Legendrian submanifold $\mathcal{L}$ defined by (19) and given by the integral curves of the vector field $\hat{X}$ defined by (20), then the integral curves of the contact field X_H along $\mathcal{L}$ are precisely the quasi-static processes of the system under consideration.

Remark 2. Note that the term in H containing the function a^0 [see Eq. (17)] plays no role in the physical interpretation of X_H we made above. In particular, we can still make such an interpretation even though $a^0 = 0$. We add such a term just for later convenience.

2.2.2 Iso-energetic surfaces and Reeb vector flows

Let (M, ω) be a $(2n + 2)$-dimensional symplectic manifold. If we have a vector field Δ on M such that $\mathsf{L}_\Delta \omega = \omega$, we say that Δ is a **Liouville vector field** for ω. (Note that $\mathsf{d} i_\Delta \omega = \omega$). In such a case (see Ref. [19]), the 1-form $\eta := i_\Delta \omega$ determines a contact form on any hyper-surface S of M transverse to Δ, i.e. any submanifold S such that

$$T_x M = T_x S \oplus \langle \Delta(x) \rangle, \quad \forall x \in S.$$

For instance, if the transverse hyper-surface S is given by a level set of a function $H : M \to \mathbb{R}$ (*ipso facto* $\Delta(H)$ is non-vanishing along S), it is easy to show that the Reeb vector field ξ associated to $\left(S, i_\Delta \omega_{|S}\right)$ is

$$\xi = -\frac{1}{\Delta(H)_{|S}} (X_H)_{|S}. \tag{21}$$

Thus, the trajectories of the (symplectic) Hamiltonian system defined by H, along the given level set S, are in essence the integral curves of a Reeb vector field: the **Reeb vector flow** (up to a re-parametrization). In particular, if ξ has closed integral curves, so does X_H along S (see Ref. [25] for more details).

In order to give a concrete example (see Ref. [19]), suppose that $M = \mathbb{R}^{2n+2}$. Denote by $\left(q^1, ..., q^{n+1}, p_1, ..., p_{n+1}\right)$ its global coordinates and consider the symplectic form ω such that last coordinates are canonical. Then,

$$\Delta = \frac{1}{2} \left(q^i \frac{\partial}{\partial q_i} + p^i \frac{\partial}{\partial p_i} \right)$$

is a Liouville vector field for ω. Moreover, the $(2n + 1)$-sphere $\mathbb{S}^{2n+1}$ is a contact manifold with contact form

$$\eta = \frac{1}{2} \left(q^i \, \mathsf{d} p_i - p_i \, \mathsf{d} q^i \right)_{|\mathbb{S}^{2n+1}}$$

and Reeb vector field

$$\xi = 2 \left(q^i \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial q^i} \right)_{|\mathbb{S}^{2n+1}}.$$

Since $\mathbb{S}^{2n+1}$ can be described as the level set $H = \sum_{i=1}^{n+1} \left(\left(q^i\right)^2 + p_i^2 \right) = 1$, and $\Delta(H) = H$, we have from (21) that $\xi = X_{H|\mathbb{S}^{2n+1}}$. In other words, the integral curves of X_H along $\mathbb{S}^{2n+1}$ define exactly a Reeb vector flow.

3 The Π-HJE for contact Hamiltonian systems

Consider a contact manifold (M, η), a contact vector field X_H and a fibration $\Pi : M \to N$.

3.1 Partial solutions

According to Eq. (2), having a (partial) solution of the Π-HJE for X_H is the same as having a section $\sigma : N \to M$ of Π satisfying

$$\sigma_* \circ X_H^\sigma = X_H \circ \sigma, \tag{22}$$

with $X_H^\sigma := \Pi_* \circ X_H \circ \sigma$.

Remark 3. In Ref. [18], a HJE is defined in the restricted context of the contact manifolds described in Remark 1 (the contact form is the same, up to a sign). Such an equation corresponds exactly to the Π-HJE with $\Pi = \pi_Q \times \mathsf{id}_{\mathbb{R}}$ (in particular, $N = Q \times \mathbb{R}$), where $\pi_Q : T^*Q \to Q$ is the canonical cotangent projection.

In the following, we shall study Eq. (22) in some particular situations.

Theorem 1. *Let $\sigma : N \to M$ be a section of Π. If σ is a solution of the Π-HJE for X_H, then*

$$i_{X_H^\sigma} \sigma^* \mathsf{d}\eta = \sigma^* \left(\xi \left(H \right) \eta - \mathsf{d}H \right) \quad and \quad i_{X_H^\sigma} \sigma^* \eta = \sigma^* H. \tag{23}$$

On the other hand, if σ satisfies above equations, then

$$\mathsf{Im} \left(\sigma_* \circ X_H^\sigma - X_H \circ \sigma \right) \subseteq \mathsf{Ker}\, \Pi_* \cap \left(\mathsf{Im}\, \sigma_* \right)^\perp \cap \mathsf{Ker}\, \eta, \tag{24}$$

where $\perp$ means the orthogonal[1] w.r.t. the 2-form $\mathsf{d}\eta$.

Proof. Given $p \in N$ and a vector $v \in T_p N$,

$$\left(\mathsf{d}\eta \right)_{\sigma(p)} \left(\sigma_{*,p} \left(X_H^\sigma \left(p \right) \right), \sigma_{*,p} \left(v \right) \right) = \left(\sigma^* \mathsf{d}\eta \right)_p \left(X_H^\sigma \left(p \right), v \right) = \left(i_{X_H^\sigma} \left(\sigma^* \mathsf{d}\eta \right) \right)_p \left(v \right) \tag{26}$$

and, using the first part of (15),

$$\left(\mathsf{d}\eta \right)_{\sigma(p)} \left(X_H \left(\sigma \left(p \right) \right), \sigma_{*,p} \left(v \right) \right) = \left(\sigma^* \left(i_{X_H} \mathsf{d}\eta \right) \right)_p \left(v \right) = \left(\sigma^* \left(\xi \left(H \right) \eta - \mathsf{d}H \right) \right)_p \left(v \right). \tag{27}$$

Thus, if Eq. (22) holds, the first part of (23) immediately follows from above equations. On the other hand, since

$$\eta_{\sigma(p)} \left(\sigma_{*,p} \left(X_H^\sigma \left(p \right) \right) \right) = \left(i_{X_H^\sigma} \sigma^* \eta \right)_p \tag{28}$$

and, using the second part of (15),

$$\eta_{\sigma(p)} \left(X_H \left(\sigma \left(p \right) \right) \right) = \left(i_{X_H} \eta \right) \left(\sigma \left(p \right) \right) = H \left(\sigma \left(p \right) \right) = \sigma^* H \left(p \right), \tag{29}$$

then the second part of (23) is obtained if Eq. (22) holds.

Let us show the other affirmation of the theorem. First note that, since σ is a section of Π, then $\Pi_* \circ \sigma_* = \mathsf{id}_{TN}$ and consequently, using the definition of X_H^σ,

$$\Pi_* \circ \left(\sigma_* \circ X_H^\sigma - X_H \circ \sigma \right) = 0.$$

This means that $\mathsf{Im} \left(\sigma_* \circ X_H^\sigma - X_H \circ \sigma \right) \subseteq \mathsf{Ker}\, \Pi_*$. Also, subtracting (26) and (27), we have from the first part of (23) that

$$\left(\mathsf{d}\eta \right)_{\sigma(p)} \left(\sigma_{*,p} \left(X_H^\sigma \left(p \right) \right) - X_H \left(\sigma \left(p \right) \right), \sigma_{*,p} \left(v \right) \right) = 0,$$

what implies that $\mathsf{Im} \left(\sigma_* \circ X_H^\sigma - X_H \circ \sigma \right) \subseteq \left(\mathsf{Im}\, \sigma_* \right)^\perp$. Finally, subtracting (28) and (29), from the second part of (23) we have that

$$\eta_{\sigma(p)} \left(\sigma_{*,p} \left(X_H^\sigma \left(p \right) \right) - X_H \left(\sigma \left(p \right) \right) \right) = 0,$$

which ends our proof. $\triangle$

It is worth mentioning that, in general, Eqs. (22) and (23) are not equivalent. Some additional conditions must be fulfilled to have an equivalence. The following corollaries contain some of those conditions.

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[1] Given a vector space V, a subspace $A \subseteq V$ and a (possibly degenerate) bilinear $\beta : V \times V \to \mathbb{R}$, the orthogonal related to β is the set $A^\perp = \{ v \in V : \beta \left(v, a \right) = 0, \, \forall a \in A \}$. It is well-known that, for any two subspaces $A, B \subseteq V$,

$$A^\perp \cap B^\perp = \left(A + B \right)^\perp, \quad A^\perp + B^\perp \subseteq \left(A \cap B \right)^\perp \quad and \quad A \subseteq A^{\perp\perp}. \tag{25}$$

Corollary 1. *Eqs.* (22) *and* (23) *are equivalent if*[2]

$$\mathsf{Ker}\,\Pi_* \cap \mathsf{Ker}\,\eta \subseteq (\mathsf{Ker}\,\Pi_*)^{\perp}. \tag{30}$$

Proof. It is clear from the last theorem that we only must prove that (23) implies (22). Also from the last theorem, we know that (23) implies (24). So, it is enough to prove that (24) implies (22). The hypothesis of the present corollary says that [see Eq. (25)]

$$\mathsf{Ker}\,\Pi_* \cap (\mathsf{Im}\,\sigma_*)^{\perp} \cap \mathsf{Ker}\,\eta \subseteq (\mathsf{Ker}\,\Pi_*)^{\perp} \cap (\mathsf{Im}\,\sigma_*)^{\perp} = \left((\mathsf{Ker}\,\Pi_*)_{|\mathsf{Im}\sigma} + \mathsf{Im}\,\sigma_* \right)^{\perp}.$$

Since σ is a section of Π, we know that

$$TM_{|\mathsf{Im}\sigma} = (\mathsf{Ker}\,\Pi_*)_{|\mathsf{Im}\sigma} \oplus \mathsf{Im}\,\sigma_*, \tag{31}$$

and consequently $\mathsf{Ker}\,\Pi_* \cap (\mathsf{Im}\,\sigma_*)^{\perp} \cap \mathsf{Ker}\,\eta = \{0\}$. This fact combined with Eq. (24) drives us precisely to Eq. (22). $\triangle$

Corollary 2. *Eqs.* (22) *and* (23) *are equivalent if*

$$(\mathsf{Im}\,\sigma_*)^{\perp} \cap \mathsf{Ker}\,\eta \subseteq \mathsf{Im}\,\sigma_*. \tag{32}$$

Proof. The hypothesis says that [recall Eq. (25)]

$$\mathsf{Ker}\,\Pi_* \cap (\mathsf{Im}\,\sigma_*)^{\perp} \cap \mathsf{Ker}\,\eta \subseteq \mathsf{Ker}\,\Pi_* \cap \mathsf{Im}\,\sigma_* \subseteq \left((\mathsf{Ker}\,\Pi_*)_{|\mathsf{Im}\sigma}^{\perp} + (\mathsf{Im}\,\sigma_*)^{\perp} \right)^{\perp}.$$

So, our result easily follows by using (31) and the same reasoning as in the proof of the previous corollary. $\triangle$

Conditions (30) and (32) are similar to the isotropy and co-isotropy conditions, respectively, of symplectic geometry. In fact, condition (30) is usually called **pre-isotropy**, and implies the usual isotropy condition for the subbundle $\mathsf{Ker}\,\Pi_* \cap \mathsf{Ker}\,\eta \subseteq \mathsf{Ker}\,\eta$ w.r.t. the symplectic fibration on $\mathsf{Ker}\,\eta$ given by $\mathsf{d}\eta_{|\mathsf{Ker}\,\eta}$.

Definition 2. We shall say that a section $\sigma : N \to M$ is **isotropic** (resp. **Legendrian**) if so is its image $\mathsf{Im}\sigma$, i.e. if $\mathsf{Im}\,\sigma_* \subseteq \mathsf{Ker}\,\eta$, or equivalently $\sigma^*\eta = 0$ (resp. and if in addition $2\dim N + 1 = \dim M$).

Corollary 3. *If σ is an isotropic solution of the Π-HJE for X_H, then $H \circ \sigma = 0$.*

Proof. If σ is a solution of the Π-HJE for X_H, we know from Theorem 1 that (23) holds. Then, if $\sigma^*\eta = 0$, the corollary follows from the second part of (23). $\triangle$

As a direct consequence of the above corollaries we have the next result.

Corollary 4. *If some of the conditions* (30) *or* (32) *is satisfied and in addition σ is isotropic, then Eq.* (22) *as well as Eq.* (23) *are equivalent to the equation $H \circ \sigma = 0$.*

Let us describe a particular, but very important case. Given a manifold Q, consider the contact manifold $(T^*Q \times \mathbb{R}, \eta_Q)$ defined in Remark 1. Take $\Pi : T^*Q \times \mathbb{R} \to Q$ such that $\Pi = \pi_Q \circ p_1$ (recall that $\pi_Q : T^*Q \to Q$ is the canonical cotangent projection). It is easy to see that Π is a fibration and $\mathsf{Ker}\,\Pi_* = (\mathsf{Ker}\,\Pi_*)^{\perp}$. In particular,

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[2]This condition is satisfied by the submersion Π used in Ref. [18] (see Remark 3).

we have from this last identity that Eq. (30) holds. Let $\sigma : Q \to T^*Q \times \mathbb{R}$ be a section of Π. In the canonical coordinates $(\mathbf{x}, \mathbf{y}, z)$ (see Remark 1 again), the local expression of σ is given by the formula

$$\sigma(\mathbf{x}) = (\mathbf{x}, \mathbf{F}(\mathbf{x}), \Phi(\mathbf{x}))$$

for some $\mathbb{R}^n$-valued (resp. $\mathbb{R}$-valued) function $\mathbf{F}$ (resp. Φ). Suppose that σ is isotropic (see Definition 2), and consequently Legendrian (for dimensional reasons). The condition $\sigma^* \eta_Q = 0$ means that

$$F_i(\mathbf{x}) + \frac{\partial \Phi}{\partial x^i}(\mathbf{x}) = 0.$$

Since the hypothesis of Corollary 4 are fulfilled in this case, the Π-HJE for any contact field X_H reduces to $H \circ \sigma = 0$. In local terms, this means that

$$H(\mathbf{x}, -\nabla \Phi(\mathbf{x}), \Phi(\mathbf{x})) = 0.$$

So, restricted to Legendrian sections of Π, the Π-HJE represents a general first order PDE (partial differential equation) defined by the function H.

It is well-known that the characteristic field of above PDE is precisely the contact field X_H. In particular, if we know the trajectories of X_H, then we can construct the solutions of such an equation (see Ref. [8]). But we are actually interested in the opposite problem (not only in the present example, but also in the general case): finding the trajectories of X_H from the solutions of its Π-HJE. To do that, it is not enough to have only one solution, but we need a "big family" of them, namely a **complete solution** (see the Section 1). Moreover, we need a particular kind of complete solutions, which we shall present in Section 4.

3.2 Complete solutions

In this section we study the complete solutions in the case of contact systems. Let us begin with an example.

3.2.1 Solutions related to thermodynamic processes

Let us go back to Section 2.2.1. Consider the submersion $\Pi : \mathbb{R}^{2n+1} \to \mathbb{R}^n$ given by $\Pi(\mathbf{x}, \mathbf{y}, z) = \mathbf{x}$ and the related Π-HJE for X_H [see Eq. (22)]. It is easy to show that

$$X_H^\sigma = a^i \frac{\partial}{\partial x^i} \tag{33}$$

for every section $\sigma : \mathbb{R}^n \to \mathbb{R}^{2n+1}$ of Π. If we write

$$\sigma(\mathbf{x}) = (\mathbf{x}, \sigma^y(\mathbf{x}), \sigma^z(\mathbf{x})),$$

then Eq. (22) translates to

$$a^j \frac{\partial \sigma_i^y}{\partial x^j} = a^0 \left(\sigma_i^y + \frac{\partial \Phi}{\partial x^i} \right) - \frac{\partial a^0}{\partial x^i} (\sigma^z - \Phi) - \left[\frac{\partial a^j}{\partial x^i} \left(\sigma_j^y + \frac{\partial \Phi}{\partial x^j} \right) + a^j \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \right] \tag{34}$$

and

$$a^j \frac{\partial \sigma^z}{\partial x^j} = a^0 (\sigma^z - \Phi) + a^j \frac{\partial \Phi}{\partial x^j}. \tag{35}$$

Defining $\alpha_i(\mathbf{x}) := \sigma_i^y(\mathbf{x}) + \frac{\partial \Phi}{\partial x^i}(\mathbf{x})$ and $\beta(\mathbf{x}) := \sigma^z(\mathbf{x}) - \Phi(\mathbf{x})$, the Eqs. (34) and (35) transform into

$$a^j \left(\frac{\partial \alpha_i}{\partial x^j} - \frac{\partial \alpha_j}{\partial x^i} \right) = a^0 \alpha_i - \frac{\partial a^0}{\partial x^i} \beta - \frac{\partial}{\partial x^i} \left(a^j \alpha_j \right)$$

and

$$a^j \frac{\partial \beta}{\partial x^j} = a^0 \beta,$$

respectively. If we think of the functions a_i and α_i as the components of a vector field and a 1-form

$$\hat{X} := a^i \frac{\partial}{\partial x^i} \quad \text{and} \quad \alpha := \alpha_i \, \mathsf{d} x^i, \tag{36}$$

respectively, we can write above equations as

$$\mathsf{L}_{\hat{X}} \alpha = a^0 \, \alpha - \beta \, \mathsf{d} a^0 \tag{37}$$

and

$$\mathsf{L}_{\hat{X}} \beta = a^0 \beta. \tag{38}$$

Let us suppose that a^0 is **constant** and take $\alpha = e^{-f} \, \mathsf{d} g$. Then, Eq. (37) reduces to

$$\mathsf{L}_{\hat{X}} \left(e^{-f} \right) \, \mathsf{d} g + e^{-f} \, \mathsf{L}_{\hat{X}} \left(\mathsf{d} g \right) = a^0 \, e^{-f} \, \mathsf{d} g,$$

or equivalently

$$-\mathsf{L}_{\hat{X}} \left(f \right) \, \mathsf{d} g + \mathsf{L}_{\hat{X}} \left(\mathsf{d} g \right) = a^0 \, \mathsf{d} g.$$

Choosing f such that

$$\mathsf{L}_{\hat{X}} \left(f \right) = \mathsf{i}_{\hat{X}} \mathsf{d} f = -a^0, \tag{39}$$

the function g must satisfy

$$\mathsf{L}_{\hat{X}} \left(\mathsf{d} g \right) = \mathsf{d} \left(\mathsf{i}_{\hat{X}} \mathsf{d} g \right) = 0. \tag{40}$$

Remark 4. Note that, if $a^0 = 0$, then we can take $f = 0$.

Since $\hat{X}$ is a non-null vector field, it is easy to show that given n constant $c_1, \ldots, c_n$, some c_k non-null, then we can find n independent functions $\{g_1, ..., g_n\}$ satisfying

$$\mathsf{i}_{\hat{X}} \mathsf{d} g_k = c_k, \quad k = 1, ..., n. \tag{41}$$

Hence, we have n independent functions solving Eq. (40).

Remark 5. Suppose that $c_1 \neq 0$. Then the $n - 1$ functions $g_i - \frac{c_i}{c_1} \, g_1$, $i = 2, ..., n$, are first integrals for the vector field $\hat{X}$ [see Eq. (41)].

Finally, given constant $\lambda^1, ..., \lambda^n$, it easily follows that $\alpha := e^{-f} \lambda^k \, \mathsf{d} g_k$ solves (37). Also, given any constant λ^0, the function $\beta := \lambda^0 \, e^{-f}$ solves (38). In this way, we have a family of solutions of the Π-HJE for X_H given by

$$\sigma_\lambda \left(\mathbf{x} \right) := \left(\mathbf{x}, e^{-f(\mathbf{x})} \left(\lambda^k \, \nabla g_k \left(\mathbf{x} \right) \right) - \nabla \Phi \left(\mathbf{x} \right), \Phi \left(\mathbf{x} \right) + \lambda^0 \, e^{-f(\mathbf{x})} \right), \tag{42}$$

where $\lambda = \left(\lambda^0, \lambda^1, ..., \lambda^n \right) \in \mathbb{R}^{n+1}$. Such a family defines a complete solution

$$\Sigma : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n+1} : (\mathbf{x}, \lambda) \mapsto \sigma_\lambda \left(\mathbf{x} \right). \tag{43}$$

In fact, the independence of the functions g_i's ensures that Σ is a surjective local diffeomorphism.

Remark 6. According to Remark 4, if $a^0 = 0$ we have a complete solution given by

$$\sigma_\lambda \left(\mathbf{x} \right) := \left(\mathbf{x}, \lambda^k \, \nabla g_k \left(\mathbf{x} \right) - \nabla \Phi \left(\mathbf{x} \right), \Phi \left(\mathbf{x} \right) + \lambda^0 \right).$$

Note that the form of $X_H^{\sigma_\lambda}$ does not depend on λ [see Eq. (33)]. On the other hand, for $\lambda = \left(\lambda^0, \lambda^1, ..., \lambda^n \right) = (0, 0, ..., 0) \equiv 0$, we have that $\mathsf{Im} \sigma_0 = \mathcal{L}$ [see Eq. (19)] and $X_H^{\sigma_0} = X_{H|\mathcal{L}} = \hat{X}$ [see Eqs. (20) and (36)]. So, one of the partial solutions defines a Legendrian submanifold.

Remark. A relationship between thermodynamical systems and the Hamilton-Jacobi Theory was also explored in Ref. [24], but from another point of view.

3.2.2 A useful characterization

In the following, we shall give an alternative description of the complete solutions. Fix a contact manifold (M, η) and a fibration $\Pi : M \to N$.

Theorem 2. *A surjective local diffeomorphism* $\Sigma : N \times \Lambda \to M$, *satisfying* $\Pi \circ \Sigma = \mathfrak{p}_N$, *is a complete solution of the* Π-*HJE for a contact vector* X_H *if and only if*

$$i_{X_H^\Sigma} \Sigma^* \mathsf{d}\eta = \Sigma^* \left(\xi \left(H \right) \eta - \mathsf{d}H \right) \quad and \quad i_{X_H^\Sigma} \Sigma^* \eta = \Sigma^* H, \tag{44}$$

i.e. X_H^Σ *is a contact field for the contact form* $\Sigma^* \eta$ *and for the function* $\Sigma^* H = H \circ \Sigma$.

Proof. Suppose that Σ is a complete solution. Since

$$\left(i_{X_H^\Sigma} \Sigma^* \eta \right) (p, \lambda) = \eta_{\Sigma(p,\lambda)} \left(\Sigma_* \circ X_H^\Sigma (p, \lambda) \right), \quad (p, \lambda) \in N \times \Lambda, \tag{45}$$

the second part of (4) tells us that

$$\left(i_{X_H^\Sigma} \Sigma^* \eta \right) (p, \lambda) = \eta_{\Sigma(p,\lambda)} \left(X_H \circ \Sigma (p, \lambda) \right) = \left(i_{X_H} \eta \right) \left(\Sigma (p, \lambda) \right).$$

Then, we obtain the second part of (44) from the corresponding one of (15). On the other hand, given any vector $v \in T_p N \times T_\lambda \Lambda$,

$$\left(i_{X_H^\Sigma} \Sigma^* \mathsf{d}\eta \right)_{(p,\lambda)} (v) = (\mathsf{d}\eta)_{\Sigma(p,\lambda)} \left(\Sigma_* \circ X_H^\Sigma (p, \lambda), \Sigma_{*,(p,\lambda)} (v) \right), \tag{46}$$

and, using the second part of (4),

$$\left(i_{X_H^\Sigma} \Sigma^* \mathsf{d}\eta \right)_{(p,\lambda)} (v) = (\mathsf{d}\eta)_{\Sigma(p,\lambda)} \left(X_H \circ \Sigma (p, \lambda), \Sigma_{*,(p,\lambda)} (v) \right) = (i_{X_H} \mathsf{d}\eta)_{\Sigma(p,\lambda)} \left(\Sigma_{*,(p,\lambda)} (v) \right).$$

Also,

$$\left[\Sigma^* \left(\xi \left(H \right) \eta - \mathsf{d}H \right) \right]_{(p,\lambda)} (v) = \left(\xi \left(H \right) \eta - \mathsf{d}H \right)_{\Sigma(p,\lambda)} \left(\Sigma_{*,(p,\lambda)} (v) \right). \tag{47}$$

So, from the first part of (15) we obtain the first one of (44).

Now, let us show the converse. Suppose that a local diffeomorphism Σ satisfies (44). From (45) and the second parts of (15) and (44) it follows that

$$\eta_{\Sigma(p,\lambda)} \left(\Sigma_* \circ X_H^\Sigma (p, \lambda) - X_H \circ \Sigma (p, \lambda) \right) = 0. \tag{48}$$

On the other hand, given again $v \in T_p N \times T_\lambda \Lambda$, Eqs. (46) and (47) and the first parts of (15) and (44) tell us that

$$(\mathsf{d}\eta)_{\Sigma(p,\lambda)} \left(\Sigma_* \circ X_H^\Sigma (p, \lambda) - X_H \circ \Sigma (p, \lambda), \Sigma_{*,(p,\lambda)} (v) \right) = 0.$$

Since v is arbitrary and $\Sigma_{*,(p,\lambda)}$ is bijective, above equation implies that there is $\alpha \in \mathbb{R}$ such that

$$\Sigma_* \circ X_H^\Sigma (p, \lambda) - X_H \circ \Sigma (p, \lambda) = \alpha \, \xi \left(\Sigma (p, \lambda) \right). \tag{49}$$

Combining (48) and (49), we have that $\alpha = 0$, from which the converse of the theorem follows. $\triangle$

It is worth noticing that being a complete solution is (in essence) the same as fulfilling Eq. (44) (as we showed above), but being a partial solution is not the same as satisfying Eq. (23) (see the corollaries of the previous section).

4 An integration procedure

Fix again a contact manifold (M, η), a contact vector field X_H and a fibration $\Pi : M \to N$. We shall see in this section that, under some additional assumptions, having a complete solution of the Π-HJE for X_H ensures that the integral curves of X_H can be constructed up to quadratures. This extends to contact manifolds the results obtained in [10] (and briefly reviewed in Section 1) for symplectic and Poisson manifolds.

4.1 Pseudo-isotropy condition

Assume that we have a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for some vector field X, non-necessarily Hamiltonian.

Definition 3. We shall say that Σ is **pseudo-isotropic** if

$$\sigma_\lambda^* \mathrm{d}\eta = 0, \quad \forall \lambda \in \Lambda, \tag{50}$$

and that it is **isotropic** if each partial solution σ_λ is isotropic, i.e.

$$\sigma_\lambda^* \eta = 0, \quad \forall \lambda \in \Lambda. \tag{51}$$

Note that isotropy implies pseudo-isotropy. Also, the condition (50) is the same as saying that the partial solutions satisfy $\mathsf{Im}\,(\sigma_\lambda)_* \subseteq (\mathsf{Im}\,(\sigma_\lambda)_*)^\perp$.

Remark. It is worth mentioning that isotropy condition does not hold, generically, for contact Hamiltonian systems. In fact, if σ_λ were isotropic for all λ, the second part of Eq. (23) would say that H is the null function. Nevertheless, see Definition 7.

Let us go back to the case of contact Hamiltonian systems. An immediate consequence of Eq. (50) is the fact that, for all $\lambda \in \Lambda$, we can construct up to quadratures a function W_λ such that $\sigma_\lambda^* \eta = \mathrm{d}W_\lambda$, at least locally. From now on, assume for simplicity that N is connected and simply connected.

Remark 7. If this is not the case, for any $p \in N$ consider a connected and simply connected open neighborhood V of p and an open subset $\Lambda_V \subseteq \Lambda$ such that Σ restricted to $V \times \Lambda_V$ is a diffeomorphism. Then, in what follows, replace N by V, Λ by Λ_V, and M by $\Sigma\,(V \times \Lambda_V)$.

Under above assumption, W_λ is globally defined and the function $W : N \times \Lambda \to \mathbb{R} : (p, \lambda) \to W_\lambda\,(p)$ satisfies

$$i_X\,(\mathrm{d}W - \Sigma^*\eta) = 0, \tag{52}$$

for all $X \in \mathfrak{X}\,(N \times \Lambda)$ such that $\mathsf{Im}X \subseteq TN \times \{0\}$. In particular, since $\mathsf{Im}X_H^\Sigma \subseteq TN \times \{0\}$ [recall Eq. (6)], we have from (44) and (52) that

$$\mathsf{L}_{X_H^\Sigma} W = i_{X_H^\Sigma} \mathrm{d}W = i_{X_H^\Sigma} \Sigma^*\eta = \Sigma^* H \tag{53}$$

and

$$i_{X_H^\Sigma} \Sigma^* \mathrm{d}\eta = i_{X_H^\Sigma} \mathrm{d}\Sigma^*\eta = -i_{X_H^\Sigma} \mathrm{d}\,(\mathrm{d}W - \Sigma^*\eta) = -\mathsf{L}_{X_H^\Sigma}\,(\mathrm{d}W - \Sigma^*\eta)\,. \tag{54}$$

Then, on the one hand, for every $\lambda \in \Lambda$ and every integral curve γ of $X_H^{\sigma_\lambda}$, it follows from (53) that

$$\frac{d}{dt} W_\lambda\,(\gamma\,(t)) = H\,(\Sigma\,(\gamma\,(t)\,, \lambda))\,. \tag{55}$$

On the other hand, consider the function $\varphi : N \times \Lambda \to T^*\Lambda$ such that $\varphi\,(p, \lambda) \in T_\lambda^*\Lambda$, for all $(p, \lambda) \in N \times \Lambda$, and

$$\langle \varphi\,(p, \lambda)\,, z \rangle = \langle (\mathrm{d}W - \Sigma^*\eta)\,(p, \lambda)\,, (0, z) \rangle\,, \quad \forall z \in T_\lambda\Lambda. \tag{56}$$

(Recall Eq. (8) of Section 1). Consider also the related functions $\varphi_\lambda : p \in N \mapsto \varphi(p, \lambda) \in T_\lambda^* \Lambda$. Then, using (52) and the fact that $\operatorname{Im}\left[X_H^\Sigma, Z\right] \subseteq TN \times \{0\}$ for any vector field $Z \in \mathfrak{X}(\Lambda)$, we have that

$$\mathsf{L}_{X_H^\Sigma} \circ i_Z \left(\mathsf{d}W - \Sigma^* \eta\right) = i_Z \circ \mathsf{L}_{X_H^\Sigma} \left(\mathsf{d}W - \Sigma^* \eta\right) \tag{57}$$

and consequently [see Eq. (54)]

$$\left\langle \frac{d}{dt}\varphi_\lambda\left(\gamma\left(t\right)\right), z \right\rangle = \left\langle \left(\mathsf{L}_{X_H^\Sigma}\left(\mathsf{d}W - \Sigma^* \eta\right)\right)\left(\gamma\left(t\right), \lambda\right), (0, z) \right\rangle = -\left\langle \left(i_{X_H^\Sigma}\Sigma^* \mathsf{d}\eta\right)\left(\gamma\left(t\right), \lambda\right), (0, z) \right\rangle, \tag{58}$$

for all $z \in T_\lambda \Lambda$. Identities (55) and (58), together with the next result, will be very useful in the rest of the paper.

Proposition 1. *Each function* $(\varphi_\lambda, W_\lambda) : N \to T_\lambda^* \Lambda \times \mathbb{R}$ *is an immersion. As a consequence,*

$$(\varphi, W) : N \times \Lambda \to T^* \Lambda \times \mathbb{R}$$

is an immersion also.

Proof. Since φ_λ takes values in the vector space $T_\lambda^* \Lambda$, let us identify its differential $(\varphi_\lambda)_*$ with a function from TN to $T_\lambda^* \Lambda$. Under this identification, it is easy to show that

$$\left\langle (\varphi_\lambda)_{*,p}(x), z \right\rangle = \left[\mathsf{d}\left(\mathsf{d}W - \Sigma^* \eta\right)\right]_{(p,\lambda)}\left((x, 0), (0, z)\right) = -\left(\Sigma^* \mathsf{d}\eta\right)_{(p,\lambda)}\left((x, 0), (0, z)\right),$$

for all $p \in N$, $x \in T_p N$ and $z \in T_\lambda \Lambda$. Note that, by the pseudo-isotropy condition,

$$\left(\Sigma^* \mathsf{d}\eta\right)_{(p,\lambda)}\left((x, 0), (0, z)\right) = \left(\Sigma^* \mathsf{d}\eta\right)_{(p,\lambda)}\left((x, 0), (y, z)\right),$$

for all $y \in T_p N$. Then, if $(\varphi_\lambda)_{*,p}(x) = 0$, we must have that $\Sigma_{*,(p,\lambda)}(x, 0) = \alpha\,\xi(p, \lambda)$, for some number α. On the other hand, also by pseudo-isotropy [see for instance Eq. (52)],

$$\left(W_\lambda\right)_{*,p}(x) = \left(\Sigma^* \eta\right)_{(p,\lambda)}(x, 0) = \alpha.$$

So, if we also ask that $(W_\lambda)_{*,p}(x) = 0$, then $\alpha = 0$ and consequently x must be zero. This ends our proof. $\triangle$

The theorem below is the analogous for contact manifolds of Proposition 3.17 of Ref. [10] (for symplectic manifolds), and reflects the geometric meaning of the function (φ, W).

Theorem. *Let* (M, η) *be a contact manifold and* $\Sigma : N \times \Lambda \to M$ *a pseudo-isotropic complete solution of the* Π-*HJE for some vector field X. Consider on $N \times \Lambda$ the contact form $\Sigma^* \eta$ and on $T^* \Lambda \times \mathbb{R}$ the one given by the canonical 1-form on $T^* \Lambda$ (see Remark 1). Then, $(\varphi, W) : N \times \Lambda \to T^* \Lambda \times \mathbb{R}$ is a anti-morphism of contact manifolds.*[3]

Proof. We must show that $(\varphi, W)^* \eta_\Lambda = -\Sigma^* \eta$. Given a vector $(x, y) \in T_p N \times T_\lambda \Lambda$, we have that

$$
\begin{aligned}
\left[(\varphi, W)^* \eta_\Lambda\right](x, y) &= \eta_\Lambda\left(\varphi_{*,(p,\lambda)}(x, y), \mathsf{d}W(p, \lambda)(x, y)\right) \\
&= \left\langle \varphi_\lambda(p), y \right\rangle + \left\langle \mathsf{d}W(p, \lambda), (x, y) \right\rangle \\
&= \left\langle \left(\mathsf{d}W - \Sigma^* \eta\right)(p, \lambda), (0, y) \right\rangle + \left\langle \mathsf{d}W(p, \lambda), (x, y) \right\rangle,
\end{aligned}
$$

and, using that $\left\langle \left(\mathsf{d}W - \Sigma^* \eta\right)(p, \lambda), (x, 0) \right\rangle = 0$ [see Eq. (52)], the wanted identity follows. $\triangle$

---

[3]Given two contact manifolds (M_1, η_1) and (M_2, η_2), a smooth function $f : M_1 \to M_2$ is a **morphism** (resp. **anti-morphism**) of contact manifolds if $f^* \eta_2 = \eta_1$ (resp. $f^* \eta_2 = -\eta_1$).

4.2 Contact systems satisfying $\xi(H) = 0$

Now suppose that we have a contact system with H such that $\xi(H) = 0$. Of course, this condition is fulfilled by the Reeb vector field.

Remark. In the example of Section 2.2.1, we have that $\xi(H) = a^0$ [see Eqs. (13) and (17)]. So, taking $a^0 = 0$, we have from (17) a big family of examples satisfying the condition $\xi(H) = 0$, all of them with a precise physical meaning (see Remark 2).

Let Σ be a pseudo-isotropic complete solution. When $\xi(H) = 0$, Eq. (44) says that $i_{X_H^\Sigma}\Sigma^*\mathsf{d}\eta = -\Sigma^*\mathsf{d}H$. So, by the pseudo-isotropy condition we have that

$$i_X\Sigma^*\mathsf{d}H = 0,$$

for all $X \in \mathfrak{X}(N \times \Lambda)$ such that $\mathsf{Im}X \subseteq TN \times \{0\}$, or equivalently

$$\sigma_\lambda^*\mathsf{d}H = 0, \quad \forall \lambda \in \Lambda.$$

Assuming again that N is connected, it follows that each function $H \circ \sigma_\lambda \in C^\infty(N)$ is constant. In other words, we can construct a function $h : \Lambda \to \mathbb{R}$ such that

$$h \circ \mathfrak{p}_\Lambda = H \circ \Sigma. \tag{59}$$

This implies that

$$i_{X_H^\Sigma}\Sigma^*\mathsf{d}\eta = -\mathfrak{p}_\Lambda^*\mathsf{d}h.$$

As a consequence, given an integral curve γ of $X_H^{\sigma_\lambda}$, Eq. (58) tells us in this case that

$$\frac{d}{dt}\varphi_\lambda(\gamma(t)) = \mathsf{d}h(\lambda),$$

or equivalently,

$$\varphi_\lambda(\gamma(t)) = t\,\mathsf{d}h(\lambda) + \varphi_\lambda(\gamma(0)). \tag{60}$$

The latter is exactly the equation that appeared in Section 1 [see Eq. (9)] in the context of symplectic manifolds. On the other hand, using (55) and (59),

$$\frac{d}{dt}W_\lambda(\gamma(t)) = h(\lambda),$$

i.e.

$$W_\lambda(\gamma(t)) = t\,h(\lambda) + W_\lambda(\gamma(0)). \tag{61}$$

Accordingly, if we want to find the integral curves γ of each vector field $X_H^{\sigma_\lambda}$, we just need to solve the algebraic equations given by (60) and (61), which can be solved univocally because the function $(\varphi_\lambda, W_\lambda)$ is an immersion (see Proposition 1). Summing up, we have the next result.

Theorem 3. *Given a contact manifold (M, η), a fibration $\Pi : M \to N$ and a function $H : M \to \mathbb{R}$ such that $\xi(H) = 0$, if we know a pseudo-isotropic complete solution Σ of the Π-HJE for X_H, then the integral curves of X_H can be constructed up to quadratures. In the particular case when $H = 1$, we have that the integral curves of $X_H = \xi$, i.e. the Reeb vector flow, can be constructed up to quadratures.*

Its proof follows directly from above calculations when N is connected and simply connected. Otherwise, we can replicate such calculations around every point $p \in N$, as emphasized in Remark 7.

The previous theorem extends to contact systems (M, X_H) (with $\xi(H) = 0$) the result obtained in Ref. [10] for Hamiltonian systems on symplectic manifolds. When the condition $\xi(H) = 0$ is not satisfied, we need another kind of complete solutions. They will be studied in the next subsection.

4.3 The g-pseudo-isotropy condition

Consider again a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for some contact field X_H.

Definition 4. We shall say that Σ is g-**pseudo-isotropic for** H if

$$\sigma_\lambda^* \mathsf{d} \left(g \, \eta \right) = 0, \quad \forall \lambda \in \Lambda, \tag{62}$$

for some non-vanishing function $g \in C^\infty \left(M \right)$ such that

$$X_H \left(g \right) + g \, \xi \left(H \right) = 0. \tag{63}$$

A pseudo-isotropic solution is also g-pseudo-isotropic for every H such that $\xi \left(H \right) = 0$. We just must take $g = 1$ above.

The next result try to explain the meaning of Eq. (63). The proof is left to the reader.

Proposition 2. *Given two functions H and g on M, we have that*

$$i_{X_H} \mathsf{d} \left(g \eta \right) = \left(X_H \left(g \right) + g \, \xi \left(H \right) \right) \eta - \mathsf{d} \left(g H \right) \quad and \quad i_{X_H} g \eta = g H. \tag{64}$$

In addition, if g is a non-vanishing function, then X_H is also a contact field w.r.t. to the contact form $\eta' = g \eta$, but with Hamiltonian function $H' = g H$. And denoting ξ' the Reeb vector field for η', we have that

$$\xi' \left(H' \right) = \frac{X_H \left(g \right) + g \, \xi \left(H \right)}{g}. \tag{65}$$

So, if we have a g-pseudo-isotropic complete solution for H, we can see it as a pseudo-isotropic complete solution for the contact manifold (M, η') and the contact field with Hamiltonian H'. Moreover, from Eqs. (63) and (65) we have that $\xi' \left(H' \right) = 0$. Consequently, we are in the situation of the Theorem 3, and the next result easily follows.

Theorem 4. *Given a contact manifold (M, η), a fibration $\Pi : M \to N$ and a function $H : M \to \mathbb{R}$, if we know a g-pseudo-isotropic complete solution Σ of the Π-HJE for X_H, then the integral curves of X_H can be constructed up to quadratures.*

Suppose now that we have a contact vector field X_H with $H \left(p \right) \neq 0$ for all p. It is easy to see that $g = 1/H$ is a solution of (63). So, according to Eq. (64) of Proposition 2, X_H is the Reeb vector field of the contact form η/H, and the next result follows form the last part of Theorem 3.

Theorem 5. *Given a contact manifold (M, η), a fibration $\Pi : M \to N$ and a non-vanishing function $H : M \to \mathbb{R}$, if we know a complete solution Σ of the Π-HJE for X_H such that*

$$\sigma_\lambda^* \mathsf{d} \left(\frac{\eta}{H} \right) = 0, \quad \forall \lambda \in \Lambda, \tag{66}$$

then the integral curves of X_H can be constructed up to quadratures.

Remark 8. If H is a non-vanishing function and Σ is a complete solution, it is easy to show that the following statements are equivalent:

- Σ satisfies Eq. (66).

- Σ is pseudo-isotropic w.r.t. the contact form η/H.

- Σ is $1/H$-pseudo isotropic for H (w.r.t. the contact form η).

4.3.1 Thermodynamic processes revisited

Let us go back to Section 3.2.1. Recall we assumed there that a^0 is constant. The complete solution given by Eqs. (42) and (43) is g-pseudo-isotropic with $g = e^f$. To show it, first note that

$$X_H(f) = \hat{X}(f) = -a^0 = -\xi(H),$$

where we have used Eqs. (18) and (39) and the fact that $\xi(H) = a^0$. Then $X_H(e^f) + e^f \xi(H) = 0$ [recall Eq. (63)]. Secondly,

$$(\sigma_\lambda)_{*,\mathbf{x}}(\mathsf{v}) = \left(\mathsf{v}, \left(\lambda^k \nabla g_k\right)\left(\nabla e^{-f}\right)\cdot\mathsf{v} + \left(e^{-f}\left(\lambda^k\,\mathsf{Hess}\,(g_k)\right) - \mathsf{Hess}\,(\Phi)\right)\cdot\mathsf{v}, \nabla\left(\Phi + \lambda^0\,e^{-f}\right)\cdot\mathsf{v}\right),$$

for all $\mathsf{v} \in \mathbb{R}^n$, and consequently

$$\eta\left((\sigma_\lambda)_{*,\mathbf{x}}(\mathsf{v})\right) = \left(e^{-f}\left(\lambda^k\,\nabla g_k\right) - \nabla\Phi\right)\cdot\mathsf{v} + \nabla\left(\Phi + \lambda^0\,e^{-f}\right)\cdot\mathsf{v} = e^{-f}\left(\lambda^k\,\nabla g_k - \lambda^0\,\nabla f\right)\cdot\mathsf{v},$$

that is to say,

$$\sigma_\lambda^*\eta = e^{-f}\,\mathsf{d}\left(-\lambda^0\,f + \lambda^k\,g_k\right). \tag{67}$$

This clearly implies that $\sigma_\lambda^*\mathsf{d}\left(e^f\,\eta\right) = 0$, and the g-pseudo-isotropy of Σ is proved. In particular, Eq. (67) says that the function $W : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}$ can be taken as

$$W(\mathbf{x}, \lambda) := -\lambda^0\,f(\mathbf{x}) + \lambda^k\,g_k(\mathbf{x}). \tag{68}$$

Remark. As mentioned in Remark 6, if $a^0 = 0$, or equivalently $\xi(H) = 0$, we have a solution in which $f = 0$. Then, the functions σ_λ define a pseudo-isotropic complete solution in that case.

On the other hand, using (41), we have that

$$H \circ \Sigma(\mathbf{x}, \lambda) \quad = \quad H\left(\mathbf{x}, e^{-f(\mathbf{x})}\left(\lambda^k\,\nabla g_k(\mathbf{x})\right) - \nabla\Phi(\mathbf{x}), \Phi(\mathbf{x}) + \lambda^0\,e^{-f}\right)$$

$$= \quad e^{-f}\left(a^0\,\lambda^0 + \lambda^k\,c_k\right).$$

This means that

$$\left(e^f H\right) \circ \Sigma(\mathbf{x}, \lambda) = h(\lambda)$$

with $h : \mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$h\left(\lambda^0, \lambda^1, ..., \lambda^n\right) = a^0\,\lambda^0 + \lambda^k\,c_k.$$

Now, let us calculate the application $\varphi : \mathbb{R}^n \times \mathbb{R}^{n+1} \to T^*\mathbb{R}^{n+1}$. If we identify $T\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1} \times (\mathbb{R} \times \mathbb{R}^n)$, given $(\varsigma, \mathsf{w}) \in \mathbb{R} \times \mathbb{R}^n$, it easily follows that [see Eq. (68)]

$$\langle\mathsf{d}W(\mathbf{x}, \lambda), (0, (\varsigma, \mathsf{w}))\rangle = -f(\mathbf{x})\,\varsigma + G(\mathbf{x})\cdot\mathsf{w}$$

and

$$\Sigma_{*,(\mathbf{x},\lambda)}(0, (\varsigma, \mathsf{w})) = \left(0, e^{-f(\mathbf{x})}\,DG(\mathbf{x})\cdot\mathsf{w}, e^{-f(\mathbf{x})}\,\varsigma\right),$$

where $G = (g_1, ..., g_n)$. Then

$$\eta\left(\Sigma_{*,(\mathbf{x},\lambda)}(0, (\varsigma, \mathsf{w}))\right) = e^{-f(\mathbf{x})}\,\varsigma$$

and

$$\langle\varphi(\mathbf{x}, \lambda), (\varsigma, \mathsf{w})\rangle = \langle(\mathsf{d}W)(\mathbf{x}, \lambda) - \Sigma^*\left(e^f\eta\right), (0, (\varsigma, \mathsf{w}))\rangle = -(1 + f(\mathbf{x}))\,\varsigma + G(\mathbf{x})\cdot\mathsf{w},$$

what implies that

$$\varphi(\mathbf{x}, \lambda) = (-1 - f(\mathbf{x}), G(\mathbf{x})).$$

Here we are identifying $T^*_\lambda \mathbb{R}^{n+1}$ with $\mathbb{R} \times \mathbb{R}^n$. With all that, the equations of motion reduce to

$$\frac{d}{dt} W_\lambda \left(\mathbf{x} \left(t \right) \right) = a^0 \, \lambda^0 + \lambda^k \, c_k$$

and

$$\frac{d}{dt} \varphi_\lambda \left(\mathbf{x} \left(t \right) \right) = a^0 \, \mathsf{d}\lambda^0 + c_k \, \mathsf{d}\lambda^k,$$

being both of them equivalent to

$$\frac{d}{dt} f \left(\mathbf{x} \left(t \right) \right) = a^0$$

and

$$\frac{d}{dt} g_i \left(\mathbf{x} \left(t \right) \right) = -c_i, \quad i = 1, ..., n. \tag{69}$$

Since the g_i's are functionally independent functions, it is enough to consider Eq. (69) in order to find $\mathbf{x}\left(t\right)$. The latter define the integral curves of each vector field $X_H^{\sigma_\lambda} \in \mathfrak{X}\left(\mathbb{R}^n\right)$, and in particular of $\hat{X} = X_{H|\mathcal{L}}$ [see Eq. (20)]. The fact that they can be found up to quadratures can also be explained from the following: we are dealing with a vector field on an n-dimensional manifold and with $n-1$ first integrals (see Remark 5).

4.3.2 A unifier point of view

In this paper we are actually working with a restricted notion of the concept of contact manifold. The more general one (see Ref. [2]) is given by a pair $(M, \mathcal{H})$, where M is an $(2n+1)$-dimensional manifold and $\mathcal{H}$ is a distribution on M of dimension $2n$: the **contact structure**, which satisfy a certain "non-degeneracy condition." Such a condition can be stated as follows: around every point of M there exists an open neighborhood U and a local 1-form α such that $\mathsf{Ker}\,\alpha = \mathcal{H}_{|U}$ and $\left(\mathsf{d}\alpha\right)^n \wedge \alpha \neq 0$. Our notion corresponds to the case in which the contact structure can be described by a global 1-form. Such subclass of contact manifolds are sometime called **co-oriented contact manifolds**. Given two of them, say (M, η) and (M, η'), it is clear that they give rise to the same contact structure $\mathcal{H}$ if and only if there exists a non-vanishing function g such that $\eta' = g\,\eta$. This defines an equivalence relation between co-oriented contact manifolds. On the other hand, we can say that a vector field $X \in \mathfrak{X}(M)$ is a (global) Hamiltonian contact field w.r.t. the contact structure $\mathcal{H}$ if it is a Hamiltonian contact field w.r.t. some global contact form η (if exists) defining $\mathcal{H}$. As stated in Proposition 2, if $X = X_H$ for some contact form η, then $X = X_{gH}$ for the equivalent contact form $g\,\eta$.

So, if we think of our contact systems as triples $(M, \mathcal{H}, X)$, where $\mathcal{H}$ is a contact structure and X is a Hamiltonian contact field w.r.t. $\mathcal{H}$, we can unify the Theorems 3, 4 and 5 as follows.

Theorem 6. *If we know a contact form η defining $\mathcal{H}$, with Reeb vector field ξ, such that the Hamiltonian function H of X w.r.t. η satisfies $\xi(H) = 0$, and we know a pseudo-isotropic (w.r.t. η) complete solution of the Π-HJE for X, then X is integrable by quadratures.*

4.4 Comments on contact non-commutative integrability

In the context of symplectic manifolds, the notion of non-commutative integrability was originally presented in [21]. See also [11] and references therein. For contact manifolds we shall consider the following notion, based on Refs. [12, 13]. Let (M, η) be a contact manifold with Reeb vector field ξ.

Definition 5. A contact system (M, X_H) is **contact non-commutative integrable (CNCI)** if

1. $\xi(H) = 0$ and

2. there exists a fibration $F : M \to \Lambda$ such that:

(a) (first integrals) $\mathsf{Im}X_H \subset \mathsf{Ker}F_*$,

(b) (pseudo-isotropy) $\mathsf{Ker}F_* \subset (\mathsf{Ker}F_*)^\perp$,

(c) $(\mathsf{Ker}F_*)^\perp$ is integrable and

(d) $\mathsf{Im}\xi \subset \mathsf{Ker}F_*$,

If all above conditions hold, the compact and connected leaves of F are invariant tori, and action-angle-like coordinates can be constructed for M around such leaves [12, 13]. In particular, the system is integrable by quadratures.

Let us explain the meaning of the items $2\,(a)$ and $2\,(b)$. If Λ is an open subset of $\mathbb{R}^l$, the item $2\,(a)$ says that the components $F_1, ..., F_l$ of F must satisfy $X_H\,(F_i) = 0$, which means that such components are constant along the integral curves of X_H. The item $2\,(b)$, on the other hand, says that the leaves of F are pseudo-isotropic submanifolds. This is why we shall say that a submersion F satisfying $2\,(a)$ and $2\,(b)$ defines a **set of pseudo-isotropic first integrals**.

In Reference [10], it is shown a *duality* involving (local) complete solutions of the HJE and (local) first integrals. Concretely, it is shown that, given a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for X_H, we can construct around every point of M a neighborhood U and a submersion $F : U \to \Lambda$ such that

$$\mathsf{Im}\,X_{H|U} \subset \mathsf{Ker}\,F_* \quad \text{and} \quad TU = \mathsf{Ker}\left(\Pi_{|U}\right)_* \oplus \mathsf{Ker}\,F_*.$$

U and F are given by the formulae

$$U := \Sigma\,(V) \quad \text{and} \quad F := p_{\Lambda|U} \circ \left(\Sigma_{|V}\right)^{-1}, \tag{70}$$

where $V \subseteq N \times \Lambda$ is an open subset for which $\Sigma_{|V}$ is a diffeomorphism with its image. Note that

$$\mathsf{Ker}\,F_{*,\Sigma(p,\lambda)} = \mathsf{Im}\,(\sigma_\lambda)_{*,p}, \tag{71}$$

for all $(p, \lambda) \in V$. This means that, if Σ is pseudo-isotropic, the same is true for the leaves of F. Reciprocally, from a submersion $F : M \to \Lambda$ satisfying

$$\mathsf{Im}\,X_H \subset \mathsf{Ker}\,F_* \quad \text{and} \quad TM = \mathsf{Ker}\,\Pi_* \oplus \mathsf{Ker}\,F_*,$$

we can construct, around every point of M, a neighborhood U and a local complete solution Σ of the Π-HJE. The involved subset U is one for which $(\Pi, F)_{|U}$ is a diffeomorphism with its image, and Σ is given by

$$\Sigma = (\Pi, F)_{|U}^{-1} : \Pi\,(U) \times F\,(U) \to U. \tag{72}$$

Also, it is easy to see that (71) holds for all $(p, \lambda) \in \Pi\,(U) \times F\,(U)$. Accordingly, if the leaves of F are pseudo-isotropic, so is Σ.

Summarizing, a pseudo-isotropic complete solution gives rise to a set of local pseudo-isotropic first integrals *via* Eq. (70), and a set of pseudo-isotropic first integrals gives rise to a local pseudo-isotropic complete solution *via* Eq. (72).

From this duality, given a contact system (M, X_H) such that $\xi\,(H) = 0$, if we know a set of pseudo-isotropic first integrals F, i.e. a submersion satisfying the items $2\,(a)$ and $2\,(b)$ of the last definition, then, according to Theorem 3, such a system is integrable up to quadratures. In particular, the other two items $[2\,(c)$ and $2\,(d)]$ are not needed to ensure integrability by quadratures!

Remark. An analogous result was proved in Ref. [10], but in the context of symplectic and Poisson manifolds (see also Ref. [9]).

Moreover, if we consider the notions of contact manifolds and contact systems given in Section 4.3.2, from Theorem 6 and the above discussion we have the next result.

Theorem 7. *Given a contact system $(M, \mathcal{H}, X)$, if we know a contact form η defining $\mathcal{H}$, with Reeb vector field ξ, such that the Hamiltonian function H of X w.r.t. η satisfies $\xi(H) = 0$, and we know a set of pseudo-isotropic first integrals F (for the contact form η), then X is integrable by quadratures.*

As an example, consider the system of Section 2.2.1. It can be described as a triple $(M, \mathcal{H}, X)$ with $M = \mathbb{R}^{2n+1}$, $\mathcal{H} = \mathsf{Ker}\,\eta$, $\eta \coloneqq y_i\,\mathsf{d}x^i + \mathsf{d}z$ and X given by (18). We have shown in Section 4.3.1 that the complete solution Σ given by (42) and (43) is e^f-pseudo-isotropic. So, defining $\eta' = e^f \eta$ and calling ξ' its Reeb vector field, form the Proposition 2 (and the comment below it) we have that the Hamiltonian function of X w.r.t. η' is $H' = e^f H$ [see Eq. (17)] and satisfies $\xi'(H') = 0$. Also, we have that Σ is pseudo-isotropic w.r.t. η' and that $\mathcal{H} = \mathsf{Ker}\,\eta'$. Now, let us find the first integrals related to Σ. It is easy to see that

$$\Sigma^{-1}(\mathbf{x}, \mathbf{y}, z) = \left(\mathbf{x}, e^{f(\mathbf{x})}\,(z - \Phi(\mathbf{x})), e^{f(\mathbf{x})}\,(\mathbf{y} - \nabla\Phi(\mathbf{x}))^t \cdot [DG(\mathbf{x})]^{-1}\right),$$

where $DG(\mathbf{x})$ is the differential of the function $G \coloneqq (g_1, ..., g_n) : \mathbb{R}^n \to \mathbb{R}^n$. Then, Eq. (70) (and the duality above discussed) ensures that

$$F(\mathbf{x}, \mathbf{y}, z) \coloneqq (F_0, ..., F_n)(\mathbf{x}, \mathbf{y}, z) \coloneqq \left(e^{f(\mathbf{x})}\,(z - \Phi(\mathbf{x})), e^{f(\mathbf{x})}\,(\mathbf{y} - \nabla\Phi(\mathbf{x}))^t \cdot [DG(\mathbf{x})]^{-1}\right)$$

defines a set of $n + 1$ pseudo-isotropic first integrals of X (w.r.t. η'). So we have a contact form and a fibration satisfying the theorem above.

Remark. Using Eq. (65), it is easy to show that $\xi'(F_0) = 1$. This means that $\mathsf{Im}\,\xi' \nsubseteq \mathsf{Ker}\,F_*$. Thus, the item $2(d)$ of above definition does not hold for F. Also, it can be shown that $(\mathsf{Ker}\,F_*)^\perp$, which in this case is equal to $\mathsf{Ker}\,F_* + \langle\xi'\rangle$, is not integrable, i.e. item $2(c)$ does not hold either.

5 Another integration procedure

As we have seen in Section 4, if we have a contact system (M, X_H) such that $\xi(H) = 0$, and we know a pseudo-isotropic complete solution for it, then X_H can be integrated up to quadratures. If condition $\xi(H) = 0$ does not hold, to ensure integrability up to quadratures we can try to find a solution g of Eq. (63) and a g-pseudo-isotropic complete solution. When H is a non-vanishing function, we can take $g = 1/H$. Of course, we could also do that in the open submanifold where $H \neq 0$. This is good because, in such a case, we do not need to solve Eq. (63). But, what can we do on the subset of M where $\xi(H) \neq 0$ and $H = 0$? Do we have an alternative to the g-pseudo-isotropic complete solutions? To give an answer to these questions, we first need some definitions.

5.1 The $\xi(H) \neq 0$ case and the bi-isotropy condition

Let (M, X_H) be a contact system such that $\xi(H) \neq 0$. For simplicity, suppose that

$$M_0 \coloneqq H^{-1}(0) = \{m \in M : H(m) = 0\}$$

is a closed regular submanifold of M (i.e. 0 is a regular value of H) and denote by M_1 the (open) complement of M_0, i.e.

$$M_1 \coloneqq \left(H^{-1}(0)\right)^c = \{m \in M' : H(m) \neq 0\}. \tag{73}$$

Consider the related inclusions $i_{0,1} : M_{0,1} \to M$.

Proposition 3. *Under above conditions and notation,*

1. *the pair $(M_0, d\eta_0)$, with $\eta_0 := i_0^* \eta$, is a symplectic manifold. Also, M_0 is an X_H-invariant submanifold. In particular, there exists a vector field $Y \in \mathfrak{X}(M_0)$ such that*

$$i_{0*} \circ Y = X_H \circ i_0; \tag{74}$$

2. *the pair (M_1, η_1), with $\eta_1 := i_1^* \eta$, is a contact manifold and, defining $H_1 := i_1^* H$,*

$$i_{1*} \circ X_{H_1} = X_H \circ i_1.$$

Proof. 1. The condition $\xi(H) \neq 0$ is equivalent to

$$\xi(p) \notin \mathsf{Ker}(dH(p)) = T_p M_0, \quad \forall p \in M_0. \tag{75}$$

So, since $\mathsf{Ker}(d\eta)_p = \langle \xi(p) \rangle$, it follows that $i_0^* d\eta$ is a symplectic form on M_0. On the other hand, since $X_H(H) = H \xi(H)$ [see the first part of Eq. (15)], we have that

$$X_H(p) \in \mathsf{Ker}(dH(p)) = T_p M_0, \quad \forall p \in M_0,$$

what says precisely that M_0 is X_H-invariant.

2. Since M_1 is a open submanifold, it is clear that $i_1^* \eta$ is a contact form. For the same reason, if we restrict (12) to a point $p \in M_1$, we shall find that the Reeb vector field ξ_1 of (M_1, η_1) is given $\xi_1(p) = \xi(p)$ (i.e. the restriction of ξ to M_1). Moreover, if we make the same restriction on Eq. (15), we shall have that

$$d\eta(X_H(p), v) = \left\langle -dH_1(p) + \xi_1(H_1)(p)(\eta_1)_p, v \right\rangle, \quad \forall v \in T_p M_1,$$

and

$$\eta(X_H(p)) = H_1(p).$$

This means, omitting the inclusion i_1, that $X_H(p) = X_{H_1}(p)$ for all $p \in M_1$, as we wanted to show. $\triangle$

It is clear that, if we have the integral curves of the vector fields $Y = (X_H)_{|M_0} \in \mathfrak{X}(M_0)$ and $X_{H_1} = (X_H)_{|M_1} \in \mathfrak{X}(M_1)$ (see proposition above), then we have those of X_H. So, if we want to find such curves by means of a complete solution, we need:

1. to restrict, in some sense, the complete solutions to the X-invariant submanifolds M_0 and M_1;

2. to impose on the resulting restrictions appropriate conditions to ensure integrability by quadratures.

For the first point, we have the next definition (based on Definition 4.17 of Ref. [10]). Consider an arbitrary manifold M, a vector field $X \in \mathfrak{X}(M)$, a fibration $\Pi : M \to N$ and a complete solution $\Sigma : N \times \Lambda \to M$.

Definition 6. Given an X-invariant submanifold $M' \subset M$, we shall say that Σ **restricts to** M' if there exist submanifolds N', Λ' of N and Λ, respectively such that $\Pi_{|M'} : M' \to N'$ is a fibration, $\Sigma(N' \times \Lambda') \subset M'$ and

$$\Sigma_{M'} := \Sigma_{|N' \times \Lambda'} : N' \times \Lambda' \to M'$$

is a complete solution of the $\Pi_{|M'}$-HJE for $X_{|M'}$. We shall call $\Sigma_{M'}$ a **restriction** of Σ to M'. On the other hand, if for every $p \in M'$ there exists an open neighborhood $U \subseteq M'$ of p such that Σ **restricts to** U, then we shall say that Σ **locally restricts to** M', and we shall call **local restriction** of Σ to M' any related map Σ_U.

Remark 9. If M' is a open submanifold, it is always true that M is X-invariant and that Σ locally restricts to M'.

For the second point, we shall consider the following conditions. Let us go back to the contact context.

Definition 7. Under the conditions of Proposition 3, we shall say that a complete solution Σ is **bi-isotropic** if:

1. Σ restricts to M_0 and for some restriction $\Sigma_0 := \Sigma_{M_0}$ we have that

$$(\sigma_{0\lambda})^* \eta_0 = 0 \tag{76}$$

for all its partial solutions $\sigma_{0\lambda}$.

2. Σ restricts to M_1 and for some restriction $\Sigma_1 := \Sigma_{M_1}$ we have that

$$(\sigma_{1\lambda})^* d\left(\frac{\eta_1}{H_1}\right) = 0 \tag{77}$$

for all its partial solutions $\sigma_{1\lambda}$.

The **locally bi-isotropic** version consists in replacing above restrictions Σ_0 and Σ_1 by local restrictions around every point of the corresponding manifolds.

Remark 10. Note that Eq. (76) implies that $(\sigma_{0\lambda})^* d\eta_0 = 0$, i.e. Σ_0 is isotropic in the sense of symplectic manifolds (see Ref. [10]). On the other hand, Eq. (76) says exactly that Σ_1 is $1/H_1$-pseudo-isotropic (w.r.t. the contact form η_1) or, equivalently, pseudo-isotropic w.r.t. η_1/H_1 (see Remark 8).

5.2 The integration process for the $\xi(H) \neq 0$ case

Let Σ be a bi-isotropic complete solution. Then, in particular, we have a complete solution $\Sigma_1 : N_1 \times \hat{\Lambda} \to M_1$ of the $\Pi_{|M_1}$-HJE for the vector field $(X_H)_{|M_1} = X_{H_1}$, which is pseudo-isotropic w.r.t. η_1/H_1. According to Theorem 5, this implies that X_{H_1} is integrable up to quadratures. So, the vector field X_H can be explicitly integrated (up to quadratures) along M_1. Let us see what happens along M_0.

Bi-isotropy condition also ensures that we have a complete solution $\Sigma_0 : N_0 \times \hat{\Lambda} \to M_0$ of the $\Pi_{|M_0}$-HJE for the vector field $(X_H)_{|M_0} = Y \in \mathfrak{X}(M_0)$. Thus, Σ_0 satisfies the identity

$$\Sigma_{0*} \circ Y^{\Sigma_0} = Y \circ \Sigma_0. \tag{78}$$

Also, Eq. (76) ensures that

$$i_{Y^{\Sigma_0}} \Sigma_0^* \eta_0 = 0. \tag{79}$$

[This identity could also be deduced from the second part of (44)]. In addition, using the first part of (44) and the Eqs. (74) and (78), for every vector $v \in T_p N_0 \times T_\lambda \hat{\Lambda}$ we have that

$$
\begin{aligned}
\left(i_{Y^{\Sigma_0}} \Sigma_0^* d\eta_0\right)_{(p,\lambda)} (v) &= (d\eta)_{\Sigma_0(p,\lambda)} \left(i_{0*} \circ \Sigma_{0*} \circ Y^{\Sigma_0}(p,\lambda), (i_0 \circ \Sigma_0)_{*,(p,\lambda)}(v)\right) \\
&= (d\eta)_{\Sigma_0(p,\lambda)} \left(X_H \circ i_0 \circ \Sigma_0(p,\lambda), (i_0 \circ \Sigma_0)_{*,(p,\lambda)}(v)\right) \\
&= \left\langle (\xi(H)\,\eta - dH) \circ i_0 \circ \Sigma_0(p,\lambda), (i_0 \circ \Sigma_0)_{*,(p,\lambda)}(v)\right\rangle,
\end{aligned}
$$

and consequently,

$$i_{Y^{\Sigma_0}} \Sigma_0^* d\eta_0 = \Sigma_0^* i_0^* (\xi(H)\,\eta - dH) = \Sigma_0^* ((\xi(H) \circ i_0)\,\eta_0),$$

where we have used that $i_0^* dH = 0$. Combining (79) and the last equation, we have that

$$\mathsf{L}_{Y^{\Sigma_0}} \Sigma_0^* \eta_0 = \varsigma\, \Sigma_0^* \eta_0, \tag{80}$$

with

$$\varsigma := \xi(H) \circ \mathfrak{i}_0 \circ \Sigma_0. \tag{81}$$

From Eq. (80) we shall construct the integral curves of Y. To do that, as in previous sections, consider the function

$$\hat{\varphi} : N_0 \times \hat{\Lambda} \to T^*\hat{\Lambda}$$

such that $\hat{\varphi}(p, \lambda) \in T^*_\lambda \hat{\Lambda}$, for all $(p, \lambda) \in N_0 \times \hat{\Lambda}$, and

$$\langle \hat{\varphi}(p, \lambda), z \rangle = - \langle (\Sigma_0^* \eta_0)(p, \lambda), (0, z) \rangle, \quad \forall z \in T_\lambda \hat{\Lambda}, \tag{82}$$

with its related functions $\hat{\varphi}_\lambda : N_0 \to T^*_\lambda \hat{\Lambda}$. Using the isotropy property of Σ_0 in the sense of symplectic manifolds (see Remark 10), the next result can be shown as in Ref. [10] (see Proposition 3.16 there).

Proposition 4. *Each function $\hat{\varphi}_\lambda$ (and consequently $\hat{\varphi}$) is an immersion.*

Let us go back to Eq. (80). Given an integral curve $(\gamma(t), \lambda)$ of Y^{Σ_0}, in terms of $\hat{\varphi}_\lambda$ such an equation reads

$$\frac{d}{dt}\hat{\varphi}_\lambda(\gamma(t)) = \varsigma_\lambda(\gamma(t))\,\hat{\varphi}_\lambda(\gamma(t)), \tag{83}$$

with $\varsigma_\lambda(\gamma(t)) := \varsigma(\gamma(t), \lambda)$. [This can be proved by using Eq. (57), but replacing X_H^Σ, η and W by Y^{Σ_0}, η_0 and 0, respectively, and using property (76)]. On the other hand, last proposition implies that, around every point of N_0, and fixing a basis of $T^*_\lambda \hat{\Lambda}$, there exists a number $k = \dim N_0$ of coordinates of $\hat{\varphi}_\lambda$ (w.r.t. that basis) that defines a coordinate system $\psi = (\psi_1, ..., \psi_k)$ for N_0. Then, in terms of ψ, equation above translates to

$$\dot{\psi}_i(t) = \varsigma_\lambda\left(\psi^{-1}(\psi_1(t), ..., \psi_k(t))\right)\psi_i(t), \tag{84}$$

where for simplicity we are writing $\psi_i(\gamma(t)) = \psi_i(t)$. Let us show that above equations can be solved by quadratures. Note first that, since $\xi(H) \neq 0$, the function ς_λ is non-vanishing. Then, beside the trivial solution, we can look for a solution where, for instance, $\psi_1(t)$ is not identically zero. For such a solution we would have that

$$\dot{\psi}_i(t) = \frac{\dot{\psi}_1(t)}{\psi_1(t)}\psi_i(t),$$

and consequently $\psi_i(t) = c_i\psi_1(t)$ for some constant c_i. Replacing this in (84) for $i = 1$, we obtain the equation

$$\dot{\psi}_1(t) = F(\psi_1(t)) \quad \text{with} \quad F(x) = \varsigma_\lambda\left(\psi^{-1}(x, c_2 x, ..., c_k x)\right)x.$$

It is clear that if we find a function ψ_1 satisfying the equation above, which can be constructed up to one quadrature, we have the solution we are looking for. Concluding, we have the next result.

Theorem 8. *Given a contact manifold (M, η), a fibration $\Pi : M \to N$ and a function $H : M \to \mathbb{R}$ such that $\xi(H)$ is a non-vanishing function, if we know a bi-isotropic complete solution Σ of the Π-HJE for X_H, then the integral curves of X_H can be constructed up to quadratures.*

Remark. Of course, if instead of a bi-isotropic complete solution on M we have a complete solution satisfying Eq. (76) on M_0 and a pseudo-isotropic complete solution on M_1 (independent of each other), then we can proceed exactly as we did above. We have introduced the concept of bi-isotropy just with the aim of presenting a set of sufficient conditions (that ensure integrability by quadratures) involving the whole phase space M of the system.

5.3 An illustrative example

Consider the manifold $\mathbb{R}^3$, denote its global coordinates as (q, p, s), and define on it the function H such that

$$H(q, p, s) = \frac{p^2 + q^2}{2} - \alpha s,$$

with $\alpha \neq 0$. We shall consider on $\mathbb{R}^3$ the contact form $\eta = p\, \mathsf{d}q + \mathsf{d}s$. The related contact system, with

$$X_H(q, p, s) = p\, \frac{\partial}{\partial q} - (q + \alpha\, p)\, \frac{\partial}{\partial q} + \left(\frac{q^2 - p^2}{2} - \alpha\, s \right) \frac{\partial}{\partial s},$$

is sometimes known as the 1-*dimensional damped oscillator*.

Remark. This system has also been studied in Ref. [18] (see also Ref. [5]). Here, besides finding a complete solution of its Π-HJE, for a given fibration Π (see below), we also find an expression for some of its trajectories.

Note that

$$\xi(H) = \frac{\partial}{\partial s} H = -\alpha \neq 0. \tag{85}$$

Consider also the fibration $\Pi : \mathbb{R}^3 \to \mathbb{R}$ such that $\Pi(q, p, s) = q$. A section

$$\sigma : \mathbb{R} \to \mathbb{R}^3 \ : \ q \mapsto (q, \phi(q), \chi(q))$$

is a solution of the Π-HJE for above system if and only if

$$\phi'(q)\, \phi(q) = -q - \alpha\, \phi(q) \tag{86}$$

and

$$\chi'(q)\, \phi(q) = \frac{q^2 - (\phi(q))^2}{2} - \alpha\, \chi(q). \tag{87}$$

Since the first equation is homogeneous, a solution ϕ can be found up to quadratures, unless locally. In fact, for $\alpha \neq 2$, the general solution of (86) is given by the algebraic equation

$$\frac{(\phi(q) - a_+\, q)^{b_+}}{(\phi(q) - a_-\, q)^{b_-}} = \frac{\lambda_1}{\sqrt{q}}, \tag{88}$$

with $a_\pm = (-\alpha/2) \pm \sqrt{1 - (\alpha/2)^2}$, $b_\pm = a_\pm / \sqrt{4 - \alpha^2}$ and λ_1 arbitrary. For $\alpha = 2$, the general solution is given by

$$(\phi(q) + q)\, \exp\left(\frac{q}{\phi(q) + q} \right) = \lambda_1.$$

In any case, given a solution ϕ of (86), the general solution of (87) is

$$\chi(q) = \frac{q^2 + (\phi(q))^2}{2\alpha} + \lambda_2\, \exp\left(\int \frac{-1}{\phi(r)} dr \right), \tag{89}$$

with λ_2 also arbitrary. Taking into account the dependence on λ_1 and λ_2 of above solutions, we shall refer to them as ϕ^{λ_1} and $\chi^{\lambda_1, \lambda_2}$. It can be shown that the function $\Sigma : N \times \Lambda \to \mathbb{R}^3$, with $N = \mathbb{R} - \{0\}$ and $\Lambda = (\mathbb{R} - \{0\}) \times \mathbb{R}$, such that

$$\Sigma(q, \lambda_1, \lambda_2) := \left(q, \phi^{\lambda_1}(q), \chi^{\lambda_1, \lambda_2}(q) \right),$$

is a global diffeomorphism with its (open dense) image $M := \Sigma(N \times \Lambda) \subseteq \mathbb{R}^3$, and consequently a complete solution along M. Moreover, since for $\lambda_2 = 0$ we have that [see (89)]

$$H\left(q, \phi^{\lambda_1}(q), \chi^{\lambda_1, 0}(q) \right) = \frac{q^2 + \left(\phi^{\lambda_1}(q) \right)^2}{2} - \alpha\, \chi^{\lambda_1, 0}(q) = 0,$$

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then Σ restricts to $M_0 := H^{-1}(0) \cap M$ (which is a closed regular submanifold of M). Its restriction Σ_0 is defined on $N \times \hat{\Lambda}$, with $\hat{\Lambda} = (\mathbb{R} - \{0\}) \times \{0\} \simeq \mathbb{R} - \{0\}$. Also, related to $M_1 := \left(H^{-1}(0)\right)^c \cap M$, we have an open restriction Σ_1 with $\Lambda' = (\mathbb{R} - \{0\}) \times (\mathbb{R} - \{0\})$. (Recall Definition 6). For dimensional reasons, both Σ_0 and Σ_1 satisfy the conditions of Definition 7. Let us concentrate on M_0. According to Eq. (82),

$$\hat{\varphi}_{\lambda_1} : \mathbb{R} - \{0\} \to T^*_{\lambda_1}(\mathbb{R} - \{0\})$$

is given by

$$\hat{\varphi}_{\lambda_1}(q) = \frac{1}{2} \frac{\partial}{\partial \lambda_1} \left(\phi^{\lambda_1}(q)\right)^2 \mathsf{d}\lambda_1.$$

Then, integrating (83) for a curve $q(t)$ on N (taking into account that in this case $\varsigma = -\alpha$ -see (81) and (85)-), we have that

$$\frac{\partial}{\partial \lambda_1}\left(\phi^{\lambda_1}(q(t))\right)^2 = \frac{\partial}{\partial \lambda_1}\left(\phi^{\lambda_1}(q(0))\right)^2 e^{-\alpha t}.$$

So, in order to find the integral curves of X_H along M_0, it only rests to solve the above algebraic equation for $q(t)$. Once this equation is solved, such integral curves can be written as

$$\left(q(t), \phi^{\lambda_1}(q(t)), \frac{(q(t))^2 + \left(\phi^{\lambda_1}(q(t))\right)^2}{2\alpha}\right).$$

5.4 Addressing the general situation

Consider a contact Hamiltonian system (M, X_H). In order to ensure the integrability up to quadratures of its equations of motion, we can look for a g-pseudo-isotropic solution, as we have seen in Section 4.3, or we can combine the results of Sections 4.2 and 5.2. For the second option, define

$$M' := \{m \in M : \xi(H) \neq 0\} \quad \text{and} \quad M'' := M \setminus M' = \{m \in M : \xi(H) = 0\}.$$

Inside M', consider the subsets

$$M_0 := H^{-1}(0) \cap M' \quad \text{and} \quad M_1 := M' \setminus M_0,$$

and inside M'' consider its interior $M_2 := \mathsf{int}\, M''$. Note that on the open submanifold M' we have, as in the previous sections, a symplectic manifold (M_0, η_0) and a contact manifold (M_1, η_1). In the latter, we have that $(X_H)_{|M_1} = X_{H_{|M_1}}$. Also, since M_2 is an open submanifold, we have another contact manifold (M_2, η_2) where $(X_H)_{|M_2} = X_{H_{|M_2}}$. With these observations in mind, we can enunciate the following theorem. Its proof is contained in those of Theorems 3 and 8.

Theorem 9. *Given a contact manifold (M, η), a fibration $\Pi : M \to N$ and a function $H : M \to \mathbb{R}$, if we know:*

1. *a bi-isotropic solution on $M_0 \cup M_1 = M'$;*

2. *a pseudo-isotropic solution on $M_2 = \mathsf{int}\, M''$.*

then the integral curves of X_H can be constructed up to quadratures on the open dense subset $M \setminus \partial M''$.

Since the topological boundary $\partial M''$ of M'' is not in general an invariant submanifold, the trajectories with initial conditions inside such a subset must be studied separately (case by case).

Acknowledgements

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