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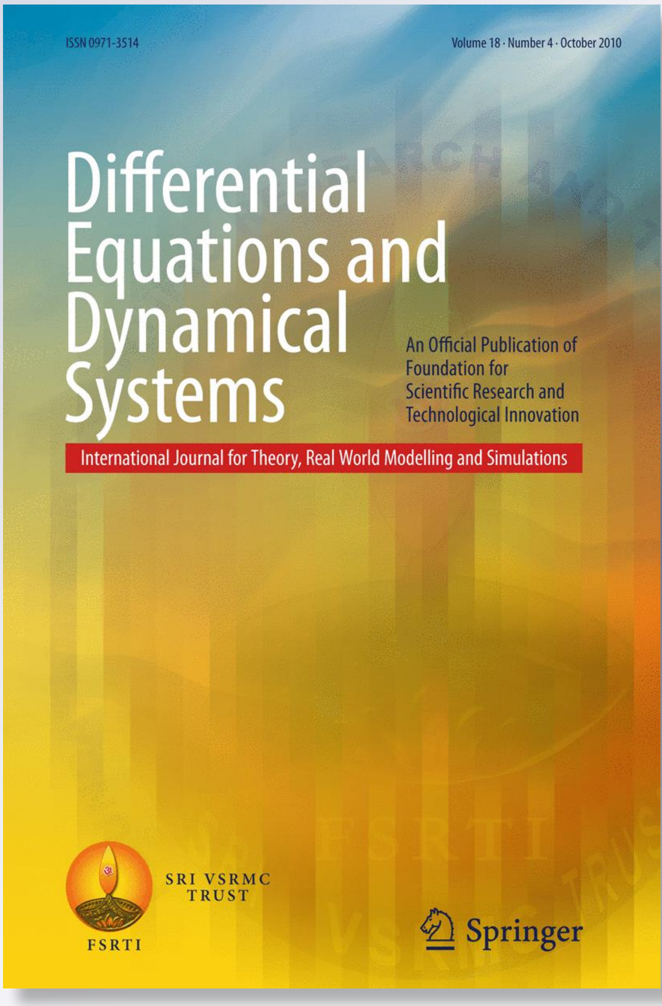

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ORIGINAL RESEARCH

A Nonlinear Problem Depending on the Unknown Dirichlet Values of the Solution

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Abstract A Neumann boundary value problem for a general two ion electro-diffusion model is studied. Unlike classical second order Neumann problems, the nonlinear equation considered in this work has the particularity that it depends on the unknown Dirichlet values of the solution. Using Leray–Schauder topological degree, we prove the existence of at least one solution under non-asymptotic conditions of Landesman–Lazer type.

Keywords Two-ion electro-diffusion models · Landesman–Lazer conditions · Topological degree

Mathematics Subject Classification (2000) 34B15 · 34B99

Introduction

In the last years, there has been an increasing interest in multi-ion electro-diffusion problems. The model equations are derived in [4,5] and, in a more general context, in [8]. Different boundary value problems for these equations have been studied; for example, some particular cases of the two and three ions equations are solved in [2,3]. The Painlevé structure of the equations has been described in [6].

However, a problem of a different type is studied in [10], where the author discusses a two-point boundary value problem arising in the study of two ions with the same valency

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diffusing and migrating across a liquid junction under the influence of an electric field E. Elimination of the ionic concentrations leads to the following equation for the unknown function y, which is proportional to the electric field E in the rescaled interval [0, 1]:

$$y''(x) = y\left\{\lambda - \frac{y_0^2 - y(x)^2}{2} + \left[l\lambda + \frac{y_0^2 - y_1^2}{2}\right]x\right\} - \left[l\lambda + \frac{y_0^2 - y_1^2}{2}\right]D. \tag{1}$$

Here, y_0 and y_1 denote the (unknown) values of the solution at the points $x = 0$ and $x = 1$, respectively. The constants $\lambda > 0$, $l > 0$ and $D \in (-1, 1)$ depend on the physical parameters, such as the diffusion constant. Replacing y by $-y$ if necessary, it may be assumed that $D \geq 0$. Electrical neutrality in the reservoirs yields the homogeneous Neumann boundary conditions

$$y'(0) = y'(1) = 0. \tag{2}$$

Thus, the problem is unconventional, since the equation depends on the yet-to-be determined Dirichlet values of the solution at the boundary of the interval.

Using degree theory and the well-known method of upper and lower solutions, existence of a positive solution of this problem when $D > 0$ has been obtained in [10, Theorem 2], provided that the following relation holds:

$$\lambda \geq 2l\left(1 - \frac{1}{(1+l)^2}\right)D^2. \tag{3}$$

In the recent work [1], it has been proved that condition (3) can be dropped. The proof is based in a two-dimensional shooting method and a careful computation of some accurate estimates, which allow to prove that a certain mapping defined over an appropriate subset of $\mathbb{R}^2$ has nonzero Brouwer degree.

However, although the particular case (1) and (2) has been solved, it is a problem of mathematical interest to determine whether or not existence of solutions can be proved in the more general situation

$$y''(x) = f(x, y(x), y(0), y(1)), \tag{4}$$

where $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary continuous function.

It is worthy to observe that, beside the specific form of the nonlinearity f, the main difficulty of problem (4) relies on the dependence on the unknown Dirichlet boundary values of the solution y. A way of avoiding this consists in considering, instead, the following four-dimensional system of first order equations:

$$\begin{cases} y'(x) = u(x) \\ u'(x) = f(x, y(x), v(x), w(x)) \\ v'(x) = 0 \\ w'(x) = 0, \end{cases} \tag{5}$$

together with the following boundary conditions:

$$\begin{cases} u(0) = u(1) = 0 \\ y(0) = v(0) \\ y(1) = w(1). \end{cases} \tag{6}$$

In this setting, the problem can be interpreted in the context of the so-called *resonant problems*. Indeed, the kernel of the linear operator L given by $L(y, u, v, w) := (y' - u, u', v', w')$

over the space of functions in $C^1([0,T],\mathbb{R}^4)$ satisfying (6) is the non-trivial subspace spanned by the vector $\mathbb{V} := (1, 0, 1, 1)$.

Using the Leray–Schauder topological degree, we shall prove the solvability of (4)–(2) under Landesman–Lazer type conditions (see e.g. [7,9]). Roughly speaking, when $f = p(x) + h(y)$ these conditions ensure the existence of solutions when h is bounded, provided that the average of p lies between the limits of h at $\pm\infty$. In this work, we present an extension of this result for a general f; further, our conditions shall not require any specific assumption on the asymptotic behavior of f.

A first result in this direction is the following straightforward extension of the standard Landesman–Lazer theorem:

Theorem 1 *Assume that f is bounded, and that for every $x \in [0,1]$ the limits*

$$\lim_{s\to\pm\infty} f(x, s + A, s, s + B) := f^{\pm}(x)$$

exist uniformly for $|A|, |B| \le \|f\|_\infty$. Then (4)–(2) admits a solution, provided that one of the following conditions holds:

$$\int_0^1 f^-(x)\,dx < 0 < \int_0^1 f^+(x)\,dx \tag{7}$$

or

$$\int_0^1 f^+(x)\,dx < 0 < \int_0^1 f^-(x)\,dx. \tag{8}$$

It is observed that, in the preceding result, the boundedness condition on the nonlinearity f plays the role of avoiding the interference with the spectrum of the linear operator $Ly := -y''$ under Neumann conditions. However, this goal can be achieved under less restrictive conditions.

In first place, observe that L is nonnegative; in consequence, interference with the spectrum might occur when (8) holds, but it should not be expected when (7) holds, provided that f satisfies some appropriate conditions. In second place, we might consider also that f is sublinear and, again, interference at infinity is avoided. Finally, as we shall prove, the boundedness restriction can be also dropped if we assume that f is bounded, but only from one side. Regarding these situations, we shall consider three different cases (see cases 1–3 in "Existence of solutions").

On the other hand, it may be observed that Theorem 1 is restrictive also in the sense that existence of limits for f is assumed. As in the standard Landesman–Lazer result, it is easily seen that if Fatou's Lemma is used instead of dominated convergence, then an analogous result can be obtained in terms of the asymptotic upper and lower limits for f. However, we shall introduce even more general conditions, that are non-asymptotic, in the sense that they assume a specific behavior of f only over a bounded region of the space. More precisely, for some constants $a < b$ and $r > 0$ to be determined, we shall define the functions

$$f^a_{sup}(x) := \sup_{a-r\le y,y_1\le a+r} f(x,y,a,y_1), \quad f^b_{sup}(x) := \sup_{b-r\le y,y_1\le b+r} f(x,y,b,y_1),$$

and

$$f_{inf}^{a}(x) := \inf_{a-r\le y,y_1\le a+r} f(x,y,a,y_1), \quad f_{inf}^{b}(x) := \inf_{b-r\le y,y_1\le b+r} f(x,y,b,y_1),$$

and establish our main existence result, namely:

Theorem 2 *Assume that one of the situations given by the cases* 1, 2 *or* 3 *in "Existence of solutions" holds, and that*

$$\int_0^1 f_{sup}^{a}(x)\,dx < 0 < \int_0^1 f_{inf}^{b}(x)\,dx \tag{9}$$

or

$$\int_0^1 f_{sup}^{b}(x)\,dx < 0 < \int_0^1 f_{inf}^{a}(x)\,dx. \tag{10}$$

Then (4)–(2) *admits at least one classical solution.*

The paper is organized as follows. In "An abstract continuation theorem", we give an abstract continuation theorem for (5)–(6), that shall be applied for proving our results in "Existence of solutions". Finally, in "Some examples" we provide some simple examples for which the different cases of Theorem 2 are satisfied.

An Abstract Continuation Theorem

Let us consider the following Banach Space

$$E := \left\{\mathbb{X} := (y,u,v,w) \in C([0,1])^4 : \mathbb{X} \text{ satisfies } (6)\right\},$$

equipped with the standard norm

$$\|\mathbb{X}\| := \max\{\|y\|_\infty, \|u\|_\infty, \|v\|_\infty, \|w\|_\infty\}.$$

Thus, we look for an element of E satisfying (5). In order to apply the Leray–Schauder degree method to the problem, we shall introduce a homotopy in the following way: let $\sigma \in [0,1]$, and consider the functional equation in E given by

$$\mathbb{X} = (y(0) + T_{\mathbb{X}}(1))\mathbb{V} + \sigma S(\mathbb{X}) \tag{11}$$

with $\mathbb{V}$ as before,

$$T_{\mathbb{X}}(x) := \int_0^x f(s,y(s),v(s),w(s))\,ds,$$

and

$$S(\mathbb{X})(x) := \left(\int_0^x T_{\mathbb{X}}(s)\,ds,\ T_{\mathbb{X}}(x) - xT_{\mathbb{X}}(1),\ 0,\ \int_0^1 T_{\mathbb{X}}(s)\,ds\right).$$

We claim that $\mathbb{X} \in E$ is a solution of (5) if and only if $\mathbb{X}$ solves (11) with $\sigma = 1$. More generally, we have:

Lemma 1 *Let* $\mathbb{X} \in E$ *and* $0 < \sigma \leq 1$. *Then* $\mathbb{X}$ *is a solution of* (11) *if and only if* $\mathbb{X}$ *satisfies:*

$$\begin{cases} y'(x) = u(x) \\ u'(x) = \sigma f(x, y(x), y(0), y(1)) \\ v'(x) = 0 \\ w'(x) = 0 \end{cases} \tag{12}$$

Proof Let $\mathbb{X}$ be a solution of (11), then its first coordinate y is given by

$$y(x) = y(0) + T_{\mathbb{X}}(1) + \sigma \int_0^x T_{\mathbb{X}}(s)\, ds.$$

It follows that $T_{\mathbb{X}}(1) = 0$, and $y'(x) = \sigma T_{\mathbb{X}}(x) = u(x)$. Moreover, $y''(x) = u'(x) = \sigma f(x, y(x), v(x), w(x))$, and from the last two coordinates in (11) we deduce:

$$v \equiv y(0), \quad w \equiv y(0) + \sigma \int_0^1 T_{\mathbb{X}}(s)\, ds = y(1).$$

Conversely, let $\mathbb{X}$ satisfy (12), then $v \equiv y(0)$, $w \equiv y(1)$ and

$$u' = \sigma f(x, y(x), v, w).$$

As $u(0) = u(1) = 0$, it is seen that $T_{\mathbb{X}}(1) = 0$. Moreover,

$$u(x) = \sigma \int_0^x f(s, y(s), v, w)\, ds = \sigma T_{\mathbb{X}}(x),$$

and as $y' = u$ we obtain that $y(x) = y(0) + \sigma \int_0^x T_{\mathbb{X}}(s)\, ds$. Hence $w = y(0) + \sigma \int_0^1 T_{\mathbb{X}}(s)\, ds$, and the proof is complete. □

The preceding lemma motivates us to define the operator $F_\sigma : E \to E$ given by

$$F_\sigma(\mathbb{X}) = \mathbb{X} - (y(0) + T_{\mathbb{X}}(1))\mathbb{V} - \sigma S(\mathbb{X}).$$

It is easy to see that F_σ is well defined, and it can be written as $F_\sigma = I - K_\sigma$, where $K_\sigma : E \to E$ is compact for any $\sigma \in [0, 1]$. Furthermore, the range of K_0 is contained in the one-dimensional subspace $\mathcal{V} \subset E$ spanned by $\mathbb{V}$: this implies that if $\Omega \subset E$ is a bounded domain such that F_0 does not vanish on $\partial\Omega$ (i.e. on $\partial\Omega \cap \mathcal{V}$), then

$$deg_{LS}(F_0, \Omega, 0) = deg_B(F_0|_{\mathcal{V}}, \Omega \cap \mathcal{V}, 0).$$

It is clear that $\Omega \cap \mathcal{V} = \{s.\mathbb{V} : s \in J\}$ for some open and bounded set $J \subset \mathbb{R}$, and $F_0(s.\mathbb{V}) = -T_{s.\mathbb{V}}(1).\mathbb{V}$.

Thus, $deg_{LS}(F_0, \Omega, 0)$ may be identified with the Brouwer degree of the scalar function $\phi : J \to \mathbb{R}$ given by

$$\phi(s) := -T_{s.\mathbb{V}}(1) = -\int_0^1 f(x, s, s, s)\, dx.$$

If we assume for simplicity that $J = (a, b)$ for some $a < b$ (for example, this is the case when Ω is convex), then by Lemma 1 and the homotopy invariance of the Leray–Schauder degree, we obtain the following continuation theorem:

Theorem 3 *With the previous notations, assume that*

(1) (12) *has no solutions on* $\partial\Omega$ *for any* $\sigma \in (0, 1)$.
(2) $\phi(a).\phi(b) < 0$.

Then (5) *has at least one solution* $\mathbb{X} \in \overline{\Omega}$.

Existence of Solutions

In this section, we shall give a proof of our main result by applying the previous continuation theorem. For a better understanding of the difficulties (and also for historical reasons), we shall give first an elementary proof of Theorem 1, and then proceed with the more general non-asymptotic conditions.

Proof of Theorem 1 In order to apply the Theorem 3, let us firstly observe that if

$$y''(x) = \sigma f(x, y(x), y(0), y(1)), \quad y'(0) = y'(1) = 0$$

for some $\sigma \in (0, 1)$, then

$$\|y - y(0)\|_\infty \leq \|y'\|_\infty \leq \|y''\|_\infty < r := \|f\|_\infty.$$

Thus, we may consider

$$\Omega := \{\mathbb{X} \in E : \|y - v\|_\infty, \|w - v\|_\infty, \|u\|_\infty < r, \|v\|_\infty < R\}$$

for some $R > 0$ to be specified. If $\mathbb{X} = (y, u, v, w) \in \partial\Omega$ solves (12) for some $\sigma \in (0, 1)$, then $\|y - v\|_\infty, \|w - v\|_\infty, \|u\|_\infty < r$, $|y(0)| = R$ and

$$\int_0^1 f(x, y(0) + A(x), y(0), y(0) + B)\, dx = 0, \tag{13}$$

where

$$A := y - y(0) = y - v \quad B := y(1) - y(0) = w - v.$$

From the assumptions, letting $y(0) \to \pm\infty$ we deduce that

$$f(x, y(0) + A(x), y(0), y(0) + B) \to f^{\pm}(x)$$

for each x, and by dominated convergence this contradicts (13). Thus, if $R \gg 0$, then (12) has no solutions on $\partial\Omega$ for $\sigma \in (0, 1)$. Finally, let us note that in this case $J = (-R, R)$, and

$$\phi(s) = -\int_0^1 f(x, s, s, s)\, dx \to -\int_0^1 f^{\pm}(x)\, dx$$

as $s \to \pm\infty$. Hence, the second condition in Theorem 3 is satisfied when R is large enough. □

Next, we shall introduce three specific conditions, regarding the different situations that have been discussed in the introduction.

Let $a < b$ be fixed constants, and consider the following cases:

- Case 1: There exists $k \in L^1(0, 1)$ such that

$$yf(x, y, y_0, y_1) \geq -k(x)$$

for any $x \in [0, 1]$, $y_0 \in [a, b]$ and $|y - y_0|, |y_1 - y_0| \leq r$, for some positive constant r such that $r^2 \geq \int_0^1 k(x)\,dx$.

- Case 2: There exists a constant $r > 0$ such that

$$|f(x, y, y_0, y_1)| \leq r$$

for every $x \in [0, 1]$, $y_0 \in [a, b]$ and $|y - y_0|, |y_1 - y_0| \leq r$.
For example, if f grows at most linearly, namely

$$|f(x, y, y_0, y_1)| \leq \alpha|y| + |y_0| + \beta\gamma|y_1| + \delta,$$

with $2(\alpha + \gamma) + \beta < 1$, then it suffices to take $a = -cr$ and $b = cr$ for some appropriate constants $c > 1$, and r large enough.

- Case 3: There exists $k \in L^1(0, 1)$ such that either

$$f(x, y, y_0, y_1) \geq -k(x)$$

or

$$f(x, y, y_0, y_1) \leq k(x)$$

for every $x \in [0, 1]$, $y_0 \in [a, b]$ and $|y - y_0|, |y_1 - y_0| \leq r$, for some positive constant r such that $r \geq \|k\|_{L^1} + \int_0^1 k(x)\,dx$

Now, we are ready to prove our main result.

Proof of Theorem 3 We shall apply the previous continuation theorem over the set

$$\Omega := \{\mathbb{X} \in E : \|y - v\|_\infty, \|w - v\|_\infty < r, \|u\|_\infty < \tilde{r}, a < v < b\},$$

for some positive constant $\tilde{r}$ to be specified. Let us suppose that $\mathbb{X} \in \partial\Omega$ solves (12) for some $\sigma \in (0, 1)$, and consider the three different cases:

- In case 1, using the fact that $\int_0^1 f(x, y(x), y(0), y(1))\,dx = 0$, we obtain:

$$\int_0^1 y'(x)^2\,dx = -\int_0^1 y'(x)(y(x) - y(0))\,dx$$

$$= -\sigma \int_0^1 y(x) f(x, y(x), y(0), y(1))\,dx \leq \sigma \int_0^1 k(x)\,dx < r^2.$$

- In case 2, it is directly seen that

$$|y''(x)| = \sigma|f(x, y(x), y(0), y(1))| < r,$$

and hence $\|y'\|_\infty < r$.

- In case 3, if for example the first condition holds we obtain:

$$|y''(x)| \leq \sigma\left([f(x, y(x), y(0), y(1)) + k(x)] + |k(x)|\right),$$

and hence

$$\|y'\|_\infty \le \|y''\|_{L^1} \le \sigma \left(\int_0^1 k(x)\,dx + \|k\|_{L^1} \right) < r.$$

Thus, in all cases we obtain that $\|y - y(0)\|_\infty < r$. In cases 2 and 3, it also holds that $\|y'\|_\infty < r$, so we may set $\tilde{r} = r$; in case 1, we deduce that

$$\|y'\|_\infty \le \|y''\|_\infty < \max_{(x,y,y_0,y_1)\in K} |f(x, y, y_0, y_1)| := \tilde{r},$$

where $K := [0, 1] \times [a - r, b + r] \times [a, b] \times [a - r, b + r]$. Hence, $y(0) = a$ or $y(0) = b$. If for example $y(0) = a$, and (9) holds,

$$0 = \int_0^1 f(x, y(x), y(0), y(1))\,dx \le \int_0^1 f^a_{sup}(x)\,dx < 0,$$

a contradiction. The other cases follow in a similar way. Finally, observe that if (9) holds, then

$$\phi(a) = -\int_0^1 f(x, a, a, a)\,dx \ge -\int_0^1 f^a_{sup}(x)\,dx > 0,$$

$$\phi(b) = -\int_0^1 f(x, b, b, b)\,dx \le -\int_0^1 f^b_{inf}(x)\,dx < 0,$$

and the second condition of Theorem 3 is satisfied. The same conclusion yields when (10) holds.

Remark 1 It is easy to verify that Theorem 1 can be deduced as a consequence of Theorem 2.

Remark 2 It is interesting to observe that, in the original problem (1), the (unbounded) nonlinearity has the limits

$$\lim_{s\to\pm\infty} f(x, s, s, s) := \pm\infty.$$

Thus, one might believe, at first sight, that condition (9) is fulfilled under case 1 for some appropriate choices of a, b and r. However, this is not true, since for s large the sign of f changes sign in small neighborhoods of the point (x, s, s, s). Still, the results in [1] and [10] indicate that it could be possible to apply Theorem 3 directly over $\Omega := \{\mathbb{X} \in E : 0 < y, v, w < R, \|u\|_\infty < r\}$ for some constants $r, R > 0$. In this case, the second condition of Theorem 3 is trivially satisfied, and the difficulty relies on the existence of a priori bounds for the solutions of the homotopy problem (12) with $y > 0$. It is not clear whether or not existence of solutions can be proved by means of degree methods without imposing restrictions on the parameters of the problem.

Some Examples

In this section, we give some examples for which the different assumptions of our existence results hold. For simplicity, let us consider $f = p(x) + h(y) + \varphi(y_0, y_1)$, where p, h and

φ are continuous. Without loss of generality, we may assume that $\int_0^1 p(t)\,dt = 0$; then, (9) and (10) now read

$$h^a_{sup} + \varphi^a_{sup} < 0 < h^b_{inf} + \varphi^b_{inf} \tag{14}$$

or

$$h^b_{sup} + \varphi^b_{sup} < 0 < h^a_{inf} + \varphi^a_{inf}, \tag{15}$$

where

$$h^a_{sup} := \sup_{a-r\le y\le a+r} h(y), \quad \varphi^a_{sup} := \sup_{a-r\le y_1\le a+r} \varphi(a, y_1),$$

and h^b_{sup}, φ^b_{sup} and $h^{a,b}_{inf}$, $\varphi^{a,b}_{inf}$ are defined in an analogous way.

In the spirit of the original Landesman–Lazer result, an elementary example satisfying the conditions in Theorem 1 is obtained if we choose $h(y) = \arctan y$ and φ such that $\varphi(y_0, y_1) \to 0$ as $|(y_0, y_1)| \to \infty$. More general situations are considered in the following examples, that correspond to the respective cases 1, 2 and 3 in Theorem 2 (it is easy to check that all conditions are fulfilled):

1. $$h(y) = |y|^\tau y, \quad |\varphi(y_0, y_1)| \le A + B|(y_0, y_1)|,$$
   for some constants $A, B, \tau > 0$.
2. $$h(y) = -\frac{y}{|y|^\tau}, \quad \varphi \text{ bounded},$$
   with $0 < \tau < 1$.
3. $$h(y) = -e^y, \quad \varphi \text{ bounded from above},$$
   $$\liminf_{y_0, y_1\to-\infty} \varphi(y_0, y_1) > 0,$$
   and
   $$\lim_{y_0, y_1\to+\infty} \frac{\varphi(y_0, y_1)}{|(y_0, y_1)|^\gamma} = 0$$
   for some $\gamma > 0$.

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