THE INTIMATE RELATIONSHIP BETWEEN THE MCNAUGHTON AND THE CHINESE REMAINDER THEOREMS FOR MV-ALGEBRAS

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We show the intimate relationship between McNaughton Theorem and the Chinese Remaindner Theorem for MV-algebras. We develop a very short and simple proof of McNaughton Theorem. The arguing is elementary and right out of the definitions. We exhibit the theorem as just an instance of the Chinese theorem. Since the variety of MV-algebras is arithmetic, the Chinese theorem holds for MV-algebras. However, to make this paper self-contained and entirely elementary, we include a simple proof of this theorem inspired in [2].

Recall the following adjunction and equation that hold in any MV-algebra:

(1) $\quad a)\ x \leq y \oplus z \iff x \ominus z \leq y\,, \qquad b)\ (x \ominus y) \oplus (x \wedge y) = x.$

2. **Lemma.** *Given any MV-algebra A and ideals I_1, I_2 of A, we have:*
If $a_1, a_2 \in A$ are such that $a_1 \equiv a_2\,(I_1,\, I_2)$, then there is $a \in A$ such that $a \equiv a_1\,(I_1)$, and $a \equiv a_2\,(I_2)$.

Proof. We know that $a_1 \ominus a_2$ and $a_2 \ominus a_1 \in (I_1,\, I_2)$. Thus there are $c_1, d_1 \in I_1$ and $c_2, d_2 \in I_2$ such that $a_1 \ominus a_2 \leq c_1 \oplus c_2$ and $a_2 \ominus a_1 \leq d_2 \oplus d_1$. From (1) a) above we have that $(a_1 \ominus a_2) \ominus c_2 \leq c_1$, and $(a_2 \ominus a_1) \ominus d_1 \leq d_2$. Then $(a_1 \ominus a_2) \ominus c_2 \in I_1$ and $(a_2 \ominus a_1) \ominus d_1 \in I_2$.

Set $\quad a = (a_1 \ominus a_2 \ominus c_2) \oplus (a_2 \ominus a_1 \ominus d_1) \oplus (a_1 \wedge a_2).$

Then $\quad [a]_{I_1} = ([a_1]_{I_1} \ominus [a_2]_{I_1}) \oplus ([a_1]_{I_1} \wedge [a_2]_{I_1}) = [a_1]_{I_1}\,,$

the second equation by (1) b) above. Similarly for $[a]_{I_2} = [a_2]_{I_2}$. □

Taking into account the equation $\bigcap_{i=1}^{n-1}(I_i,\, I_n) = (\bigcap_{i=1}^{n-1} I_i,\, I_n)$, this lemma easily generalize by induction to a finite number of ideals.

3. **Theorem** ([2] 2.6.). *Given any MV-algebra A and a finite number of ideals $I_1, \cdots, I_n$ of A, we have:*
If $a_1, \cdots, a_n \in A$ are such that $a_i \equiv a_j(I_i,\, I_j)$ for $i, j \in \{1, \cdots, n\}$, then there exists $a \in A$ such that $a \equiv a_i(I_i)$ for $i \in \{1, \cdots, n\}$.

Recall that a function $[0,\, 1]^n \xrightarrow{f} [0,\, 1]$ is called a McNaughton function iff f is continuous with respect to the natural topology of $[0,\, 1]^n$, and there are linear polynomials $g_1, \ldots g_k$ with integer coefficients such that for each point $x \in [0,\, 1]^n$ there is an index $j \in \{1, \ldots k\}$ with $f(x) = g_j(x)$.

Let F_n be the free MV-algebra on n generators, that is, the algebra of terms in n variables. Each term $a \in F_n$ determines a function $f_a : [0,\, 1]^n \longrightarrow [0,\, 1]$ called a *term function*. Let A_n be the algebra of term

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functions. By definition there is a surjective morphism $F_n \to A_n$, which we do not assume to be an isomorphism (that is, we do not assume Chang's completeness theorem).

Recall that for any linear polynomial in n variables with integer coefficients g there is a term $a \in F_n$ such that $f_a = g^\sharp$ as functions $[0,\, 1]^n \longrightarrow [0,\, 1]$, where $g^\sharp = (g \wedge 0) \vee 1$ ([1, 3.1.9]).

We will apply theorem 3 to the algebra A_n.

4. **Theorem** (McNaughton)**.** *Given any McNaughton function f, there is a term $a \in F_n$ such that $f = f_a$.*

Proof. Let $g_1, \ldots\, g_k$ be linear constituents for f, and let $h_1, \ldots\, h_k$, $h_i \in A_n$, be term functions such that $h_i = g_i^\sharp$. For each permutation σ of the set $\{1, \ldots\, k\}$, let $Z_\sigma = \{x \,|\, h_{\sigma(1)}(x) \leq h_{\sigma(2)} \leq \ldots\, h_{\sigma(k)}\}$. The sets Z_σ cover $[0,\, 1]^n$, Z_σ is the Zero set of the ideal $T_\sigma = (h_{\sigma(1)} \ominus h_{\sigma(2)}\,,\, \ldots\,, h_{\sigma(k-1)} \ominus h_{\sigma(k)})$, $Z_\sigma = Z(T_\sigma)$, and $Z_\sigma \cap Z_\mu = Z(T_\sigma, T_\mu)$. Let $p \in Z_\sigma$, and let $u_\sigma \in \{1, \ldots\, k\}$ be an index such $f(p) = h_{u_\sigma}(p)$. Then as in [1, page 67], $(f = h_{u_\sigma})|_{Z_\sigma}$. Thus, for any two permutations σ, μ, $(h_{u_\sigma} = h_{u_\mu})|_{Z(T_\sigma, T_\mu)}$. It follows from [1, 3.4.8] [1] that $h_{u_\sigma} \equiv h_{u_\mu}(T_\sigma, T_\mu)$. We apply Theorem 3 above to the collection of term functions h_{u_σ} and ideals T_σ in A_n. It follows that there is a term function $h \in A_n$ such that $h \equiv h_{u_\sigma}(T_\sigma)$. Then for any $x \in [0,\, 1]^n$, $x \in Z_\sigma$ fore some σ, and $h(x) = h_{u_\sigma}(x) = f(x)$. Take any $a \in F_n$ with $h = f_a$. □

References

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