MATRIX-VALUED ORTHOGONAL POLYNOMIALS ASSOCIATED TO $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$, II

ERIK KOELINK, MAARTEN VAN PRUIJSSEN, PABLO ROMÁN

ABSTRACT. In a previous paper we have introduced matrix-valued analogues of the Chebyshev polynomials by studying matrix-valued spherical functions on $\mathrm{SU}(2) \times \mathrm{SU}(2)$. In particular the matrix-size of the polynomials is arbitrarily large. The matrix-valued orthogonal polynomials and the corresponding weight function are studied. In particular, we calculate the LDU-decomposition of the weight where the matrix entries of L are given in terms of Gegenbauer polynomials. The monic matrix-valued orthogonal polynomials P_n are expressed in terms of Tirao's matrix-valued hypergeometric function using the matrix-valued differential operator of first and second order to which the P_n's are eigenfunctions. From this result we obtain an explicit formula for coefficients in the three-term recurrence relation satisfied by the polynomials P_n. These differential operators are also crucial in expressing the matrix entries of P_nL as a product of a Racah and a Gegenbauer polynomial. We also present a group theoretic derivation of the matrix-valued differential operators by considering the Casimir operators corresponding to $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

1. INTRODUCTION

Matrix-valued orthogonal polynomials have been studied from different perspectives in recent years. Originally they have been introduced by Krein [17], [18]. Matrix-valued orthogonal polynomials have been related to various different subjects, such as higher-order recurrence equations, spectral decompositions, and representation theory. The matrix-valued orthogonal polynomials studied in this paper arise from the representation theory of the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ with the compact subgroup $\mathrm{SU}(2)$ embedded diagonally, see [15] for this particular case and Gangolli and Varadarajan [8], Tirao [23], Warner [25] for general group theoretic interpretations of matrix-valued spherical functions. An important example is the study of the matrix-valued orthogonal polynomials for the case $(\mathrm{SU}(3), \mathrm{U}(2))$, which has been studied by Grünbaum, Pacharoni and Tirao [9] mainly exploiting the invariant differential operators. In [15] we have studied the matrix-valued orthogonal operators related to the case $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$, which lead to the matrix-valued orthogonal polynomial analogues of Chebyshev polynomials of the second kind U_n, in a different fashion. In the current paper we study these matrix-valued orthogonal polynomials in more detail.

Date: April 26, 2013.

In order to state the most important results for these polynomials we recall the weight function [15, Thm. 5.4]

$$\begin{aligned} W(x)_{n,m} &= \sqrt{1-x^2}\sum_{t=0}^{m} \alpha_t(m,n)\, U_{n+m-2t}(x), \\ \alpha_t(m,n) &= \frac{(2\ell+1)}{n+1}\frac{(2\ell-m)!m!}{(2\ell)!}(-1)^{m-t}\frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}}\frac{(2\ell+2-t)_t}{t!} \end{aligned} \tag{1.1}$$

if $n \geq m$ and $W(x)_{n,m} = W(x)_{m,n}$ otherwise. Here and elsewhere in this paper $\ell \in \frac{1}{2}\mathbb{N}$, $n, m \in \{0, 1, \cdots, 2\ell\}$, and U_n is the Chebyshev polynomial of the second kind. Note that the sum in (1.1) actually starts at $\min(0, n+m-2\ell)$. It follows that $W\colon [-1,1] \to M_{2\ell+1}(\mathbb{C})$, $W(x) = \big(W(x)_{n,m}\big)_{n,m=0}^{2\ell}$, is a $(2\ell+1)\times(2\ell+1)$-matrix-valued integrable function such that all moments $\int_{-1}^{1} x^n W(x)\, dx$, $n \in \mathbb{N}$, exist. From the construction given in [15, §5] it follows $W(x)$ is positive definite almost everywhere. By general considerations, e.g. [10], we can construct the corresponding monic matrix-valued orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$, so

$$\langle P_n, P_m\rangle_W = \int_{-1}^{1} P_n(x)\, W(x)\, \big(P_m(x)\big)^*\, dx = \delta_{nm} H_n, \qquad 0 < H_n \in M_{2\ell+1}(\mathbb{C}) \tag{1.2}$$

where $H_n > 0$ means that H_n is a positive definite matrix, and $P_n(x) = \sum_{k=0}^{n} x^k P^n_k$ with $P^n_k \in M_{2\ell+1}(\mathbb{C})$ and $P^n_n = I$, the identity matrix. The polynomials P_n are the monic variants of the matrix-valued orthogonal polynomials constructed in [15] from representation theoretic considerations. Note that (1.2) defines a matrix-valued inner product $\langle\cdot,\cdot\rangle_W$ on the matrix-valued polynomials. Using the orthogonality relations for the Chebyshev polynomials U_n it follows that

$$(H_0)_{nm} = \delta_{nm}\frac{\pi}{2}\frac{(2\ell+1)^2}{(n+1)(2\ell-n+1)} \tag{1.3}$$

which is accordance with [15, Prop. 4.6]. From [15] we can also obtain an expression for H_n by translating the result of [15, Prop. 4.6] to the monic case in [15, (4.6)], but since the matrix Υ_d in [15, (4.6)] is relatively complicated this leads to a complicated expression for the squared norm matrix H_n in (1.2). In Corollary 5.4 we give a simpler expression for H_n from the three-term recurrence relation.

These polynomials have a group theoretic interpretation as matrix-valued spherical functions associated to $(SU(2)\times SU(2), SU(2))$, see [15] and Section 7. In particular, in [15, §5] we have shown that the corresponding orthogonal polynomials are not irreducible, but can be written as a 2-block-diagonal matrix of irreducible matrix-valued orthogonal polynomials. Indeed, if we put $J \in M_{2\ell}(\mathbb{C})$, $J_{nm} = \delta_{n+m,2\ell}$ we have $JW(x) = W(x)J$ for all $x \in [-1,1]$, and by [15, Prop. 5.5] J and I span the commutant $\{Y \in M_{2\ell}(\mathbb{C}) \mid [Y, W(x)] = 0\, \forall\, x \in [-1,1]\}$. Note that J is a self-adjoint involution, $J^2 = I$, $J^* = J$. It is easier to study the polynomials P_n, and we discuss the relation to the irreducible cases when appropriate.

In this paper we continue the study of the matrix-valued orthogonal polynomials and the related weight function. Let us discuss in some more detail the results we obtain in this paper. Some of these results are obtained employing the group theoretic interpretation and some are obtained using special functions. Essentially, we obtain the following results for the weight function:

(a) explicit expression for $\det(W(x))$, hence proving [15, Conjecture 5.8], see Corollary 2.3;
(b) an LDU-decomposition for W in terms of Gegenbauer polynomials, see Theorem 2.1.

Part (a) can be proved by a group theoretic consideration, and gives an alternative proof for a related statement by Koornwinder [16], but we actually calculate it directly from (b). The LDU-decomposition hinges on expressing the integral of the product of two Gegenbauer polynomials and a Chebyshev polynomial as a Racah polynomial, see Lemma 2.7.

For the matrix-valued orthogonal polynomials we obtain the following results:

(i) P_n as eigenfunctions to a second-order matrix-valued differential operator $\tilde{D}$ and a first-order matrix-valued differential operator $\tilde{E}$, compare [15, §7], see Theorem 3.1 and Section 3;
(ii) the group-theoretic interpretation of $\tilde{D}$ and $\tilde{E}$ using the Casimir operators for $\mathrm{SU}(2) \times \mathrm{SU}(2)$, see Section 7, for which the paper by Casselman and Miličić [5] is essential;
(iii) explicit expressions for the matrix entries of the polynomials P_n in terms of matrix-valued hypergeometric series using the matrix-valued differential operators, see Theorem 4.5;
(iv) explicit expressions for the matrix entries of the polynomials P_nL in terms of (scalar-valued) Gegenbauer polynomials and Racah polynomials using the LDU-decomposition of the weight W and differential operators, see Theorem 6.2;
(v) explicit expression for the three-term recurrence satisfied by P_n, see Theorem 5.3.

In particular, (i) and (ii) follow from group theoretic considerations, see Section 3 and 7. This then gives the opportunity to link the polynomials to the matrix-valued hypergeometric differential operator, leading to (iii). The explicit expression in (iv) involving Gegenbauer polynomials is obtained by using the LDU-decomposition of the weight matrix and the differential operator $\tilde{D}$. The expression of the coefficients as Racah polynomials involves involves the first order differential operator as well. Finally, in [15, Thm. 4.8] we have obtained an expression for the coefficients of the three-term recurrence relation where the matrix entries of the coefficient matrices are given as sums of products of Clebsch-Gordan coefficients, and the purpose of (v) is to give a closed expression for these matrices. The case $\ell = 0$, or the spherical case, corresponds to the Chebyshev polynomials $U_n(x)$, which occur as spherical functions for $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$ or equivalently as characters on $\mathrm{SU}(2)$. For these cases almost all of the statements above reduce to well-known statements for Chebyshev polynomials, except that the first order differential operator has no meaning for this special case.

The structure of the paper is as follows. In Section 2 we discuss the LDU-decomposition of the weight, but the main core of the proof is refererred to Appendix A. In Section 3 we discuss the matrix-valued differential operators to which the matrix-valued orthogonal polynomials are eigenfunctions. We give a group theoretic proof of this result in Section 7. In [15, §7] we have derived the same operators by a judicious guess and next proving the result. In order to connect to Tirao's matrix-valued hypergeometric series, we switch to another variable. The connection is made precise in Section 4. This result is next used in Section 5 to derive a simple expression for the coefficients in the three-term recurrence of the monic orthogonal polynomials, improving a lot on the corresponding result [15, Thm. 4.8]. In Section 6 we explicitly establish that the entries of the matrix-valued orthogonal polynomials times the L-part of the LDU-decomposition of the weight W can be given explicitly as a product of a Racah polynomial and a Gegenbauer polynomial, see Theorem 6.2. Some of the above statements require somewhat lengthy and/or tedious manipulations, and in order to deal with these computations and also for various other checks we have used computer algebra.

As mentioned before, we consider the matrix-valued orthogonal polynomials studied in this paper as matrix-valued analogues of the Chebyshev polynomials of the second kind. As is well known, the group theoretic interpretation of the Chebyshev polynomials, or more generally of spherical functions, leads to more information on these special functions, and it remains to study which of these properties can be extended in this way to the explicit set of matrix-valued orthogonal polynomials studied in this paper. This paper is mainly analytic in nature, and we only use the group theoretic interpretation to give a new way on how to obtain the first and second order matrix-valued differential operator which have the matrix-valued orthogonal polynomials as eigenfunctions. We note that all differential operators act on the right. The fact that we have both a first and a second order differential operator makes it possible to consider linear combinations, and this is useful in Section 4 to link to Tirao's matrix-valued differential hypergeometric function and Section 6 in order to diagonalise (or decouple) a suitable matrix-valued differential operator. The techniques in Section 4 and Section 6 are based on the techniques developed in [19], [20], [21], [22].

We finally remark that J.A. Tirao has informed us that I. Zurrián, I. Pacharoni and J. Tirao have obtained results of a similar nature by considering matrix-valued orthogonal polynomials for the closely related pair $(\mathrm{SO}(4), \mathrm{SO}(3))$, see [26], along the lines of [9]. We stress that our results and the results by Zurrián, Tirao and Pacharoni have been obtained independently.

Acknowledgement. We thank Juan A. Tirao for his suggestion on how to obtain the first order differential operator from the Casimir operators in Section 7. We also thank Erik van den Ban for pointing out to one of us (MvP) the paper [5] and for explaining the result of this paper. The work of Pablo Román on this paper was done while employed by the Katholieke Universiteit Leuven, Belgium, through grant OT/08/33 of the KULeuven and grant P06/02 of the Belgian Interuniversity Attraction Pole.

2. LDU-decomposition of the weight

In this section we state the LDU-decomposition of the weight matrix W in (1.1). The details of the proof, involving summation and transformation formulas for hypergeometric series (up to ${}_7F_6$-level), is presented in Appendix A. Some direct consequences of the LDU-decomposition are discussed. The explicit decomposition is a crucial ingredient in Section 6, where the matrix-valued orthogonal polynomials are related to the classical Gegenbauer and Racah polynomials.

In order to formulate the result we need the Gegenbauer, or ultraspherical, polynomials, see e.g. [2], [11], [14], defined by

$$C_n^{(\alpha)}(x) = \frac{(2\alpha)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+2\alpha \\ \alpha+\frac{1}{2}\end{matrix}; \frac{1-x}{2}\right). \tag{2.1}$$

The Gegenbauer polynomials are orthogonal polynomials;

$$\int_{-1}^{1}(1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(x) C_m^{(\alpha)}(x)dx = \delta_{nm}\frac{(2\alpha)_n\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}{n!\,(n+\alpha)\,\Gamma(\alpha)} = \delta_{nm}\frac{\pi\,\Gamma(n+2\alpha)\,2^{1-2\alpha}}{\Gamma(\alpha)^2\,(n+\alpha)\,n!} \tag{2.2}$$

Theorem 2.1. *The weight matrix W has the following LDU-decomposition;*

$$W(x) = \sqrt{1-x^2}\, L(x)\, T(x)\, L(x)^t, \qquad x \in [-1,1],$$

where $L\colon [-1,1] \to M_{2\ell+1}(\mathbb{C})$ is the unipotent lower triangular matrix

$$L(x)_{mk} = \begin{cases} 0, & k > m \\ \dfrac{m!\,(2k+1)!}{(m+k+1)!\,k!} C^{(k+1)}_{m-k}(x), & k \le m \end{cases}$$

and $T\colon [-1,1] \to M_{2\ell+1}(\mathbb{C})$ is the diagonal matrix

$$T(x)_{kk} \;=\; c_k(\ell)(1-x^2)^k, \quad c_k(\ell) \;=\; \frac{4^k (k!)^4 (2k+1)}{((2k+1)!)^2} \frac{(2\ell+k+1)!\,(2\ell-k)!}{((2\ell)!)^2}.$$

Note that the matrix-entries of L are independent of ℓ, hence of the size of the matrix-valued weight W. Using

$$\frac{d^k}{dx^k} C^{(\alpha)}_n(x) = 2^k (\alpha)_k\, C^{(\alpha+k)}_{n-k}(x)$$

we can write uniformly $L(x)_{mk} = \frac{m!\,2^{-k}\,(2k+1)!}{(k!)^2\,(m+k+1)!} \frac{d^k U_m}{dx^k}(x)$. In Theorem 6.2 we extend Theorem 2.1, but Theorem 2.1 is an essential ingredient in Theorem 6.2.

Since $W(x)$ is symmetric, it suffices to consider the (n,m)-matrix-entry for $m \le n$ of Theorem 2.1. Hence Theorem 2.1 follows directly from Proposition 2.2 using the explicit expression (1.1) for the weight W.

Proposition 2.2. *The following relation*

$$\sum_{t=0}^{m} \alpha_t(m,n) U_{n+m-2t}(x) = \sum_{k=0}^{m} \beta_k(m,n)(1-x^2)^k\, C^{(k+1)}_{n-k}(x) C^{(k+1)}_{m-k}(x)$$

with the coefficients $\alpha_t(m,n)$ given by (1.1) and

$$\beta_k(m,n) = \frac{m!}{(m+k+1)!} \frac{n!}{(n+k+1)!} k!\,k!\,2^{2k}(2k+1) \frac{(2\ell+k+1)!\ (2\ell-k)!}{(2\ell)!\,(2\ell)!}$$

holds for all integers $0 \le m \le n \le 2\ell$, and all $\ell \in \frac{1}{2}\mathbb{N}$.

Before discussing the proof we list some corollaries of Theorem 2.1. First of all, we can use Theorem 2.1 to prove [15, Conjecture 5.8], see (a) of Section 1.

Corollary 2.3. $\det\bigl(W(x)\bigr) = (1-x^2)^{2(\ell+\frac{1}{2})^2} \prod_{k=0}^{2\ell} c_k(\ell)$.

Remark 2.4. We also have another proof of this fact using a group theoretic approach to calculate $\det(\Phi_0(x))$, see [15] and Section 7 for the definition of Φ_0, and W is up to trivial factors equal to $(\Phi_0)(\Phi_0)^*$. This proof is along the lines of Koornwinder [16].

Secondly, using $J \in M_{2\ell+1}(\mathbb{C})$, $J_{nm} = \delta_{n+m,2\ell}$ and $W(x) = JW(x)J$, see [15, Prop. 5.5, §6.2], we obtain from Theorem 2.1 the UDL-decomposition for W. For later reference we also recall $JP_n(x)J = P_n(x)$, since both are the monic matrix-valued orthogonal polynomials with respect to $W(x) = JW(x)J$.

Corollary 2.5. *$W(x) = \sqrt{1-x^2}\,\big(JL(x)J\big)\big(JT(x)J\big)\big(JL(x)J\big)^t$, $x \in [-1,1]$ gives the UDL-decomposition of the weight W.*

Thirdly, considering the Fourier expansion of the weight function $W(\cos\theta)$, and using the expression of the weight in terms of Clebsch-Gordan coefficients, see [15, (5.4), (5.6), (5.7)] we obtain a Fourier expansion, which is actually equivalent to Theorem 2.1.

Corollary 2.6. *We have the following Fourier expansion*

$$\sum_{k=0}^{m\wedge n}(-4)^k(2k+1)\frac{(m-k+1)_k\,(n-k+1)_k}{(m+1)_{k+1}\,(n+1)_{k+1}}\frac{(2\ell+k+1)!\;(2\ell-k)!}{(2\ell)!\,(2\ell)!}e^{-i(n+m)t}(1-e^{2it})^{2k}$$
$$\times\ {}_2F_1\left(\begin{matrix}k-n,k+1\\ -n\end{matrix};e^{2it}\right){}_2F_1\left(\begin{matrix}k-m,k+1\\ -m\end{matrix};e^{2it}\right)=$$
$$\sum_{j=0}^{2\ell}\sum_{j_1=0}^{n}\sum_{\substack{j_2=0\\ j_1+j_2=j}}^{2\ell-n}\sum_{i_1=0}^{m}\sum_{\substack{i_2=0\\ i_1+i_2=j}}^{2\ell-m}\frac{\binom{n}{j_1}\binom{2\ell-n}{j_2}}{\binom{2\ell}{j}}\frac{\binom{m}{i_1}\binom{2\ell-m}{i_2}}{\binom{2\ell}{j}}e^{i((n-j_1+j_2)-(m-i_1+i_2))t}$$

Proof. In [15, §5, 6] the weight function $W(\cos t)$ was initially defined as a Fourier polynomial with the coefficients given in terms of Clebsch-Gordan coefficients. After relabeling this gives

$$\sum_{j=0}^{2\ell}\sum_{j_1=0}^{n}\sum_{\substack{j_2=0\\ j_1+j_2=j}}^{2\ell-n}\sum_{i_1=0}^{m}\sum_{\substack{i_2=0\\ i_1+i_2=j}}^{2\ell-m}\frac{\binom{n}{j_1}\binom{2\ell-n}{j_2}}{\binom{2\ell}{j}}\frac{\binom{m}{i_1}\binom{2\ell-m}{i_2}}{\binom{2\ell}{j}}e^{i((n-j_1+j_2)-(m-i_1+i_2))t}$$
$$=\big(L(\cos t)T(\cos t)L(\cos t)^t\big)_{nm}=\sum_{k=0}^{\min(m,n)}\beta_k(m,n)\sin^{2k}t\,C_{n-k}^{(k+1)}(\cos t)C_{m-k}^{(k+1)}(\cos t)$$

where we have used [15, (5.10)] to express the Clebsch-Gordan coefficients in terms of binomial coefficients.

Using the result [3, Cor. 6.3] by Koornwinder and Badertscher together with the Fourier expansion of the Gegenbauer polynomial, see [3, (2.8)], [2, (6.4.11)], [11, (4.5.13)], we find the Fourier expansion of $\sin^k t\,C_{n-k}^{(k+\lambda)}(\cos t)$ in terms of Hahn polynomials defined by

$$Q_k(j;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-k,k+\alpha+\beta+1,-j\\ \alpha+1,-N\end{matrix};1\right),\qquad k\in\{0,1,\cdots,N\},\tag{2.3}$$

see [2, p. 345], [11, §6.2], [14, §1.5]. For $\lambda=1$ the explicit formula is

$$\begin{aligned}\frac{i^k(n+1)_{k+1}\,(n-k)!}{2^k(\frac{3}{2})_k\,(2k+2)_{n-k}}\sin^k t\,C_{n-k}^{(k+1)}(\cos t) &= \sum_{j=0}^{n}Q_k(j;0,0,n)e^{i(2j-n)t}\\ &= e^{-int}(1-e^{2it})^k\,{}_2F_1\left(\begin{matrix}k-n,k+1\\ -n\end{matrix};e^{2it}\right)\end{aligned}\tag{2.4}$$

using the generating function [14, (1.6.12)] for the Hahn polynomials in the last equality. Plugging this in the identity gives the required result. ☐

In the proof of Proposition 2.2 and Theorem 2.1 given in Appendix A we use a somewhat unusual integral representation of a Racah polynomial. Recall the Racah polynomials, [2, p. 344], [14, §1.2], defined by

$$R_k(\lambda(t);\alpha,\beta,\gamma,\delta) \;=\; {}_4F_3\left(\begin{matrix}-k, k+\alpha+\beta+1, -t, t+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix};1\right) \tag{2.5}$$

where $\lambda(t) = t(t+\gamma+\delta+1)$, and one out of $\alpha+1$, $\beta+\delta+1$, $\gamma+1$ equals $-N$ with a non-negative integer N. The Racah polynomials with $0 \leq k \leq N$ form a set of orthogonal polynomials for $t \in \{0,1,\cdots,N\}$ for suitable conditions on the parameters. For the special case of the Racah polynomials in Lemma 2.7 the orthogonality relations are given in Appendix A.

Lemma 2.7. *For integers $0 \leq t, k \leq m \leq n$ we have*

$$\int_{-1}^{1} (1-x^2)^{k+\frac12} C_{n-k}^{(k+1)}(x) C_{m-k}^{(k+1)}(x) U_{n+m-2t}(x)\,dx = \frac{\sqrt{\pi}\,\Gamma(k+\frac32)}{(k+1)} \frac{(k+1)_{m-k}}{(m-k)!} \frac{(k+1)_{n-k}}{(n-k)!}$$
$$\times \frac{(-1)^k\,(2k+2)_{m+n-2k}\,(k+1)!}{(n+m+1)!} R_k(\lambda(t);0,0,-n-1,-m-1)$$

Remark 2.8. Lemma 2.7 can be extended using the same method of proof to

$$\int_{-1}^{1} (1-x^2)^{\alpha+k+\frac12} C_{n-k}^{(\alpha+k+1)}(x) C_{m-k}^{(\alpha+k+1)}(x) C_{n+m-2t}^{(\beta)}(x)\,dx =$$
$$\frac{(\alpha+k+1)_{m-k}\,(2k+2\alpha+2)_{n-k}}{(m-k)!\,(n-m)!} \frac{(-m+\beta-\alpha-1)_{m-t}}{(m-t)!} \frac{(\beta)_{n-t}\sqrt{\pi}\,\Gamma(\alpha+k+\frac32)}{\Gamma(\alpha+n+m-t+2)} \tag{2.6}$$
$$\times\; {}_4F_3\left(\begin{matrix}k-m, -m-2\alpha-k-1, t-m, \beta+n-t\\ \beta-\alpha-1-m, -m-\alpha, n-m+1\end{matrix};1\right)$$

assuming $n \geq m$. Lemma 2.7 corresponds to the case $\alpha = 0$, $\beta = 1$ after using a transformation for a balanced ${}_4F_3$-series. Note that ${}_4F_3$-series can be expressed as a Racah polynomial orthogonal on $\{0,1,\cdots,m\}$ in case $\alpha = 0$ or $\beta = \alpha+1$, which corresponds to Lemma 2.7. We do not use (2.6) in the paper, and a proof follows the lines of the proof of Lemma 2.7 as given in Appendix A.

In [15, Thm. 6.5], see Section 1, we have proved that the weight function W is not irreducible, meaning that there exists $Y \in M_{2\ell+1}(\mathbb{C})$ so that

$$YW(x)Y^t \;=\; \begin{pmatrix} W_1(x) & 0\\ 0 & W_2(x)\end{pmatrix}, \qquad YY^t \;=\; I \;=\; Y^tY \tag{2.7}$$

and that there is no further reduction.

We can then obtain results combining the reducibility and the LDU-decomposition. E.g. assuming $2\ell+1$ even and writing

$$Y \;=\; \begin{pmatrix} A & B\\ C & D\end{pmatrix}, \quad L(x) \;=\; \begin{pmatrix} l_1(x) & 0\\ r(x) & l_2(x)\end{pmatrix}, \quad T(x) \;=\; \begin{pmatrix} t_1(x) & 0\\ 0 & t_2(x)\end{pmatrix},$$

with A, D diagonal and B, C antidiagonal, see [15, Cor. 5.6], $l_1(x)$, $l_2(x)$ lower-diagonal matrices and $r(x)$ a full matrix, we can work out the block-diagonal structure of $YL(x)T(x)L(x)^tY^t$. It follows that the off-diagonal blocks being zero is equivalent to

$$(Al_1(x)t_1(x) + Br(x)t_1(x))\left(l_1(x)^tC^t + r(x)^tD^t\right) + Bl_2(x)t_2(x)l_2(x)^tD^t = 0. \tag{2.8}$$

This can be rewritten as an identity for four sums of products of two Gegenbauer polynomials involving the weight function and the constants in Theorem 2.1 being zero. We do not write the explicit results, since we do not need them.

3. Matrix-valued orthogonal polynomials as eigenfunctions of matrix-valued differential operators

In [15, §7] we have derived that the matrix-valued orthogonal polynomials are eigenfunctions for a second and a first order matrix-valued differential operator by looking for suitable matrix-valued differential operators self-adjoint with respect to the matrix-valued inner product $\langle\cdot,\cdot\rangle_W$ by establishing relations between the coefficients of the differential operators and the weight W, and next judiciously guessing the general result and next proving it by a verification. In this paper we show that essentially these operators can be obtained from the group theoretic interpretation by establishing that the matrix-valued differential operators are obtainable from the Casimir operators for $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Since the paper is split into a first part of analytic nature and a second part of group theoretic nature, we state the result in this section whereas the proofs are given in Section 7. The Sections 4 and 6 depend strongly on the matrix-valued differential operators in Theorem 3.1.

Recall that all differential operators act on the right, so for a matrix-valued polynomial $P\colon \mathbb{R}\to M_N(\mathbb{C})$ depending on the variable x, the s-th order differential operator $D = \sum_{i=0}^{s}\frac{d^i}{dx^i}F_i(x)$, $F_i\colon \mathbb{R}\to M_N(\mathbb{C})$, acts by

$$\big(PD\big)(x) \,=\, \sum_{i=0}^{s}\frac{d^iP}{dx^i}(x)F_i(x), \qquad PD\colon \mathbb{R}\to M_N(\mathbb{C})$$

where $\left(\frac{d^iP}{dx^i}(x)\right)_{nm} = \frac{d^iP_{nm}}{dx^i}(x)$ is a matrix which is multiplied from the right by the matrix $F_i(x)$. The matrix-valued orthogonal polynomial is an eigenfunction of a matrix-valued differential operator if there exists a matrix $\Lambda\in M_N(\mathbb{C})$, the eigenvalue matrix, so that $PD=\Lambda P$ as matrix-valued functions. Note that the eigenvalue matrix is multiplied from the left. For more information on differential operators for matrix-valued functions, see e.g. [10], [24].

We denote by E_{ij} the standard matrix units, i.e. E_{ij} is the matrix with all matrix entries equal to zero, except for the (i,j)-the entry which is 1. By convention, if either i or j is not in the appropriate range, the matrix E_{ij} is zero.

Theorem 3.1. *Define the second order matrix-valued differential operator*

$$\tilde{D} \,=\, (1-x^2)\,\frac{d^2}{dx^2} \,+\, \left(\frac{d}{dx}\right)(\tilde{C}-x\tilde{U}) \,-\, \tilde{V}$$

$$\tilde{C} = \sum_{i=0}^{2\ell}(2\ell-i)E_{i,i+1} + \sum_{i=0}^{2\ell} iE_{i,i-1}, \qquad \tilde{U} = (2\ell+3)I, \qquad \tilde{V} = -\sum_{i=0}^{2\ell} i(2\ell-i)E_{ii}$$

and the first order matrix-valued differential operator

$$\tilde{E} \,=\, \left(\frac{d}{dx}\right)(\tilde{B}_0 + x\tilde{B}_1) \,+\, \tilde{A}$$

$$\tilde{B}_0 = -\sum_{i=0}^{2\ell} \frac{(2\ell-i)}{4\ell} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(\ell-i)}{2\ell} E_{ii} + \sum_{i=0}^{2\ell} \frac{i}{4\ell} E_{i,i-1},$$

$$\tilde{B}_1 = -\sum_{i=0}^{2\ell} \frac{(\ell-i)}{\ell} E_{i,i}, \qquad \tilde{A} = \sum_{i=0}^{2\ell} \frac{(2\ell+2)(i-2\ell)}{-4\ell} E_{i,i},$$

then the monic matrix-valued orthogonal polynomials P_n satisfy

$$P_n\tilde{D} \,=\, \Lambda_n(\tilde{D})P_n, \qquad \Lambda_n(\tilde{D}) = \sum_{i=0}^{2\ell} \left(-n(n-1) - n(2\ell+3) + i(2\ell-i)\right) E_{ii},$$

$$P_n\tilde{E} \,=\, \Lambda_n(\tilde{E})P_n, \qquad \Lambda_n(\tilde{E}) = \sum_{i=0}^{2\ell} \left(\frac{n(\ell-i)}{2\ell} - \frac{(2\ell+2)(i-2\ell)}{4\ell}\right) E_{ii}.$$

and the operators $\tilde{D}$ and $\tilde{E}$ commute. The operators are symmetric with respect to W.

The group theoretic proof of Theorem 3.1 is given in Section 7. Theorem 3.1 has been proved in [15, Thms. 7.5, 7.6] analytically. The symmetry of the operators with respect to W means that $\langle PD, Q\rangle_W = \langle P, QD\rangle_W$ and $\langle PE, Q\rangle_W = \langle P, QE\rangle_W$ for all matrix-valued polynomials with respect to the matrix-valued inner product $\langle\cdot,\cdot\rangle_W$ defined in (1.2). The last statement follows immediately from the first by the results of Grünbaum and Tirao [10]. Also, $[D, E] = 0$ follows from the fact that the eigenvalue matrices commute. In the notation of [10] we have $\tilde{D}, \tilde{E} \in \mathcal{D}(W)$, where $\mathcal{D}(W)$ is the $*$-algebra of matrix-valued differential operators having the matrix-valued orthogonal polynomials as eigenfunctions.

Note that $\tilde{E}$ has no analogue in case $\ell = 0$, whereas $\tilde{D}$ reduces to the hypergeometric differential operator for the Chebyshev polynomials U_n.

The matrix-differential operator $\tilde{D}$ is J-invariant, i.e. $J\tilde{D}J = \tilde{D}$. The operator $\tilde{E}$ is almost J-anti-invariant, up to a multiple of the identity. This is explained in Theorem 7.15 and the discussion following this theorem. In particular, $\tilde{D}$ descends to the corresponding irreducible matrix-valued orthogonal polynomials, but $\tilde{E}$ does not, see also [15, §7].

4. Matrix-valued orthogonal polynomials as matrix-valued hypergeometric functions

The polynomial solutions to the hypergeometric differential equation, see (6.5), are uniquely determined. Many classical orthogonal polynomials, such as the Jacobi, Hermite, Laguerre and Chebyshev, can be written in terms of hypergeometric series. For matrix-valued valued functions Tirao [24] has introduced a matrix-valued hypergeometric differential operator and its solutions. The purpose of this section is to link the monic matrix-valued orthogonal polynomials to Tirao's matrix-valued hypergeometric functions.

We want to use Theorem 3.1 in order to express the matrix-valued orthogonal polynomials as matrix-valued hypergeometric functions using Tirao's approach [24]. In order to do so we have to switch from the interval $[-1,1]$ to $[0,1]$ using $x = 1-2u$. We define

$$R_n(u) \,=\, (-1)^n 2^{-n} P_n(1-2u), \qquad Z(u) \,=\, W(1-2u) \tag{4.1}$$

so that the rescaled monic matrix-valued orthogonal polynomials R_n satisfy

$$Z(u) = W(1-2u), \qquad \int_0^1 R_n(u) Z(u) R_m(u)^* \, du \,=\, 2^{-1-2n} H_n. \tag{4.2}$$

In the remainder of Section 4 we work with the polynomials R_n on the interval $[0,1]$. It is a straightforward check to rewrite Theorem 3.1.

Corollary 4.1. *Let D and E be the matrix-valued differential operators*

$$D \,=\, u(1-u)\frac{d^2}{du^2} + \left(\frac{d}{du}\right)(C - uU) - V, \qquad E \,=\, \left(\frac{d}{du}\right)(uB_1 + B_0) + A_0,$$

where the matrices C, U, V, B_0, B_1 and A_0 are given by

$$C = -\sum_{i=0}^{2\ell} \frac{(2\ell-i)}{2} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(2\ell+3)}{2} E_{ii} - \sum_{i=0}^{2\ell} \frac{i}{2} E_{i,i-1}, \qquad U = (2\ell+3)I,$$

$$V = -\sum_{i=0}^{2\ell} i(2\ell-i) E_{i,i} \qquad A_0 = \sum_{i=0}^{2\ell} \frac{(2\ell+2)(i-2\ell)}{2\ell} E_{i,i},$$

$$B_0 = -\sum_{i=0}^{2\ell} \frac{(2\ell-i)}{4\ell} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(\ell-i)}{2\ell} E_{ii} + \sum_{i=0}^{2\ell} \frac{i}{4\ell} E_{i,i-1}, \qquad B_1 = -\sum_{i=0}^{2\ell} \frac{(\ell-i)}{\ell} E_{i,i}.$$

Then D and E are symmetric with respect to the weight W, and D and E commute. Moreover for every integer $n \geq 0$,

$$R_n D = \Lambda_n(D) R_n, \qquad \Lambda_n(D) = \sum_{i=0}^{2\ell} \left(-n(n-1) - n(2\ell+3) + i(2\ell-i)\right) E_{ii},$$

$$R_n E = \Lambda_n(E) R_n, \qquad \Lambda_n(E) = \sum_{i=0}^{2\ell} \left(-\frac{n(\ell-i)}{\ell} + \frac{(2\ell+2)(i-2\ell)}{2\ell}\right) E_{ii}.$$

It turns out that it is more convenient to work with $D_\alpha = D + \alpha E$ for $\alpha \in \mathbb{R}$, so that $R_n D_\alpha = \Lambda_n(D_\alpha) R_n$ with diagonal eigenvaluematrix $\Lambda_n(D_\alpha) = \Lambda_n(D) + \alpha \Lambda_n(E)$. By [10, Prop. 2.6] we have $\Lambda_n(D_\alpha) = -n^2 - n(U_\alpha - 1) - V_\alpha$. Since the eigenvalue matrix $\Lambda_n(D_\alpha)$ is diagonal, the matrix-valued differential equation $R_n D_\alpha = \Lambda_n(D_\alpha) R_n$ can be read as $2\ell+1$ differential equations for the rows of R_n. The i-th row of R_n is a solution to

$$u(1-u)p''(u) + p'(u)(C_\alpha - uU_\alpha) - p(u)(V_\alpha + \lambda) = 0, \qquad \lambda = \big(\Lambda_n(D_\alpha)\big)_{ii} \tag{4.3}$$

for $p \colon \mathbb{C} \to \mathbb{C}^{2\ell+1}$ a (row-)vector-valued polynomial function. Here $C_\alpha = C + \alpha B_0$, $U_\alpha = U - \alpha B_1$, $V_\alpha = V - \alpha A_0$ using the notation of Corollary 4.1. Now (4.3) allows us to connect to Tirao's matrix-valued hypergeometric function [24], which we briefly recall in Remark 4.2.

Remark 4.2. Given $d \times d$ matrices C, U and V we can consider the differential equation

$$z(1-z)F''(z) + (C - zU)F'(z) - VF(z) = 0, \quad z \in \mathbb{C}, \tag{4.4}$$

where $F\colon \mathbb{C} \to \mathbb{C}^d$ is a (column-)vector-valued function which is twice differentiable. It is shown by Tirao [24] that if the eigenvalues of C are not in $-\mathbb{N}$, then the matrix-valued hypergeometric function ${}_2H_1$ defined as the power series

$$\begin{gathered} {}_2H_1\left(\begin{matrix} U, V \\ C \end{matrix} ; z\right) = \sum_{i=0}^{\infty} \frac{z^i}{i!} [C, U, V]_i, \\ [C,U,V]_0 = 1, \quad [C,U,V]_{i+1} = (C+i)^{-1}\big(i^2 + i(U-1) + V\big)[C,U,V]_i \end{gathered} \tag{4.5}$$

converges for $|z| < 1$ in $M_d(\mathbb{C})$. Moreover, for $F_0 \in \mathbb{C}^d$ the (column-)vector-valued function

$$F(z) = {}_2H_1\left(\begin{matrix} U, V \\ C \end{matrix} ; z\right) F_0$$

is a solution to (4.4) which is analytic for $|z| < 1$, and any analytic (on $|z| < 1$) solution to (4.4) is of this form.

Comparing Tirao's matrix-valued hypergeometric differential equation (4.4) with (4.3) and using Remark 4.2, we see that

$$p(u) = \left({}_2H_1\left(\begin{matrix} U_\alpha^t, V_\alpha^t + \lambda \\ C_\alpha^t \end{matrix} ; u\right) P_0 \right)^t = P_0^t \left({}_2H_1\left(\begin{matrix} U_\alpha^t, V_\alpha^t + \lambda \\ C_\alpha \end{matrix} ; u\right) \right)^t, \qquad P_0 \in \mathbb{C}^{2\ell+1}, \tag{4.6}$$

are the solutions to (4.3) which are analytic in $|u| < 1$ assuming that eigenvalues of C_α^t are not in $-\mathbb{N}$. We first verify this assumption. Even though V_α and U_α are symmetric, we keep the notation for transposed matrices for notational esthetics.

Lemma 4.3. *For every $\ell \in \frac{1}{2}\mathbb{N}$, the matrix C_α is a diagonalisable matrix with eigenvalues $(2j+3)/2$, $j \in \{0, \ldots, 2\ell\}$.*

Proof. Note that C_α is tridiagonal, so that $v_\lambda = \sum_{n=0}^{2\ell} p_n(\lambda) e_n$ is an eigenvector for C_α for the eigenvalue λ if and only if

$$\begin{aligned} -(\lambda - \frac{3}{2})\, p_n(\lambda) = {} & \frac{(2\ell - n)(2\ell + \alpha)}{4\ell}\, p_{n+1}(\lambda) \\ & - \left(\frac{(2\ell+\alpha)(2\ell - n) + n(2\ell - \alpha)}{4\ell} \right) p_n(\lambda) + \frac{(2\ell - \alpha)n}{4\ell}\, p_{n-1}(\lambda). \end{aligned}$$

The three-term recurrence relation corresponds precisely to the three-term recurrence relation for the Krawtchouk polynomials for $N \in \mathbb{N}$,

$$K_n(x; p, N) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} ; \frac{1}{p}\right), \qquad n, x \in \{0, 1, \cdots, N\},$$

see e.g. [2, p. 347], [11, §6.2], [14, §1.10], with $N = 2\ell$, $p = \frac{2\ell+\alpha}{4\ell}$. The Krawtchouk polynomials are orthogonal with respect to the binomial distribution for $0 < p < 1$, or $\alpha \in (-2\ell, 2\ell)$, and we find

$$p_n(\lambda) = K_n(\lambda; \frac{2\ell+\alpha}{4\ell}, \ell)$$

and the eigenvalues of C_α are $\frac{3}{2}+x$, $x \in \{0,1,\cdots,2\ell\}$. This proves the statement for $\alpha \in (-2\ell, 2\ell)$.

Note that for $\alpha \neq \pm 2\ell$, the matrix C_α is tridiagonal, and the eigenvalue equation is solved by the same contiguous relation for the ${}_2F_1$-series leading to the same statement for $|\alpha| > 2\ell$. In case $\alpha = \pm 2\ell$ the matrix $C_{\pm 2\ell}$ is upper or lower triangular, and the eigenvalues can be read off from the diagonal. $\square$

In particular, we can give the eigenvectors of C_α explicitly in terms of terminating ${}_2F_1$-hypergeometric series, but we do not use the result in the paper.

So (4.6) is valid and this gives a series representation for the rows of the monic polynomial R_n. Since each row is polynomial, the series has to terminate. This implies that there exists $n \in \mathbb{N}$ so that $[C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_{n+1}$ is singular and $0 \neq P_0 \in \operatorname{Ker}([C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_{n+1})$.

Suppose that n is the least integer for which $[C^t_\alpha, U^t_\alpha, V^t_\alpha+\lambda]_{n+1}$ is singular, i.e. $[C^t_\alpha, U^t_\alpha, V^t_\alpha+\lambda]_i$ is regular for all $i \leq n$. Since

$$[C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_{n+1} = (C^t_\alpha + n)^{-1}\left(n^2 + n(U^t_\alpha - 1) + V^t_\alpha + \lambda\right)[C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_n \tag{4.7}$$

and since the matrix $(C_\alpha + n)$ is invertible by Lemma 4.3, $[C_\alpha, U_\alpha, V_\alpha + \lambda]_{n+1}$ is a singular matrix if and only if the diagonal matrix

$$M^\alpha_n(\lambda) \;=\; \left(n^2 + n(U^t_\alpha - 1) + V^t_\alpha + \lambda\right) \;=\; \left(n^2 + n(U_\alpha - 1) + V_\alpha + \lambda\right) \;=\; \lambda - \Lambda_n(D_\alpha) \tag{4.8}$$

is singular. Note that the diagonal entries of $M^\alpha_n(\lambda)$ are of the form $\lambda - \lambda^\alpha_j(n)$, so that $M_n(\lambda)$ is singular if and only if $\lambda = \lambda^\alpha_j(n)$ for some $j \in \{0,1,\cdots,2\ell\}$. We need that the eigenvalues are sufficiently generic.

Lemma 4.4. *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Then $(j,n) = (i,m) \in \{0,1\cdots,2\ell\}\times\mathbb{N}$ if and only if $\lambda^\alpha_j(n) = \lambda^\alpha_i(m)$.*

Proof. Assume $\lambda^\alpha_j(n) = \lambda^\alpha_i(m)$ and let $(j,n),(i,m) \in \{0,1\cdots,2\ell\}\times\mathbb{N}$, then

$$0 \;=\; \lambda^\alpha_j(n) - \lambda^\alpha_i(m) \;=\; (m-n)(n+m+2+2\ell) + (j-i)\left(\frac{\alpha(2\ell+4)}{2\ell} - j - i + 2\ell\right).$$

If $j \neq i$, then we solve for $\alpha = \frac{2\ell}{(2\ell+4)}\left(-\frac{(m-n)(m+n+2\ell+2)}{j-i} + j + i - 2\ell\right)$ which is rational.

Assume next that $j = i$, then $(m-n)(n+m+2+2\ell) = 0$. Since $n,m,\ell \geq 0$, it follows that $n = m$ and hence $(j,n) = (i,m)$. $\square$

Assume α irrational, so that Lemma 4.4 shows that $M_n\big(\lambda^\alpha_i(m)\big)$ is singular if and only if $n = m$. So in the series (4.6) the matrix $[C^t_\alpha, U^t_\alpha, V^t_\alpha+\lambda]_{n+1}$ is singular and $[C^t_\alpha, U^t_\alpha, V^t_\alpha+\lambda]_i$ is non-singular for $0 \leq i \leq n$. Furthermore, by Lemma 4.4 we see that the kernel of $[C^t_\alpha, U^t_\alpha, V^t_\alpha+\lambda]_{n+1}$ is one-dimensional if and only if $\lambda = \lambda^\alpha_n(i)$, $i \in \{0,1,\cdots,2\ell\}$. In case $\lambda = \lambda^\alpha_n(i)$ we see that (4.6) is polynomial for

$$P_0 \;=\; [C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda^\alpha_n(i)]^{-1}_n e_i$$

determined uniquely up to a scalar, where e_i is the standard basis vector.

We can now state the main result of this section, expressing the monic polynomials R_n as a matrix-valued hypergeometric function.

Theorem 4.5. *With the notation of Remark 4.2 the monic matrix-valued orthogonal polynomials are given by*

$$\big(R_n(u)\big)_{ij} \;=\; \left({}_2H_1 \left(\begin{matrix} U_\alpha^t, V_\alpha^t + \lambda_n^\alpha(i) \\ C_\alpha^t \end{matrix} ; u \right) n!\, [C_\alpha^t, U_\alpha^t, V_\alpha^t + \lambda_n^\alpha(i)]_n^{-1} e_i \right)_j^t$$

for all $\alpha \in \mathbb{R}$.

Note that the left hand side is independent of α, which is not obvious for the right hand side.

Proof. Let us first assume that α is irrational, so that the result follows from the considerations in this section using that the i-th row of $R_n(u)$ is a polynomial of (precise) degree n. The constant follows from monocity of R_n, so that $(R_n(u))_{ii} = u^n$.

Note that the left hand side is independent of α, and the right hand side is continuous in α. Hence the result follows for $\alpha \in \mathbb{R}$. ☐

In the scalar case $\ell = 0$ Theorem 4.5 reduces to

$$R_n(u) \;=\; (-4)^{-n}(n+1)\; {}_2F_1 \left(\begin{matrix} -n,\, n+2 \\ \frac{3}{2} \end{matrix} ; u \right), \tag{4.9}$$

which is the well-known hypergeometric expression for the monic Chebyshev polynomials, see [2, §2.5], [11, (4.5.21)], [14, (1.8.31)].

5. Three-term recurrence relation

Matrix-valued orthogonal polynomials satisfy a three-term recurrence relation, see e.g. [6], [10]. In [15, Thm. 4.8] we have determined the three-term recurrence relation for the closely related matrix-valued orthogonal polynomials explicitly in terms of Clebsch-Gordan coefficients. The matrix entries of the matrices occurring in the three-term recurrence relation have been given explicitly as sums of products of Clebsch-Gordan coefficients. The purpose of this section is to give simpler expressions for the monic matrix-valued orthogonal polynomials using the explicit expression in terms of Tirao's matrix-valued hypergeometric functions as established in Theorem 4.5.

From general theory the monic orthogonal polynomials $R_n\colon \mathbb{R} \to M_N(\mathbb{C})$ satisfy a three-term recurrence relation $uR_n(u) = R_{n+1}(u) + X_nR_n(u) + Y_nR_{n-1}(u)$, $n \geq 0$, where $R_{-1} = 0$ and $X_n, Y_n \in M_{2\ell+1}(\mathbb{C})$ are matrices depending on n and not on x. Lemma 5.1 should be compared to [6, Lemma 2.6].

Lemma 5.1. *Let $\{R_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials and write $R_n(u) = \sum_{k=0}^n R_k^n\, u^k$, $R_k^n \in M_N(\mathbb{C})$, and $R_n^n = I$. Then the coefficients X_n, Z_n of the three-term recurrence relation are given by*

$$X_n \;=\; R_{n-1}^n - R_n^{n+1}, \qquad Y_n \;=\; R_{n-2}^n - R_{n-1}^{n+1} - X_nR_{n-1}^n.$$

Proof. Let $\langle\cdot,\cdot\rangle$ denote the matrix-valued inner product for which the monic polynomials are orthogonal. Using the three-term recursion, orthogonality relations and expanding the monic polynomial of degree $n+1$ gives

$$\langle uR_n - X_nR_n - Y_nR_{n-1}, R_n\rangle \;=\; \langle R_{n+1}, R_n\rangle \;=\; \langle u^{n+1}, R_n\rangle \;+\; R_n^{n+1}\langle u^n, R_n\rangle.$$

By the orthogonality relations the left hand side can be evaluated as

$$\langle uR_n - X_nR_n - Y_nR_{n-1}, R_n\rangle \;=\; \langle u^{n+1}, R_n\rangle \;+\; R^n_{n-1}\langle u^n, R_n\rangle \;-\; X_n\langle R_n, R_n\rangle$$

and comparing the two right hand sides gives the required expression for X_n, since $\langle R_n, R_n\rangle = \langle u^n, R_n\rangle$ is invertible.

The expression for Y_n follows by considering on the one hand

$$\begin{aligned}\langle uR_n - X_nR_n - Y_nR_{n-1}, R_{n-1}\rangle \;&=\; \langle R_{n+1}, R_{n-1}\rangle\\ &=\; \langle u^{n+1}, R_{n-1}\rangle \;+\; R^{n+1}_{n}\langle u^n, R_{n-1}\rangle \;+\; R^{n+1}_{n-1}\langle u^{n-1}, R_{n-1}\rangle.\end{aligned}$$

while on the other hand the left hand side also equals

$$\begin{aligned}&\langle u^{n+1}, R_{n-1}\rangle \;+\; R^n_{n-1}\langle u^n, R_{n-1}\rangle \;+\; R^n_{n-2}\langle u^{n-1}, R_{n-1}\rangle\\ &\qquad -\; X_n\langle u^n, R_{n-1}\rangle \;-\; X_nR^n_{n-1}\langle u^{n-1}, R_{n-1}\rangle \;-\; Y_n\langle R_{n-1}, R_{n-1}\rangle\end{aligned}$$

and using the expression for X_n and cancelling common terms gives the required expression, since $\langle R_{n-1}, R_{n-1}\rangle = \langle u^{n-1}, R_{n-1}\rangle$ is invertible. □

In order to apply Lemma 5.1 for the explicit monic polynomials in this paper we need to calculate the coefficients, which is an application of Theorem 4.5.

Lemma 5.2. *Let $\{R_n\}_{n\geq 0}$ be the monic polynomials with respect to Z on $[0,1]$. Then*

$$\begin{aligned}R^n_{n-1} &= \sum_{j=0}^{n} \frac{jn}{4(n+j)}E_{j,j-1} - \sum_{j=0}^{n}\frac{n}{2}E_{j,j} + \sum_{i=0}^{n}\frac{n(2\ell-j)}{4(2\ell-j+n)}E_{j,j+1}\\ R^n_{n-2} &= \sum_{j=0}^{n}\frac{n(n-1)j(j-1)}{32(n+j)(n+j-1)}E_{j,j-2} - \sum_{j=0}^{n}\frac{n(n-1)j}{8(n+j)}E_{j,j-1}\\ &\quad + \sum_{j=0}^{n}\frac{n(n-1)(3j^2-6\ell j-2n^2+n-4n\ell)}{16(n+j)(i-2\ell-n)}E_{j,j} - \sum_{j=0}^{n}\frac{n(n-1)(2\ell-j)}{8(2\ell+n-j)}E_{j,j+1}\\ &\quad + \sum_{j=0}^{n}\frac{n(n-1)(2\ell-j)(2\ell-j-1)}{32(2\ell-j+n-1)(2\ell+n-j)}E_{j,j+2}\end{aligned}$$

Proof. We can calculate R^n_{n-1} by considering the coefficients of u^{n-1} using the expression in Theorem 4.5. This gives

$$\begin{aligned}(R^n_{n-1})_{ij} \;&=\; \frac{n!}{(n-1)!}\left([C^t_\alpha, U^t_\alpha, V^t_\alpha+\lambda^\alpha_n(i)]_{n-1}[C^t_\alpha, U^t_\alpha, V^t_\alpha+\lambda^\alpha_n(i)]_n^{-1}e_i\right)^t_j\\ &=\; n\left(M^\alpha_{n-1}\big(\lambda^\alpha_n(i)\big)^{-1}(C^t_\alpha+n-1)e_i\right)^t_j\end{aligned}$$

using the recursive definition (4.7) of $[C^t_\alpha, U^t_\alpha, V^t_\alpha+\lambda^\alpha_n(i)]_n$. Note that $M^\alpha_{n-1}\big(\lambda^\alpha_n(i)\big)$ is indeed invertible by Lemma 4.4 for irrational α. The explicit expression of the right hand side gives the result after a straightforward computation, since the resulting matrix is tridiagonal.

We can calculate R^n_{n-2} analogously,

$$(R^n_{n-2})_{ij} \;=\; n(n-1)\;\left(M^\alpha_{n-2}\big(\lambda^\alpha_n(i)\big)^{-1}(C^t_\alpha+n-2)M^\alpha_{n-1}\big(\lambda^\alpha_n(i)\big)^{-1}(C^t_\alpha+n-1)e_i\right)^t_j$$

and a straightforward but tedious calculation gives the result. Note that R^n_{n-2} is a five-diagonal matrix, since it is the product of two tridiagonal matrices. ☐

Note that even though we have used the additional degree of freedom α in the proof of Lemma 5.2, the resulting expressions are indeed independent of α.

Now we are ready to obtain the coefficients in the recurrence relation satisfied by the polynomials R_n.

Theorem 5.3. *For any $\ell \in \frac{1}{2}\mathbb{N}$ the monic orthogonal polynomials R_n satisfy the three-term recurrence relation*

$$u\, R_n(u) \;=\; R_{n+1}(u) \;+\; X_n\, R_n(u) \;+\; Y_n\, R_{n-1}(u),$$

where the matrices X_n, Y_n are given by

$$X_n \;=\; -\sum_{i=0}^{2\ell}\left[\frac{i^2 E_{i,i-1}}{4(n+i)(n+i+1)} - \frac{E_{i,i}}{2} + \frac{(2\ell-i)^2 E_{i,i+1}}{4(2\ell+n-i)(2\ell+n-i+1)}\right],$$

$$Y_n \;=\; \sum_{i=0}^{2\ell}\frac{n^2(2\ell+n+1)^2}{16(n+i)(n+i+1)(2\ell+n-i)(2\ell+n-i+1)}E_{i,i}.$$

Proof. This is a straightforward computation using Lemma 5.1 and Lemma 5.2. The calculation of X_n is straightforward from Lemma 5.1 and Lemma 5.2. In order to calculate Y_n we need X_nR_{n-1}. A calculation shows

$$X_nR^n_{n-1} = -\sum_{j=0}^{2\ell}\frac{nj^2(j-1)}{16(n+j-1)(n+j)(n+j+1)}E_{j,j-2} + \sum_{j=0}^{2\ell}\frac{nj(n+2j+1)}{8(n+j)(n+j+1)}E_{j,j-1}+$$

$$\sum_{j=0}^{2\ell}\left(\frac{-i^2n(2\ell-i+1)}{16(n+i)(n+i+1)(2\ell-i+1+n)} - \frac{n}{4} - \frac{(2\ell-i)^2(i+1)n}{16(2\ell-i+1+n)(2\ell-i+n)(n+i+1)}\right)E_{jj}$$

$$-\sum_{j=0}^{2\ell}\frac{n(2\ell-i)(4\ell-2j+n+1)}{8(2\ell+n-j)(2\ell+n-j+1)}E_{j,j+1}$$

$$+\sum_{j=0}^{2\ell}\frac{n(2\ell-j)^2(2\ell-j+1)}{16(2\ell-j+n-1)(2\ell-j+n)(2\ell-j+n+1)}E_{j,j+2}$$

Now Lemma 5.1 and a computation show that Y_n reduces to a tridiagonal matrix. ☐

Now (4.1) and Theorem 5.3 give the three-term recurrence

$$x\, P_n(x) \;=\; P_{n+1}(x) \;+\; (1-2X_n)\, P_n(x) \;+\; 4Y_nP_{n-1}(x) \tag{5.1}$$

for the monic orthogonal polynomials with respect to the matrix-valued weight W on $[-1,1]$. The case $\ell=0$ corresponds to the three-term recurrence for the monic Chebyshev polynomials U_n. Note moreover, that $\lim_{n\to\infty} X_n = \frac{1}{2}$ and $\lim_{n\to\infty} Y_n = \frac{1}{16}$, so that the monic matrix-valued orthogonal polynomials fit in the Nevai class, see [7]. Note the matrix-valued orthogonal polynomials P_n in this paper are considered as matrix-valued analogues of the Chebyshev polynomials of the second kind, because of the group theoretic interpretation [15] and Section

7, but that these polynomials are not matrix-valued Chebyshev polynomials in the sense of [7, §3].

Using the three-term recurrence relation (5.1) and (1.2) we get

$$4Y_nH_{n-1} \;=\; \int_{-1}^{1} xP_n(x)W(x)P_{n-1}(x)^*dx \;=\; \int_{-1}^{1} P_n(x)W(x)(xP_{n-1}(x))^*dx \;=\; H_n \tag{5.2}$$

analogous to the scalar-valued case. Since H_0 is determined in (1.3) we obtain H_n.

Corollary 5.4. *The squared norm matrix H_n is*

$$(H_n)_{ij} \;=\; \delta_{ij}\frac{\pi}{2}\frac{(n!)^2\,(2\ell+1)_{n+1}^2}{(i+1)_n^2\,(2\ell-i+1)_n^2}\frac{2^{-2n}}{(n+i+1)(2\ell-i+n+1)}$$

and $JH_nJ = H_n$.

In [15, Thm. 4.8] we have stated the three-term recurrence relation for the polynomials $Q^\ell_n(a)$, $a \in A_*$, see also Section 7 of this paper. Apart from a relabeling of the orthonormal basis the monic polynomials corresponding to Q^ℓ_n are precisely the polynomials P_n, see [15, §6.2, (6.4)] for the precise identification

$$P_d(x)_{n,m} = \Upsilon_d^{-1}Q_d(a_{\arccos x})_{-\ell+n,-\ell+m}, \quad n,m \in \{0,1,\ldots,2\ell\}, \tag{5.3}$$

see also Section 7, where Υ_d in (5.3) is the leading coefficient of Q^ℓ_d.

Corollary 5.5. *The polynomials Q^ℓ_n as in* [15, §4] *satisfy the recurrence*

$$\phi(a)\,Q^\ell_n(a) \;=\; A_n\,Q^\ell_{n+1}(a) \;+\; B_n\,Q^\ell_n(a) \;+\; C_n\,Q^\ell_{n-1}(a)$$

where

$$A_n \;=\; \sum_{q=-\ell}^{\ell} \frac{(n+1)(2\ell+n+2)}{2(\ell-q+n+1)(\ell+q+n+1)}E_{q,q},$$

$$B_n \;=\; \sum_{q=-\ell}^{\ell} \frac{(\ell-q+1)(\ell+q)}{2(\ell-q+n+1)(\ell+q+n+1)}E_{q,q-1} + \sum_{q=-\ell}^{\ell} \frac{(\ell+q+1)(\ell-q)}{2(\ell-q+n+1)(\ell+q+n+1)}E_{q,q+1}$$

$$C_n \;=\; \sum_{q=-\ell}^{\ell} \frac{n(2\ell+n+1)}{2(n+q+\ell+1)(n-q+\ell+1)}E_{q,q}.$$

Proof. We use $A_n = \Upsilon_n\Upsilon_{n+1}^{-1}$, $B_n = \Upsilon_n(1-2X_n)\Upsilon_n^{-1}$, $C_n = \Upsilon_n(4Y_n)\Upsilon_{n-1}^{-1}$ and Theorem 5.3 to obtain the result from a straighforward computation. The matrices Υ_n are given by

$$(\Upsilon_n)_{p,q} \;=\; \delta_{pq}\,2^n\frac{(\ell-q+1)_n\,(\ell+q+1)_n}{n!\,(2\ell+2)_n} \tag{5.4}$$

which follows from [16, (3.10), (3.16)] where we have to bear in mind that the polynomials in [16] differ from ours by an application of J. Note that [15] does not give this value for Υ_n. $\square$

Recall from [15, Thm. 4.8, (3.2)] that the matrix entries of the matrices A_n, B_n and C_n are explicitly known as a square of a double sum with summand the product of four Clebsch-Gordan coefficients, hence Corollary 5.5 leads to an explicit expression for this square.

6. The matrix-valued orthogonal polynomials related to Gegenbauer and Racah polynomials

The LDU-decomposition of the weight W of Theorem 2.1 has the weight functions of the Gegenbauer polynomials in the diagonal T, so we can expect a link between the matrix-valued polynomials $P_n(x)L(x)$ and the Gegenbauer polynomials. We cannot do this via the orthogonality relations and the weight function, since the matrix L also depends on x. Instead we use an approach based on the differential operators $\tilde{D}$ and $\tilde{E}$ of Section 3, and because of the link to the matrix-valued hypergeometric differential operator as in Theorem 4.5 we switch to the matrix-valued orthogonal polynomials R_n and $x = 1-2u$. It turns out that the matrix entries of $P_n(x)L(x)$ can be given as a product of a Racah polynomial times a Gegenbauer polynomial, see Theorem 6.2.

We use the differential operators D and E of Corollary 4.1, and as in Section 4 it is handier to work with the second-order differential operator $D_{-2\ell} = D - 2\ell E$. By Theorem 2.1 we have $W(x) = L(x)T(x)L(x)^t$, hence $Z(u) = L(1-2u)T(1-2u)L(1-2u)^t$. For this reason we look at the differential operator conjugated by $M(u) = L(1-2u)$.

In general, for $D = \frac{d^2}{du^2}F_2(u) + \frac{d}{du}F_1(u) + F_0(u)$ a second order matrix-valued differential operator, conjugation with the matrix-valued function M, which we assume invertible for all u, gives

$$\begin{aligned} M^{-1}DM \;=\; \frac{d^2}{du^2}M^{-1}F_2M + \frac{d}{du}\left(M^{-1}F_1M + 2\frac{dM^{-1}}{du}F_0M\right) \\ + \left(M^{-1}F_0M + \frac{dM^{-1}}{du}F_1M + \frac{d^2M^{-1}}{du^2}F_0M\right). \end{aligned}$$

Note that differentiating $M^{-1}M = I$ gives $\frac{dM^{-1}}{du} = -M^{-1}\frac{dM}{du}M^{-1}$, and similarly we find $\frac{d^2M^{-1}}{du^2} = -M^{-1}\frac{d^2M}{du^2}M^{-1} + 2M^{-1}\frac{dM}{du}M^{-1}\frac{dM}{du}M^{-1}$. We are investigating the possibility of $M^{-1}DM$ being a diagonal matrix-valued differential operator. We now assume that $F_2(u) = u(1-u)$, so that $M^{-1}F_2M = u(1-u)$. A straightforward calculation using this assumption and the calculation of the derivatives of M^{-1} shows that $M^{-1}DM = u(1-u)\frac{d^2}{du^2} + \frac{d}{du}T_1 + T_0$ with T_0 and T_1 matrix-valued functions if and only if the following equations (6.1), (6.2) hold:

$$F_0M \;-\; \frac{dM}{du}T_1 \;-\; u(1-u)\,\frac{d^2M}{du^2} \;=\; MT_0 \tag{6.1}$$

$$F_1M \;-\; 2u(1-u)\frac{dM}{du} \;=\; MT_1. \tag{6.2}$$

Of course, T_0 and T_1 need not be diagonal in general, but this is the case of interest.

Proposition 6.1. *The differential operator $\mathcal{D} = M^{-1}D_{-2\ell}M$ is the diagonal differential operator*

$$\mathcal{D} \;=\; u(1-u)\frac{d^2}{du^2} + \left(\frac{d}{du}\right)T_1(u) + T_0,$$

where

$$T_1(u) = \frac{1}{2}T_1^1 - u\,T_1^1, \quad T_1^1 \;=\; \sum_{i=0}^{2\ell}(2i+3)E_{i,i}, \qquad T_0 \;=\; \sum_{i=0}^{2\ell}(2\ell-i)(2\ell+i+2)E_{i,i}$$

Moreover, $\mathcal{R}_n(u) = R_n(u)M(u)$ *satisfies*

$$\mathcal{R}_n\,\mathcal{D} \,=\, \Lambda_n(\mathcal{D})\,\mathcal{R}_n, \qquad \Lambda_n(\mathcal{D}) \,=\, \Lambda_n(D) - 2\ell\,\Lambda_n(E).$$

The proof shows that $M^{-1}D_\alpha M$ can only be a diagonal differential operator for $\alpha = -2\ell$. Note that $\mathcal{D}$ is a matrix-valued differential operator as considered by Tirao, see Remark 4.2 and [24], and diagonality of $\mathcal{D}$ implies that the matrix-valued hypergeometric ${}_2H_1$-series can be given explicitly in terms of (usual) hypergeometric series. In particular, we find as in the proof of Theorem 4.5 that

$$\big(\mathcal{R}_n(u)\big)_{kj} = \left({}_2H_1 \left(\begin{matrix} T_1^1,\, \lambda_n(k) - T_0 \\ \frac{1}{2}T_1^1 \end{matrix} ; u \right) v \right)^t_j, \quad v_k = \big(\mathcal{R}_n(0)\big)_{kj}, \quad \lambda_n(k) = \Lambda_n(\mathcal{D})_{kk}, \tag{6.3}$$

since the condition $\sigma(\frac{1}{2}T_1^1) \not\subset -\mathbb{N}$ is satisfied.

Proof. Consider $D_\alpha = D + \alpha E$, so that $F_2(u) = u(1-u)$ and the above considerations apply and $F_1(u) = C_\alpha - uU_\alpha$, and $F_0 = -V_\alpha$. We want to find out if we can obtain matrix-valued functions T_1 and T_0 satisfying (6.1), (6.2) for this particular F_1, F_2 and $M(u) = L(1-2u)$. Since F_0 is diagonal, and assuming that T_0, T_1 can be taken diagonal it is clear that taking the (k,l)-th entry of (6.1) leads to

$$(F_0)_{kk}M_{kl} - \frac{dM_{kl}}{du}(T_1)_{ll} - u(1-u)\frac{d^2M_{kl}}{du^2} \,=\, M_{kl}(T_0)_{ll}. \tag{6.4}$$

By Theorem 2.1 we have $M_{kl} = 0$ for $l > k$ and for $l \leq k$

$$M_{kl}(u) \,=\, \binom{k}{l}\, {}_2F_1 \left(\begin{matrix} l-k, k+l+2 \\ l+\frac{3}{2} \end{matrix} ; u \right)$$

so that (6.4) has to correspond to the second order differential operator

$$u(1-u)f''(u) \,+\, \big(c - (a+b+1)u\big)f'(u) - abf(u) = 0, \qquad f(u) \,=\, {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; u \right) \tag{6.5}$$

for the hypergeometric function. This immediately gives

$$(T_1)_{ll} \,=\, l + \frac{3}{2} - (2l+3)u, \qquad (T_0)_{ll} - (F_0)_{kk} \,=\, k^2 + 2k - (l^2 + 2l).$$

Since $(F_0)_{kk} = (-V_\alpha)_{kk} = -k^2 + (2\ell + \alpha\frac{(2\ell+2)}{2\ell})k - \alpha(2\ell+2)$, this is only possible for $\alpha = -2\ell$, and in that case

$$(T_1)_{ll} \,=\, l + \frac{3}{2} - (2l+3)u, \qquad (T_0)_{ll} \,=\, -l^2 - 2l + 2\ell(2\ell+2). \tag{6.6}$$

It remains to check that for $\alpha = -2\ell$ the condition (6.2) is valid with the explicit values (6.6). For $\alpha = -2\ell$ the matrix-valued function F_1 is lower triangular instead of tridiagonal, so that (6.2) is an identity in the subalgebra of lower triangular matrices. With the explicit expression for M we have to check that

$$\big((\frac{3}{2} + k) - u(3+2k)\big)M_{kl} - kM_{k-1,l} \,-\, 2u(1-u)\frac{dM_{kl}}{du} \,=\, M_{kl}\big(\frac{3}{2} + l - u(3+2l)\big)$$

which can be identified with the identity

$$(1-x^2)\,\frac{dC^{(l+1)}_{k-l}}{dx}(x) \;=\; (k+l+1)C^{(l+1)}_{k-l-1}(x) - x(k-l)C^{(l+1)}_{k-l}(x).$$

In turn, this identity can be easily obtained from [1, (22.7.21)] or from [11, (4.5.3), (4.5.7)]. □

Since R_n and M are polynomial, Proposition 6.1 and the explicit expression for the eigenvalue matrix in Corollary 4.1 imply that $(\mathcal{R}_n)_{kj}$ is a polynomial solution to

$$\begin{aligned} u(1-u)f''(u) &+ \big((j+\tfrac{3}{2}) - u(2j+3)\big)\, f'(u) + (2\ell-j)(2\ell+j+2)f(u) \\ &= \Big(-n(n-1) - n(2\ell+3) + k(2\ell-k) + 2n(\ell-k) - (2\ell+2)(k-2\ell)\Big)\, f(u) \end{aligned}$$

which can be rewritten as

$$u(1-u)f''(u) + \big((j+\tfrac{3}{2}) - u(2j+3)\big)\, f'(u) - (j-k-n)(n+k+j+2)f(u) \;=\; 0$$

which is the hypergeometric differential operator for which the polynomial solutions are uniquely determined up to a constant. This immediately gives

$$\mathcal{R}_n(u)_{kj} \;=\; c_{kj}(n)\; {}_2F_1\left(\begin{matrix} j-k-n,\; n+k+j+2 \\ j+\frac{3}{2} \end{matrix} ; u \right). \tag{6.7}$$

for $j-k-n \le 0$ and $\mathcal{R}_n(u)_{kj} = 0$ otherwise. The case $n=0$ corresponds to Theorem 2.1 and we obtain $c_{kj}(0) = \binom{k}{j}$. It remains to determine the constants $c_{kj}(n)$ in (6.7).

First, switching to the variable x, we find

$$\big(\mathcal{P}_n(x)\big)_{kj} \;=\; \big(P_n(x)L(x)\big)_{kj} \;=\; (-2)^n\, c_{kj}(n)\, \frac{(n+k-j)!}{(2j+2)_{n+k-j}} C^{(j+1)}_{n+k-j}(x) \tag{6.8}$$

so that by (6.8) the orthogonality relations (1.2) and (2.2) give

$$\delta_{nm}(H_n)_{kl} \;=\; (-2)^{n+m} \sum_{j=0}^{2\ell\wedge(n+k)} c_{kj}(n)\overline{c_{lj}(m)}c_j(\ell)\, \frac{(n+k-j)!}{(2j+2)_{n+k-j}}\, \delta_{n+k,m+l} \frac{\sqrt{\pi}\,\Gamma(j+\frac{3}{2})}{(n+k+1)\,j!}.$$

Using the explicit value for $c_j(\ell)$ as in Theorem 2.1 and Corollary 5.4 we find orthogonality relations for the coefficients $c_{kj}(n)$:

$$(H_n)_{kk}\, 2^{-2n}\, \delta_{nm} \;=\; \sum_{j=0}^{2\ell\wedge(n+k)} c_{kj}(n)\overline{c_{k+m-n,j}(m)} \frac{(j!)^2\,(2j+1)\,(2\ell+j+1)!\,(2\ell-j)!\,(n+k-j)!}{(n+k+j+1)!\,(n+k+1)\,(2\ell)!^2} \tag{6.9}$$

Note that we can also obtain recurrence relations for the coefficients $c_{kj}(n)$ using the three-term recurrence relation of Theorem 5.3.

Theorem 6.2. *The polynomials $\mathcal{R}_n(u) = R_n(u)M(u)$ satisfy*

$$\mathcal{R}_n(u)_{kj} = c_{k,0}(n)(-1)^j \frac{(-2\ell)_j\,(-k-n)_j}{j!\,(2\ell+2)_j}\, {}_4F_3\left(\begin{matrix}-j, j+1, -k, -2\ell-n-1\\ 1, -k-n, -2\ell\end{matrix};1\right) \times {}_2F_1\left(\begin{matrix}j-k-n,\, n+k+j+2\\ j+\frac{3}{2}\end{matrix};u\right)$$

with $\mathcal{R}_n(u)_{kj} = 0$ for $j-k-n>0$ and

$$c_{k,0}(n) = (-1)^n 4^{-n} \frac{n!\,(2\ell+2)_n}{(k+1)_n\,(2\ell-k+1)_n}$$

We view Theorem 6.2 as an extension of Theorem 2.1, but Theorem 2.1 is instrumental in the proof of Theorem 6.2. Since the inverse of $M(u)$, or of $L(x)$, does not seem to have a nice explicit expression we do not obtain an interesting expression for the matrix elements of the matrix-valued monic orthogonal polynomials $R_n(u)$ or of $P_n(x)$. Note also that the case $\ell = 0$ gives back the hypergeometric representation of the Chebyshev polynomials of the second kind U_n, see (4.9).

Comparing with (2.5) we see that we can view the ${}_4F_3$-series in Theorem 6.2 as a Racah polynomial $R_k(\lambda(j); -2\ell-1, -k-n-1, 0, 0)$, respectively $R_{k+n-m}(\lambda(j); -2\ell-1, -k-n-1, 0, 0)$, see (2.5), where the N of the Racah polynomials equals 2ℓ in case $2\ell \leq k+n$ and N equals $k+n$ in case $2\ell \geq k+n$. Using the first part of Theorem 6.2 we see that the orthogonality relations (6.9) lead to

$$\begin{aligned}(H_n)_{kk}\, 2^{-2n}\, \delta_{nm} &= \frac{\pi}{2} \frac{2\ell+1}{(n+k+1)^2} |c_{k0}(n)|^2 \sum_{j=0}^{2\ell\wedge(n+k)} \frac{(2j+1)\,(-2\ell)_j\,(-n-k)_j}{(2\ell+2)_j\,(n+k+2)_j} \\ &\times R_k(\lambda(j); -2\ell-1, -k-n-1, 0, 0)\, R_{k+n-m}(\lambda(j); -2\ell-1, -k-n-1, 0, 0),\end{aligned} \tag{6.10}$$

which corresponds to the orthogonality relations for the corresponding Racah polynomials, see [2, p. 344], [14, §1.2]. From this we find that the sum in (6.10) equals

$$\delta_{nm} \frac{(2\ell+1)(n+k+1)}{(2\ell+1+n-k)}.$$

Hence,

$$|c_{k0}(n)|^2 = (H_n)_{kk}\, 2^{-2n} \frac{2}{\pi} \frac{(n+k+1)(2\ell+1+n-k)}{(2\ell+1)^2} = 4^{-2n} \frac{(n!)^2\,(2\ell+2)_n^2}{(k+1)_n^2\,(2\ell-k+1)_n^2} \tag{6.11}$$

using Corollary 5.4.

We end this section with the proof of Theorem 6.2. The idea of the proof is to obtain a three-term recurrence for the coefficients $c_{kj}(n)$ with explicit initial conditions, and to compare the resulting three-term recurrence with well-known recurrences for Racah polynomials, see [2], [11], [14]. The three-term recurrence relation is obtained using the first-order differential operator E and the fact that the $\mathcal{R}_n$, being analytic eigenfunctions to $\mathcal{D}$, are completely determined by the value at 0, see Remark 4.2.

Proof of Theorem 6.2. Since the matrix-valued differential operators D and E commute and have the matrix-valued orthogonal polynomials R_n as eigenfunctions by Corollary 4.1, we see that $\mathcal{E} = M^{-1}EM$ satisfies

$$\mathcal{E}\,\mathcal{R}_n = \Lambda_n(\mathcal{E})\,\mathcal{R}_n, \quad \Lambda_n(\mathcal{E}) = \Lambda_n(E), \qquad \mathcal{E}\,\mathcal{D} = \mathcal{D}\,\mathcal{E} \tag{6.12}$$

Moreover, in the same spirit as the proof of Proposition 6.1 we obtain

$$\begin{aligned} \mathcal{E} &= \left(\frac{d}{du}\right) S_1(u) + S_0(u), \\ S_1(u) &= u(1-u)\sum_{i=0}^{2\ell} \frac{i^2(2\ell+i+1)}{\ell(2i-1)(2i+1)}\, E_{i,i-1} - \sum_{i=0}^{2\ell} \frac{(2\ell-i)}{4\ell}\, E_{i,i+1}, \\ S_0(u) &= (1-2u)\sum_{i=0}^{2\ell} \frac{i^2(2\ell+i+1)}{2\ell(2i-1)}\, E_{i,i-1} + \sum_{i=0}^{2\ell} \frac{i(i+1)-4\ell(\ell+1)}{2\ell}\, E_{i,i} \end{aligned} \tag{6.13}$$

by a straightforward calculation.

Define the vector space of (row-)vector valued functions

$$V(\lambda) = \{F \text{ analytic at } u = 0 \mid F\mathcal{D} = \lambda F\},$$

and $\nu\colon V(\lambda) \to \mathbb{C}^{2\ell+1}$, $F \mapsto F(0)$, is an isomorphism, see Remark 4.2 and [24]. Because of (6.12) we have the following commutative diagram

$$\begin{array}{ccc} V(\lambda) & \xrightarrow{\mathcal{E}} & V(\lambda) \\ {\scriptstyle\nu}\big\downarrow & & \big\downarrow{\scriptstyle\nu} \\ \mathbb{C}^{2\ell+1} & \xrightarrow{N(\lambda)} & \mathbb{C}^{2\ell+1} \end{array}$$

with $N(\lambda)$ a linear map. In order to determine $N(\lambda)$ we note that $F \in V(\lambda)$ can be written as, cf (6.3),

$$F_j(u) = \left({}_2H_1\left(\begin{matrix} T_1^1,\ \lambda - T_0 \\ \frac{1}{2}T_1^1 \end{matrix} ; u \right) F(0)^t \right)_j^t,$$

so that $\frac{dF_j}{du}(0) = F(0)(\lambda - T_0)(\frac{1}{2}T_1^1)^{-1}$ by construction of the ${}_2H_1$-series, see Remark 4.2. Now (6.12) gives

$$N(\lambda) = (\lambda - T_0)(\frac{1}{2}T_1^1)^{-1}S_1(0) + S_0(0)$$

acting from the right on row-vectors from $\mathbb{C}^{2\ell+1}$.

By Proposition 6.1 we have that the k-th row $\big((\mathcal{R}_n)_{kj}(\cdot)\big)_{j=0}^{2\ell}$ of R_n is contained in $V(\lambda_n(k))$, see (6.3). On the other hand, the k-th row of $\mathcal{R}_n$ is an eigenfunction of $\mathcal{E}$ for the eigenvalue $\mu_n(k) = \Lambda_n(\mathcal{E})_{kk}$. Since $\nu\Big(\big((\mathcal{R}_n)_{kj}\big)_{j=0}^{2\ell}\Big) = (c_{kj}(n))_{j=0}^{2\ell}$ we see that the row-vector $c_k =$

$(c_{kj}(n))_{j=0}^{2\ell}$ satisfies $c_k N(\lambda_n(k)) = \mu_n(k)\, c_k$, which gives the recurrence relation

$$
\begin{aligned}
-\frac{(i+k+n+1)(i-k-n-1)(2\ell-i+1)}{(2i+1)} c_{k,i-1}(n) + (i(i+1) - 4\ell(\ell+1))c_{k,i}(n) \\
+ \frac{(i+1)^2(2\ell+i+2)}{(2i+1)} c_{k,i+1}(n) \;=\; (-2n(\ell-k) + (2\ell+2)(k-2\ell))c_{k,i}(n), \quad (6.14)
\end{aligned}
$$

with the convention $c_{k,-1}(n) = 0$. Note that $c_{jk}(0) = \binom{k}{j}$ indeed satisfies (6.14). Comparing (6.14) with the three-term recurrence relation for the Racah polynomials or the corresponding contiguous relation for balanced ${}_4F_3$-series, see e.g. [2, p. 344], [14, §1.2], gives

$$
c_{kj}(n) \;=\; c_{k,0}(n)(-1)^j \frac{(-2\ell)_j\,(-k-n)_j}{j!\,(2\ell+2)_j}\; {}_4F_3\left(\begin{matrix} -j, j+1, -k, -2\ell-n-1 \\ 1, -k-n, -2\ell \end{matrix}\,;1\right)
$$

and $c_{kj}(n) = 0$ for $j > k+n$.

It remains to determine the constants $c_{k0}(n)$, and we have already determined its absolute value in (6.11) by matching it to the orthogonality relations for Racah polynomials. From the three-term recurrence relation Theorem 5.3 we see that the constants $c_{kj}(n)$ are all real, so it remains to determine the sign of $c_{k0}(n)$. Theorem 5.3 gives a three-term recurrence for $\mathcal{R}_n(u)$, and taking the $(k,0)$-th matrix entry gives a polynomial identity in u using (6.7). Next taking the leading coefficient gives the recursion

$$
c_{k0}(n+1) = -\frac{(n+k+2)}{4(n+k+1)}\, c_{k0}(n) + \frac{(2\ell-k)^2}{4(2\ell+n-k)(2\ell+n-k+1)}\, c_{k+1,0}(n)
$$

and plugging in $c_{k0}(n) = \mathrm{sgn}(c_{k0}(n))|c_{k0}(n)|$ and using the explicit value for $|c_{k0}(n)|$ gives

$$
\begin{aligned}
&\mathrm{sgn}(c_{k0}(n+1))\,(n+1)(2\ell+n+2) = \\
&\qquad -\mathrm{sgn}(c_{k0}(n))(n+k+2)(2l-k+n+1) + \mathrm{sgn}(c_{k+1,0}(n))(2\ell-k)(k+1).
\end{aligned}
$$

This gives $\mathrm{sgn}(c_{k0}(n)) = \mathrm{sgn}(c_{k+1,0}(n))$ for the right hand side to factorise as in the left hand side, and then $\mathrm{sgn}(c_{k0}(n+1)) = -\mathrm{sgn}(c_{k0}(n))$. Since $c_{k0}(0) = 1$, we find $\mathrm{sgn}(c_{k0}(n)) = (-1)^n$. □

Remark 6.3. Theorem 6.2 can now be plugged into the three-term recurrence relation for $\mathcal{R}_n$ of Theorem 5.3, and this then gives a intricate three-term recurrence relation for Gegenbauer polynomials involving coefficients which consist of sums of two Racah polynomials. We leave this to the interested reader.

Remark 6.4. We sketch another approach to the proof of the value of $c_{k0}(n)$ by calculating the value $c_{k,2\ell}(n)$ in case $k+n \geq 2\ell$ or $c_{k,n+k}(n)$ in case $k+n < 2\ell$. For instance, in case $k+n \geq 2\ell$ we have

$$
(\mathcal{R}_n(u))_{k,2\ell} \;=\; (R_n(u)M(u))_{k,2\ell} \;=\; (R_n(u))_{k,2\ell} \;=\; R_n(u))_{2\ell-k,0}
$$

using that M is a unipotent lower-triangular matrix-valued polynomial and the symmetry $JR_n(u)J = R_n(u)$, see [15, §5]. Now the leading coefficient of the right hand side can be calculated using Theorem 4.5, and combining with (6.7), the value $c_{k,2\ell}(n)$ follows. Then the recurrence (6.14) can be used to find $c_{k0}(n)$.

7. Group theoretic interpretation

The purpose of this section is to give a group theoretic derivation of Theorem 3.1 complementing the analytic derivation of [15, §7]. For this we need to recall some of the results of [15].

7.1. **Group theoretic setting of the matrix-valued orthogonal polynomials.**

In this subsection we recall the construction of the matrix-valued orthogonal polynomials and the corresponding weight starting from the pair $(\mathrm{SU}(2)\times\mathrm{SU}(2),\mathrm{SU}(2))$ and an $\mathrm{SU}(2)$-representation T^ℓ. Then we discuss how the differential operators come into play and what their relation is with the matrix-valued orthogonal polynomials. The goal of this section is to provide a map of the relevant differential operators in the group setting to the relevant differential operators for the matrix-valued orthogonal polynomials in Theorem 7.8.

Let $U=\mathrm{SU}(2)\times\mathrm{SU}(2)$ and $K=\mathrm{SU}(2)$ diagonally embedded in U. Note that K is the set of fixed points of the involution $\theta: U\to U:(x,y)\mapsto(y,x)$. The irreducible representations of U and K are denoted by T^{ℓ_1,ℓ_2} and T^ℓ as is explained in [15, §2]. The representation space of T^ℓ is denoted by H^ℓ which is a $2\ell+1$-dimensional vector space. If T^ℓ occurs in T^{ℓ_1,ℓ_2} upon restriction to K we defined the spherical function $\Phi^\ell_{\ell_1,\ell_2}$ in [15, Def. 2.2] as the T^ℓ-isotypical part of the matrix T^{ℓ_1,ℓ_2}. Let $A\subset U$ be the subgroup

$$A=\left\{a_t=\left(\begin{pmatrix} e^{it/2} & 0\\ 0 & e^{-it/2}\end{pmatrix},\begin{pmatrix} e^{-it/2} & 0\\ 0 & e^{it/2}\end{pmatrix}\right),0\le t<4\pi\right\}$$

and let $M=Z_K(A)$. Recall the decomposition $U=KAK$, [12, Thm. 7.38]. The restricted spherical functions $\Phi^\ell_{\ell_1,\ell_2}|_A$ take values in $\mathrm{End}_M(H^\ell)$, see [15, Prop. 2.3]. Since $\mathrm{End}_M(H^\ell)\cong\mathbb{C}^{2\ell+1}$ this allows allows us to view the restricted spherical functions as being $\mathbb{C}^{2\ell+1}$-valued. The parametrization of the U-representations that contain T^ℓ indicates how to gather the restricted spherical functions. Following [15, Fig. 3] we write $(\ell_1,\ell_2)=\zeta(d,h)$ with $\zeta(d,h)=(\frac12(d+\ell+h),\frac12(d+\ell-h))$. Here $d\in\mathbb{N}$ and $h\in\{-\ell,-\ell+1,\dots,\ell\}$. We recall the definition of the full spherical functions of type ℓ, [15, Def. 4.2].

Definition 7.1. *The full spherical function of type ℓ and degree d is the matrix-valued function $\Phi^\ell_d: A\to\mathrm{End}(\mathbb{C}^{2\ell+1})$ whose j-th row is the restricted spherical function $\Phi^\ell_{\ell_1,\ell_2}$ with $(\ell_1,\ell_2)=\zeta(d,j)$.*

The full spherical function of degree zero has the remarkable property of being invertible on the subset $A_{reg}:=\{a_t|t\in(0,\pi)\cup(\pi,2\pi)\cup(2\pi,3\pi)\cup(3\pi,4\pi)\}$, which was first proved by Koornwinder [16, Prop. 3.2]. The invertibility follows also from Corollary 2.3. Let $\phi=\Phi^0_{1/2,1/2}$ be the minimal nontrivial zonal spherical function [15, §3]. Together with the recurrence relations for the full spherical functions with ϕ [15, Prop. 3.1] this gives rise to the full spherical polynomials [15, Def. 4.4].

Definition 7.2. *The full spherical polynomial $Q^\ell_d: A\to\mathrm{End}(\mathbb{C}^{2\ell+1})$ is defined by $Q^\ell_d(a)=(\Phi^\ell_0(a))^{-1}\Phi^\ell_d(a)$.*

The name full spherical polynomial comes from the fact that the Q^ℓ_d are polynomials in ϕ. The full spherical polynomials Q^ℓ_d are orthogonal with respect to

$$\langle Q, P\rangle_{V^\ell} = \int_A Q(a)V^\ell(a)P(a)da, \quad V^\ell(a_t) = (\Phi^\ell_0(a_t))^*\Phi^\ell_0(a_t)\sin^2 t,$$

see [15, Cor. 5.7].

In [15, §5] we studied the weight functions V^ℓ extensively. It turns out that the matrix entries are polynomials in the function ϕ, apart from the common factor $\sin t$. Upon changing the variable $x = \phi(a)$ we obtain the following system of matrix-valued orthogonal polynomials.

Definition 7.3. *Let $R^\ell_d : [0, 1] \to \mathrm{End}(\mathbb{C}^{2\ell+1})$ be the polynomial defined by $R^\ell_d(\phi(a)) = Q^\ell_d(a)$. The degree of R^ℓ_d is d. The polynomials are orthogonal with respect to*

$$\langle R, P\rangle_{W^\ell} = \int_{-1}^{1} R(x)W^\ell(x)P(x)\, dx,$$

where W^ℓ is defined by $W^\ell(\phi(a))d\phi = V^\ell(a)da$.

The weight $W^\ell(x)$ from Definition 7.3 is the same as the weight defined in (1.1) where we have to bear in mind that the basis is parametrised differently. The matrix-valued polynomials R^ℓ_d correspond to the family $\{P_d\}_{d\geq 0}$ from Theorem 3.1 by means of making the R^ℓ_d monic. Given a system of matrix-valued orthogonal polynomials as in Definition 7.3 it is of great interest to see whether there are interesting differential operators. More precisely we define the algebra $\mathcal{D}(W^\ell)$ as the algebra of differential operators that are self adjoint with respect to the weight W^ℓ and that have the R^ℓ_d as eigenfunctions. We define a map that associates to a certain left invariant differential operator on the group U an element in $\mathcal{D}(W)$.

Before we go into the construction we observe that the spherical functions may also be defined on the complexification $A^{\mathbb{C}}$, using Weyl's unitary trick. Indeed, all the representations that we consider are finite dimensional and unitary, so they give holomorphic representations of the complexifications $U^{\mathbb{C}}$ and $K^{\mathbb{C}}$.

A great part of the constructions that we are about to consider follows from Casselman and Miličić [5], where the differential operators act from the left. In this section we follow this convention, except that we transpose the results at the end in order to obtain the proof of Theorem 3.1 where the differential operators act from the right.

Let $U(\mathfrak{u}^{\mathbb{C}})$ be the universal enveloping algebra for the complexification $\mathfrak{u}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{u}$ of the group $U = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Let $\theta\colon U(\mathfrak{u}^{\mathbb{C}}) \to U(\mathfrak{u}^{\mathbb{C}})$ be the flip on simple tensors extending the Cartan involution $\theta\colon \mathfrak{su}(2) \times \mathfrak{su}(2) \to \mathfrak{su}(2) \times \mathfrak{su}(2)$, $(X, Y) \mapsto (Y, X)$. Recall $\mathfrak{k} \cong \mathfrak{su}(2)$ is the fixed-point set of θ. Let $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ denote the subalgebra of elements that commute with $\mathfrak{k}^{\mathbb{C}}$. Let $\mathfrak{Z}$ denote the center of $U(\mathfrak{k}^{\mathbb{C}})$.

Lemma 7.4. *$U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \cong \mathfrak{Z} \otimes \mathfrak{Z}$.*

Proof. From [13, Satz 2.1 and Satz 2.3] it follows that $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \cong \mathfrak{Z} \otimes (\mathfrak{Z} \otimes \mathfrak{Z})$ where $\mathfrak{Z} \otimes \mathfrak{Z}$ is a $\mathfrak{Z}$-algebra via $\mathfrak{Z} \to \mathfrak{Z} \otimes \mathfrak{Z} : \omega \mapsto \omega \otimes \omega$. Hence $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \cong \mathfrak{Z} \otimes \mathfrak{Z}$. □

Proposition 7.5. *The elements of the algebra $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ have the spherical functions $\Phi^\ell_{\ell_1,\ell_2}$ as eigenfunctions. This remains true when we extend $\Phi^\ell_{\ell_1,\ell_2}$ to $U^{\mathbb{C}}$.*

Proof. See [25, Thm. 6.1.2.3]. The second statement follows from Weyl's unitary trick. □

The spherical functions $\Phi^{\ell}_{\ell_1,\ell_2}$ have T^{ℓ}-transformation behaviour:

$$\Phi^{\ell}_{\ell_1,\ell_2}(k_1uk_2) = T^{\ell}(k_1)\Phi^{\ell}_{\ell_1,\ell_2}(u)T^{\ell}(k_2) \tag{7.1}$$

for all $k_1, k_2 \in K$ and $u \in U$, see [15, Def. 2.2]. Let $C(A)$ denote the set of continuous ($\mathbb{C}$-valued) functions on A. Casselman and Miličić [5] define the map

$$\Pi_{\ell} : U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \to C(A) \otimes U(\mathfrak{a}^{\mathbb{C}}) \otimes \mathrm{End}(\mathrm{End}_M(H^{\ell}))$$

and prove the following properties [5, Thm. 3.1,Thm. 3.3].

Theorem 7.6. *Let $F : U \to \mathrm{End}(H^{\ell})$ be a smooth function that satisfies (7.1). Then $(DF)|_A = \Pi_{\ell}(D)(F|_A)$ for all $D \in U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$. Moreover, Π_{ℓ} is an algebra homomorphism.*

We call $\Pi_{\ell}(D)$ the T^{ℓ}-radial part of $D \in U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$. In particular we have

$$\Pi_{\ell}(D)(\Phi^{\ell}_{\ell_1,\ell_2}|_A) = \lambda^{\ell}_{D,\ell_1,\ell_2}\Phi^{\ell}_{\ell_1,\ell_2}|_A, \quad \lambda^{\ell}_{D,\ell_1,\ell_2} \in \mathbb{C}.$$

Upon identifying $\mathrm{End}_M(H^{\ell}) \cong \mathbb{C}^{2\ell+1}$ we observe that we may view $\Pi_{\ell}(D)$ as a differential operator of the $\mathrm{End}(\mathbb{C}^{2\ell+1})$-valued functions that act on from the left. In particular, let $C(A, \mathrm{End}(\mathbb{C}^{2\ell+1}), T^{\ell})$ denote the vector space generated by the $\Phi^{\ell}_d, d \geq 0$. The following lemma follows immediately from the construction.

Lemma 7.7. *Let $D \in U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ be self-adjoint and consider $\Pi_{\ell}(D)$ as a differential operator acting on $C(A, \mathrm{End}(\mathbb{C}^{2\ell+1}), T^{\ell})$ from the left. Then $\Pi_{\ell}(D)$ is self-adjoint for $\langle\cdot,\cdot\rangle_{V^{\ell}}$.*

Definition 7.8. *Let $f : U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \to \mathcal{D}(W^{\ell})$ be defined by sending D to the conjugation of the differential operator $\Pi_{\ell}(D)$ by Φ^{ℓ}_0 followed by changing the variable $x = \phi(a)$.*

The map $f : U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \to \mathcal{D}(W^{\ell})$ is an algebra homomorphism. It gives an abstract description of a part of $\mathcal{D}(W^{\ell})$. Note that f is not surjective because in [15, Prop. 8.1] we have found a differential operator that does not commute with some of the other. However, the algebra $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ is commutative [13, Satz 2.3].

By means of Lemma 7.4 we know that $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ is generated by $\Omega_1 = \Omega_{\mathfrak{k}} \otimes 1$ and $\Omega_2 = 1 \otimes \Omega_{\mathfrak{k}}$ where $\Omega_{\mathfrak{k}} \in \mathfrak{Z}$ is the Casimir operator. In the following subsection we calculate $f(\Omega_1 + \Omega_2)$ and $f(\Omega_1 - \Omega_2)$ explicitly. Upon transposing and taking suitable linear combinations we find the differential operators $\tilde{D}$ and $\tilde{E}$ from Theorem 3.1.

7.2. **Calculation of the Casimir operators.** The goal of this subsection is to calculate $f(\Omega)$ and $f(\Omega')$ where f is the map described in Definition 7.8 and where $\Omega = \Omega_1 + \Omega_2$ and $\Omega' = \Omega_1 - \Omega_2$. We proceed in a series of six steps. (1) First we provide expressions for the Casimir operators Ω and Ω' which (2) we rewrite according to the infinitesimal Cartan decomposition defined by Casselman and Miličić [5, §2]. These calculations are similar to those in [25, Prop. 9.1.2.11]. (3) From this expression we can easily calculate the T^{ℓ}-radial parts, see Theorem 7.6. The radial parts are differential operators for $\mathrm{End}_M(H^{\ell})$-valued functions on A. At this point we see that we can extend matters to the complexification $A^{\mathbb{C}}$ of A as in [5, Ex. 3.7]. (4) We identify $\mathrm{End}_M(H^{\ell}) \cong \mathbb{C}^{2\ell+1}$ and rewrite the radial parts of step 3 accordingly. (5) We conjugate these differential operators with Φ_0 and (6) we make a change of variables to obtain two matrix-valued differential operators $f(\Omega)$ and $f(\Omega')$. Along the way we keep track of the differential equations for the spherical functions. Finally we give expressions for the eigenvalues Λ_d and Γ_d of $f(\Omega)$ and $f(\Omega')$ such that the full spherical polynomials Q_d are the corresponding eigenfunctions. Following Casselman and Miličić [5, §2] the roots are considered as characters, hence written multiplicatively.

(1). First we concentrate on one factor $K \cong \mathrm{SU}(2)$, with Lie algebra $\mathfrak{k}$ and standard Cartan subalgebra $\mathfrak{t}$. The complexifications are denoted by $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{t}^{\mathbb{C}}$ and we use the standard basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{\alpha^{-1}} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for $\mathfrak{k}^{\mathbb{C}}$. The Casimir of K is given by $\Omega_{\mathfrak{k}} = \frac{1}{2}H^2 + 4\{E_\alpha E_{\alpha^{-1}} + E_{\alpha^{-1}} E_\alpha\}$. It is well-known that the matrix-elements of the irreducible unitary representation T^ℓ of $\mathrm{SU}(2)$ are eigenfunctions of the Casimir operator $\Omega_{\mathfrak{k}}$ for the eigenvalue $\frac{1}{2}(\ell^2+\ell)$, see e.g. [12, Thm. 5.28]. The roots of the pair $(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{t}^{\mathbb{C}})$ are given by $R = \{(\alpha,1), (\alpha^{-1},1), (1,\alpha), (1,\alpha^{-1})\}$. The positive roots are choosen as $R^+ = \{(\alpha,1),(1,\alpha^{-1})\}$, so that the two positive roots restrict to the same root $R(\mathfrak{u}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$ which we declare positive. The corresponding root vectors are $E_{(\alpha,1)} = (E_\alpha, 0)$, etc. Define

$$E = (E_\alpha,0)(E_{\alpha^{-1}},0) + (E_{\alpha^{-1}},0)(E_\alpha,0).$$

Then we have $\Omega_1 = \frac{1}{2}(H,0)^2 + 4E$ and $\Omega_2 = \theta(\Omega_1)$. In particular, the spherical function $\Phi^\ell_{\ell_1,\ell_2}$ is an eigenfunction of Ω_i for the eigenvalue $\frac{1}{2}(\ell_i^2 + \ell_i)$ for $i = 1, 2$.

We have $(H,0) = \frac{1}{2}((H,-H) + (H,H))$ and $(0,H) = \frac{1}{2}((H,H) - (H,-H))$ and from this we find in $U(\mathfrak{u}^{\mathbb{C}})$

$$\begin{aligned} \Omega &= \Omega_1 + \Omega_2 = \frac{1}{4}(H,H)^2 + \frac{1}{4}(H,-H)^2 + 4(E + \theta(E)), \\ \Omega' &= \Omega_1 - \Omega_2 = \frac{1}{2}(H,-H)(H,H) + 4(E - \theta(E)). \end{aligned} \tag{7.2}$$

(2). Following Casselman and Miličić [5, §2] we can express Ω and Ω' according to the infinitesimal Cartan decomposition of $U(\mathfrak{u}^{\mathbb{C}})$. Let $\beta \in R$ and denote $X_\beta = E_\beta + \theta(E_\beta) \in \mathfrak{k}^{\mathbb{C}}$. Denote $Y^a = \mathrm{Ad}(a^{-1})Y$ for $a \in A$. In [5, Lemma 2.2] it is proved that the equality

$$(1-\beta(a)^2)X_\beta = \beta(a)(E^a_\beta - \beta(a)E_\beta)$$

holds for all $a \in A_{reg}$. This is the key identity in a straightforward but tedious calculation to prove the following proposition which we leave to the reader.

Proposition 7.9. *Let $a \in A_{reg}$ and $\beta \in R^+$. Then*

$$\begin{aligned} \Omega = \frac{1}{16}\left((H,-H)^2 + (H,H)^2\right) - \frac{2}{(\beta(a)^{-1}-\beta(a))^2}\{X^a_\beta X^a_{\beta^{-1}} + \\ X^a_{\beta^{-1}} X^a_\beta + X_\beta X_{\beta^{-1}} + X_{\beta^{-1}} X_\beta - (\beta(a)+\beta(a)^{-1})(X^a_\beta X_{\beta^{-1}} + X^a_{\beta^{-1}} X_\beta)\} + \\ \frac{1}{4}\frac{\beta(a)+\beta(a)^{-1}}{\beta(a)-\beta(a)^{-1}}(H,-H). \end{aligned} \tag{7.3}$$

and

$$\Omega' = \frac{1}{8}(H,H)(H,-H) + \frac{1}{4}\frac{\beta(a)+\beta(a)^{-1}}{\beta(a)-\beta(a)^{-1}}(H,H) + \frac{2}{\beta(a)-\beta(a)^{-1}}(X^a_{\beta^{-1}}X_\beta - X^a_\beta X_{\beta^{-1}}), \tag{7.4}$$

The calculation of (7.3) is completely analogous to that of [25, Prop. 9.1.2.11] and it is clear that the (7.3) is invariant for interchanging β and β^{-1}. The expression in (7.4) is also invariant for interchanging β and β^{-1} albeit that it is less clear in this case. In either case the expressions (7.3) and (7.4) do not depend on the choice of $\beta \in R^+$.

(3). Following Casselman and Miličić [5, §3] we calculate the T^ℓ-radial parts of Ω and Ω'. This is a matter of applying the map Π_ℓ from Theorem 7.6 to the expressions (7.3) and (7.4). At the same time we note that the coefficients in (7.3) and (7.4) are analytic functions on A_{reg}. They extend to meromorphic functions on the complexification $A^{\mathbb{C}}$ of A which we identify with $\mathbb{C}^\times$ using the map

$$a : \mathbb{C}^\times \to A^{\mathbb{C}} : w \mapsto a(w) = \left(\begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}, \begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix} \right).$$

Under this isomorphism the differential operator $(H,-H)$ translates to $w\frac{d}{dw}$. To see this let $g : A^{\mathbb{C}} \to \mathbb{C}$ be holomorphic and consider $(H,-H)g(a(w))$ which is equal to

$$(H,-H)g(a(w)) = \left\{ \frac{d}{dt} g(a(e^t w)) \right\}_{t=0} = w\frac{d}{dw}(g \circ a)(w).$$

Following [5], [25] we find the following expressions for the T^ℓ-radial parts of Ω and Ω';

$$\begin{aligned}\Pi_\ell(\Omega) = \frac{1}{16}\left(w\frac{d}{dw}\right)^2 + \frac{1}{4}\frac{w^2+w^{-2}}{w^2-w^{-2}}w\frac{d}{dw} + \frac{1}{16}T^\ell(H)^2 + \\ -\frac{2}{(w^2-w^{-2})^2}\left\{T^\ell(E_\alpha)T^\ell(E_{\alpha^{-1}})\bullet + T^\ell(E_{\alpha^{-1}})T^\ell(E_\alpha)\bullet + \right. \\ \left. \bullet T^\ell(E_\alpha)T^\ell(E_{\alpha^{-1}}) + \bullet T^\ell(E_{\alpha^{-1}})T^\ell(E_\alpha)\right\} + \\ 2\frac{w^2+w^{-2}}{(w^2-w^{-2})^2}\left\{T^\ell(E_\alpha)\bullet T^\ell(E_{\alpha^{-1}}) + T^\ell(E_{\alpha^{-1}})\bullet T^\ell(E_\alpha)\right\},\end{aligned} \tag{7.5}$$

and

$$\begin{aligned}\Pi_\ell(\Omega') = \frac{1}{8}T^\ell(H)w\frac{d}{dw} + \frac{1}{4}\frac{w^2+w^{-2}}{w^2-w^{-2}}T^\ell(H) + \\ \frac{2}{w^2-w^{-2}}\left\{T^\ell(E_{\alpha^{-1}})\bullet T^\ell(E_\alpha) + T^\ell(E_\alpha)\bullet T^\ell(E_{\alpha^{-1}})\right\}\end{aligned} \tag{7.6}$$

where the bullet ($\bullet$) indicates where to put the restricted spherical function. The matrices $T^\ell(E_\alpha)$ and $T^\ell(H)$ are easily calculated in the basis of weight vectors. Note that $T^\ell(E_{\alpha^{-1}}) = JT^\ell(E_\alpha)J$. We give the entries of $T^\ell(E_\alpha)$ in the proof of Lemma 7.11.

The following proposition is a direct consequence of Theorem 7.6 and Proposition 7.5.

Proposition 7.10. *The restricted spherical functions are eigenfunctions of the radial parts of Ω and Ω',*

$$\Pi_\ell(\Omega)(\Phi_\ell^{\ell_1,\ell_2}|_{A^{\mathbb{C}}}) = \frac{1}{2}(\ell_1^2+\ell_1+\ell_2^2+\ell_2)\Phi_\ell^{\ell_1,\ell_2}|_{A^{\mathbb{C}}}, \quad \Pi_\ell(\Omega')(\Phi_\ell^{\ell_1,\ell_2}|_{A^{\mathbb{C}}}) = \frac{1}{2}(\ell_1^2+\ell_1-\ell_2^2-\ell_2)\Phi_\ell^{\ell_1,\ell_2}|_{A^{\mathbb{C}}}.$$

(4). The spherical functions $\Phi^\ell_{\ell_1,\ell_2}$ restricted to the torus $A^{\mathbb{C}}$ take their values in $\mathrm{End}_M(H^\ell)$ and this is a $2\ell+1$-dimensional vector space. We identify

$$\mathrm{End}_M(H^\ell) \to \mathbb{C}^{2\ell+1} : Y \mapsto Y^{\mathrm{up}}$$

to obtain functions $(\Phi^\ell_{\ell_1,\ell_2}|_{A^{\mathbb{C}}})^{\mathrm{up}}$. The reason for putting the diagonals up is that we want to write the differential operators as differential operators with coefficients in the function algebra on A with values in $\mathrm{End}(\mathbb{C}^{2\ell+1})$ instead of the way $\Pi_\ell(\Omega)$ and $\Pi_\ell(\Omega')$ are defined. The differential operators that are conjugated to act on $\mathbb{C}^{2\ell+1}$-valued functions are also denoted by $(\cdot)^{\mathrm{up}}$. The differential operators (7.5) and (7.6) that are defined for $\mathrm{End}_M(H^\ell)$-valued functions conjugate to differential operators $\Pi_\ell(\Omega)^{\mathrm{up}}$ and $\Pi_\ell(\Omega')^{\mathrm{up}}$ for $\mathbb{C}^{2\ell+1}$-valued functions. All the terms except for the last ones in (7.5) and (7.6) transform straightforwardly.

Lemma 7.11. *The linear isomorphism* $\mathrm{End}_M(H^\ell) \to \mathbb{C}^{2\ell+1} : D \mapsto D^{\mathrm{up}}$ *conjugates the linear map* $\mathrm{End}_M(H^\ell) \to \mathrm{End}_M(H^\ell) : D \mapsto T^\ell(E_\alpha)DT^\ell(E_{\alpha^{-1}})$ *to* $\mathbb{C}^{2\ell+1} \to \mathbb{C}^{2\ell+1} : D^{\mathrm{up}} \mapsto C^\ell D^{\mathrm{up}}$, *where* $C^\ell \in \mathrm{End}(\mathbb{C}^{2\ell+1})$ *is the matrix given by*

$$C^\ell_{p,j} = \frac{1}{4}(\ell+j)(\ell-j+1)\delta_{j-p,1}, \quad \ell \le p, j \le \ell.$$

Likewise, $D \mapsto T^\ell(E_{\alpha^{-1}})DT^\ell(E_\alpha)$ *transforms to* $D^{\mathrm{up}} \mapsto JC^\ell JD^{\mathrm{up}}$, *where J is the anti-diagonal defined by $J_{ij} = \delta_{i,-j}$ with $-\ell \le i, j \le \ell$.*

Proof. Working with the normalized weight-basis as in [16, §1] we see that $T^\ell(E_\alpha)$ is the matrix given by

$$T^\ell(E_\alpha)_{ij} = \delta_{i,i+1}\frac{\ell+i+1}{2}\sqrt{\frac{(\ell-i-2)!(\ell+i+2)!}{(\ell-i-1)!(\ell+i+1)!}}$$

and $T^\ell(E_{\alpha^{-1}}) = JT^\ell(E_\alpha)J$. The lemma follows from elementary manipulations. ☐

We collect the expressions for the conjugation of the differential operators (7.5) and (7.6) by the linear map $Y \mapsto Y^{\mathrm{up}}$ where we have used Lemma 7.11.

$$\begin{aligned}\Pi_\ell(\Omega_1+\Omega_2)^{\mathrm{up}} = \frac{1}{16}\left(w\frac{d}{dw}\right)^2 + \frac{1}{4}\frac{w^2+w^{-2}}{w^2-w^{-2}}w\frac{d}{dw} + \frac{1}{16}T^\ell(H)^2+ \\ -\frac{4}{(w^2-w^{-2})^2}\left\{T^\ell(E_\alpha)T^\ell(E_{\alpha^{-1}})+T^\ell(E_{\alpha^{-1}})T^\ell(E_\alpha)\right\}+ \\ 2\frac{w^2+w^{-2}}{(w^2-w^{-2})^2}\left\{JC^\ell J + C^\ell\right\},\end{aligned} \tag{7.7}$$

$$\Pi_\ell(\Omega_1-\Omega_2)^{\mathrm{up}} = \frac{1}{8}T^\ell(H)w\frac{d}{dw} + \frac{1}{4}\frac{w^2+w^{-2}}{w^2-w^{-2}}T^\ell(H) + \frac{2}{w^2-w^{-2}}\left\{JC^\ell J - C^\ell\right\}. \tag{7.8}$$

The differential operators (7.7) and (7.8) also act on the full spherical functions $\Phi_d^{\ell,t}$. Collecting the eigenvalues of the columns in $\Phi_d^{\ell,t}$ in diagonal matrices we obtain the following differential equations:

$$\Pi_\ell(\Omega_1+\Omega_2)^{\mathrm{up}}\Phi_d = \Phi_d\Lambda_d, \tag{7.9}$$
$$\Pi_\ell(\Omega_1-\Omega_2)^{\mathrm{up}}\Phi_d = \Phi_d\Gamma_d, \tag{7.10}$$

where $(\Lambda_d)_{pj}=\frac{1}{4}\delta_{p,j}(d^2+j^2+2d(\ell+1)+\ell(\ell+2))$ and $(\Gamma_d)_{pj}=\frac{1}{2}\delta_{p,j}j(\ell+d+1)$. For further reference we write

$$\Pi_\ell(\Omega_1+\Omega_2)^{\mathrm{up}} = a_2(w)\frac{d^2}{dw^2}+a_1(w)\frac{d}{dw}+a_0(w), \tag{7.11}$$
$$\Pi_\ell(\Omega_1-\Omega_2)^{\mathrm{up}} = b_1(w)\frac{d}{dw}+b_0(w). \tag{7.12}$$

(5). Recall from Definition 7.1 that the full spherical polynomials $Q_d^{\ell,t}$ are obtained from the full spherical functions $\Phi_d^{\ell,t}$ by the description $Q_d^{\ell,t}=(\Phi_0^{\ell,t})^{-1}\Phi_d^{\ell,t}$. We conjugate the differential operators (7.7) and (7.8) with Φ_0 to obtain differential operators to which the polynomials Q_d are eigenfunctions. We need a technical lemma.

Lemma 7.12.
- *Let $\sigma^\ell:\mathbb{C}^\times\to\mathrm{End}(\mathbb{C}^{2\ell+1})$ be the map given by $\sigma^\ell(w)=\ell(w^2+w^{-2})I+S^\ell$ where S^ℓ is defined by $\left(S^\ell\right)_{p,j}=-(\ell-j)\delta_{p-j,1}-(\ell+j)\delta_{j-p,1}$. Then*

$$\frac{1}{2}w(w^2-w^{-2})\frac{d}{dw}\Phi_0^{\ell,t}(w)=\Phi_0^{\ell,t}(w)\sigma^\ell(w). \tag{7.13}$$

- *Let $\upsilon^\ell:\mathbb{C}^\times\to\mathrm{End}(\mathbb{C}^{2\ell+1})$ be the map given by $\upsilon^\ell(w)=\frac{1}{8}\frac{w^3}{w^4-1}\left(\frac{1+w^4}{w^2}U^\ell_{\mathrm{diag}}+U^\ell_{lu}\right)$, where $\left(U^\ell_{lu}\right)_{i,j}=(-2\ell+2j)\delta_{i,j+1}+(2\ell+2j)\delta_{i+1,j}$ and $\left(U^\ell_{\mathrm{diag}}\right)_{i,j}=-2i\delta_{ij}$. Then*

$$b_1(w)\Phi_0^{\ell,t}(a(w))=\Phi_0^{\ell,t}(a(w))\upsilon^\ell(w). \tag{7.14}$$

Proof. The matrix coefficients of $\Phi_0^{\ell,t}(a(w))$ are given by

$$(\Phi_0^{\ell,t}(a(w)))_{p,j}=\frac{(\ell-j)!(\ell+j)!(\ell-p)!(\ell+p)!}{(2\ell)!}\times \sum_{r=\max(0,-p-j)}^{\min(\ell-p,\ell-j)}\frac{w^{4r-2\ell+2p+2j}}{r!(\ell-p-r)!(\ell-j-r)!(p+j+r)!}, \tag{7.15}$$

see [16, Prop. 3.2]. The matrix-valued function $b_1(w)$ is equal to the constant matrix $\frac{1}{8}T^\ell$ where $T^\ell(H)_{ij}=2\delta_{ij}j$. We can now express the matrix coefficients of the matrices in equations (7.13) and (7.14) in Laurent polynomials in the variable w and comparing coefficients of these polynomials shows that the equalities hold. ☐

Definition 7.13. *Define $\Omega_\ell=(\Phi_0^{\ell,t})^{-1}\circ\Pi_\ell(\Omega)^{\mathrm{up}}\circ\Phi_0^{\ell,t}$ and $\Delta_\ell=(\Phi_0^{\ell,t})^{-1}\circ\Pi_\ell(\Omega')^{\mathrm{up}}\circ\Phi_0^{\ell,t}$.*

Theorem 7.14. *The differential operators Ω_ℓ and Δ_ℓ are given by*

$$\Omega_\ell = \frac{1}{16}\left(w\frac{d}{dw}\right)^2 + \frac{1}{4}\left\{(\ell+1)(w^2+w^{-2}) + S^\ell\right\}\frac{w}{w^2-w^{-2}}\frac{d}{dw} + \Lambda_0, \tag{7.16}$$

$$\Delta_\ell = \upsilon^\ell(w)\frac{d}{dw} + \Gamma_0. \tag{7.17}$$

Proof. This is a straightforward calculation using the expressions (7.11) and (7.12), bearing in mind that the coefficients are matrix-valued. In both calculations the difficult parts are taken care of by Lemma 7.12. $\square$

(6). The elementary zonal spherical function $\Phi_0^{\frac{1}{2},\frac{1}{2}}$ is denoted by ϕ and we have $\phi(a(w)) = \frac{1}{2}(w^2+w^{-2})$. In this final step we note that the differential operators Ω_ℓ and Δ_ℓ are invariant under the maps $w \mapsto -w$ and $w \mapsto w^{-1}$. This shows that the differential operators can be pushed forward by $\phi \circ a$ to obtain differential operators on $\mathbb{C}$ in a coordinate $z = \phi(a(w))$. Using the identities $w\frac{d}{dw}(h\circ\phi)(w) = (w^2-w^{-2})h'(\phi(w))$, $(w\frac{d}{dw})^2(h\circ\phi)(w) = (w^2-w^{-2})^2h''(\phi(a(w))) + 2(w^2+w^{-2})h'(\phi(a(w)))$ and $(w^2-w^{-2})^2 = 4(\phi(a(w))^2-1)$ we transform (7.16) and (7.17) into

$$\widetilde{\Omega_\ell} = \frac{1}{4}(z^2-1)\left(\frac{d}{dz}\right)^2 + \frac{1}{4}\left\{(2\ell+3)z + S^\ell\right\}\frac{d}{dz} + \Lambda_0, \tag{7.18}$$

$$\widetilde{\Delta_\ell} = \frac{1}{8}\left(2zU^\ell_{\mathrm{diag}} + U^\ell_{ul}\right)\frac{d}{dz} + \Gamma_0. \tag{7.19}$$

Recall that the $\mathrm{End}(\mathbb{C}^{2\ell+1})$-valued polynomials $R^{\ell,t}_d$ are defined by pushing forward the $\mathrm{End}(\mathbb{C}^{2\ell+1})$-valued functions $Q^{\ell,t}_d$ over $\phi\circ a$, see Definition 7.3.

Theorem 7.15. *The members of the family $\{R^{\ell,t}_d\}_{d\geq 0}$ of $\mathrm{End}(\mathbb{C}^{2\ell+1})$-valued polynomials of degree d are eigenfunctions of the differential operators $\widetilde{\Omega_\ell}$ and $\widetilde{\Delta_\ell}$ with eigenvalues Λ_d and Γ_d respecively. The transposed differential operators $(\widetilde{\Omega_\ell})^t$ and $(\widetilde{\Delta_\ell})^t$ satisfy*

$$-4(\widetilde{\Omega_\ell})^t + 2(\ell^2+\ell) = \tilde{D}, \tag{7.20}$$

$$-\frac{2}{\ell}(\widetilde{\Delta_\ell})^t - (\ell+1) = \tilde{E}, \tag{7.21}$$

where $\tilde{D}$ and $\tilde{E}$ are defined in Theorem 3.1.

Proof. The only things that need proofs are the equalities of the differential operators. These follow easily upon comparing coefficients where one has to bear in mind the different labeling of the matrices involved in the two cases. $\square$

Note that the differential operators $\tilde{D}$ and $\widetilde{\Omega_\ell}$ are invariant under conjugation by the matrix J, where $J_{i,j} = \delta_{i,-j}$. The differential operator $\widetilde{\Delta_\ell}$ is anti-invariant for this conjugation. The differential operator $\tilde{E}$ does not have this nice property.

Appendix A. Proof of Theorem 2.1

The purpose of this appendix is to prove the LDU-decomposition of Theorem 2.1. We prove instead the equivalent Proposition 2.2, and we start with proving Lemma 2.7.

Note that the integral in Lemma 2.7 is zero by (2.2) in case $t > m$, since $C^{(k+1)}_{m-k}(x)U_{n+m-2t}(x)$ is a polynomial of degree $n + 2m - k - 2t < n - k$.

We start by proving Lemma 2.7 in the remaining case for which we use the following well-known formulas for connection and linearisation formulas of Gegenbauer polynomials, see e.g. [2, Thm. 6.8.2], [11, Thm. 9.2.1];

$$
\begin{aligned}
C^{(\gamma)}_n(x) &= \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(\gamma-\beta)_k(\gamma)_{n-k}}{k!\,(\beta+1)_{n-k}} \binom{\beta+n-2k}{\beta} C^{(\beta)}_{n-2k}(x),\\
C^{(\alpha)}_n(x)C^{(\alpha)}_m(x) &= \sum_{k=0}^{m\wedge n} \frac{(n+m-2k+\alpha)(n+m-2k)!(\alpha)_k}{(n+m-k+\alpha)k!}\\
&\qquad\times \frac{(\alpha)_{n-k}(\alpha)_{m-k}(2\alpha)_{n+m-k}}{(n-k)!(m-k)!(\alpha)_{n+m-k}(2\alpha)_{n+m-2k}} C^{(\alpha)}_{n+m-2k}(x).
\end{aligned}
\tag{A.1}
$$

Proof of Lemma 2.7. We indicate the proof of Lemma 2.7, so that the reader can easily fill in the details. Calculating the product of two Gegenbauer polynomials as a sum using the linearisation formula of (A.1) and expanding the Chebyshev polynomial $U_{n+m-2t}(x) = C^{(1)}_{n+m-2t}(x)$ in terms of Chebyshev polynomials with parameter $k+1$ using the linearisation formula of (A.1), we can rewrite the integral as a double sum with an integral of Chebyshev polynomials that can be evaluated using the orthogonality relations (2.2) reducing the integral of Lemma 2.7 to the single sum

$$
\begin{aligned}
\sum_{r=\max(0,t-k)}^{\min(t,m-k)} &\frac{(m+n-k+1-2r)}{(m+n-k+1-r)} \frac{(k+1)_r(k+1)_{n-k-r}(k+1)_{m-k-r}(2k+2)_{m+n-2k-r}}{r!\,(m-k-r)!\,(n-k-r)!\,(k+1)_{m+n-2k-r}}\\
&\times \frac{(-k)_{k+r-t}(n+m-t-k-r)!}{(k-t+r)!\,(k+2)_{n+m-t-k-r}} \frac{\sqrt{\pi}\,\Gamma(k+\frac{3}{2})}{(k+1)\,\Gamma(k+1)}.
\end{aligned}
$$

Assuming for the moment that $k \geq t$, so the sum is $\sum_{r=0}^{\min(t,m-k)}$. Then this sum can be written as a very-well-poised ${}_7F_6$-series

$$
\begin{aligned}
&\frac{\sqrt{\pi}\,\Gamma(k+\frac{3}{2})}{(k+1)\,\Gamma(k+1)} \frac{(k+1)_{m-k}}{(m-k)!} \frac{(k+1)_{n-k}}{(n-k)!} \frac{(2k+2)_{m+n-2k}}{(k+1)_{m+n-2k}} \frac{(-k)_{k-t}}{(k-t)!} \frac{(n+m-t-k)!}{(k+2)_{n+m-t-k}}\\
&\times {}_7F_6\left(\begin{matrix} \frac{1}{2}(k-m-n+1), k+1, k-m, k-n, k-m-n-1, -t, t-m-n-1 \\ \frac{1}{2}(k-m-n-1), -m, -n, -m-n-1, k-t+1, -n-m+k+t \end{matrix} ;1\right).
\end{aligned}
$$

Using Whipple's transformation [2, Thm. 3.4.4], [4, §4.3] of a very-well-poised ${}_7F_6$-series to a balanced ${}_4F_3$-series, we find that the ${}_7F_6$-series can be written as

$$
\frac{(k-m-n)_t\,(-t)_t}{(k-t+1)_t\,(-m-n-1)_t}\,{}_4F_3\left(\begin{matrix} -k, k+1, -t, t-m-n-1 \\ -n,\ -m,\ 1 \end{matrix} ;1\right).
$$

Simplifying the shifted factorials and recalling the definition of the Racah polynomials (2.5) in terms of a balanced ${}_4F_3$-series gives the result in case $k \geq t$.

In case $k \leq t$ we have to relabel the sum, which turns out again to be a very-well-poised ${}_7F_6$-series which can be transformed to a balanced ${}_4F_3$-series. The resulting balanced ${}_4F_3$-series is not a Racah polynomial as in the statement of Lemma 2.7, but it can be transformed to a Racah polynomial using Whipple's transformation for balanced ${}_4F_3$-series [2, Thm. 3.3.3]. Keeping track of the constants proves Lemma 2.7 in this case. $\square$

As remarked in Section 2, Theorem 2.1 follows from Proposition 2.2. In order to prove Proposition 2.2 we assume $\alpha_t(m,n)$ to be given by (1.1) and we want to find $\beta_k(m,n)$. Given the explicit expression for $\alpha_t(m,n)$, we see that multiplying by $\sqrt{1-x^2}\, U_{n+m-2t}(x)$, integrating over $[-1,1]$ and using Lemma 2.7 we find

$$\alpha_t(m,n)\frac{\pi}{2} = \sum_{k=0}^{m} \beta_k(m,n) C_k(m,n) R_k(\lambda(t); 0, 0, -m-1, -n-1) \tag{A.2}$$

where

$$C_k(m,n) = \frac{\sqrt{\pi}\,\Gamma(k+\frac{3}{2})}{(k+1)} \frac{(k+1)_{m-k}}{(m-k)!} \frac{(k+1)_{n-k}}{(n-k)!} \frac{(-1)^k\,(2k+2)_{m+n-2k}\,(k+1)!}{(n+m+1)!}.$$

Using the orthogonality relations for the Racah polynomials, see [2, p. 344], [14, §1.2],

$$\sum_{t=0}^{m} (m+n+1-2t)\, R_k(\lambda(t); 0, 0, -m-1, -n-1)\, R_l(\lambda(t); 0, 0, -m-1, -n-1) = \delta_{k,l} \frac{(n+1)(m+1)}{(2k+1)} \frac{(m+2)_k (n+2)_k}{(-m)_k (-n)_k}$$

we find the following explicit expression for $\beta_k(m,n)$

$$\begin{aligned} \beta_k(m,n) =& \frac{1}{C_k(m,n)} \frac{(2k+1)}{(n+1)(m+1)} \frac{(-m)_k(-n)_k}{(m+2)_k(n+2)_k} \\ &\times \sum_{t=0}^{m} (m+n+1-2t) R_k(\lambda(t); 0, 0, -m-1, -n-1) \alpha_t(m,n) \frac{\pi}{2} \end{aligned} \tag{A.3}$$

Now Proposition 2.2, and hence Theorem 2.1, follows from the following summation and simplifying the result.

Lemma A.1. *For $\ell \in \frac{1}{2}\mathbb{N}$, $n, m, k \in \mathbb{N}$ with $0 \leq k \leq m \leq n$ we have*

$$\sum_{t=0}^{m} (-1)^t \frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}} \frac{(2\ell+2-t)_t}{t!} (m+n+1-2t) R_k(\lambda(t); 0, 0, -m-1, -n-1) = (-1)^{m+k} \frac{(2\ell+k+1)!\,(2\ell-k)!}{(2\ell+1)!} \frac{(n+1)}{m!\,(2\ell-m)!}$$

Proof. Start with the left hand side and insert the ${}_4F_3$-series for the Racah polynomial and interchange summations to find

$$\sum_{j=0}^{k} \frac{(-k)_j\,(k+1)_j}{j!\,j!\,(-m)_j\,(-n)_j} \frac{(n-2\ell)_m}{(n+2)_m}$$

$$\times \sum_{t=j}^{m} (-1)^t \frac{(-1-n-m)_t}{(2\ell-n-m+1)_t} \frac{(-2\ell-1)_t}{t!} (-1)^t (m+n+1-2t)(-t)_j (t-m-n-1)_j$$

Relabeling the inner sum $t = j+p$ shows that the inner sum equals

$$(-1)^j \frac{(-1-n-m)_{2j}}{(2\ell-n-m+1)_j} (-2\ell-1)_j (1+m+n-2j)$$

$$\times \sum_{p=0}^{m-j} \frac{(-1-n-m+j)_p}{(2\ell-n-m+1+j)_p} \frac{(-2\ell-1+j)_p}{p!} \frac{(1+\frac{1}{2}(-1-m-n+2j))_p}{(\frac{1}{2}(-1-m-n+2j))_p} \frac{(-1-m-n+2j)_p}{(-1-m-n+j)_p}$$

and the sum over p is a hypergeometric sum. Multiplying by $\frac{(j-m)_p(j-n)_p}{(j-m)_p(j-n)_p}$ the sum can be written as a very-well-poised ${}_5F_4$-series

$${}_5F_4\left(\begin{matrix} 1+\frac{1}{2}(-1-m-n+2j),\, -1-m-n+2j,\, -1-2\ell+j,\, j-m,\, j-n \\ \frac{1}{2}(-1-m-n+2j), 2\ell-n-m+j+1,\, j-n,\, j-m \end{matrix} ;1\right) =$$

$$\frac{(-m-n+2j)_{m-j}}{(-m-n+j+1+2\ell)_{m-j}} \frac{(-m+1+2\ell)_{m-j}}{(-m+j)_{m-j}}$$

by the terminating Rogers-Dougall summation formula [4, §4.4].

Simplifying shows that the left hand side of the lemma is equal to the single sum

$$\frac{(n-2\ell)_m}{(n+2)_m} (-1)^m (n+m+1) \frac{(-n-m)_m}{(2\ell-n-m+1)_m} \frac{(2\ell+1-m)_m}{m!} \sum_{j=0}^{k} \frac{(-k)_j\,(k+1)_j}{j!\,j!} \frac{(-2\ell-1)_j}{(-2\ell)_j}$$

which can be summed by the Pfaff-Saalschütz summation [2, Thm. 2.2.6], [11, (1.4.5)]. This proves the lemma after some simplifications. ☐

Appendix B. Moments

In this appendix we give an explicit sum for the generalised moments for W. By the explicit expression

$$U_r(x) \;=\; (r+1)\; {}_2F_1\left(\begin{matrix} -r,\, r+2 \\ \frac{3}{2} \end{matrix} ; \frac{1-x}{2}\right)$$

we find

$$\int_{-1}^{1} (1-x)^n U_r(x) \sqrt{1-x^2}\, dx \;=\; (r+1) \sum_{k=0}^{r} \frac{(-r)_k (r+2)_k}{k!\,(\frac{3}{2})_k} 2^{-k} 2^{n+k+2} \frac{\Gamma(n+k+\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(n+k+3)}$$

$$= (r+1) 2^{n+2} \frac{\Gamma(n+\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(n+3)} {}_3F_2\left(\begin{matrix} -r, r+2, n+\frac{3}{2} \\ \frac{3}{2},\, n+3 \end{matrix} ;1\right) = (r+1) 2^{n+2} \frac{\Gamma(n+\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(n+3)} \frac{(-n)_r}{(n+3)_r}$$

using the beta-integral in the first equality and the Pfaff-Saalschütz summation [2, Thm. 2.2.6], [11, (1.4.5)] in the last equality. For $m \leq n$, the explicit expression (1.1) gives the following generalised moments

$$\begin{aligned}\int_{-1}^{1}(1-x)^p W(x)_{nm}\,dx &= 2^{p+2}\frac{\Gamma(p+\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(p+3)}\frac{(2\ell+1)}{n+1}\frac{(2\ell-m)!m!}{(2\ell)!}\\ &\times\sum_{t=0}^{m}(-1)^{m-t}\frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}}\frac{(2\ell+2-t)_t}{t!}(n+m-2t+1)\frac{(-p)_{n+m-2t}}{(p+3)_{n+m-2t}}\end{aligned} \tag{B.1}$$

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, 1992.

[2] G.E. Andrews, R.A. Askey, R. Roy, *Special Functions*, Cambridge Univ. Press, 1999.

[3] E. Badertscher, T.H. Koornwinder, *Continuous Hahn polynomials of differential operator argument and analysis on Riemannian symmetric spaces of constant curvature*, Canad. J. Math. 44 (1992), 750–773.

[4] W.N. Bailey, *Generalized Hypergeometric Series*, Hafner, 1964.

[5] W. Casselman, D. Miličić, *Asymptotic behavior of matrix coefficients of admissible representations*, Duke Math. J. 49 (1982), 869–930.

[6] D. Damanik, A. Pushnitski, B. Simon, *The analytic theory of matrix orthogonal polynomials*, Surveys in Approx. Th. 4 (2008), 1–85.

[7] A.J. Durán, *Ratio asymptotics for orthogonal matrix polynomials*, J. Approx. Theory 100 (1999), 304–344.

[8] R. Gangolli, V.S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Ergebnisse Math. Grenzgebiete 101, Springer, 1988.

[9] F.A. Grünbaum, I. Pacharoni, J. Tirao, *Matrix valued spherical functions associated to the complex projective plane*, J. Funct. Anal. 188 (2002), 350–441.

[10] F.A. Grünbaum, J. Tirao, *The algebra of differential operators associated to a weight matrix*, Integral Eq. Operator Theory 58 (2007), 449–475.

[11] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, paperback ed., Cambridge Univ. Press, 2009.

[12] A.W. Knapp, *Lie Groups Beyond an Introduction*, Prog. Math. 140, Birkhäuser, 2002.

[13] F. Knop, *Der Zentralisator einer Liealgebra in einer einhüllenden Algebra*, J. Reine Angew. Math. 406 (1990), 5–9.

[14] R. Koekoek, R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, online at `http://aw.twi.tudelft.nl/~koekoek/askey.html`, Report 98-17, Technical University Delft, 1998.

[15] E. Koelink, M. van Pruijssen, P. Román, *Matrix-valued orthogonal polynomials related to* $(\mathrm{SU}(2)\times\mathrm{SU}(2),\mathrm{diag})$, IMRN, to appear, arXiv:1012.2719v2.

[16] T.H. Koornwinder, *Matrix elements of irreducible representations of* $\mathrm{SU}(2)\times\mathrm{SU}(2)$ *and vector-valued orthogonal polynomials*, SIAM J. Math. Anal. 16 (1985), 602–613.

[17] M.G. Krein, *Fundamental aspects of the representation theory of hermitian operators with deficiency index* (m,m), AMS Translations ser. 2 97 (1971), 75–143.

[18] M.G. Krein, *Infinite J-matrices and a matrix moment problem*, Dokl. Akad. Nauk SSSR 69 (1949), 125–128.

[19] I. Pacharoni, P. Román, *A sequence of matrix valued orthogonal polynomials associated to spherical functions*, Constr. Approx. 28 (2008), 127–147.

[20] I. Pacharoni, J. Tirao, *Three term recursion relation for spherical functions associated to the complex hyperbolic plane*, J. Lie Theory 17 (2007), 791–828.

[21] P. Román, *Análisis armónico de funciones matriciales, en el plano hiperbólico complejo*, PhD. Thesis, Universidad Nacional de Córdoba, 2007.

[22] P. Román, J. Tirao, *Spherical functions, the complex hyperbolic plane and the hypergeometric operator*, Internat. J. Math. 17 (2006), 1151–1173.
[23] J.A. Tirao, *Spherical functions*, Rev. Un. Mat. Argentina 28 (1976/77), 75–98.
[24] J. Tirao, *The matrix-valued hypergeometric equation*, Proc. Natl. Acad. Sci. USA 100 (2003), 8138–8141.
[25] G. Warner, *Harmonic Analysis on Semi-simple Lie Groups II*, Grundlehren math. Wissenschaften 189, Springer, 1972.
[26] I.N. Zurrián, *Funciones esféricas matriciales asociadas a la esfera tridimensional*, Trabajo final de licenciatura en matemática, direction Juan A. Tirao, 2008, see `http://www.famaf.unc.edu.ar/institucional/biblioteca/trabajos/601/14519.pdf`

EK, MvP: Radboud Universiteit, IMAPP, Heyendaalseweg 135, 6525 GL Nijmegen, The Netherlands

E-mail address: `e.koelink@math.ru.nl` `m.vanpruijssen@math.ru.nl`

PR: CIEM, FaMAF, Universidad Nacional de Córdoba, Medina Allende s/n Ciudad Universitaria, Córdoba, Argentina

E-mail address: `roman@famaf.unc.edu.ar`