

Asymptotic Stabilizability of Underactuated Hamiltonian Systems With Two Degrees of Freedom*

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April 29, 2019

Abstract

For an underactuated (simple) Hamiltonian system with two degrees of freedom and one degree of underactuation, a rather general condition that ensures its stabilizability, by means of the existence of a (simple) Lyapunov function, was found in a recent paper by D.E. Chang within the context of the energy shaping method. Also, in the same paper, some additional assumptions were presented in order to ensure also asymptotic stabilizability. In this paper we extend these results by showing that above mentioned condition is not only sufficient, but also a necessary one. And, more importantly, we show that no additional assumption is needed to ensure asymptotic stabilizability.

Keywords Underactuated Systems, Hamiltonian Systems, Asymptotic Stability, Lyapunov Functions.

1 Introduction

Consider an underactuated Hamiltonian system with two degrees of freedom and exactly one actuator (i.e. with one degree of underactuation). Such a system can be described by a pair (H, Y) , where H is a Hamiltonian function on a 4-dimensional phase space and Y is a (vertical) vector field defining the direction of the actuator. Fix a critical point α_0 for the related Hamiltonian vector field X_H and assume from now on that:

1. H is *simple*, i.e. in any canonical coordinate chart (x, y, p_x, p_y)

$$H(x, y, p_x, p_y) = \frac{1}{2} (p_x, p_y) \begin{bmatrix} a(x, y) & b(x, y) \\ b(x, y) & c(x, y) \end{bmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} + h(x, y)$$

for some function h and with

$$\mathbb{H}(x, y) := \begin{bmatrix} a(x, y) & b(x, y) \\ b(x, y) & c(x, y) \end{bmatrix}$$

a positive-definite matrix;

2. there exist canonical coordinates, that we shall call **adapted coordinates**, in which:
 - (a) Y is given by the constant vector $(0, 0, 0, 1)$;
 - (b) the critical point α_0 is represented by the vector $\bar{\mathbf{0}} := (0, 0, 0, 0)$.

A system satisfying above conditions will be called **underactuated simple Hamiltonian system with two degrees of freedom**. In Ref. [7], D.E. Chang found, among other things,

*S.D. Grillo and L.M. Salomone thank CONICET for its financial support.

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a sufficient condition that ensures the stabilizability¹ of such systems at the given critical point $\bar{\mathbf{0}}$. The mentioned condition can be written (in adapted coordinates) as

$$\left(b \frac{\partial^2 h}{\partial x^2} + c \frac{\partial^2 h}{\partial x \partial y} \right) (0, 0) \neq 0 \quad \text{or} \quad \frac{\partial^2 h}{\partial x^2} (0, 0) > 0. \quad (1)$$

In coordinate-free terms, according to [7], above inequalities mean that: either the linearization of the system at the given critical point is controllable, or it is uncontrollable with uncontrollable modes given by a purely imaginary pair.

Chang's work was done within the framework of the *energy shaping* method (see for instance [1, 2, 3, 4, 5, 10, 14, 15, 16, 17]), or more precisely, within his version of the method, developed in [6, 8, 9]. Let us briefly review such a method in the present context. Its main idea is to construct, for a given pair (H, Y) and a given critical point of X_H , a state feedback controller u and a simple Lyapunov function \hat{H} for the resulting closed-loop system. Note that \hat{H} (to be simple) must have the form

$$\hat{H}(x, y, p_x, p_y) = \frac{1}{2} (p_x, p_y) \begin{bmatrix} f(x, y) & g(x, y) \\ g(x, y) & l(x, y) \end{bmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \hat{h}(x, y)$$

with

$$\hat{\mathbb{H}}(x, y) := \begin{bmatrix} f(x, y) & g(x, y) \\ g(x, y) & l(x, y) \end{bmatrix} \quad (2)$$

positive-definite. To find the controller u , a set of partial differential equations (PDEs), known as *matching conditions*, must be solved. Such PDEs have the pair (H, Y) as datum and the aforementioned Lyapunov function \hat{H} as their unknown. According to Ref. [11], for pairs (H, Y) as described above, and using adapted coordinates, the mentioned PDEs are

$$\sum_{i,j,k,l=1}^2 \left(\partial_k \hat{\mathbb{H}}^{ij} \mathbb{H}^{kl} - \partial_k \mathbb{H}^{ij} \hat{\mathbb{H}}^{kl} \right) p_i p_j p_l = 0, \quad (3)$$

the *kinetic matching condition*, and

$$\sum_{k,l=1}^2 \left(\partial_k \hat{h} \mathbb{H}^{kl} - \partial_k h \hat{\mathbb{H}}^{kl} \right) p_l = 0, \quad (4)$$

the *potential matching condition*, and must be satisfied for all (x, y, p_x, p_y) such that

$$p_x g(x, y) + p_y l(x, y) = 0. \quad (5)$$

Here, ∂_1 (resp. ∂_2) denotes the partial derivative w.r.t. x (resp. y), $p_1 = p_x$ and $p_2 = p_y$. Note that for \hat{H} to be a Lyapunov function related to the point $\bar{\mathbf{0}}$, we need that \hat{h} is a positive-definite function w.r.t. $(0, 0)$, i.e.

$$\hat{h}(0, 0) = 0 \quad \text{and} \quad \hat{h}(x, y) > 0 \quad \text{for all} \quad (x, y) \neq (0, 0). \quad (6)$$

Once a solution \hat{H} of the matching conditions is given, the method provide a concrete procedure to construct a state feedback controller. In the case under consideration, such a controller has the form (see Ref. [11])

$$u(x, y, p_x, p_y) = (0, 0, 0, \lambda(x, y, p_x, p_y)), \quad (7)$$

with

$$\lambda(x, y, p_x, p_y) := - \frac{\left(\mu + \{ \hat{H}, H \} \right) (x, y, p_x, p_y)}{p_x g(x, y) + p_y l(x, y)}. \quad (8)$$

¹By *stabilizable* at a point α_0 we mean that there exists a state feedback controller such that the related closed-loop system is stable at α_0 .

Here $\{\cdot, \cdot\}$ denotes the canonical Poisson bracket and μ is any non-negative function such that

$$\frac{\mu(x, y, p_x, p_y)}{p_x g(x, y) + p_y l(x, y)} \quad (9)$$

is smooth. Thus, if a solution \hat{H} of the matching conditions (3) (4) and (5) is found, and satisfies above mentioned positivity requirements [see (6)], the system in question can be stabilized at the point $\bar{\mathbf{0}}$ by means of the controller (7). And, as we said above, such stability is ensured by the existence of a Lyapunov function for the related closed-loop system: the solution \hat{H} .

It was shown in [11] that the Chang’s version of the energy shaping method [6, 8, 9] is *maximal* among the so-called “Lyapunov based methods.” More precisely, if an underactuated simple Hamiltonian system (with any number of degrees of freedom and any degree of underactuation) is stabilized by a method that gives rise to a closed-loop system with a simple Lyapunov function, then such a function must be a solution of the matching conditions and the related controller is exactly the one given by the energy shaping method.

Coming back to (1), what Chang showed in [7] was actually that (1) is a necessary and sufficient condition for finding a simple solution of the matching conditions (with the above mentioned positive requirements), and consequently a sufficient condition for stabilizability. Moreover, in the same paper, two additional assumptions to ensure not only stabilizability, but also asymptotic stabilizability, were presented.

In the present paper, we further study the stabilizability condition (1) and show:

- a. A (slightly) different stabilizability characterization: (1) is a sufficient and a necessary condition to stabilize an underactuated simple Hamiltonian system with two degrees of freedom by any method (not only the energy shaping) that guarantees such stability by exhibiting (or at least by ensuring the existence of) a simple Lyapunov function.
- b. The main result of the paper: (1) not only ensures stabilizability, but also asymptotic stabilizability. That is to say, no additional condition is needed, other than (1), in order to prove the asymptotic stabilizability for an underactuated simple Hamiltonian system with two degrees of freedom.

The paper is organized as follows. In §2 we write down a more convenient expression of the matching conditions (3), (4) and (5). Then, studying the existence of their solutions, we give an alternative derivation of the condition (1) and show the point **a** above. We could show that point simply by combining the results of Chang in [7] and the above mentioned *maximal character* of the energy shaping method (showed in Ref. [11]). Nevertheless, we decided to make a detailed proof because of the involved reasoning and calculations, which are necessary to prove the second result of the paper. The latter is done in §3, where, by combining the LaSalle’s invariance principle, a Dirac-like algorithm and the Morse Lemma, we show that condition (1) also implies asymptotic stabilizability (i.e. we prove the claim of point **b**). Finally, we illustrate our results with an example.

2 Stabilizability of systems with two degrees of freedom

In this section we prove that, given an underactuated simple Hamiltonian system with two degrees of freedom, condition (1) ensures its stabilizability. Reciprocally, if such a system is stabilizable and that stability can be ensured by a simple Lyapunov function, we prove that condition (1) must hold. All that will be done by using the Chang version of the energy shaping method [6, 8, 9]. To begin with, we shall write down (3), (4) and (5) in a way which is more appropriate for our purposes.

We want to emphasize that the results of the present section represent a slightly modification of those contained in the work of Chang [7] and, as we said in the Introduction, they can be proved just by using the maximal character of the energy shaping method [11]. However, we decide to give an alternative proof here because some of the intermediate steps are crucial for showing the main result of this paper, developed in §3.

2.1 Re-writing the matching conditions

Consider a pair (H, Y) and a set of adapted coordinates (x, y, p_x, p_y) as those described in the Introduction.

Remark 1. Since H is simple, it is easy to show that $\bar{\mathbf{0}}$ is critical for the Hamiltonian vector field

$$X_H = \left(\frac{\partial H}{\partial p_x}, \frac{\partial H}{\partial p_y}, -\frac{\partial H}{\partial x}, -\frac{\partial H}{\partial y} \right) \quad (10)$$

if and only if $(0, 0)$ is critical for the function h .

Consider also the matching conditions (3), (4) and (5) for the unknowns $\hat{\mathbb{H}}$ and \hat{h} . Note first that, since \mathbb{H} is positive-definite (see (1) in the Introduction), then

$$a, c > 0 \quad \text{and} \quad \Delta := ac - b^2 > 0. \quad (11)$$

Analogously, regarding $\hat{\mathbb{H}}$, we have that

$$f > 0 \quad \text{and} \quad fl - g^2 > 0, \quad (12)$$

which implies that $l > 0$. To further simplify the notation, define

$$\delta := \frac{fl - g^2}{l}, \quad \gamma := \frac{g}{l}, \quad (13)$$

and

$$B := a_x - 2\gamma b_x + \gamma^2 c_x. \quad (14)$$

(From now on, the subindices x and y denote partial differentiation). In terms of these new variables, it can be shown that the kinetic matching condition (3), combined with (5), and the positivity conditions (12) are equivalent to

$$(a - b\gamma)\delta_x + (b - c\gamma)\delta_y = B\delta, \quad \delta > 0 \quad (15)$$

and $l > 0$, while the potential matching condition (4), combined with (5), adopt the form

$$(a - b\gamma)\hat{h}_x + (b - c\gamma)\hat{h}_y = h_x\delta. \quad (16)$$

Summing up, the matching and positivity conditions can be described by (15), (16) and (6) for the unknowns $(\delta, \gamma, \hat{h})$, plus the condition $l > 0$. To go back to the original variables, we just must use the formulae [see (13)]

$$f = \delta + l\gamma^2 \quad \text{and} \quad g = l\gamma. \quad (17)$$

2.2 A sufficient condition for stabilizability

As we said in the Introduction about the energy shaping method, if we find a solution to (3), (4), (5) and (6), then we can construct a vector field u [see (7) and (8)] and a simple Lyapunov function that ensures the stability of the related closed-loop system at $\bar{\mathbf{0}}$ (at least locally around $\bar{\mathbf{0}}$). So, according to the last subsection, the stabilizability of (H, Y) around $\bar{\mathbf{0}}$ can be analyzed by studying the existence of solutions $(\delta, \gamma, \hat{h})$ of (15), (16) and (6). To do that, let us consider the next two lemmas. In what follows, we shall call $U \subset \mathbb{R}^4$ the neighborhood of $\bar{\mathbf{0}}$ where the adapted coordinates take their values. Also, for simplicity, we shall write $\mathbf{0} := (0, 0)$.

Lemma 1. *Given a function γ satisfying*

$$\gamma(\mathbf{0}) \neq \frac{b(\mathbf{0})}{c(\mathbf{0})} \quad (18)$$

and

$$[(a - b\gamma)h_{xx} + (b - c\gamma)h_{xy}](\mathbf{0}) > 0, \quad (19)$$

there exist functions δ and \hat{h} such that $(\delta, \gamma, \hat{h})$ is a solution of (15), (16) and (6).

Proof. Let us begin with (15). This is a first-order PDE, so we can use the *Method of Characteristics* to find a solution around $\mathbf{0}$. But, in order for this to make sense, we need a suitable boundary condition on a non-characteristic submanifold Γ . Let V be the projection of U onto the first two coordinates. Observe that the characteristic vector field is $A = (a - b\gamma, b - c\gamma)$. Then, we may take the submanifold $\Gamma \subset \mathbb{R}^2$ to be the x -axis, i.e. to take

$$\Gamma = \{(x, 0) : x \in \mathbb{R}\} \cap V,$$

so long as we ensure that the second component of A is nonzero around $\mathbf{0}$. But this amounts to choose γ such that (18) holds. Since we need $\delta > 0$, we can impose the boundary condition $\delta|_{\Gamma} = s$, where $s : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $s(0) > 0$. In this case, the Theorem of Characteristics states that there is a unique solution δ of (15) such that $\delta(x, 0) = s(x)$, which implies, by continuity, that $\delta(x, y) > 0$ around $\mathbf{0}$. We can shrink V (together with U), if necessary, in order to ensure that $\delta > 0$ along all of V . From now on, we shall use this shrinking process implicitly (finitely many times).

Let us continue with (16) and (6). The former is also a first-order PDE, and with the same characteristic vector field A . Assuming (18) again, the x -axis is a non-characteristic submanifold and we can impose $\hat{h}|_{\Gamma} = r$, where $r : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that $r(0) = 0$. This implies that $\hat{h}(\mathbf{0}) = 0$, which is the first part of (6). The second part says that $\mathbf{0}$ is an isolated minimum for \hat{h} , or equivalently, $\mathbf{0}$ is critical for \hat{h} and the Hessian of \hat{h} is positive-definite at $\mathbf{0}$. Let us analyze these conditions. Since $\mathbf{0}$ is critical for h (see Remark 1), i.e. $(h_x(\mathbf{0}), h_y(\mathbf{0})) = \mathbf{0}$, it follows from (16) that \hat{h} must satisfy

$$\left[(a - b\gamma) \hat{h}_x + (b - c\gamma) \hat{h}_y \right] (\mathbf{0}) = 0.$$

Thus, since $b(\mathbf{0}) - c(\mathbf{0})\gamma(\mathbf{0}) \neq 0$ [see (18)], in order to have that $(\hat{h}_x(\mathbf{0}), \hat{h}_y(\mathbf{0})) = \mathbf{0}$, it suffices to ask that $\hat{h}_x(\mathbf{0}) = r'(0) = 0$. So far, we have that r and s must satisfy

$$s(0) > 0, \quad r(0) = r'(0) = 0. \quad (20)$$

On the other hand, the Hessian of \hat{h} is positive-definite at $\mathbf{0}$ if and only if

$$\hat{h}_{xx}(\mathbf{0}) > 0 \quad \text{and} \quad \left(\hat{h}_{xx} \hat{h}_{yy} - \hat{h}_{xy}^2 \right) (\mathbf{0}) > 0. \quad (21)$$

It is easy to compute the second partial derivatives of \hat{h} at $\mathbf{0}$ using (16) and the boundary conditions $\delta|_{\Gamma} = s$ and $\hat{h}|_{\Gamma} = r$. This gives, omitting the evaluation point $\mathbf{0}$,

$$\begin{cases} \hat{h}_{xx} = r''(0), \\ \hat{h}_{xy} = \frac{h_{xx} s(0) - (a-b\gamma) r''(0)}{(b-c\gamma)}, \\ \hat{h}_{yy} = \frac{h_{xy} s(0)}{(b-c\gamma)} - \frac{(a-b\gamma) h_{xx} s(0) + (a-b\gamma)^2 r''(0)}{(b-c\gamma)^2}. \end{cases} \quad (22)$$

Then, we must have

$$\hat{h}_{xx} = r''(0) > 0 \quad (23)$$

and

$$\hat{h}_{xx} \hat{h}_{yy} - \hat{h}_{xy}^2 = \frac{s(0) r''(0)}{(b-c\gamma)^2} \left((a-b\gamma) h_{xx} + (b-c\gamma) h_{xy} - \frac{h_{xx}^2 s(0)}{r''(0)} \right) > 0. \quad (24)$$

Accordingly, since (19) holds by hypothesis, in order to ensure (21) it is enough to take

$$r''(0) > \frac{h_{xx}^2(\mathbf{0}) s(0)}{[(a-b\gamma) h_{xx} + (b-c\gamma) h_{xy}](\mathbf{0})}. \quad (25)$$

□

The next lemma gives a necessary and sufficient condition, in terms of H , for the existence of a function γ fulfilling (19). The proof can be found in the Appendix.

Lemma 2. *There exists a function γ such that (19) holds if and only if*

$$[b h_{xx} + c h_{xy}] (\mathbf{0}) \neq 0 \quad \text{or} \quad h_{xx} (\mathbf{0}) > 0. \quad (26)$$

Moreover, in such a case, $\gamma (\mathbf{0})$ can be chosen such that

$$\begin{cases} |\gamma (\mathbf{0})| > \left| \left[\frac{a h_{xx} + b h_{xy}}{b h_{xx} + c h_{xy}} \right] (\mathbf{0}) \right| \\ [\gamma (b h_{xx} + c h_{xy})] (\mathbf{0}) < 0 \end{cases} \quad (27)$$

if $[b h_{xx} + c h_{xy}] (\mathbf{0}) \neq 0$, and using the following table

	$h_{xy} (\mathbf{0}) = 0$	$h_{xy} (\mathbf{0}) < 0$	$h_{xy} (\mathbf{0}) > 0$
$b (\mathbf{0}) = 0$	any	$\gamma (\mathbf{0}) > 0$	$\gamma (\mathbf{0}) < 0$
$b (\mathbf{0}) > 0$	$\gamma (\mathbf{0}) < \frac{a(\mathbf{0})}{b(\mathbf{0})}$	$\frac{b(\mathbf{0})}{c(\mathbf{0})} < \gamma (\mathbf{0}) < \frac{a(\mathbf{0})}{b(\mathbf{0})}$	$\gamma (\mathbf{0}) < \min \left(\frac{a(\mathbf{0})}{b(\mathbf{0})}, \frac{b(\mathbf{0})}{c(\mathbf{0})} \right)$
$b (\mathbf{0}) < 0$	$\gamma (\mathbf{0}) > \frac{a(\mathbf{0})}{b(\mathbf{0})}$	$\gamma (\mathbf{0}) > \max \left(\frac{a(\mathbf{0})}{b(\mathbf{0})}, \frac{b(\mathbf{0})}{c(\mathbf{0})} \right)$	$\frac{a(\mathbf{0})}{b(\mathbf{0})} < \gamma (\mathbf{0}) < \frac{b(\mathbf{0})}{c(\mathbf{0})}$

(28)

if $h_{xx} (\mathbf{0}) > 0$. All of these conditions are compatible with (18).

Summarizing, if (26) holds, in order to find a solution $(\delta, \gamma, \hat{h})$ of (15), (16) and (6), it is enough to take γ satisfying (18) and also (27) or (28), as explained in the last lemma. Thus, we have proved the following.

Theorem 1. *Consider an underactuated simple Hamiltonian system with two degrees of freedom and a set of adapted coordinates related to it. Then if (26) holds, the system is stabilizable at $\bar{\mathbf{0}}$, i.e. there exists a state feedback controller u , defined at least around $\bar{\mathbf{0}}$, such that the related closed-loop system is stable at $\bar{\mathbf{0}}$. Moreover, such a stability can be ensured by the existence of a simple Lyapunov function.*

2.3 A necessary condition for the existence of a simple Lyapunov function

Using the same notation as above, suppose that an underactuated simple Hamiltonian system with two degrees of freedom (H, Y) can be stabilized at $\bar{\mathbf{0}}$, and that such stabilization is ensured by the existence of a simple Lyapunov function. More precisely, suppose that there exists a controller $u = \lambda Y$ and a simple Lyapunov function \hat{H} , both of them defined at least around $\bar{\mathbf{0}}$, ensuring the stability of the related closed-loop system. Then, it was shown in [11] that u and \hat{H} must be given by the energy shaping method (or more precisely, by the so-called *simple CH method*). In particular, \hat{H} must be locally given, in adapted coordinates, by a solution $(\delta, \gamma, \hat{h})$ of (15), (16) and (6) [and by (2) and (17) and some function $l > 0$]. We want to show from this fact that (26) must be satisfied. To do that, let us consider two cases.

1. [γ does not satisfy (18)] If $\gamma (\mathbf{0}) = b(\mathbf{0})/c(\mathbf{0})$, then [recall (11)]

$$[a - b \gamma] (\mathbf{0}) = a(\mathbf{0}) - b(\mathbf{0}) \frac{b(\mathbf{0})}{c(\mathbf{0})} = \frac{\Delta(\mathbf{0})}{c(\mathbf{0})} > 0.$$

On the other hand, if we differentiate (16) and evaluate the result at $\mathbf{0}$, we obtain

$$(a(\mathbf{0}) - b(\mathbf{0}) \gamma(\mathbf{0})) \hat{h}_{xx}(\mathbf{0}) = h_{xx}(\mathbf{0}) \delta(\mathbf{0}).$$

(Recall that $\mathbf{0}$ is critical for \hat{h} and h). As a consequence, using that $\delta(\mathbf{0}) > 0$ and $\hat{h}_{xx}(\mathbf{0}) > 0$,

$$h_{xx}(\mathbf{0}) = \frac{[a - b \gamma] (\mathbf{0}) \hat{h}_{xx}(\mathbf{0})}{\delta(\mathbf{0})} > 0.$$

In other words, condition (26) must hold.

2. [γ satisfies (18)] Let us call V the neighborhood of $\mathbf{0}$ where the functions δ, γ, \hat{h} are defined. Define $r(x) := \hat{h}(x, 0)$ and $s(x) := \delta(x, 0)$ for all x such that $(x, 0) \in V$. It is clear that the domain of the last functions is an open neighborhood of $\mathbf{0}$. Then, as we saw in the previous section, differentiating (16) and evaluating at $\mathbf{0}$ (and using that $\mathbf{0}$ is critical for h), we arrive at (22). Thus, the positivity conditions (21) for \hat{h} can be studied in terms of (23) and (24). From the latter, and from the fact that $s(0) > 0$ (since δ must be positive), it easily follows that (19) is a necessary condition. But according to Lemma 2, this says again that condition (26) must be satisfied.

Combining above discussion with Theorem 1, we have the following characterization.

Theorem 2. *Under the conditions of Theorem 1, (H, Y) is stabilizable at $\bar{\mathbf{0}}$, and such stability can be ensured by the existence of a simple Lyapunov function, if and only if (26) holds.*

3 Asymptotic stabilizability

In Reference [7], it was shown that the condition² $[b h_{xx} + c h_{xy}](\mathbf{0}) \neq 0$ also implies asymptotic stability (as previously affirmed in [12], without a proof). In any other case, in the same reference, an additional condition is proposed to ensure this kind of stability. We show in the next subsection that no condition other than (26) is needed to this end.

3.1 No additional assumptions are needed

Let $(\delta, \gamma, \hat{h})$ be a solution of (15), (16) and (6) defined around $\mathbf{0}$, with γ satisfying (18) and (19) and with *boundary conditions* given by functions s and r , as described in the proof of Lemma 1. That is to say, δ and \hat{h} must satisfy

$$\delta(x, 0) = s(x) \quad \text{and} \quad \hat{h}(x, 0) = r(x), \quad (29)$$

with s and r fulfilling (20) and (25). To ensure the existence of such a solution, we only must ask that (26) hold. Let \hat{H} be given by (2) and (17), i.e.

$$\begin{aligned} \hat{H}(x, y, p_x, p_y) = & \left[p_x^2 \left(\frac{\delta(x, y)}{l(x, y)} - \gamma^2(x, y) \right) + 2\gamma(x, y) p_x p_y + p_y^2 \right] l(x, y) \\ & + \hat{h}(x, y), \end{aligned} \quad (30)$$

for some positive function l . To write down an explicit expression for the controller, we must choose a non-negative function μ fulfilling (9). To that effect, it suffices to take

$$\mu(x, y, p_x, p_y) = \varkappa (\gamma(x, y) p_x + p_y)^2 l^2(x, y),$$

for some positive constant \varkappa .

Remark 2. According to the results of Ref. [11], the subset $\mu^{-1}(0)$, which in this case is given by

$$\mu^{-1}(0) = \{(x, y, p_x, p_y) : p_x \gamma(x, y) + p_y = 0\}, \quad (31)$$

is the *LaSalle surface* related to the Lyapunov function \hat{H} (see [13]). Note also that $\bar{\mathbf{0}} \in \mu^{-1}(0)$.

With all these elements, the state feedback controller u adopts the form

$$u = (0, 0, 0, \lambda), \quad (32)$$

where λ is locally given as [see (8)]

$$\lambda(x, y, p_x, p_y) = -\varkappa (\gamma(x, y) p_x + p_y) l(x, y) - \frac{\{\hat{H}, H\}(x, y, p_x, p_y)}{(\gamma(x, y) p_x + p_y) l(x, y)}. \quad (33)$$

²Actually, a weaker condition is considered there (see Theorem III.3).

Our next step will be to prove that the functions δ , γ , l and \hat{h} can be chosen such that the closed-loop system defined by u is asymptotically stable around $\bar{\mathbf{0}}$, i.e. the origin $\bar{\mathbf{0}}$ is an asymptotically stable equilibrium point of the vector field $X = X_H + u$. More precisely, we are going to show, without any additional assumption other than (26), that boundary conditions s and r [see (29)] and an open subset T containing $\bar{\mathbf{0}}$ can be chosen in such a way that the largest X -invariant³ submanifold of $\mathcal{S}_0 := \mu^{-1}(0) \cap T$ is the singleton $\{\bar{\mathbf{0}}\}$. Taking into account Remark 2, this would imply, *via* the LaSalle's invariance principle, that $\bar{\mathbf{0}}$ is (locally) asymptotically stabilizable for X (see [13]). The proof will be based on the next two lemmas (the proof of the first one is easy to derive, so we omit it for brevity).

Lemma 3. *Given a manifold P , a vector field X on P , a critical point α_0 of X , and a submanifold $\mathcal{S}_0 \subset P$ containing α_0 , let us define⁴*

$$\mathcal{S}_n := \{\alpha \in \mathcal{S}_{n-1} : X(\alpha) \in T\mathcal{S}_{n-1}\}, \quad n \in \mathbb{N}, \quad (34)$$

where we are assuming that each \mathcal{S}_n is a submanifold of \mathcal{S}_{n-1} . Then, the largest X -invariant subset I of \mathcal{S}_0 satisfies

$$\{\alpha_0\} \subset I \subset \bigcap_{n \in \mathbb{N}} \mathcal{S}_n.$$

In particular, if $\mathcal{S}_k = \{\alpha_0\}$ for some $k \in \mathbb{N}$, then $I = \{\alpha_0\}$.

Lemma 4. *There exist boundary conditions s and r , a function γ and an open subset $T \ni \bar{\mathbf{0}}$ such that [see (34)]:*

- the subset \mathcal{S}_1 corresponding to $\mathcal{S}_0 = \mu^{-1}(0) \cap T$ is a submanifold of \mathcal{S}_0 ;
- \mathcal{S}_2 is a submanifold of \mathcal{S}_1 ;
- $\mathcal{S}_3 = \{\bar{\mathbf{0}}\}$.

It is enough to take s and r such that [besides (20) and (25)]

$$\frac{s'(0)}{s(0)} \neq - \left[\frac{2(b-c\gamma)}{\Delta} \left(b_x - \gamma c_x - \frac{Bc}{2(b-c\gamma)} \right) \right] (\mathbf{0}), \quad (35)$$

and choose $\gamma(\mathbf{0})$ according to (18), (19) and the additional restriction

$$\gamma(\mathbf{0}) \neq \frac{(a(\mathbf{0}), b(\mathbf{0})) \mathbb{M} \begin{pmatrix} b(\mathbf{0}) \\ c(\mathbf{0}) \end{pmatrix}}{(b(\mathbf{0}), c(\mathbf{0})) \mathbb{M} \begin{pmatrix} b(\mathbf{0}) \\ c(\mathbf{0}) \end{pmatrix}}, \quad (36)$$

where \mathbb{M} is a positive-definite matrix given by

$$\mathbb{M} = \begin{bmatrix} r''(0) & \frac{h_{xx}s(0) - (a-b\gamma)r''(0)}{(b-c\gamma)} \\ \frac{h_{xx}s(0) - (a-b\gamma)r''(0)}{(b-c\gamma)} & \frac{h_{xy}s(0)(b-c\gamma) - (a-b\gamma)h_{xx}s(0) + (a-b\gamma)^2 r''(0)}{(b-c\gamma)^2} \end{bmatrix}. \quad (37)$$

Proof. According to (31), $\mu^{-1}(0)$ can be described as the zero set of the function

$$\mathfrak{F}(x, y, p_x, p_y) := \gamma(x, y) p_x + p_y.$$

We shall proceed in three steps.

1. Let us consider the subset $Z_1 \subset \mu^{-1}(0)$ such that $\mathfrak{F}_*(X)(x, y, p_x, p_y) = 0$, where \mathfrak{F}_* is the tangent map of \mathfrak{F} and [see (10) and (32)]

$$X = X_H + u = \left(\frac{\partial H}{\partial p_x}, \frac{\partial H}{\partial p_y}, -\frac{\partial H}{\partial x}, -\left(\frac{\partial H}{\partial y} - \lambda \right) \right). \quad (38)$$

³Recall that, given a manifold P and a vector field X on P , a subset $S \subset P$ is *X-invariant* if every integral curve of X with initial condition in S is contained in S .

⁴Given a manifold P , by TP we are denoting, as usual, its tangent bundle.

That is to say, Z_1 is given by the equation

$$\frac{\partial H}{\partial p_x} \frac{\partial \mathfrak{F}}{\partial x} + \frac{\partial H}{\partial p_y} \frac{\partial \mathfrak{F}}{\partial y} - \frac{\partial H}{\partial x} \frac{\partial \mathfrak{F}}{\partial p_x} - \left(\frac{\partial H}{\partial y} - \lambda \right) \frac{\partial \mathfrak{F}}{\partial p_y} = 0,$$

or equivalently by

$$\frac{\partial H}{\partial p_x} \gamma_x p_x + \frac{\partial H}{\partial p_y} \gamma_y p_x - \frac{\partial H}{\partial x} \gamma - \frac{\partial H}{\partial y} + \lambda = 0, \quad (39)$$

and $p_y = -\gamma(x, y) p_x$. Using the explicit forms of H and \hat{H} , it is easy to see that, on $\mu^{-1}(0)$,

$$\begin{cases} \frac{\partial H}{\partial x} = \frac{1}{2} B p_x^2 + h_x, & \frac{\partial H}{\partial y} = \frac{1}{2} C p_x^2 + h_y, \\ \frac{\partial H}{\partial p_x} = (a - b \gamma) p_x, & \frac{\partial H}{\partial p_y} = (b - c \gamma) p_x, \end{cases} \quad (40)$$

where B is given by (14) and

$$C := a_y - 2\gamma b_y + \gamma^2 c_y.$$

On the other hand, and according to (33), the values of λ on points of the form $(x, y, p_x, -\gamma(x, y) p_x) \in \mu^{-1}(0)$, are given by

$$\begin{aligned} \lambda &= - \lim_{p_y \rightarrow -\gamma p_x} \frac{\{\hat{H}, H\}(x, y, p_x, p_y)}{(\gamma(x, y) p_x + p_y) l(x, y)} \\ &= - \frac{1}{l(x, y)} \frac{\partial \{\hat{H}, H\}}{\partial p_y}(x, y, p_x, -\gamma(x, y) p_x). \end{aligned}$$

So, by lengthy, but straightforward calculations, from (30), (33) and (40) we have that

$$\begin{aligned} \lambda &= \left[\frac{1}{2} (B \gamma + C) - \gamma_x (a - b \gamma) - \gamma_y (b - c \gamma) \right. \\ &\quad \left. + \frac{2(b_x - \gamma c_x) \delta - b \delta_x - c \delta_y}{2l} \right] p_x^2 + \gamma h_x + h_y - \frac{b \hat{h}_x + c \hat{h}_y}{l}. \end{aligned} \quad (41)$$

We are omitting, for simplicity, the evaluation point for the involved functions. Finally, combining (39), (40) and (41), we have at $\mu^{-1}(0)$ that

$$\mathfrak{F}_*(X) = \frac{1}{l} \left[(b_x - \gamma c_x) \delta - \frac{b \delta_x + c \delta_y}{2} \right] p_x^2 - \frac{b \hat{h}_x + c \hat{h}_y}{l}.$$

Thus, Z_1 is given by the equations

$$\begin{cases} \gamma(x, y) p_x + p_y = 0, \\ K(x, y) p_x^2 - L(x, y) = 0, \end{cases} \quad (42)$$

where

$$K = (b_x - \gamma c_x) \delta - \frac{b \delta_x + c \delta_y}{2} \quad (43)$$

and

$$L = b \hat{h}_x + c \hat{h}_y. \quad (44)$$

In consequence, Z_1 can be defined by the zero set of the function

$$\mathfrak{G}(x, y, p_x, p_y) := (\mathfrak{G}_1(x, y, p_x, p_y), \mathfrak{G}_2(x, y, p_x, p_y)).$$

with

$$\mathfrak{G}_1(x, y, p_x, p_y) := \gamma(x, y) p_x + p_y,$$

$$\mathfrak{G}_2(x, y, p_x, p_y) := K(x, y) p_x^2 - L(x, y).$$

We want to see that its related tangent map (omitting the evaluation point of the involved functions)

$$\mathfrak{G}_* = \begin{pmatrix} \gamma_x p_x & \gamma_y p_x & \gamma & 1 \\ K_x p_x^2 - L_x & K_y p_x^2 - L_y & 2K p_x & 0 \end{pmatrix} \quad (45)$$

has maximal rank around $\bar{\mathbf{0}}$. In that case, the Implicit Function Theorem would ensure that this zero locus is the graph of a smooth function (and hence a submanifold) when restricted to some open neighborhood of $\bar{\mathbf{0}}$. Such tangent map is given at $\bar{\mathbf{0}}$ by

$$\mathfrak{G}_{*,\bar{\mathbf{0}}} = \begin{pmatrix} 0 & 0 & \gamma(\mathbf{0}) & 1 \\ -L_x(\mathbf{0}) & -L_y(\mathbf{0}) & 0 & 0 \end{pmatrix}. \quad (46)$$

Note that the gradient of L can be written

$$\begin{pmatrix} L_x \\ L_y \end{pmatrix} = \begin{pmatrix} \hat{h}_{xx} & \hat{h}_{xy} \\ \hat{h}_{xy} & \hat{h}_{yy} \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} + \begin{pmatrix} b_x & c_x \\ b_y & c_y \end{pmatrix} \begin{pmatrix} \hat{h}_x \\ \hat{h}_y \end{pmatrix}. \quad (47)$$

Since $\mathbf{0}$ is critical for \hat{h} , then, at $\mathbf{0}$,

$$\begin{pmatrix} L_x \\ L_y \end{pmatrix} = \begin{pmatrix} \hat{h}_{xx} & \hat{h}_{xy} \\ \hat{h}_{xy} & \hat{h}_{yy} \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}.$$

But we know that the Hessian matrix of \hat{h} is positive-definite and the function c is always positive. So, the gradient of L cannot vanish at $\mathbf{0}$. This implies that (46) has maximal rank at $\bar{\mathbf{0}}$. Consequently, there exists an open subset T_1 containing $\bar{\mathbf{0}}$ such that $Z_1 \cap T_1$ is a submanifold of $\mu^{-1}(0) \cap T_1$.

2. Consider now the subset $Z_2 \subset Z_1 \cap T_1$ given by $\mathfrak{G}_*(X)(x, y, p_x, p_y) = 0$. Easy calculations show that Z_2 is given by the points of $Z_1 \cap T_1$ such that

$$\begin{aligned} & \left[(a - b\gamma)(K_x p_x^2 - L_x) + (b - c\gamma)(K_y p_x^2 - L_y) \right. \\ & \left. - 2K \left(\frac{1}{2} B p_x^2 + h_x \right) \right] p_x = 0. \end{aligned} \quad (48)$$

We only need to evaluate the second row of \mathfrak{G}_* [see (45)] on the components of X [see (38), (40) and (41)]. In the following, we assume that $K(\mathbf{0}) \neq 0$. Observe that, since $\delta(x, 0) = s(x)$, we have $\delta_x(\mathbf{0}) = s'(0)$, and using (16) at the origin

$$\delta_y(\mathbf{0}) = \left[\frac{B s(0) - (a - b\gamma) s'(0)}{b - c\gamma} \right] (\mathbf{0}).$$

So [see (43)]

$$K(\mathbf{0}) = \left[\left(b_x - \gamma c_x - \frac{Bc}{2(b - c\gamma)} \right) s(0) + \frac{1}{2} \left(\frac{\Delta}{b - c\gamma} \right) s'(0) \right] (\mathbf{0}),$$

and consequently, the condition $K(\mathbf{0}) \neq 0$ is equivalent to (35). Under such an assumption, we can replace p_x^2 by $\frac{L}{K}$ in (48) [see (42)], and we get

$$\begin{aligned} & \left[(a - b\gamma) \left(K_x \frac{L}{K} - L_x \right) + (b - c\gamma) \left(K_y \frac{L}{K} - L_y \right) \right. \\ & \left. - 2K \left(\frac{1}{2} B \frac{L}{K} + h_x \right) \right] p_x = 0, \end{aligned} \quad (49)$$

on some open subset containing $\mathbf{0}$ (where K is non vanishing). Moreover, since $L(\mathbf{0}) = 0$ [see (44)] and $h_x(\mathbf{0}) = 0$, we have at $\mathbf{0}$ that

$$\frac{L_x K - L K_x}{K^2} = \frac{L_x}{K}, \quad \frac{L_y K - L K_y}{K^2} = \frac{L_y}{K},$$

and then, the bracketed expression in (49) takes the following form at $\mathbf{0}$

$$(a - b\gamma) L_x + (b - c\gamma) L_y.$$

Using (47), this in turn may be written as

$$(a, b) \begin{bmatrix} \hat{h}_{xx} & \hat{h}_{xy} \\ \hat{h}_{xy} & \hat{h}_{yy} \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix} - \gamma (b, c) \begin{bmatrix} \hat{h}_{xx} & \hat{h}_{xy} \\ \hat{h}_{xy} & \hat{h}_{yy} \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix}.$$

Then, if we assume that, at $\mathbf{0}$,

$$\gamma \neq \frac{(a, b) \begin{bmatrix} \hat{h}_{xx} & \hat{h}_{xy} \\ \hat{h}_{xy} & \hat{h}_{yy} \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix}}{(b, c) \begin{bmatrix} \hat{h}_{xx} & \hat{h}_{xy} \\ \hat{h}_{xy} & \hat{h}_{yy} \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix}}$$

it follows that (49) will hold only if $p_x = 0$ around $(x, y) = \mathbf{0}$. It is worth mentioning that this condition is compatible with (18) and (19). Note that, using (22), the condition above is given precisely by (36) and (37). In conclusion, there exists an open neighborhood T'_2 (which contains the point $\bar{\mathbf{0}}$) such that the subset $Z_2 \cap T'_2$ is given by

$$\gamma p_x + p_y = 0, \quad K p_x^2 - L = 0, \quad p_x = 0,$$

or equivalently

$$p_x = p_y = L = 0. \tag{50}$$

This means that $Z_2 \cap T'_2$ can be described as the zero set of the function

$$\mathfrak{H}(x, y, p_x, p_y) := (p_y, p_x, L(x, y)).$$

The tangent map of \mathfrak{H} at $\bar{\mathbf{0}}$ is given by

$$\mathfrak{H}_{*, \bar{\mathbf{0}}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L_x(\mathbf{0}) & L_y(\mathbf{0}) & 0 & 0 \end{pmatrix}.$$

Again, since $L_x(\mathbf{0})$ and $L_y(\mathbf{0})$ cannot be both zero, we conclude that $\mathfrak{H}_{*, \bar{\mathbf{0}}}$ has maximal rank. Thus, there exists inside T'_2 an open neighborhood T_2 of $\bar{\mathbf{0}}$ such that $Z_2 \cap T_2$ is a submanifold of $Z_1 \cap T_1 \cap T_2$.

3. Now, consider the subset $Z_3 \subset Z_2 \cap T_2$ defined by $\mathfrak{H}_*(X)(x, y, p_x, p_y) = 0$. Using (38), (40) and (41), it follows that, along $Z_2 \cap T_2$ [see (50)]

$$X = (0, 0, -h_x, \gamma h_x),$$

so, in order for $\mathfrak{H}_*(X)$ to vanish, it is necessary that $h_x = 0$. But, if this is the case, using the potential matching condition

$$(a - b\gamma) \hat{h}_x + (b - c\gamma) \hat{h}_y = \delta h_x,$$

or equivalently, $a \hat{h}_x + b \hat{h}_y - \gamma L = \delta h_x$, we have on Z_3 that

$$L = b \hat{h}_x + c \hat{h}_y = 0 \quad \text{and} \quad a \hat{h}_x + b \hat{h}_y = 0,$$

i.e.

$$\mathbb{H} \begin{pmatrix} \hat{h}_x \\ \hat{h}_y \end{pmatrix} = 0.$$

Calling π the projection of \mathbb{R}^4 onto the first two components, we can say that above identity holds if and only if all the points of $\pi(Z_3)$ are critical for \hat{h} . By the *Morse Lemma*, since $\mathbf{0}$ is a non-degenerate critical point of \hat{h} [recall (6)], there exists a neighborhood V of $\mathbf{0}$ such that $\pi(Z_3) \cap V = \{\mathbf{0}\}$. But $p_x = p_y = 0$ on Z_3 , which implies that $Z_3 \cap T_3 = \{\bar{\mathbf{0}}\}$ for $T_3 := \pi^{-1}(V)$.

Summing up, if we define $T := T_1 \cap T_2 \cap T_3$ and $\mathcal{S}_0 := \mu^{-1}(0) \cap T$, from (34) we obtain $\mathcal{S}_1 = Z_1 \cap T$, which is a submanifold of \mathcal{S}_0 , $\mathcal{S}_2 = Z_2 \cap T$, which is a submanifold of \mathcal{S}_1 , and $\mathcal{S}_3 = \{\bar{\mathbf{0}}\}$. Hence, the three points of the lemma follow. \square

Concluding, if (26) holds, asymptotic stabilizability is ensured. Reciprocally, if we can ensure asymptotic stabilizability by the existence of a simple Lyapunov function, then we can also ensure stabilizability, and Theorem 2 implies that (26) holds. In other terms,

Theorem 3. *Under the conditions of Theorem 1, (H, Y) is asymptotically stabilizable at $\bar{\mathbf{0}}$, and such a stability can be ensured by the existence of a simple Lyapunov function, if and only if H satisfies (26).*

3.2 Example: the inertia wheel pendulum

Now, we illustrate our results with a concrete underactuated system (H, Y) with two degrees of freedom, the *inertia wheel pendulum*:

- the configuration space is $Q = S^1 \times S^1$, whose natural almost-global coordinates will be denoted (θ, ψ) ;
- the Hamiltonian is

$$H(\theta, \psi, p_\theta, p_\psi) = \frac{1}{2} (p_\theta, p_\psi) \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{pmatrix} p_\theta \\ p_\psi \end{pmatrix} + M(1 + \cos \theta),$$

where a, b, c, M are constants and $a, b, M, ac - b^2$ are strictly greater than zero;

- and the space of actuators is given by the subbundle spanned by the vector field Y taking the constant value $(0, 0, 0, 1)$.

We shall find, by using the energy shaping method, a state feedback controller u for this system and a related simple Lyapunov function \hat{H} which make the closed-loop system $X_H + u$ asymptotically stable at $(\theta, \psi, p_\theta, p_\psi) = (0, 0, 0, 0) = \bar{\mathbf{0}}$.

Replacing x by θ and y by ψ , (26) in this case says that (because $h_{\theta\psi}(\mathbf{0}) = 0$)

$$b h_{\theta\theta}(\mathbf{0}) \neq 0 \quad \text{or} \quad h_{\theta\theta}(\mathbf{0}) > 0,$$

which is equivalent to $h_{\theta\theta}(\mathbf{0}) \neq 0$, since $b \neq 0$. And it does hold, because $h_{\theta\theta}(\mathbf{0}) = -M \neq 0$. Then, the inertia wheel pendulum can be asymptotically stabilized around $\bar{\mathbf{0}}$, as it is well known. On the other hand, according to (14), we have that $B = 0$. So, the kinetic and potential matching conditions read [see (15) and (16)]

$$(a - b\gamma) \delta_\theta + (b - c\gamma) \delta_\psi = 0, \tag{51}$$

and

$$(a - b\gamma) \hat{h}_\theta + (b - c\gamma) \hat{h}_\psi = -M \delta \sin \theta, \tag{52}$$

respectively. Let us construct a solution $(\delta, \gamma, \hat{h})$ of above equations, with $\delta > 0$ and \hat{h} positive-definite w.r.t. $\mathbf{0}$. We shall take γ constant. Following the steps of §2.2, it is enough to take γ such that $\gamma \neq b/c$ [see (18)] and, using (27) of Lemma 2 [since $h_{\theta\theta}(\mathbf{0}) = -M < 0$], also ask that

$$|\gamma| > \frac{a}{b} \quad \text{and} \quad -\gamma b M < 0.$$

The second inequality says that γ is positive, so, above equations only impose the condition $\gamma > a/b$. Note also that, since $a, b, c, ac - b^2 > 0$, we have that $a/b > b/c$. Hence, all the conditions on γ reduce to

$$\gamma > \frac{a}{b}.$$

Remark 3. Note that, for this system, we cannot take $\gamma = b/c$. In fact, in such a case, according to the calculations we made in §2.3, the positivity of δ and \hat{h} would impose that $h_{\theta\theta}(\mathbf{0}) > 0$, which is not true.

Regarding the boundary conditions defining δ and \hat{h} , i.e. the functions s and r , respectively, we must ask [see (20) and (25)]

$$\begin{cases} s(0) > 0, \\ r(0) = r'(0) = 0 \quad \text{and} \\ r''(0) > \frac{h_{\theta\theta}^2(\mathbf{0}) s(0)}{(a-b\gamma) h_{\theta\theta}(\mathbf{0})} = -\frac{M s(0)}{a-b\gamma} \end{cases} \quad (53)$$

And to ensure asymptotic stabilizability, according to Eqs. (35), (36) and (37) of Lemma 4, we ask that

$$\frac{s'(0)}{s(0)} \neq 0, \quad \text{i.e. } s'(0) \neq 0,$$

and that γ satisfy

$$\gamma \neq \frac{\eta^2 ab r''(0) - \eta [M s(0) + \zeta r''(0)] (ac + b^2) + \zeta M s(0) + \zeta^2 r''(0) bc}{\eta^2 b^2 r''(0) - 2\eta [M s(0) + \zeta r''(0)] bc + \zeta M s(0) + \zeta^2 r''(0) c^2}, \quad (54)$$

where $\zeta := a - b\gamma$ and $\eta := b - c\gamma$. Thus, take any number $\gamma > a/b$, any function s such that $s(0) > 0$ and $s'(0) \neq 0$, and any function⁵ r such that $r(0) = r'(0) = 0$ and $r''(0)$ satisfying (53) and (54), and let us apply the Method of Characteristics to (51) and (52), with boundary conditions on $\psi = 0$ given by s and r . The characteristic equations for (51) are

$$\begin{cases} \dot{\theta} = a - b\gamma, & \theta(0) = \theta_0, \\ \dot{\psi} = b - c\gamma, & \psi(0) = 0, \\ \dot{\delta} = 0, & \delta(0) = s(\theta_0). \end{cases}$$

Then

$$\theta(t) = (a - b\gamma)t + \theta_0, \quad \psi(t) = (b - c\gamma)t,$$

and defining $\Upsilon := (a - b\gamma) / (b - c\gamma)$ we find

$$\delta(\theta, \psi) = s(\theta - \Upsilon\psi). \quad (55)$$

The characteristic equation for (52) (and for δ given above) is

$$\dot{\hat{h}} = -M s(\theta_0) \sin((a - b\gamma)t + \theta_0),$$

and integrating we obtain

$$\hat{h}(\theta, \psi) = \frac{M s(\theta - \Upsilon\psi)}{a - b\gamma} (\cos\theta - \cos(\theta - \Upsilon\psi)) + r(\theta - \Upsilon\psi). \quad (56)$$

Finally, with δ and \hat{h} given by (55) and (56), and considering any positive function l , we have from (30), (32) and (33) the controller u and the Lyapunov function \hat{H} we are looking for.

A Proof of Lemma 2

Suppose first that $h_{xx}(\mathbf{0}) \leq 0$ and $(b h_{xx} + c h_{xy})(\mathbf{0}) = 0$. Then, omitting the evaluation point $\mathbf{0}$,

$$\begin{aligned} (a - b\gamma) h_{xx} + (b - c\gamma) h_{xy} &= a h_{xx} + b h_{xy} - \gamma (b h_{xx} + c h_{xy}) \\ &= \left(a - \frac{b^2}{c}\right) h_{xx} = \frac{\Delta}{c} h_{xx} \leq 0 \end{aligned}$$

for any function γ . This proves the first implication of the lemma (by denying the second one). For the converse, suppose first that $b h_{xx} + c h_{xy} \neq 0$. Since

$$(a - b\gamma) h_{xx} + (b - c\gamma) h_{xy} = a h_{xx} + b h_{xy} - \gamma (b h_{xx} + c h_{xy}),$$

⁵Additionally, the functions s and r may be taken with period 2π in order to look for a quasi-global solution.

for any γ with sign opposite to $b h_{xx} + c h_{xy}$ and such that

$$|\gamma| > \left| \frac{a h_{xx} + b h_{xy}}{b h_{xx} + c h_{xy}} \right|,$$

we have that (19) holds. This implies (27). Now, suppose that $h_{xx} > 0$. If $h_{xy} = 0$, since $a > 0$, it is clear that it is enough to choose γ such that $a > b\gamma$. If instead $h_{xy} \neq 0$, we distinguish three cases: $b = 0$, $b > 0$ and $b < 0$.

- If $b = 0$, then $a - b\gamma = a > 0$ and $b - c\gamma = -c\gamma$, and consequently it is sufficient to choose γ with opposite sign to h_{xy} .
- If $b > 0$, we show that it is possible to take γ so as to fulfill one of the following expressions

$$a - b\gamma > 0 \quad \text{and} \quad b - c\gamma > 0,$$

or

$$a - b\gamma > 0 \quad \text{and} \quad b - c\gamma < 0.$$

In order to make $a - b\gamma > 0$, we need $\gamma < \frac{a}{b}$. If in addition $b - c\gamma > 0$, then $\gamma < \frac{b}{c}$. Hence, it suffices to take $\gamma < \min(\frac{a}{b}, \frac{b}{c})$. On the contrary, if $b - c\gamma < 0$, then we can take $\frac{b}{c} < \gamma < \frac{a}{b}$, which is always possible because

$$ac - b^2 > 0 \quad \text{and} \quad b > 0 \quad \Rightarrow \quad \frac{b}{c} < \frac{a}{b}.$$

Thus, if $h_{xy} > 0$ we choose $\gamma < \min(\frac{a}{b}, \frac{b}{c})$ and if $h_{xy} < 0$ we take $\frac{b}{c} < \gamma < \frac{a}{b}$. In both cases we get the desired result.

- If $b < 0$, then $a - b\gamma > 0$ implies $\gamma > \frac{a}{b}$. If in addition $b - c\gamma > 0$, then $\gamma < \frac{b}{c}$ and so we can take γ such that $\frac{a}{b} < \gamma < \frac{b}{c}$, which is always possible because

$$ac - b^2 > 0 \quad \text{and} \quad b < 0 \quad \Rightarrow \quad \frac{a}{b} < \gamma < \frac{b}{c}.$$

On the contrary, if $b - c\gamma < 0$, then $\gamma > \frac{b}{c}$ and it is sufficient to choose $\gamma > \max(\frac{a}{b}, \frac{b}{c})$. Again, both cases lead to the desired result. All these alternatives give the table (28).

The last assertion of the lemma is immediate, because all conditions on γ are inequalities.

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