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LAGRANGIAN REDUCTION OF NONHOLONOMIC DISCRETE MECHANICAL SYSTEMS

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Abstract. In this paper we propose a process of lagrangian reduction and reconstruction for nonholonomic discrete mechanical systems where the action of a continuous symmetry group makes the configuration space a principal bundle. The result of the reduction process is a discrete dynamical system that we call the discrete reduced system. We illustrate the techniques by analyzing two types of discrete symmetric systems where it is possible to go further and obtain (forced) discrete mechanical systems that determine the dynamics of the discrete reduced system.

1. **Introduction.** The elimination of degrees of freedom of a symmetric mechanical system, the basic goal of reduction theory, is an old subject that dates back to the mid-nineteen century. The work of Routh in the context of abelian symmetries of classical mechanical systems was extended by many others in an effort to explore different aspects of the reduction process. Currently, there are well developed theories of reduction in the Hamiltonian setting, where the emphasis is in the reduction of the symplectic and Poisson structures, and the lagrangian setting, mostly focused on the reduction of variational principles. The literature in this area is vast; modern references are, for instance, [18, 2, 22, 8].

Discrete time mechanical systems have been considered in the literature since the 1960s, mostly as a way to approximate and model the behavior of (continuous) mechanical systems (see [21] and the references therein). The dynamics of nonholonomic discrete mechanical systems has been introduced by J. Cortés and S. Martinez in [10]. Many characteristics of mechanical systems have a discrete analogue. One important feature whose discrete analogue has been exposed only partially in the literature is the reduction of symmetries. The purpose of the present work is to describe a reduction and reconstruction process for discrete time mechanical systems with nonholonomic constraints, in the lagrangian setting.

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There are many reasons for being interested in the reduction of a symmetric system, be it continuous or discrete. On the one hand, the reduced system has less degrees of freedom, which may be an advantage when trying to solve the equations of motion. On the other, in many cases, the reduced system encodes the core dynamics of the system of interest. A good example of the advantage of working on the reduced system in the discrete case is presented by S. Jalnapurkar et al. in [16] where they show how the reduced system is free from geometric phases' effects allowing them to observe interesting dynamical structures, which were hard to separate in the unreduced system.

There are already several results on the reduction and reconstruction of symmetric nonholonomic discrete mechanical systems available in the literature. The case where the configuration space of the system Q is a Lie group has been studied by Y. Fedorov and D. Zenkov in [13, 12] and by R. McLachlan and M. Perlmutter in [23]. They obtain reduced equations of motion on the Lie algebra of the Lie group of symmetries which coincide, in the unconstrained case, with those obtained by A. Bobenko and Y. Suris in [4]. The case of Chaplygin systems, that is, when the symmetry group acts in such a way that no symmetry direction is compatible with the given constraints is analyzed by J. Cortés in [11] and, in context of groupoids, by D. Iglesias et al. in [15]. Also, the unconstrained case has been treated by S. Jalnapurkar et al. in [16] and, using discrete connections, by M. Leok et al. in [19]. A different approach using groupoids is given in [15]; in this case, the abstract reduction theory of groupoids produces a general reduced system on a groupoid; our approach differs from theirs, ours being more elementary —we stay in the "pair groupoid" case— and, at the same time, more explicit because we incorporate additional information (connections) to our models.

The reduction results that we present here are modeled on the results of H. Cendra et al. in [8] for continuous systems. They consider the action of a Lie group G on a configuration manifold Q so that the quotient map $\pi : Q \to Q/G$ is a principal bundle and the other data is G-invariant. They relate the variational principle that determines the dynamics of the original system with a reduced variational principle, that determines the dynamics of the reduced system. Eventually, equations of motion are derived from both variational principles. The introduction of a connection on the principal bundle π serves them to, first, construct an isomorphic model for the natural reduced space TQ/G and, also, to split the reduced variational principle and equations of motion into horizontal and vertical parts. Appropriate choice of a connection can lead to a simplified analysis of a specific mechanical system.

At a philosophical level, a common approach to discrete mechanics consists of replacing the tangent bundle TQ by $Q \times Q$, with the idea that infinitesimal displacement —velocities— are replaced by finite displacement —pairs of points. Even though this is a powerful idea, there are some very important differences between TQ and $Q\times Q$, that make it difficult to transfer techniques developed for continuous systems to discrete systems, as we will see later. The construction of the reduced space in [8] relies on an isomorphism defined using a connection on the principal bundle $Q \to Q/G$, which is seen as a G-invariant splitting of TQ, that descends to a splitting of TQ/G. In the discrete case, we follow the same path, but using what we call an affine discrete connection, a minor generalization of the discrete connections introduced by M. Leok et al. in [19] which allows us to split $(Q \times Q)/G$. Using, in addition, a connection, we are able to derive a reduced variational principle as well as reduced equations of motion that split in horizontal and vertical parts.

The vertical part of the variational principle and equations turn out to be equivalent to what is known as the discrete nonholonomic momentum evolution equation [10] that determines how the discrete nonholonomic momentum mapping, defined in terms of the symmetry directions that are compatible with the constraints, evolves for discrete mechanical system. The so called horizontal symmetries have the property of preserving the discrete nonholonomic momentum.

We specialize the general reduction process in two different settings: the Chaplygin and horizontal systems, where we can go further to obtain reduced systems that can be described easily. The first case is already present in the literature and we derive the known results from our approach. The other hasn't been considered before in the discrete setting to our knowledge; horizontal systems have been considered in the continuous setting by Cortés in [11]. Our approach can also be used in the case where the configuration space is a Lie Group to re derive the results of [23] mentioned above.

Constraints in a mechanical system on a configuration space Q are usually given by two distributions: $\mathcal{D}$ that describes the allowed variations and $\mathcal{C}_K$ that limits the allowed trajectories. The D'Alembert Principle used to determine the dynamics of such systems requires that $\mathcal{D} = \mathcal{C}_K$. More general situations, called generalized nonholonomic systems, where no connection is made between $\mathcal{D}$ and $\mathcal{C}_K$, have been considered in [20, 7, 5] . Discrete mechanical systems as introduced in [10] should be called generalized in the same sense as above, since the setup for constrained discrete mechanical systems consists of $\mathcal{D}$, with the same meaning as in the continuous case, and a submanifold $\mathcal{D}_d \subset Q \times Q$ which restricts the allowable discrete trajectories, and no relation between the two is assumed.

The layout of the paper is as follows. In Section 2, we sketch very roughly the lagrangian reduction of classical mechanical systems, which serves the dual purpose of introducing the results in the continuous case as well as the variational approach that serves as motivation for our work in the discrete case. Section 3 introduces discrete mechanical systems and their symmetries. In Section 4 we introduce some tools, including the affine discrete connections, that will be useful in the analysis of the reduction process to be carried out in Section 5 and condensed as Theorem 5.11. The statement of this result is written in terms that are not obviously defined on the reduced system; the purpose of Section 6 is to give an intrinsic version of that result, which we achieve with Corollary 6.5. The reconstruction of the original dynamics starting from that of the reduced system is explored in Section 7. Finally, in Section 8 we study the discrete nonholonomic momentum mapping, showing that the vertical part of the reduced variational principle is equivalent to the discrete nonholonomic momentum equation. The rest of the paper deals with the specialization of the general theory developed so far to particular situations. In Section 9 we obtain the equations of motion for systems where $Q \to Q/G$ is a trivial principal bundle. In Sections 10 and 11 we study systems with Chaplygin and horizontal symmetries respectively; in the Chaplygin case we obtain intrinsic versions that specialize to the results of [11], while in the horizontal case we find the discrete version of a type of symmetry whose continuous counterpart had been studied in [11].

Last, we wish to thank Hernán Cendra for his interest and valuable comments on this work.

2. **Reduction of classical mechanical systems.** In this section we recall some basic facts of the lagrangian reduction theory of (generalized) nonholonomic mechanical systems in the presence of symmetry. We refer to [3] and [6] for further details.

2.1. Generalized nonholonomic mechanical systems.

Definition 2.1. A *generalized nonholonomic mechanical system* is a quadruple $(Q, L, \mathcal{D}, C_K)$ where Q is a differentiable manifold, the *configuration space*, $L : TQ \to \mathbb{R}$ is a smooth function on the tangent bundle of Q, the *lagrangian*, $\mathcal{D}$ is a subbundle of TQ, the *variational constraints* or *virtual displacements*, and $C_K \subset TQ$ is a submanifold, the *kinematic constraints*.

For every such system, the *action functional* is defined by

$$S(q) := \int_{t_0}^{t_1} L(q(t), \dot{q}(t))dt,$$

where $q : [t_0, t_1] \to Q$ is a smooth curve in Q and $\dot{q} : [t_0, t_1] \to TQ$ is its velocity. An *infinitesimal variation* of q is a smooth curve $\delta q : [t_0, t_1] \to TQ$. An infinitesimal variation is said to have vanishing end points if $\delta q(t_0) = 0$ and $\delta q(t_1) = 0$.

The dynamics of a generalized nonholonomic mechanical system is determined by the following Principle.

Definition 2.2 (Lagrange–D'Alembert Principle)**.** A trajectory of $(Q, L, \mathcal{D}, C_K)$ is a curve $q : [t_0, t_1] \to Q$ which

- satisfies the kinematic constraints: $(q(t), \dot{q}(t)) \in C_K$ for all $t \in [t_0, t_1]$ and
- is a critical point of S for the admissible variations: $dS(q)(\delta q) = 0$ for all infinitesimal variations δq of q with vanishing end points and such that $\delta q(t) \in \mathcal{D}_{q(t)}$ for all $t \in [t_0, t_1]$.

As usual, the Lagrange–D'Alembert Principle gives rise to a set of equations called the generalized Lagrange–D'Alembert equations for $(Q, L, \mathcal{D}, C_K)$. These equations of motion can be written in a coordinate-free way if an affine connection ∇ on Q is chosen. Assuming this additional datum, [6] proves the following result.

Theorem 2.3. *A smooth curve q in Q is a trajectory of $(Q, L, \mathcal{D}, C_K)$ if and only if $(q(t), \dot{q}(t)) \in C_K$ for all $t \in [t_0, t_1]$ and*

$$-\frac{D}{Dt}\mathbb{F}L(q(t), \dot{q}(t)) + \mathbb{B}L(q(t), \dot{q}(t)) \in (\mathcal{D}_{q(t)})^\circ \quad \text{for all} \quad t \in [t_0, t_1], \tag{1}$$

where $\mathbb{F}L$ and $\mathbb{B}L$ denote the fiber and base derivatives of L, as defined in [6, Definition 3] *and $(\mathcal{D}_{q(t)})^\circ \subset T^*_{q(t)}Q$ is the annihilator of $\mathcal{D}_{q(t)} \subset T_{q(t)}Q$.*

Condition (1) is known as the (generalized) Lagange–D'Alembert equation.

2.2. **Symmetric generalized nonholonomic mechanical systems.** In what follows, we assume that G is a Lie group that acts (on the left) on Q freely and properly, so that the quotient $\pi : Q \to Q/G$ is a principal bundle with structure group G. This action will be denoted by $l_g^Q(q)$ for all $g \in G$ and $q \in Q$; there is an induced action of G on TQ given by $l_g^{TQ}(v_q) := dl_g^Q(q)(v_q)$ for all $g \in G$ and $v_q \in T_qQ$, that is called the *lifted action*.

Definition 2.4. G is a *symmetry group* of $(Q, L, \mathcal{D}, C_K)$ if, in addition to the general assumptions stated above, L, $\mathcal{D}$ and C_K are G-invariant, that is, if $L \circ l_g^{TQ} = L$, $l_g^{TQ}(\mathcal{D}) \subset \mathcal{D}$ and $l_g^{TQ}(C_K) \subset C_K$ for all $g \in G$.

The *vertical bundle* $\mathcal{V}^G$ over Q is a subbundle of TQ with fibers $\mathcal{V}_q^G := T_q(l_G^Q(\{q\}))$. We assume that the distribution $\mathcal{S}$ with fibers $\mathcal{S}_q := \mathcal{V}_q^G \cap \mathcal{D}_q$ has locally constant rank, so that it is a subbundle of TQ. This condition, which applies to many interesting examples, is verified when, for instance, $T_qQ = \mathcal{D}_q + \mathcal{V}_q$ for all $q \in Q$ —the "dimension assumption"— holds, as is usually considered in much of the literature.

Assume that G-invariant subbundles of TQ, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{W}$ can be chosen in such a way that $\mathcal{H}$ and $\mathcal{U}$ are direct complements of $\mathcal{S}$ in $\mathcal{D}$ and $\mathcal{V}^G$ respectively, and $\mathcal{W}$ is a direct complement of $\mathcal{D} + \mathcal{V}^G$ in TQ. All together, we have the decomposition

$$TQ = \mathcal{W} \oplus \mathcal{U} \oplus \mathcal{S} \oplus \mathcal{H}, \tag{2}$$

where $\mathcal{D} = \mathcal{S} \oplus \mathcal{H}$ and $\mathcal{V}^G = \mathcal{S} \oplus \mathcal{U}$. One way to construct the complementary bundles $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{W}$ is as orthogonal complements for some G-invariant inner product on TQ. This type of inner product is usually available as the kinetic energy of symmetric mechanical systems.

Following [3] and [6], we associate a connection to the decomposition (2).

Definition 2.5. The unique connection $\mathcal{A}$ on the principal bundle $\pi : Q \to Q/G$ whose horizontal space is $Hor_{\mathcal{A}} = \mathcal{W} \oplus \mathcal{H}$ is called the *generalized nonholonomic connection* associated to the system $(Q, L, \mathcal{D}, C_K)$ and the splitting (2).

2.3. **Reduction of symmetry.** When G is a symmetry group of $(Q, L, \mathcal{D}, C_K)$ we define the *reduced lagrangian* $\ell : TQ/G \to \mathbb{R}$ by $\ell([(q, \dot{q})]_G) := L(q, \dot{q})$.

In order to establish a reduced variational principle and study the reduced equations of motion [9] and [8] find it more convenient to work on a diffeomorphic model of TQ/G as follows. Given a principal connection $\mathcal{A}$ on $\pi : Q \to Q/G$ there is an isomorphism of vector bundles over Q/G

$$\alpha_{\mathcal{A}} : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}},$$

where $\mathfrak{g}$ is the Lie algebra of G, $\tilde{\mathfrak{g}}$ is the adjoint vector bundle (see [17]) of π and $\alpha_{\mathcal{A}}([q, \dot{q}]_G) := d\pi(q, \dot{q}) \oplus [(q, \mathcal{A}(q, \dot{q}))]_G$. The connection of choice used to define $\alpha_{\mathcal{A}}$ is the generalized nonholonomic connection $\mathcal{A}$ introduced above. The reduced lagrangian ℓ can be transported to this new model space as $\hat{L} : T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$, defined by

$$\hat{L}(x, \dot{x}, \bar{v}) := \ell\big((\alpha_{\mathcal{A}})^{-1}(x, \dot{x}, \bar{v})\big) = L(q, \dot{q}),$$

where $(x, \dot{x}) = d\pi(q, \dot{q})$ and $\bar{v} = [(q, \mathcal{A}(q, \dot{q}))]_G$.

The reduced dynamics of the system is defined in [6] by the following principle.

Definition 2.6 (Generalized Lagrange–D'Alembert–Poincaré Principle)**.** A trajectory of the reduced system determined by the symmetry group G of $(Q, L, \mathcal{D}, C_K)$ is a curve $\mu := (x, \dot{x}) \oplus \bar{v} : [t_0, t_1] \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ which

- satisfies the reduced kinematic constraints: $\mu(t) \in \hat{C}_K := \alpha_{\mathcal{A}}(C_K/G)$ for all $t \in [t_0, t_1]$.
- is a critical point of the reduced action

$$\int_{t_0}^{t_1} \hat{L}(x(t), \dot{x}(t), \bar{v}(t))dt$$

for some infinitesimal variations $\delta\mu = \delta x \oplus \delta^{\mathcal{A}}\bar{v}$ of μ such that, if q is a lift of x to Q, $\delta x(t) \in \hat{\mathcal{D}}^h_{x(t)} := d\pi(q(t))(\mathcal{D}_{q(t)})$ and $\delta^{\mathcal{A}}\bar{v}(t) \in \hat{\mathcal{D}}^v := \alpha_{\mathcal{A}}(\mathcal{S}/G)$ with all variations vanishing at the end points. The precise description of the set of infinitesimal variations where the criticality condition applies requires a special form for the $\delta^{\mathcal{A}}\bar{v}$ involving the curvature of $\mathcal{A}$ (we refer to [6] for further details).

The dynamics generated by this principle can be described using equations of motion, called the reduced generalized Lagrange–D'Alembert–Poincaré equations, which are equivalent to the generalized nonholonomic Lagrange–D'Alembert equations of the system $(Q, L, \mathcal{D}, C_K)$, as shown by Theorem 9 of [6].

Theorem 2.7. *Let G be a symmetry group of $(Q, L, \mathcal{D}, C_K)$, $\mathcal{A}$ be the generalized nonholonomic connection on $\pi : Q \to Q/G$ associated to a splitting of TQ and $q : [t_0, t_1] \to Q$ a curve on Q. Then the following statements are equivalent.*

- *The curve q satisfies $(q(t), \dot{q}(t)) \in C_K$ for all $t \in [t_0, t_1]$ and (1) holds.*
- *The curve $\mu : [t_0, t_1] \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ given by $\mu := (x, \dot{x}) \oplus \bar{v} = \alpha_{\mathcal{A}}([(q, \dot{q})]_G)$ satisfies $\mu(t) \in \hat{C}_K$ and*

$$-\frac{D}{Dt}\frac{\partial \hat{L}}{\partial \bar{v}}(\mu(t)) + ad^*_{\bar{v}}\frac{\partial \hat{L}}{\partial \bar{v}}(\mu(t)) \in (\hat{\mathcal{D}}^v_{x(t)})^\circ$$

$$-\frac{D}{Dt}\frac{\partial \hat{L}}{\partial \dot{x}}(\mu(t)) + \frac{\partial \hat{L}}{\partial x}(\mu(t)) - \left\langle \frac{\partial \hat{L}}{\partial \bar{v}}, i_{x(t)}\tilde{B} \right\rangle \in (\hat{\mathcal{D}}^h_{x(t)})^\circ$$

for all $t \in [t_0, t_1]$, where $\tilde{B}$ is the reduced curvature of $\mathcal{A}$.

3. Discrete mechanical systems and symmetries.

In this section we review the notion of discrete mechanical system with nonholonomic constraints, that is the discrete time analogue of the (generalized) nonholonomic mechanical systems considered in Section 2. We also consider symmetries of such systems.

3.1. Discrete mechanical systems.

Definition 3.1. A *nonholonomic discrete mechanical system* consists of a quadruple $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ where Q and $\mathcal{D}$ are as in Definition 2.1, $L_d : Q \times Q \to \mathbb{R}$ is a smooth map, the *discrete lagrangian*, and $\mathcal{D}_d \subset Q \times Q$ is a submanifold, the *discrete kinematic constraints*.

Remark 3.2. The discrete mechanical systems defined above are slightly more general than those considered in [10] because we are not requiring that the diagonal of $Q \times Q$ be contained in $\mathcal{D}_d$. Still, in order to construct a dynamical system, we will assume that $\mathcal{D}_d$ contains the graph of a smooth map $Q \to Q$, with the case of [10] corresponding to the identity.

Following [10], discrete mechanical systems define discrete dynamical systems using a discrete Lagrange–D'Alembert Principle, roughly saying that trajectories of the dynamical system are critical points of the *discrete action functional*

$$S_d(q_\cdot) := \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$$

that satisfy the constraints. The following definition makes this notion precise.

Definition 3.3 (Discrete Lagrange–D'Alembert Principle)**.** A *discrete curve* in Q is a map $q_\cdot : \{0, 1, \ldots, N\} \to Q$ and a *variation* of a discrete curve $q_\cdot$ consists of a map $\delta q_\cdot : \{0, 1, \ldots N\} \to TQ$ such that $\delta q_k \in T_{q_k}Q$ for all k. A variation is said to have *vanishing end points* if $\delta q_0 = 0$ and $\delta q_N = 0$. A *trajectory* of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ is a discrete curve $q_\cdot$ which

- satisfies the kinematic constraints $(q_k, q_{k+1}) \in \mathcal{D}_d$ for all k and
- is a critical point of S_d for all admissible variations $\delta q.$ of $q.$: $dS_d(q.)(\delta q.) = 0$ for all infinitesimal variations with vanishing end points $\delta q_k \in \mathcal{D}_{q_k}$ for all k.

The discrete Lagrange–D'Alembert Principle leads to a set of equations. Indeed, if D_j denotes differentiation with respect to the j-th component of a Cartesian product,

$$\begin{aligned} dS_d(q.)(\delta q.) &= \sum_{k=0}^{N-1} dL_d(q_k, q_{k+1})(\delta q_k, \delta q_{k+1}) \\ &= \sum_{k=1}^{N-1} \big(D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k)\big)(\delta q_k) \\ &\quad + D_1 L_d(q_0, q_1)(\delta q_0) + D_2 L_d(q_{N-1}, q_N)(\delta q_N), \end{aligned}$$

so that $q.$ is a trajectory of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ if and only if, for all k,

$$D_1 L_d(q_k, q_{k-1}) + D_2 L_d(q_{k-1}, q_k) \in \mathcal{D}^\circ_{q_k} \quad \text{and} \quad (q_k, q_{k+1}) \in \mathcal{D}_d \tag{3}$$

or, if $\chi_\kappa(q_k, q_{k+1}) = 0$ are local equations defining $\mathcal{D}_d$,

$$D_1 L_d(q_k, q_{k-1}) + D_2 L_d(q_{k-1}, q_k) = \sum_{a=1}^{M} \lambda_{a,k}\, \omega^a(q_k) \quad \text{and} \quad \chi_\kappa(q_k, q_{k+1}) = 0,$$

for some constants $\lambda_{a,k} \in \mathbb{R}$ and $\mathcal{D}^\circ_{q_k} = \langle \omega^1(q_k), \ldots, \omega^M(q_k)\rangle$.

It can be shown that under sufficient regularity of the data, equation (3) has solutions, that are unique given sufficiently closely spaced initial data (see [23, Proposition 3] and [10]).

Sometimes it is necessary to consider systems that are forced. Such will be the case below when we consider reduced systems, even when the unreduced one is not forced. Thus, it is convenient to add the following notion.

Definition 3.4. A *forced discrete mechanical system* consists of a discrete mechanical system $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ together with a 1-form f_d on $Q \times Q$. We often write $f_d(q_0, q_1)(\delta q_0, \delta q_1) := f_d^-(q_0, q_1)(\delta q_0) + f_d^+(q_0, q_1)(\delta q_1)$ were $f_d^+ : p_2^*(TQ) \to \mathbb{R}$ and $f_d^- : p_1^*(TQ) \to \mathbb{R}$, where $p_j : Q \times Q \to Q$ denotes the projection on the j-th component and $p_j^*(TQ)$ is the pullback vector bundle (see [1]) over $Q \times Q$.

The dynamics of a forced discrete mechanical system is given by the appropriately modified discrete Lagrange–D'Alembert principle as follows.

Definition 3.5. A discrete curve $q.$ is a *trajectory of the forced discrete lagrangian mechanical system* $(Q, L_d, \mathcal{D}, \mathcal{D}_d, f_d)$ if

- the curve satisfies the kinematic constraints: $(q_k, q_{k+1}) \in \mathcal{D}_d$ for all k and
- for all variations $\delta q.$ of $q.$ that have vanishing end points and $\delta q_k \in \mathcal{D}_{q_k}$,

$$dS_d(q.)(\delta q.) + \sum_{k=0}^{N-1} f_d(q_k, q_{k+1})(\delta q_k, \delta q_{k+1}) = 0.$$

Just as we found a characterization in terms of equations in the unforced situation, $q.$ is a trajectory of a forced system if and only if, for all k,

$$\begin{cases} D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) + f_d^-(q_k, q_{k+1}) + f_d^+(q_{k-1}, q_k) \in \mathcal{D}^\circ_{q_k} \\ (q_k, q_{k+1}) \in \mathcal{D}_d. \end{cases}$$

3.2. **Symmetric discrete mechanical systems.** The main subject of this paper is the analysis of discrete mechanical systems with continuous symmetry groups. Here we continue under the assumption that the Lie group G acts on Q by l^Q making the quotient map $\pi : Q \rightarrow Q/G$ a principal bundle. In addition, we consider the *diagonal action* of G on $Q \times Q$ defined by $l_g^{Q\times Q}(q_0, q_1) := (l_g^Q(q_0), l_g^Q(q_1))$.

Definition 3.6. G is a *symmetry group* of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ if, in addition to what was stated in the previous paragraph, L_d and $\mathcal{D}_d$ are invariant by $l^{Q\times Q}$, and $\mathcal{D}$ is G-invariant by l^{TQ}. If the system is forced, we also require that the discrete force f_d be G-equivariant.

Several structures introduced in Section 2.2 remain useful in the current context. In particular, we have the subbundles $\mathcal{S}, \mathcal{V}^G, \mathcal{H}, \mathcal{U}, \mathcal{W} \subset TQ$ and the decomposition of TQ given by (2). We also have a nonholonomic connection $\mathcal{A}$ associated to that decomposition. We denote the horizontal lift associated with $\mathcal{A}$ by $h : Q \times_{Q/G} T(Q/G) \rightarrow TQ$, that is $h^q(w_{\pi(q)}) = v_q$ if $v_q \in Hor_{\mathcal{A}}(q)$ and $d\pi(q)(v_q) = w_{\pi(q)}$. The following result is straightforward.

Lemma 3.7. *Considering the G-actions given by the lifted action l^{TQ} and by $l_g^{Q\times_{Q/G}T(Q/G)}(q, t_{\pi(q)}) := (l_g^Q(q), t_{\pi(q)})$, the map h is G-equivariant.*

For each $q \in Q$, the elements $\xi \in \mathfrak{g}$ such that $\xi_Q(q) \in \mathcal{D}_q$ form a subspace in $\mathfrak{g}$, which may depend on q. For that reason, it is convenient to define the space

$$\mathfrak{g}^{\mathcal{D}} := \{(q, \xi) \in Q \times \mathfrak{g} : \xi_Q(q) \in \mathcal{D}_q\}.$$

Using the projection on the first variable, $\mathfrak{g}^{\mathcal{D}}$ is a vector bundle on Q. It is easy to see that the differential in the group direction of l^Q establishes an isomorphism of vector bundles $\mathfrak{g}^{\mathcal{D}} \simeq \mathcal{S}$.

4. **Some discrete tools.** An important step in the reduction process for classical systems is the passage from TQ/G to a model space $T(Q/G)\oplus\tilde{\mathfrak{g}}$, as seen in Section 2. This is achieved using the nonholonomic connection. In the discrete case, we follow the same philosophy but connections are not the right tool for the task. In this Section we introduce the notion of affine discrete connection and use it to construct an isomorphism that, eventually, will play the role that $\alpha_{\mathcal{A}}$ played in Section 2.3.

4.1. **Affine discrete connections.**

Definition 4.1. The *discrete vertical bundle* for the l^Q action of G is the submanifold

$$\mathcal{V}_d^G := \{(q, l_g^Q(q)) \in Q \times Q : q \in Q, g \in G\}.$$

For $q \in Q$ we define $\mathcal{V}_d^G(q) := \mathcal{V}_d^G \cap (\{q\} \times Q) \subset Q \times Q$.

Definition 4.2. The *composition* of vertical and arbitrary elements of $Q \times Q$ with the same first element is defined by $\cdot : \mathcal{V}_d^G \times_Q (Q \times Q) \rightarrow Q \times Q$ with

$$(q_0, l_g^Q(q_0)) \cdot (q_0, q_1) := (q_0, l_g^Q(q_1)).$$

We denote by l^G the conjugation action of G on itself, that is, $l_g^G(h) := ghg^{-1}$ for all $g, h \in G$.

Definition 4.3. Let $\gamma : Q \rightarrow G$ be a smooth G-equivariant map with respect to l^Q and l^G, $\Gamma := \{(q, l_{\gamma(q)}^Q(q)) : q \in Q\}$ and $Hor \subset Q \times Q$ be a G-invariant submanifold such that $\Gamma \subset Hor$. For each $q \in Q$ let $Hor(q) := Hor \cap (\{q\} \times Q)$ and $Hor^2(q) := p_2(Hor(q))$.

We say that Hor defines the *affine discrete connection* $\mathcal{A}_d$ on the principal bundle $\pi : Q \to Q/G$ if, for each $q \in Q$ and all $q_1 \in l^Q_G(\{q\}) \cap Hor^2(q)$,

$$Hor^2(q) \subset Q \text{ is a submanifold} \quad \text{and} \quad T_{q_1}Q = T_{q_1}(l^Q_G(\{q\})) \oplus T_{q_1}Hor^2(q) \tag{4}$$

(see Figure 1(a)). We denote Hor by $Hor_{\mathcal{A}_d}$, and call γ (or even Γ) the *level* of $\mathcal{A}_d$.

Remark 4.4. Recalling the notion of transversality of submanifolds (see [14]), condition (4) is equivalent to requiring that for each $q \in Q$,

$$\dim(Hor^2(q)) = \dim Q - \dim G \quad \text{and} \quad l^Q_G(\{q\}) \pitchfork Hor^2(q).$$

Remark 4.5. The notion of discrete connection introduced in [19], coincides with that of an affine discrete connection where the level Γ is the diagonal of $Q \times Q$. Since the diagonal of $Q \times Q$ plays the role of a "null element" for composition, affine discrete connections need not contain the "null element" in their horizontal space, just like affine spaces need not contain the null element of a vector space.

Remark 4.6. The idea behind the introduction of the previous definitions is that $Q \times Q$ should be a discrete version of TQ. Even though this is a powerful idea, there are several important differences between those spaces. Using the projection on the first factor and the standard projection, both spaces are fibered over Q, but TQ is a vector bundle, while $Q \times Q$ is usually not one. In particular, tangent vectors at the same base point can be added, whereas there is nothing similar for elements of $Q \times Q$. A partial fix for this problem is the composition operation introduced in Definition 4.2: it provides a way of combining vertical elements of $Q \times Q$ with arbitrary elements of $Q \times Q$ based at the same point (that is, with the same first component). Even though this is not a complete analogue of addition, it is enough to handle discrete connections.

Connections are an important tool in differential geometry. Essentially, for principal bundles $\pi : Q \to Q/G$, they provide compatible splittings

$$T_qQ = \mathcal{V}^G_q \oplus Hor(q) \quad \text{for all} \quad q \in Q. \tag{5}$$

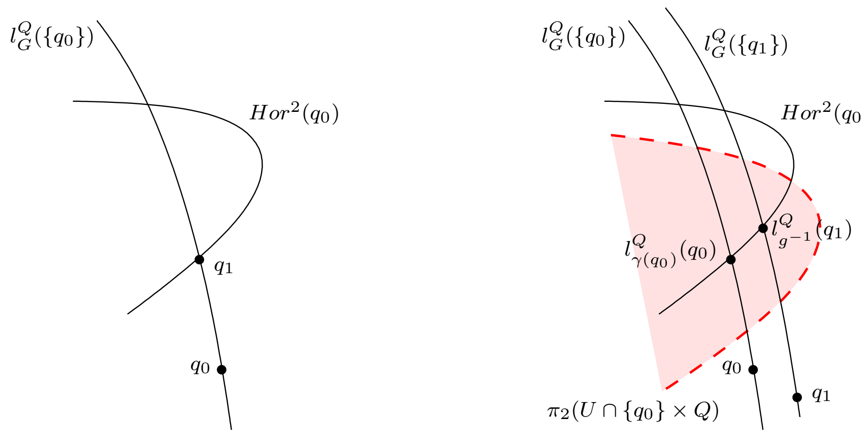
In the discrete setting, we want to be able to split $\{q\} \times Q$ in vertical and (some) complementary space. Not having an addition operation we choose a more geometric view of the problem and define affine discrete connections in terms of spaces that are complementary to $\mathcal{V}^G_d$ in the precise sense of Definition 4.3. Furthermore, this definition is equivalent to being able to decompose $\{q\} \times Q = \mathcal{V}^G_d(q) \cdot Hor(q)$ (at least in a neighborhood of Γ), which is the composition-analogue of (5).

Proposition 4.7. *Let $\mathcal{A}_d$ be an affine discrete connection of level γ. Then, there exists U, a G-invariant open neighborhood of Γ in $Q \times Q$, such that, for all $(q_0, q_1) \in Q \times Q$ with $\pi(q_1)$ sufficiently close to $\pi(q_0)$, there is a unique $g \in G$ such that*

$$(q_0, q_1) = (q_0, l^Q_g(q_0)) \cdot (q_0, l^Q_{g^{-1}}(q_1)), \tag{6}$$

with $(q_0, l^Q_{g^{-1}}(q_1)) \in Hor_{\mathcal{A}_d}(q_0) \cap U$.

Proof. Since $l^Q_{\gamma(q_0)}(q_0) \in l^Q_G(\{q_0\}) \cap Hor^2_{\mathcal{A}_d}(q_0)$, the condition $l^Q_G(\{q_0\}) \pitchfork Hor^2_{\mathcal{A}_d}(q_0)$ implies that $Hor^2_{\mathcal{A}_d}(q_0)$ intersects all orbits of G that are close to the one through q_0 (see Figure 1(b)). Therefore, for any q_1 with $\pi(q_1)$ sufficiently close to $\pi(q_0)$, there are $g \in G$ such that $l^Q_{g^{-1}}(q_1) \in Hor^2_{\mathcal{A}_d}(q_0)$. Furthermore, again by the transversality condition, there is an open set $V_{q_0} \subset Q$ where the intersection $l^Q_G(\{q_1\}) \cap Hor^2_{\mathcal{A}_d}(q_0) \cap V_{q_0}$ consists of a single point. Hence, there is a unique $g \in G$ such that $l^Q_{g^{-1}}(q_1) \in Hor^2_{\mathcal{A}_d}(q_0) \cap V_{q_0}$. Finally, taking into account the smoothness of Hor, U is constructed by gluing the sets V_{q_0}. □



(a) G orbit ($l^Q_G(q_0)$) and $Hor^2_{\mathcal{A}_d}(q_0)$

(b) G orbit through q_0 and nearby orbits

FIGURE 1. G orbit and $Hor^2_{\mathcal{A}_d}(q_0)$ in Q

Remark 4.8. The existence of a level function as a datum for an affine discrete connection may seem a bit strange at first. However, if a decomposition like (6) is expected, by taking $q_1 = q_0$, we see that $(q_0, l^Q_{g^{-1}}(q_0)) \in Hor_{\mathcal{A}_d}(q_0)$, so $\gamma(q_0) := g^{-1}$ is a level function for $\mathcal{A}_d$.

Definition 4.9. Given an affine discrete connection $\mathcal{A}_d$ we define its *discrete connection* 1*-form* $\mathcal{A}_d : Q \times Q \to G$ by $\mathcal{A}_d(q_0, q_1) := g$ where g is the element of G that appears in decomposition (6).

Remark 4.10. Notice that we will use the same letter to name both the connection $\mathcal{A}_d$ and its discrete connection 1-form. Also, calling $\mathcal{A}_d$ a 1-form can be misleading since it is not a 1-form in the usual sense but, rather, a function. Still, the name is coming from the fact that when Leok et al. introduced it in [19] they also introduce the notion of "discrete k-form", and $\mathcal{A}_d$ is a discrete 1-form in that sense.

Remark 4.11. Due to Proposition 4.7, $\mathcal{A}_d$ is only defined in a G-invariant open set $U \subset Q \times Q$. In what follows, we will abuse the notation and pretend that $\mathcal{A}_d$ is defined everywhere not to make the notation too cumbersome.

Proposition 4.12. *Let $\mathcal{A}_d$ be an affine discrete connection on the principal bundle $\pi : Q \to Q/G$. Then, for all $(q_0, q_1) \in Q \times Q$ and $g_0, g_1 \in G$,*

$$\mathcal{A}_d(l^Q_{g_0}(q_0), l^Q_{g_1}(q_1)) = g_1 \mathcal{A}_d(q_0, q_1) g_0^{-1}, \tag{7}$$

as long as both sides are defined. Conversely, given a smooth function $\mathcal{A} : Q \times Q \to G$ such that (7) holds (with $\mathcal{A}_d$ replaced by $\mathcal{A}$), then $Hor := \{(q_0, q_1) \in Q \times Q : \mathcal{A}(q_0, q_1) = e\}$ defines an affine discrete connection $\mathcal{A}_d$ with level $\gamma(q) := \mathcal{A}(q, q)^{-1}$ and whose discrete connection 1*-form is $\mathcal{A}$.*

Proof. Writing $\mathcal{A}_d(q_0, q_1) = g$ and $\mathcal{A}_d(l^Q_{g_0}(q_0), l^Q_{g_1}(q_1)) = \tilde{g}$, by definition,

$$(q_0, q_1) = (q_0, l^Q_g(q_0)) \cdot \underbrace{(q_0, l^Q_{g^{-1}}(q_1))}_{\in Hor_{\mathcal{A}_d}(q_0)}$$

$$(l^Q_{g_0}(q_0), l^Q_{g_1}(q_1)) = (l^Q_{g_0}(q_0), l^Q_{\tilde{g}}(l^Q_{g_0}(q_0))) \cdot \underbrace{(l^Q_{g_0}(q_0), l^Q_{\tilde{g}^{-1}}(l^Q_{g_1}(q_1)))}_{\in Hor_{\mathcal{A}_d}(l^Q_{g_0}(q_0))}.$$

By the G-invariance of $Hor_{\mathcal{A}_d}$, since $(q_0, l^Q_{g^{-1}}(q_1)) \in Hor_{\mathcal{A}_d}(q_0)$, it follows that $(l^Q_{g_0}(q_0), l^Q_{g_0 g^{-1}}(q_1)) \in Hor_{\mathcal{A}_d}(l^Q_{g_0}(q_0))$. We can write

$$(l^Q_{g_0}(q_0), l^Q_{g_1}(q_1)) = (l^Q_{g_0}(q_0), l^Q_{g_1 g g_0^{-1}}(l^Q_{g_0}(q_0))) \cdot \underbrace{(l^Q_{g_0}(q_0), l^Q_{g_0 g^{-1}}(q_1))}_{\in Hor_{\mathcal{A}_d}(l^Q_{g_0}(q_0))}.$$

Identity (7) follows, then, from the uniqueness of the decomposition (6).

The converse result is, mostly, routine checking and we will omit the details. That Hor is a submanifold follows from the easily proved fact that $e \in G$ is a regular value of $\mathcal{A}$. The same proof shows that $p_2(Hor(q_0))$ is a submanifold of Q of the right dimension. The G-equivariance of $\mathcal{A}$ shows that Hor is G-invariant. An application of (7) leads to establishing the transversality condition for Hor. Finally, taking $\gamma(q) := \mathcal{A}(q, q)^{-1}$, one concludes that Hor defines an affine discrete connection of level γ. □

Affine discrete connections have discrete horizontal lifts, just as regular connections do.

Definition 4.13. Let $\mathcal{A}_d$ be an affine discrete connection on the principal bundle $\pi : Q \to Q/G$. The *discrete horizontal lift* $h_d : Q \times Q/G \to Q \times Q$ is given by

$$h_d^{q_0}(r_1) := (q_0, q_1) \Leftrightarrow (q_0, q_1) \in Hor_{\mathcal{A}_d} \quad \text{and} \quad \pi(q_1) = r_1.$$

We define $\overline{h_d^{q_0}} := p_2 \circ h_d^{q_0}$.

Remark 4.14. The map $h_d^{q_0}(r_1)$ is well defined, provided that r_1 is sufficiently close to $\pi(q_0)$, since for any $q_1 \in \pi^{-1}(r_1)$,

$$h_d^{q_0}(r_1) := (q_0, l^Q_{\mathcal{A}_d(q_0,q_1)^{-1}}(q_1)). \tag{8}$$

Lemma 4.15. *Let $\mathcal{A}_d$ be an affine discrete connection on $\pi : Q \to Q/G$ and consider the natural actions of G on $Q \times (Q/G)$ (extending l^Q trivially on the second factor) and the diagonal action on $Q \times Q$. Then, h_d is G-equivariant.*

Proof. It is a direct computation using (8). □

For completeness we mention that having a smooth and G-equivariant map $\eta : Q \times (Q/G) \to Q \times Q$ such that $\pi \circ p_2 \circ \eta^q = id|_{Q/G}$ for all $q \in Q$ is equivalent to having an affine discrete connection whose discrete horizontal lift is η.

Remark 4.16. Discrete connection forms $\mathcal{A}_d$ and discrete horizontal lifts may not be defined everywhere. Indeed, if $h_d : Q \times (Q/G) \to Q \times Q$ is defined everywhere, then for any $q \in Q$, the map $r \mapsto \overline{h_d^q}(r)$ is a global section of the principal bundle $\pi : Q \to Q/G$, so that the bundle is trivial. Hence, for nontrivial principal bundles h_d and, consequently, $\mathcal{A}_d$ can only be defined in some open set of the total space.

On the other hand, if Q has a G-invariant riemannian metric —as it usually happens for mechanical systems— the following construction provides a discrete affine connection. Let γ be a level function and define

$$Hor := \{(q_0, \exp_{\tilde{\gamma}(q_0)}(v_1)) \in Q \times Q \quad \text{for some} \quad v_1 \in (\mathcal{V}^G_{\tilde{\gamma}(q_0)})^\perp \subset T_{\tilde{\gamma}(q_0)}Q\},$$

where exp and $\perp$ are those of the riemannian metric and $\tilde{\gamma}(q_0) := l^Q_{\gamma(q_0)}(q_0)$. It can be checked that Hor defines a discrete affine connection of level γ. Notice that the domain of the associated discrete affine connection form and discrete horizontal lifts are limited by the domain of the exponential mapping.

Example 4.17. Consider $Q := \mathbb{R}^2$ with the action of $G := \mathbb{R}$ given by $l^Q_g(q) := (x, y + g)$, where $q = (x, y)$. Clearly, $p_1 = \pi : Q \to Q/G$ is a (trivial) principal bundle with structure group G. In order to define an affine discrete connection on this bundle we need to find a G-invariant manifold Hor that is "complementary" to $\mathcal{V}^G_d = \{(q_0, q_1) \in Q \times Q : x_0 = x_1\}$ in $Q \times Q$. We consider complements to $l^Q_G(\{q_0\})$ for $q_0 \in Q$. A G-invariant family of curves that is complementary to the orbits is $Hor^{(2)}(q_0) = \{q_1 \in Q : y_1 - y_0 = b(x_1 + x_0)(x_1 - x_0)/2\}$ for a parameter $b \in \mathbb{R}$. Since $l^Q_G(\{q_0\}) \cap Hor^{(2)}(q_0) = \{q_0\}$, the only thing we have to prove in order to see that $Hor = \{(q_0, q_1) \in Q \times Q : y_1 - y_0 = b(x_1 + x_0)(x_1 - x_0)/2\}$ defines an affine discrete connection is that $T_{q_0}Q = T_{q_0} l^Q_G(\{q_0\}) \oplus T_{q_0} Hor^{(2)}(q_0)$, which is evident because $T_{q_0} l^Q_G(\{q_0\}) = \langle \partial_y\big|_{q_0}\rangle$ and $T_{q_0} Hor^{(2)}(q_0) = \langle \partial_x\big|_{q_0} + b x_0 \partial_y\big|_{q_0}\rangle$. We denote the discrete connection defined by Hor with $\mathcal{A}^b_d$.

Since

$$(q_0, q_1) = (q_0, l^Q_g(q_0)) \cdot \underbrace{(q_0, l^Q_{g^{-1}}(q_1))}_{\in Hor_{\mathcal{A}^b_d}} \quad \text{for} \quad g = y_1 - y_0 - \frac{1}{2} b(x_1 + x_0)(x_1 - x_0),$$

we have

$$\begin{aligned} \mathcal{A}^b_d(q_0, q_1) &= y_1 - y_0 - b(x_1 + x_0)(x_1 - x_0)/2 \\ h^{q_0}_d(r_1) &= (q_0, (r_1, y_0 + b(r_1 + x_0)(r_1 - x_0)/2)). \end{aligned} \tag{9}$$

The fact that $\mathcal{A}^b_d$ and $h^{q_0}_d$ are defined everywhere is compatible with $Q \to Q/G$ being a trivial principal bundle.

4.2. **Isomorphisms associated to an affine discrete connection.** When working with symmetric discrete mechanical systems, one is led to consider the space $(Q \times Q)/G$. It is convenient to have a different model for this space. In this section we construct such a model associated to an affine discrete connection $\mathcal{A}_d$.

We start with a special case of a general construction called the *associated bundle* of a principal bundle (see Chap. 1, Sect. 5 of [17]).

Definition 4.18. Let $\pi : Q \to Q/G$ be a principal bundle and consider the action of G on $Q \times G$ defined by $l^{Q\times G}_g(q, w) := (l^Q_g(q), l^G_g(w))$, with $l^G_g(w) := g w g^{-1}$. Being π a principal bundle, the quotient $\tilde{G} := (Q \times G)/G$ by this action is a manifold, called the *conjugate associated bundle*. The quotient map is denoted by $\rho : Q \times G \to \tilde{G}$. The projections onto each of the two components of $Q \times G$ induce smooth maps $p^{Q/G} : \tilde{G} \to Q/G$ and $p^{G/G} : \tilde{G} \to G/G$. The first, $p^{Q/G}$, turns $\tilde{G}$ into a bundle over Q/G with fiber G.

It is convenient to define

$$\tilde{F}_1 : Q \times G \times (Q/G) \to Q \quad \text{with} \quad \tilde{F}_1(q_0, w_0, r_1) := l^Q_{w_0}\big(\overline{h^{q_0}_d}(r_1)\big).$$

Extending the G action $l^{Q\times G}$ to $Q \times G \times (Q/G)$ trivially on the last component and considering the G action on Q, a simple computation using the G-equivariance of h_d shows that $\tilde{F}_1$ is G-equivariant.

Proposition 4.19. *Given an affine discrete connection $\mathcal{A}_d$ on $\pi : Q \to Q/G$, let $\tilde{\Phi}_{\mathcal{A}_d} : Q \times Q \to Q \times G \times (Q/G)$ and $\tilde{\Psi}_{\mathcal{A}_d} : Q \times G \times (Q/G) \to Q \times Q$ be defined by*

$$\tilde{\Phi}_{\mathcal{A}_d}(q_0, q_1) := (q_0, \mathcal{A}_d(q_0, q_1), \pi(q_1))$$
$$\tilde{\Psi}_{\mathcal{A}_d}(q_0, w_0, r_1) := (q_0, \tilde{F}_1(q_0, w_0, r_1)).$$

Then, $\tilde{\Phi}_{\mathcal{A}_d}$ and $\tilde{\Psi}_{\mathcal{A}_d}$ are smooth and mutually inverses (when restricted to the domain of $\mathcal{A}_d$). Furthermore, considering the diagonal action of G on $Q \times Q$ and $l_g^{Q\times G\times(Q/G)}(q_0, w_0, r_1) := (l_g^Q(q_0), l_g^G(w_0), r_1)$, both maps are G-equivariant, so that they induce diffeomorphisms $\Phi_{\mathcal{A}_d} : (Q\times Q)/G \to \tilde{G}\times(Q/G)$ and $\Psi_{\mathcal{A}_d} : \tilde{G}\times(Q/G) \to (Q \times Q)/G$.

Proof. That $\tilde{\Phi}_{\mathcal{A}_d}$ and $\tilde{\Psi}_{\mathcal{A}_d}$ are smooth is clear from the definition. Checking that they are inverses and the G-equivariance is done by direct computation. □

Remark 4.20. In fact, when $\mathcal{A}_d$ and $\overline{h_d}$ are not globally defined the maps $\Phi_{\mathcal{A}_d}$ and $\Psi_{\mathcal{A}_d}$ are diffeomorphisms between open neighborhoods of of Γ/G and $(Q \times \{e\})/G \times (Q/G)$.

The commutative diagram (10), where $\tilde{\pi}$ is the quotient map and $\Upsilon := \Phi_{\mathcal{A}_d} \circ \tilde{\pi}$, shows some of the spaces and maps we have introduced and that will be used later, when analyzing the behavior of symmetric discrete mechanical systems.

$$\begin{array}{ccc} Q \times Q & \xrightarrow[\sim]{\tilde{\Phi}_{\mathcal{A}_d}} & Q \times G \times (Q/G) \\ \downarrow{\scriptstyle \tilde{\pi}} & \searrow{\scriptstyle \Upsilon} & \downarrow{\scriptstyle \rho\times id} \\ (Q \times Q)/G & \xrightarrow[\Phi_{\mathcal{A}_d}]{\sim} & \tilde{G} \times (Q/G) \end{array} \tag{10}$$

Lemma 4.21. *Let $(q_0, q_1) \in Q \times Q$ and $(\delta q_0, \delta q_1) \in T_{(q_0,q_1)}(Q \times Q)$. Then,*

$$d\Upsilon(q_0, q_1)(\delta q_0, \delta q_1) = (d\rho(q_0, \mathcal{A}_d(q_0, q_1))(\delta q_0, d\mathcal{A}_d(q_0, q_1)(\delta q_0, \delta q_1)), d\pi(q_1)(\delta q_1)).$$

Proof. Compute $d\Upsilon$, using $\Upsilon = (\rho \times id) \circ \tilde{\Phi}_{\mathcal{A}_d}$. □

Example 4.22. In the context of Example 4.17, we have that $Q = (Q/G) \times G$ with the G-action on Q corresponding to left multiplication on G. Then $\tilde{G} \simeq (Q/G) \times G$ with $\rho((r_0, h_0), w_0) \mapsto (r_0, w_0)$. The map ρ has a section $s(r_0, w_0) := ((r_0, 0), w_0)$ (where we are using implicitly the isomorphism just defined). The isomorphisms that appear in Proposition 4.19 are

$$\tilde{\Phi}_{\mathcal{A}_d^b}(q_0, q_1) = ((x_0, y_0), y_1 - y_0 - b(x_1 + x_0)(x_1 - x_0)/2, x_1)$$
$$\tilde{\Phi}^{-1}_{\mathcal{A}_d^b}((r_0, h_0), w_0, r_1) = ((r_0, h_0), (r_1, h_0 + b(r_1 + r_0)(r_1 - r_0)/2 + w_0)).$$

Remark 4.23. The model spaces $Q \times G \times (Q/G)$ and $\tilde{G} \times (Q/G)$ are by no means unique. In fact, given an affine discrete connection $\mathcal{A}_d$ it is possible to consider the isomorphism $\alpha_{\mathcal{A}_d} : (Q \times Q)/G \to (Q/G \times Q/G) \times_{Q/G} \tilde{G}$ defined by

$$\alpha_{\mathcal{A}_d}(\tilde{\pi}(q_0, q_1)) = ((\pi(q_0), \pi(q_1)), \rho(q_0, \mathcal{A}_d(q_0, q_1))).$$

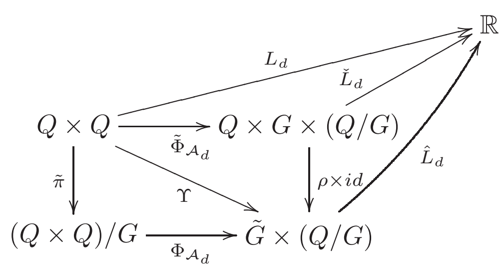
This isomorphism and the corresponding model space are very close to the data used in the analysis of the continuous case in Section 2.3 and is introduced by [19] in their proposed reduction of discrete unconstrained systems. Still, the presence of the fibered product in the model space makes it harder to work with compared to the simpler Cartesian product that appears in $\tilde{G} \times (Q/G)$.

5. **Variations and reduced variations.** In this section we analyze the relationship between the dynamics of a symmetric discrete mechanical system on $Q \times Q$, which induces a dynamics on (an open neighborhood of $Q \times \{e\} \times (Q/G)$ in) $(Q\times Q)/G$ and the dynamics on (an open neighborhood of $(Q\times\{e\})/G\times(Q/G)$ in) the isomorphic model $\tilde{G} \times (Q/G)$ of $(Q \times Q)/G$. In order to describe the dynamics on this last space we start by introducing some relevant notions.

5.1. **Reduced lagrangians.** Given $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ with a symmetry group G and an affine discrete connection $\mathcal{A}_d$ we construct a dynamical system on $Q\times G\times(Q/G)$ using the isomorphism $\tilde{\Phi}_{\mathcal{A}_d}$. In particular, the dynamics will be determined using the function $\check{L}_d := L_d \circ \tilde{\Psi}_{\mathcal{A}_d}$. Since L_d is G-invariant and, taking into account the symmetry properties of the system, $\check{L}_d$ is G-invariant, L_d and $\check{L}_d$ induce maps on the corresponding quotient spaces. We denote the map induced by $\check{L}_d$ on $\tilde{G}\times(Q/G)$ by $\hat{L}_d$. Therefore,

$$\check{L}_d(q_0, w_0, r_1) = L_d(q_0, \tilde{F}_1(q_0, w_0, r_1)) \quad \text{and} \quad \hat{L}_d(\rho(q_0, w_0), r_1) = \check{L}_d(q_0, w_0, r_1).$$

The following diagram shows all the relevant maps introduced so far.



Associated to the reduced discrete lagrangian $\hat{L}_d$ we define a reduced discrete action $\hat{S}_d$ by $\hat{S}_d(v_., r_.) := \sum_k \hat{L}_d(v_k, r_{k+1})$.

Example 5.1. Consider the discrete mechanical system $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$, where

$$Q := \mathbb{R}^2 \quad \text{with points} \quad q = (x, y),$$
$$L_d(q_0, q_1) := m\big((x_1 - x_0)^2 + (y_1 - y_0)^2\big)/2,$$
$$\mathcal{D}_q := \{\dot{x}\partial_x\big|_q + \dot{y}\partial_y\big|_q \in T_qQ : \dot{y} = x\dot{x}\} = \langle \partial_x\big|_q - x\partial_y\big|_q \rangle \subset T_qQ,$$
$$\mathcal{D}_d := \{(q_0, q_1) \in Q \times Q : y_1 - y_0 = (x_1 + x_0)(x_1 - x_0)/2\},$$

that originates as a discretization of a free particle in $\mathbb{R}^2$ subject to a nonholonomic constraint. This system can be readily solved using the discrete Lagrange–D'Alembert equations (3). Rather than following that path we will make a detailed study of its reduction in the following sections in order to illustrate some of the techniques developed so far.

The group $G := \mathbb{R}$ acts on Q as considered in Example 4.17, turning G into a symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$. Furthermore, for arbitrary $b \in \mathbb{R}$, $\mathcal{A}_d^b$ defines isomorphisms $\tilde{\Phi}_{\mathcal{A}_d^b}$ and $\Phi_{\mathcal{A}_d^b}$ that can be used to study the system. Since, by

Example 4.22, $\tilde{G} \times (Q/G) \simeq (Q/G) \times G \times (Q/G)$, the relevant lagrangians are

$$\begin{aligned}\check{L}_d((r_0, h_0), w_0, r_1) &= m\big((r_1 - r_0)^2 + (w_0 + b(r_1 + r_0)(r_1 - r_0)/2)^2\big)/2\\ \hat{L}_d(r_0, w_0, r_1) &= \check{L}_d((r_0, 0), w_0, r_1).\end{aligned}$$

Back in the general setting, we notice that, since $\hat{L}_d$ is G-invariant and $Q \times G \times (Q/G)$ is a Cartesian product (so that differentials on that space decompose in terms of differentials on each of the spaces),

$$\begin{aligned}d\hat{L}_d(v_0, r_1)(\delta v_0, \delta r_1) =& d\hat{L}_d(\rho(q_0, w_0), r_1)(d\rho(q_0, w_0)(\delta q_0, \delta w_0), \delta r_1)\\ =& d\check{L}_d(q_0, w_0, r_1)(\delta q_0, \delta w_0, \delta r_1)\\ =& D_1\check{L}_d(q_0, w_0, r_1)(\delta q_0) + D_2\check{L}_d(q_0, w_0, r_1)(\delta w_0)\\ &+ D_3\check{L}_d(q_0, w_0, r_1)(\delta r_1).\end{aligned} \tag{11}$$

In Section 5.2, we study more carefully the second term in this last equality.

Lemma 5.2. *Considering the G actions $l^{Q\times G\times(Q/G)}$, l^Q, l^G and their lifted actions to the corresponding tangent spaces, the following morphisms of vector bundles over $Q \times G \times (Q/G)$*

$$D_1\check{L}_d : p_1^*TQ \to \mathbb{R} \quad \text{and} \quad D_2\check{L}_d : p_2^*TG \to \mathbb{R}$$

are G-equivariant, where p_1 and p_2 are the projections of $Q \times G \times (Q/G)$ onto the corresponding factors.

Proof. The G-invariance of $\check{L}_d$ leads to the G-equivariance of $d\check{L}_d : T(Q \times G \times (Q/G)) \to \mathbb{R}$. The statement follows by decomposing $T(Q\times G\times(Q/G)) = p_1^*TQ \oplus p_2^*TG \oplus p_3^*T(Q/G)$. □

5.2. **Mixed curvature and reduced forces.** Given a principal bundle $\pi : Q \to Q/G$, a connection $\mathcal{A}$ and an affine discrete connection $\mathcal{A}_d$ we construct an object that, somehow, compares the notions of continuous and discrete horizontality introduced by $\mathcal{A}$ and $\mathcal{A}_d$.

Definition 5.3. The *mixed curvature* $\mathcal{B}_m$ of $\mathcal{A}$ and $\mathcal{A}_d$ is the morphism of vector bundles over $Q \times Q$, $\mathcal{B}_m : T(Q \times Q) \to {\mathcal{A}_d}^*(TG)$ defined by

$$\mathcal{B}_m(q_0, q_1)(\delta q_0, \delta q_1) := d\mathcal{A}_d(q_0, q_1)(Hor_{\mathcal{A}}(\delta q_0), Hor_{\mathcal{A}}(\delta q_1)),$$

where $Hor_{\mathcal{A}}(\delta q)$ denotes the $\mathcal{A}$-horizontal part of δq. Taking into account the isomorphism $T(Q \times Q) \simeq p_1^*(TQ) \oplus p_2^*(TQ)$, the mixed curvature decomposes as $\mathcal{B}_m = \mathcal{B}_m^+ + \mathcal{B}_m^-$, where $\mathcal{B}_m^+ : p_2^*TQ \to {\mathcal{A}_d}^*(TG)$ and $\mathcal{B}_m^- : p_1^*TQ \to {\mathcal{A}_d}^*(TG)$ are

$$\begin{aligned}\mathcal{B}_m^+(q_0, q_1)(\delta q_1) :=& \mathcal{B}_m(q_0, q_1)(0, \delta q_1) = D_2\mathcal{A}_d(q_0, q_1)(Hor_{\mathcal{A}}(\delta q_1))\\ \mathcal{B}_m^-(q_0, q_1)(\delta q_0) :=& \mathcal{B}_m(q_0, q_1)(\delta q_0, 0) = D_1\mathcal{A}_d(q_0, q_1)(Hor_{\mathcal{A}}(\delta q_0)).\end{aligned}$$

Remark 5.4. An obvious difference between the mixed curvature and the curvature of a connection is that the former can be represented as a 1-form, while the latter is represented as a 2-form.

Remark 5.5. The mixed curvature measures how $\mathcal{A}$-horizontal deformations at (q_0, q_1) depart from the level manifolds of $\mathcal{A}_d$.

Example 5.6. We continue in the context of Example 5.1. In order to compute the mixed curvature (and continue analyzing the reduction of the system) we introduce a connection on the principal bundle $\pi : Q \to Q/G$. For that purpose, the splitting (2) is

$$T_qQ = \underbrace{\{0\}}_{=\mathcal{W}_q} \oplus \underbrace{\langle \partial_y|_q \rangle}_{=\mathcal{U}_q} \oplus \underbrace{\{0\}}_{=\mathcal{S}_q} \oplus \underbrace{\langle \partial_x|_q + x\partial_y|_q \rangle}_{=\mathcal{H}_q}.$$

The connection $\mathcal{A}$ with horizontal space $Hor_{\mathcal{A}} = \mathcal{W} \oplus \mathcal{H}$ has connection 1-form and horizontal lift given by

$$\mathcal{A}(\dot{x}\partial_x|_q + \dot{y}\partial_y|_q) = \dot{y} - \dot{x}x \quad \text{ and } \quad h^q(\partial_r|_r) = \partial_x|_q + r\partial_y|_q.$$

We can now compute the mixed curvature $\mathcal{B}_m$ associated to $\mathcal{A}$ and $\mathcal{A}_d^b$:

$$\begin{aligned}
\mathcal{B}_m(q_0,q_1)(\delta q_0, \delta q_1) =& d\mathcal{A}_d^b(q_0,q_1)(Hor_{\mathcal{A}}(\delta q_0), Hor_{\mathcal{A}}(\delta q_1)) \\
=& \partial_{w_0}|_{\mathcal{A}_d^b(q_0,q_1)} \otimes \big(dy_1|_{q_1} - dy_0|_{q_0} - bx_1 dx_1|_{q_1} + bx_0 dx_0|_{q_0}\big) \\
& \big(c_0(\partial_{x_0}|_{q_0} + x_0\partial_{y_0}|_{q_0}) + c_1(\partial_{x_1}|_{q_1} + x_1\partial_{y_1}|_{q_1})\big) \\
=& (1-b)(c_1x_1 - c_0x_0)\partial_{w_0}|_{\mathcal{A}_d^b(q_0,q_1)}.
\end{aligned}$$

Notice that depending on the "relative slopes" 1 and b of $\mathcal{A}$ and $\mathcal{A}_d^b$, the mixed curvature vanishes or not.

Just as regular curvature, the mixed curvature can be seen as a "reduced object" defined on $(Q\times Q)/G$ or, using the isomorphism $\Phi_{\mathcal{A}_d}$, on $\tilde{G} \times (Q/G)$. We study this object next.

Since $\mathcal{A}_d : Q \times Q \to G$ is G-equivariant for $l^{Q\times Q}$ and l^G, considering the lifted actions on $T(Q\times Q)$ and TG, $d\mathcal{A}_d$ induces a morphism $\widetilde{d\mathcal{A}_d}$ on the quotient vector bundles:

$$\begin{array}{ccc}
T(Q\times Q)/G & \xrightarrow{\widetilde{d\mathcal{A}_d}} & TG/G \\
\downarrow & & \downarrow \\
(Q\times Q)/G & \xrightarrow{\tilde{\mathcal{A}}_d} & G/G
\end{array}$$

Let $\sigma : \tilde{G}\times(Q/G) \to (Q/G)\times(Q/G)$ be $\sigma(v_0,r_1) := (p^{Q/G}(v_0), r_1)$; then, we have the vector bundles $\sigma^*T(Q/G\times Q/G)$ over $\tilde{G}\times(Q/G)$ and $(\rho\times id)^*\sigma^*T(Q/G\times Q/G)$ over $Q\times G\times(Q/G)$. We define $\check{\chi} : (\rho\times id)^*\sigma^*T(Q/G\times Q/G) \to T(Q\times Q)$ by $\check{\chi}(q_0,w_0,r_1)(\delta r_0, \delta r_1) := (h^{q_0}(\delta r_0), h^{q_1}(\delta r_1))$ for $q_1 := \tilde{F}_1(q_0,w_0,r_1)$. As $\tilde{F}_1$ and the horizontal lift are G-equivariant, $\check{\chi}$ descends to a morphism of vector bundles χ:

$$\begin{array}{ccc}
\sigma^*T(Q/G\times Q/G) & \xrightarrow{\chi} & T(Q\times Q)/G \\
\downarrow & & \downarrow \\
\tilde{G}\times(Q/G) & \xrightarrow{\Psi_{\mathcal{A}_d}} & (Q\times Q)/G
\end{array}$$

It will be convenient to consider the composition $\widetilde{d\mathcal{A}_d}\circ\chi$ as a morphism of the vector bundles $\sigma^*T(Q/G\times Q/G) \to (\widetilde{\mathcal{A}_d}\circ\Psi_{\mathcal{A}_d})^*(TG/G)$ over the same base space $\tilde{G}\times(Q/G)$. Even better, since $\widetilde{\mathcal{A}_d}\circ\Psi_{\mathcal{A}_d} = p^{G/G}\circ p_1$ we have the following notion.

Definition 5.7. The *reduced mixed curvature* $\hat{\mathcal{B}}_m$ is the morphism of bundles over $\tilde{G}\times(Q/G)$ $\hat{\mathcal{B}}_m := \widetilde{d\mathcal{A}_d}\circ\chi : \sigma^*T(Q/G\times Q/G)\to (p^{G/G}\circ p_1)^*(TG/G)$. Explicitly,

$$\begin{aligned}\hat{\mathcal{B}}_m(v_0,r_1)(\delta r_0,\delta r_1) :=&[\mathcal{B}_m(q_0,q_1)(h^{q_0}(\delta r_0),h^{q_1}(\delta r_1))]\\ =&[d\mathcal{A}_d(q_0,q_1)(h^{q_0}(\delta r_0),h^{q_1}(\delta r_1))],\end{aligned}$$

where $v_0=\rho(q_0,w_0)$, $q_1:=\tilde{F}_1(q_0,w_0,r_1)$ and $[\cdot]$ denotes the equivalence class in TG/G. Associated to the decomposition $\mathcal{B}_m=\mathcal{B}_m^+ + \mathcal{B}_m^-$ there is a decomposition $\hat{\mathcal{B}}_m=\hat{\mathcal{B}}_m^+ + \hat{\mathcal{B}}_m^-$ where $\hat{\mathcal{B}}_m^+ : p_2^*T(Q/G)\to(p^{G/G}\circ p_1)^*(TG/G)$ and $\hat{\mathcal{B}}_m^- : (p^{Q/G}\circ p_1)^*T(Q/G)\to(p^{G/G}\circ p_1)^*(TG/G)$ are defined by

$$\begin{aligned}\hat{\mathcal{B}}_m^+(v_0,r_1)(\delta r_1) :=&[\mathcal{B}_m^+(q_0,q_1)(h^{q_1}(\delta r_1))] = [D_2\mathcal{A}_d(q_0,q_1)(h^{q_1}(\delta r_1))]\\ \hat{\mathcal{B}}_m^-(v_0,r_1)(\delta r_0) :=&[\mathcal{B}_m^-(q_0,q_1)(h^{q_0}(\delta r_0))] = [D_1\mathcal{A}_d(q_0,q_1)(h^{q_0}(\delta r_0))].\end{aligned}$$

Now we return to the discrete lagrangians introduced in section 5.1. In particular, we associate a new notion to the second term in the last equality of (11).

By Lemma 5.2, $D_2\check{L}_d : p_2^*TG\to\mathbb{R}$ is a G-equivariant morphism of vector bundles over $Q\times G\times(Q/G)$. Thus, it induces a morphism of the corresponding quotient vector bundles $\widetilde{D_2\check{L}_d}$:

$$\begin{array}{ccc}(p_2^*TG)/G & \xrightarrow{\widetilde{D_2\check{L}_d}} & \mathbb{R}\\ \downarrow & \swarrow & \\ \tilde{G}\times(Q/G) & & \end{array}$$

Notice that $(p_2^*TG)/G\simeq(p^{G/G}\circ p_1)^*TG/G$. Then, $\widetilde{D_2\check{L}_d}\circ\hat{\mathcal{B}}_m$ is a well defined morphism of vector bundles.

Definition 5.8. The *reduced discrete force* is the morphism of vector bundles over $\tilde{G}\times(Q/G)$ defined by $\hat{F}_d := \widetilde{D_2\check{L}_d}\circ\hat{\mathcal{B}}_m : \sigma^*(T(Q/G\times Q/G))\to\mathbb{R}$ (essentially a 1-form over $\tilde{G}\times(Q/G)$). Concretely,

$$\hat{F}_d(v_0,r_1)(\delta r_0,\delta r_1) := D_2\check{L}_d(q_0,w_0,r_1)d\mathcal{A}_d(q_0,q_1)(h^{q_0}(\delta r_0),h^{q_1}(\delta r_1)) \qquad (12)$$

where $v_0=\rho(q_0,w_0)$ and $q_1:=\tilde{F}_1(q_0,w_0,r_1)$. Once again, using $T(Q/G\times Q/G)\simeq p_1^*T(Q/G)\oplus p_2^*T(Q/G)$, we define $\hat{F}_d^+ := \widetilde{D_2\check{L}_d}\circ\hat{\mathcal{B}}_m^+$ and $\hat{F}_d^- := \widetilde{D_2\check{L}_d}\circ\hat{\mathcal{B}}_m^-$, so that $\hat{F}_d=\hat{F}_d^+ + \hat{F}_d^-$. Explicitly,

$$\begin{aligned}\hat{F}_d^+(v_0,r_1)(\delta r_1) :=& D_2\check{L}_d(q_0,w_0,r_1)D_2\mathcal{A}_d(q_0,q_1)(h^{q_1}(\delta r_1))\\ \hat{F}_d^-(v_0,r_1)(\delta r_0) :=& D_2\check{L}_d(q_0,w_0,r_1)D_1\mathcal{A}_d(q_0,q_1)(h^{q_0}(\delta r_0)).\end{aligned} \qquad (13)$$

Example 5.9. Continuing the analysis of the discrete mechanical system introduced in Example 5.1, we compute its reduced discrete force with respect to $\mathcal{A}$ and $\mathcal{A}_d^b$. If $(v_0,r_1)=(r_0,w_0,r_1)\in(Q/G)\times G\times(Q/G)$ we can take $q_0:=(r_0,0)$ and $q_1:=l_{w_0}^Q(\overline{h_d^{q_0}}(r_1))=(r_1,w_0+b(r_1^2-r_0^2)/2)$ in the computation. Since

$$D_2\check{L}_d((r_0,0),w_0,r_1)=m\big(w_0+b(r_1^2-r_0^2)/2\big)dw_0\big|_{w_0},$$

using the results of Example 5.6, we have

$$\begin{aligned}\hat{F}_d(r_0,w_0,r_1)(c_0\partial_{r_0}\big|_{r_0},c_1\partial_{r_1}\big|_{r_1}) = &\Big(m\big(w_0+b(r_1^2-r_0^2)/2\big)dw_0\big|_{w_0}\Big)\\ &\big((1-b)(c_1r_1-c_0r_0)\partial_{w_0}\big|_{w_0}\big)\\ =& m(1-b)(c_1r_1-c_0r_0)\big(w_0+b(r_1^2-r_0^2)/2\big).\end{aligned}$$

Equivalently,

$$\hat{F}_d(r_0,w_0,r_1) = m(1-b)\big(w_0+b(r_1^2-r_0^2)/2\big)(r_1dr_1-r_0dr_0).$$

5.3. **Reduced dynamics.** In this section we relate the dynamics of a symmetric discrete mechanical system to the dynamics of a "reduced" system, defined using a variational principle.

Remark 5.10. The analysis of the symmetric system leads us to consider some objects defined on the symmetry group G, that is a Lie group. We review very briefly a notation that is convenient and used in this subject. When $w_0,w_1\in G$ and $\delta w_0\in T_{w_0}G$ we define

$$w_1\delta w_0 := dL_{w_1}(w_0)(\delta w_0)\in T_{w_1w_0}G \quad\text{and}\quad \delta w_0 w_1 := dR_{w_1}(w_0)(\delta w_0)\in T_{w_0w_1}G,$$

where L_{w_1} and R_{w_1} denote the multiplication by w_1 on the left and right, respectively. Analogously, when $\alpha_0\in T^*_{w_0}G$,

$$\begin{aligned} w_1\alpha_0 :=&(dL_{w_1^{-1}}(w_1w_0))^*(\alpha_0)\in T^*_{w_1w_0}G \quad\text{and}\\ \alpha_0 w_1 :=&(dR_{w_1^{-1}}(w_0w_1))^*(\alpha_0)\in T^*_{w_0w_1}G.\end{aligned}$$

Last, we notice that if $\alpha_0\in T^*_{w_0}G$ and $\delta w_0\in T_{w_0}G$, the following identities hold

$$\alpha_0(\delta w_0) = (w_1\alpha_0)(w_1\delta w_0) = (\alpha_0 w_1)(\delta w_0 w_1).$$

Theorem 5.11. *Let $q_\cdot$ be a discrete curve in Q, $r_k := \pi(q_k)$, $w_k := \mathcal{A}_d(q_k,q_{k+1})$ and $v_k := \rho(q_k,w_k)$ be the corresponding discrete curves in Q/G, G and $\tilde{G}$. Then, given a discrete mechanical system $(Q,L_d,\mathcal{D},\mathcal{D}_d)$ with symmetry group G, the following statements are equivalent.*

1. *$(q_k,q_{k+1})\in\mathcal{D}_d$ for all k and $q_\cdot$ satisfies the variational principle $dS_d(q_\cdot)(\delta q_\cdot) = 0$ for all vanishing end points variations $\delta q_\cdot$ such that $\delta q_k\in\mathcal{D}_{q_k}$ for all k.*
2. *$q_\cdot$ satisfies the Lagrange–D'Alembert equations (3) for all k.*
3. *$(v_k,r_{k+1})\in\hat{\mathcal{D}}_d := \Phi_{\mathcal{A}_d}(\mathcal{D}_d/G)$ for all k and $d\hat{S}_d(r_\cdot,v_\cdot)(\delta r_\cdot,\delta v_\cdot) = 0$ for all variations $(\delta v_\cdot,\delta r_\cdot)$ with vanishing end points such that $\delta r_k\in\hat{\mathcal{D}}_{r_k} := d\pi(q_k)(\mathcal{D}_{q_k})$ and*

$$\begin{aligned}\delta v_k :=& d\rho(q_k,w_k)\big(h^{q_k}(\delta r_k), d\mathcal{A}_d(q_k,q_{k+1})(h^{q_k}(\delta r_k),h^{q_{k+1}}(\delta r_{k+1}))\big)\\ &+d\rho(q_k,w_k)\big((\xi_k)_Q(q_k),d\mathcal{A}_d(q_k,q_{k+1})((\xi_k)_Q(q_k),(\xi_{k+1})_Q(q_{q+1}))\big),\end{aligned}\tag{14}$$

   where $(q_k,\xi_k)\in\mathfrak{g}^{\mathcal{D}}$.
4. *$(v_k,r_{k+1})\in\hat{\mathcal{D}}_d$ for all k and $(v_\cdot,r_\cdot)$ satisfies the following conditions for each fixed $(v_{k-1},r_k,v_k,r_{k+1})$.*
   - *$\phi\in T^*_{r_k}(Q/G)$ defined by*

$$\begin{aligned}\phi :=& D_1\check{L}_d(q_k,w_k,r_{k+1})\circ h^{q_k} + D_3\check{L}_d(q_{k-1},w_{k-1},r_k)\\ &+\hat{F}_d^-(v_k,r_{k+1})+\hat{F}_d^+(v_{k-1},r_k)\end{aligned}\tag{15}$$

     vanishes on $\hat{\mathcal{D}}_{r_k}$, i.e.,

$$\phi\in\hat{\mathcal{D}}^\circ_{r_k}.\tag{16}$$

- $\psi \in \mathfrak{g}^*$ *defined by*

$$\psi := D_2\check{L}_d(q_{k-1}, w_{k-1}, r_k)w_{k-1}^{-1} - D_2\check{L}_d(q_k, w_k, r_{k+1})w_k^{-1} \tag{17}$$

vanishes on $\mathfrak{g}^{\mathcal{D}}_{q_k}$, i.e.,

$$\psi \in (\mathfrak{g}^{\mathcal{D}}_{q_k})^\circ. \tag{18}$$

Proof. First we study the equivalence of conditions $(q_k, q_{k+1}) \in \mathcal{D}_d$ and $(v_k, r_{k+1}) \in \hat{\mathcal{D}}_d$. If $(q_k, q_{k+1}) \in \mathcal{D}_d$, it follows that $\tilde{\pi}(q_k, q_{k+1}) \in \mathcal{D}_d/G$ and $(v_k, r_{k+1}) = \Phi_{\mathcal{A}_d}(\tilde{\pi}(q_k, q_{k+1})) \in \Phi_{\mathcal{A}_d}(\mathcal{D}_d/G)$. Conversely, if $(v_k, r_{k+1}) = \Phi_{\mathcal{A}_d}(\tilde{\pi}(q_k, q_{k+1})) \in \hat{\mathcal{D}}_d$, we have that $\tilde{\pi}(q_k, q_{k+1}) \in \mathcal{D}_d/G$, so that $(q_k, q_{k+1}) \in \mathcal{D}_d$.

Next we tackle the equivalence between the variational principles and equations.

1 $\Leftrightarrow$ 2. Is a standard computation using calculus of variations.

3 $\Rightarrow$ 1. Given a variation $\delta q.$ as in the statement, we let $\delta r_k := d\pi(q_k)(\delta q_k)$ and δv_k using (14) for $\xi_k := \mathcal{A}(q_k)(\delta q_k)$, so that $\delta q_k = h^{q_k}(\delta r_k) + (\xi_k)_Q(q_k)$. Using Lemma 4.21 to compute $d\Upsilon$,

$$\begin{aligned} dS_d(q.)(\delta q.) &= \sum_{k=0}^{N-1} dL_d(q_k, q_{k+1})(\delta q_k, \delta q_{k+1}) \\ &= \sum_{k=0}^{N-1} d\hat{L}_d(\Upsilon(q_k, q_{k+1}))d\Upsilon(\delta q_k, \delta q_{k+1}) \\ &= \sum_{k=0}^{N-1} d\hat{L}_d(v_k, r_{k+1})(\delta v_k, \delta r_{k+1}) = d\hat{S}_d(r., v.)(\delta r., \delta v.) = 0, \end{aligned} \tag{19}$$

where the last equality holds because $(r., v.)$ satisfies condition 3. Hence, $q.$ satisfies condition 1.

1 $\Rightarrow$ 3. Given a variation $(\delta r., \delta v.)$ as in the statement, let $\delta q_k := h^{q_k}(\delta r_k) + (\xi_k)_Q(q_k)$. Then, using Lemma 4.21 and the explicit form of (14), we have that $(\delta v_k, \delta r_{k+1}) = d\Upsilon(q_k, q_{k+1})(\delta q_k, \delta q_{k+1})$. A computation similar to (19) shows that

$$d\hat{S}_d(v., r.)(\delta v., \delta r.) = dS_d(q.)(\delta q.) = 0.$$

since the variation $\delta q.$ satisfies condition 1. Hence, condition 3 holds.

3 $\Leftrightarrow$ 4. For variations $(\delta v., \delta r.)$, writing $\delta v_k = d\rho(q_k, w_k)(\delta q_k, \delta w_k)$,

$$\begin{aligned} d\hat{S}_d(v., r.)(\delta v., \delta r.) &= \sum_{k=0}^{N-1} d\hat{L}_d(v_k, r_{k+1})(\delta v_k, \delta r_{k+1}) \\ &= \sum_{k=0}^{N-1} d\check{L}_d(q_k, w_k, r_{k+1})(\delta q_k, \delta w_k, \delta r_{k+1}) \\ &= \sum_{k=0}^{N-1} \big(D_1\check{L}_d(q_k, w_k, r_{k+1})(\delta q_k) + D_2\check{L}_d(q_k, w_k, r_{k+1})(\delta w_k) \\ &\quad + D_3\check{L}_d(q_k, w_k, r_{k+1})(\delta r_{k+1})\big). \end{aligned}$$

Using $\mathcal{A}$ to decompose $\delta q_k = h^{q_k}(\delta r_k) + (\xi_k)_Q(q_k)$ and rearranging indexes, for vanishing end point variations, we obtain

$$\begin{aligned} d\hat{S}_d(v.,r.)(\delta v.,\delta r.) = \sum_{k=0}^{N-1} \Big(& D_1\check{L}_d(q_k,w_k,r_{k+1}) \circ h^{q_k} + D_3\check{L}_d(q_{k-1},w_{k-1},r_k) \\ & + D_2\check{L}_d(q_k,w_k,r_{k+1}) D_1\mathcal{A}_d(q_k,q_{k+1}) \circ h^{q_k} \\ & + D_2\check{L}_d(q_{k-1},w_{k-1},r_k) D_2\mathcal{A}_d(q_{k-1},q_k) \circ h^{q_k}\Big)(\delta r_k) \\ & + \sum_{k=0}^{N-1} \big(D_1\check{L}_d(q_k,w_k,r_{k+1})((\xi_k)_Q(q_k)) \\ & \quad + D_2\check{L}_d(q_k,w_k,r_{k+1}) d\mathcal{A}_d(q_k,q_{k+1})((\xi_k)_Q(q_k),(\xi_{k+1})_Q(q_{k+1}))\big). \end{aligned} \tag{20}$$

Since the variations $\delta r.$ are independent from those generated by the $\xi.$, we conclude that condition $d\hat{S}_d(r.,v.)(\delta r.,\delta v.) = 0$ from point 3 in the statement is equivalent to the vanishing of the first and second summations in (20) independently, for all vanishing end points variations $\delta r.$ with $\delta r_k \in \hat{\mathcal{D}}_{r_k}$ for all k and for all $\xi.$ with $(q_k,\xi_k) \in \mathfrak{g}^{\mathcal{D}}$ for all k and $\xi_0 = \xi_n = 0$.

Recalling that the variations $\delta r.$ are independent and (13), it is clear that the vanishing of the first summation in (20) is equivalent to the fact that ϕ, defined in (15), satisfies condition (16).

Before we prove the equivalence of condition (18) and the vanishing of the second summation in (20) we need two auxiliary computations. On the one hand,

$$\begin{aligned} d\mathcal{A}_d(q_k,q_{k+1})((\xi_k)_Q(q_k),&(\xi_{k+1})_Q(q_{k+1})) \\ &= \frac{d}{dt}\Big|_{t=0} \big(\exp(t\xi_{k+1})\mathcal{A}_d(q_k,q_{k+1})\exp(-t\xi_k)\big) \\ &= \xi_{k+1}w_k - w_k\xi_k, \end{aligned}$$

where the notation used in the last equality is the one introduced in Remark 5.10. On the other hand, since $\check{L}_d(l_g^Q(q_k),w_k,r_{k+1}) = \check{L}_d(q_k,l_{g^{-1}}^G(w_k),r_{k+1})$,

$$\begin{aligned} D_1\check{L}_d(q_k,w_k,r_{k+1})((\xi_k)_Q(q_k)) &= \frac{d}{dt}\Big|_{t=0} \check{L}_d(l^Q_{\exp(t\xi_k)}(q_k),w_k,r_{k+1}) \\ &= D_2\check{L}_d(q_k,w_k,r_{k+1})\frac{d}{dt}\Big|_{t=0}(\exp(-t\xi_k)w_k\exp(t\xi_k)) \\ &= D_2\check{L}_d(q_k,w_k,r_{k+1})(-\xi_k w_k + w_k\xi_k). \end{aligned}$$

Using the previous computations we see that the vanishing of the second summation in (20) is equivalent to

$$\begin{aligned} 0 &= \sum_{k=1}^{N-1} \big(D_2\check{L}_d(q_k,w_k,r_{k+1})(\xi_{k+1}w_k - \xi_k w_k)\big) \\ &= \sum_{k=1}^{N-1} \big(D_2\check{L}_d(q_{k-1},w_{k-1},r_k)w_{k-1}^{-1} - D_2\check{L}_d(q_k,w_k,r_{k+1})w_k^{-1}\big)(\xi_k) \end{aligned}$$

for all $\xi.$ with $(q_k,\xi_k) \in \mathfrak{g}^{\mathcal{D}}$. It follows immediately that this last condition is equivalent to the fact that ψ, defined in (17), meets condition (18). ☐

From the proof of Theorem 5.11 we isolate the following partial result.

Lemma 5.12. *With the notation of Theorem 5.11, the following assertions are equivalent.*

1. *The discrete curve $q.$ satisfies $dS_d(q.)(\delta q.^{\mathcal{S}}) = 0$ for all vanishing end points variations such that $\delta q_k \in \mathcal{S}_{q_k}$ for all k.*
2. *The curve $(v., r.)$ satisfies condition (18) for ψ defined by (17).*

Remark 5.13. We notice that the horizontal equations that appear in item 4 of Theorem 5.11 contain force terms that, because of (12) are a composition of derivatives of the reduced lagrangian and a term involving $\mathcal{A}$ and $\mathcal{A}_d$. Since in the continuous setting, the forces that appear in the analogous equation involve the reduced curvature of the nonholonomic connection, we chose to call the corresponding term in the discrete setting the reduced mixed curvature. In this case, we recover the standard result that the vanishing of the mixed curvature implies the vanishing of the discrete forces in the reduced system. Other than that, we do not have a good reason to call $\mathcal{B}_m$ or $\hat{\mathcal{B}}_m$ "curvatures".

Remark 5.14. For classical mechanical systems, the Lagrange–D'Alembert equations (1) are second order differential equations, while their reduced counterparts that appear in Theorem 2.7 are second order in the Q/G variables but only first order in the $\tilde{\mathfrak{g}}$ ones. Analogously, for discrete mechanical systems, the discrete Lagrange–D'Alembert equations (3) are recurrence equations of second order while the reduced equations obtained in item 4 of Theorem 5.11 are second order in the Q/G variables but only first order in the $\tilde{G}$ ones.

Remark 5.15. Theorem 5.11 is similar in spirit to the reduction theorems for classical mechanical systems, like Theorem 2.7. Still there is a noticeable technical difference between both types of results. In the continuous case, even in the generalized context, the choice of a connection on the principal bundle π serves the dual purpose of constructing a model for the reduced space via the diffeomorphism $\alpha_{\mathcal{A}}$ and determines a horizontal / vertical splitting of the variational principle and equations of motion. In the discrete context a continuous connection serves the same splitting purpose but a discrete connection $\mathcal{A}_d$ is used to construct the model reduced space. It would be interesting to see if there is any advantage in using two different connections in the reduction of continuous mechanical systems.

Example 5.16. Continuing the analysis of the discrete mechanical system introduced in Example 5.1, we apply Theorem 5.11 to find the equations of motion of the reduced system. The reduced variational constraint is

$$\begin{aligned}\hat{\mathcal{D}}_{r_k} =& d\pi((r_k,0))(\mathcal{D}_{(r_k,0)}) = d\pi((r_k,0))(\langle \partial_x\big|_{(r_k,0)} - r_k\partial_y\big|_{(r_k,0)}\rangle)\\ =& \langle \partial r\big|_{r_k}\rangle = T_{r_k}(Q/G),\end{aligned}$$

and, since

$$(r_0,w_0,r_1)\in\hat{\mathcal{D}}_d \Leftrightarrow \tilde{\Phi}^{-1}_{\mathcal{A}_d^b}((r_0,0),w_0,r_1)\in\mathcal{D}_d \Leftrightarrow w_0=(1-b)(r_1^2-r_0^2)/2,$$

the reduced kinematic constraints are

$$\hat{\mathcal{D}}_d = \{(r_0,w_0,r_1)\in(Q/G)\times G\times(Q/G) : w_0=(1-b)(r_1^2-r_0^2)/2\}.$$

Next we compute the reduced equations of motion. Notice that $\mathcal{S}=\{0\}$ for this system, so that $\mathfrak{g}^{\mathcal{D}}=\{0\}$ and condition (18) is trivially satisfied, that is, there are no vertical equations.

In order to find the horizontal equations we compute

$$D_1\check{L}_d((r_k,h_k),w_k,r_{k+1}) = -m\big((r_{k+1}-r_k)+(w_k+b(r_{k+1}^2-r_k^2)/2)br_k\big)dx_k\big|_{(r_k,h_k)}$$
$$D_3\check{L}_d((r_{k-1},h_{k-1}),w_{k-1},r_k) = m\big((r_k-r_{k-1})+(w_{k-1}+b(r_k^2-r_{k-1}^2)/2)br_k\big)dr_k\big|_{r_k},$$

recall from Example 5.6 that

$$h^{(r_k,h_k)} = \left(\partial_x\big|_{(r_k,h_k)} + r_k\partial_y\big|_{(r_k,h_k)}\right) \otimes dr_k\big|_{r_k},$$

and, from Example 5.9 the expression of the reduced forces.

$$\begin{aligned}\hat{F}_d^-(r_k,w_k,r_{k+1}) &= -m(1-b)\big(w_k + b(r_{k+1}^2 - r_k^2)/2\big)r_k dr_k\big|_{r_k}\\ \hat{F}_d^+(r_{k-1},w_{k-1},r_k) &= m(1-b)\big(w_{k-1} + b(r_k^2 - r_{k-1}^2)/2\big)r_k dr_k\big|_{r_k}.\end{aligned}$$

Then, $\phi = mU_k dr_k\big|_{r_k}$ for

$$\begin{aligned}U_k := &-\big((r_{k+1}-r_k) - (r_k - r_{k-1})\big)\\ &- r_k\big(w_k - w_{k-1} + b((r_{k+1}^2 - r_k^2) - (r_k^2 - r_{k-1}^2))/2\big),\end{aligned}$$

and, since $\hat{\mathcal{D}} = T(Q/G)$, condition (16) says that $U_k = 0$, which is the horizontal equation of motion for the reduced system. Thus, the reduced evolution is determined by the system of equations

$$U_k = 0 \quad \text{and} \quad w_k = (1-b)(r_{k+1}^2 - r_k^2)/2. \tag{21}$$

Using the second equation to eliminate the w dependence in the first expression we obtain

$$0 = \big((r_{k+1}-r_k) - (r_k - r_{k-1})\big) + r_k((r_{k+1}^2 - r_k^2) - (r_k^2 - r_{k-1}^2))/2. \tag{22}$$

From this equation of degree two in r_{k+1} the evolution of this variable is obtained and, using the second equation of (21) the dynamics of w_k is determined.

Remark 5.17. Notice that, depending on the value of the parameter b, the reduced system constructed in Example 5.16 is forced or not. Besides making the reduced system unforced, the value $b = 1$ also gives a very simple dynamics to the w. variables. This will be a characteristic that we explore in more detail in Section 10 when we consider systems of Chaplygin type.

Back in the general setting, conditions (16) and (18) in Theorem 5.11 establish the equations of motion of the reduced system. However, they are explicitly written in terms of $q_\cdot$. We will see in Section 6 that, in fact, they can be defined in an intrinsic manner, in terms of objects defined on the reduced space.

6. **Intrinsic version of the reduced equations of motion.** In this section we write the horizontal and vertical equations in terms of the reduced system. In fact, the equations of motion will be given as conditions on morphisms of vector bundles on the second order reduced manifold $Q_G^{(2)} := \tilde{G} \times (Q/G) \times_{Q/G} \tilde{G} \times (Q/G)$, where the fibered product is taken over the maps $p_2 : \tilde{G} \times (Q/G) \to Q/G$ and $p^{Q/G} \circ p_1 : \tilde{G} \times (Q/G) \to Q/G$.

It is convenient to consider the space $\check{Q}_G^{(2)} := Q \times G \times (Q/G) \times_{Q/G} \tilde{G} \times (Q/G)$ with the G-action $l_g^{\check{Q}_G^{(2)}}(q_0,w_0,r_1,v_1,r_2) := (l_g^Q(q_0), l_g^G(w_0), r_1, v_1, r_2)$, so that $\check{Q}_G^{(2)}/G = Q_G^{(2)}$. Additionally, we define the maps

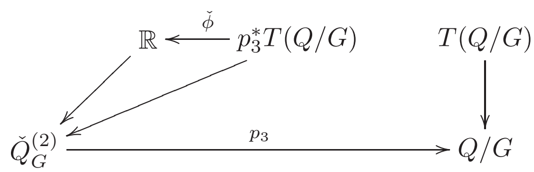
$$F_1 : \check{Q}_G^{(2)} \to Q \quad \text{by} \quad F_1(q_0,w_0,r_1,v_1,r_2) := \tilde{F}_1(q_0,w_0,r_1),$$

and $F_2 : \check{Q}_G^{(2)} \to G$ by

$$F_2(q_0,w_0,r_1,v_1,r_2) := l^G_{\tau(\tilde{F}_1(q_0,w_0,r_1),\tilde{q}_1)}(\tilde{w}_1) \quad \text{if} \quad v_1 = \rho(\tilde{q}_1,\tilde{w}_1),$$

where $\tau : Q \times_{Q/G} Q \to G$ is defined by $\tau(q,q') = g$ if $l_g^Q(q') = q$. It is easy to check that F_1 and F_2 are G-equivariant.

6.1. **Horizontal equations.** In order to give an intrinsic meaning to (16), we consider the following commutative diagram,



(23)

where $\check{\phi}$ is defined by

$$\begin{aligned}\check{\phi}(q_0,w_0,r_1,v_1,r_2,\delta r_1) :=& \big(D_1\check{L}_d(q_1,w_1,r_2)\circ h^{q_1} + D_3\check{L}_d(q_0,w_0,r_1)\\ &+ \hat{F}_d^-(v_1,r_2) + \hat{F}_d^+(\rho(q_0,w_0),r_1)\big)(\delta r_1)\end{aligned}$$

for $q_1 := \tilde{F}_1(q_0,w_0,r_1)$ and $w_1 := F_2(q_0,w_0,r_1,v_1,r_2)$. Also, G acts on $p_3^*T(Q/G)$ by $l_g^{p_3^*T(Q/G)}(q_0,w_0,r_1,v_1,r_2,\delta r_1) := (l_g^Q(q_0), l_g^G(w_0), r_1, v_1, r_2, \delta r_1)$ and trivially on $\mathbb{R}$.

Lemma 6.1. *The map $\check{\phi}$ is a G-equivariant morphism of vector bundles.*

Proof. From the explicit definition, $\check{\phi}$ is a morphism of vector bundles.

In order to check the G-equivariance, for $(q_0,w_0,r_1,v_1,r_2) \in \check{Q}_G^{(2)}$, we let $q_1 := \tilde{F}_1(q_0,w_0,r_1)$ and $w_1 := F_2(q_0,w_0,r_1,v_1,r_2)$. Then, writing

$$\begin{aligned}\check{\phi}_1(q_0,w_0,r_1,v_1,r_2,\delta r_1) &:= \big(D_1\check{L}_d(q_1,w_1,r_2)\circ h^{q_1} + D_3\check{L}_d(q_0,w_0,r_1)\big)(\delta r_1)\\ \check{\phi}_2(q_0,w_0,r_1,v_1,r_2,\delta r_1) &:= \big(\hat{F}_d^-(v_1,r_2) + \hat{F}_d^+(\rho(q_0,w_0),r_1)\big)(\delta r_1),\end{aligned}$$

we have $\check{\phi} = \check{\phi}_1 + \check{\phi}_2$. From the definition, it is immediate that $\check{\phi}_2$ is G-equivariant, so we concentrate on $\check{\phi}_1$.

$$\begin{aligned}\check{\phi}_1(l_g^{p_3^*T(Q/G)}&(q_0,w_0,r_1,v_1,r_2,\delta r_1)) = \check{\phi}_1(l_g^Q(q_0), l_g^G(w_0), r_1, v_1, r_2, \delta r_1)\\ &= D_1\check{L}_d(l_g^Q(q_1), l_g^G(w_1), r_2)\circ h^{l_g^Q(q_1)}(\delta r_1) + D_3\check{L}_d(l_g^Q(q_0), l_g^G(w_0), r_1)(\delta r_1),\end{aligned} \tag{24}$$

where we used the G-equivariance of $\tilde{F}_1$ and F_2 in the first term of the last identity. By Lemma 5.2, we have $D_1\check{L}_d(l_g^Q(q_1), l_g^G(w_1), r_2) l_g^{TQ} = D_1\check{L}_d(q_1,w_1,r_2)$, so that

$$\begin{aligned}D_1\check{L}_d(l_g^Q(q_1), l_g^G(w_1), r_2)\circ h^{l_g^Q(q_1)} &= D_1\check{L}_d(l_g^Q(q_1), l_g^G(w_1), r_2)\circ l_g^{TQ}\circ h^{q_1}\\ &= D_1\check{L}_d(q_1,w_1,r_2)\circ h^{q_1}.\end{aligned}$$

Also, from $\check{L}_d(l_g^Q(q_0), l_g^G(w_0), r_1) = \check{L}_d(q_0,w_0,r_1)$, we obtain

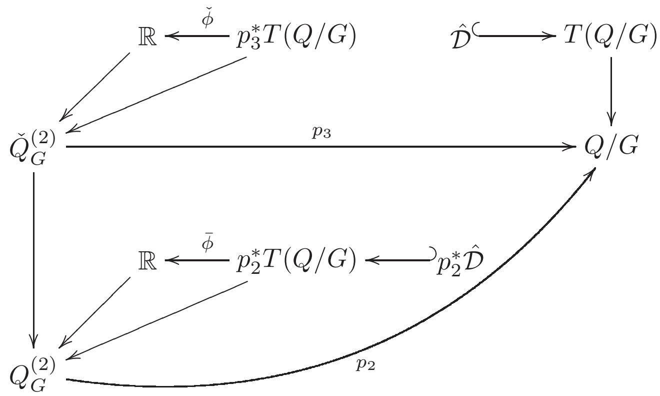
$$D_3\check{L}_d(l_g^Q(q_0), l_g^G(w_0), r_1) = D_3\check{L}_d(q_0,w_0,r_1).$$

Replacing the last two identities back in (24), the G-equivariance of $\check{\phi}_1$ follows. □

By Lemma 6.1, $\check{\phi}$ defines a morphism of vector bundles $\bar{\phi} : (p_3^*T(Q/G))/G \to \mathbb{R}$. Since $(p_3^*T(Q/G))/G = p_2^*T(Q/G)$, where $p_2 : Q_G^{(2)} \to Q/G$ is the projection, we have $\bar{\phi} : p_2^*T(Q/G) \to \mathbb{R}$. Concretely, if $(q_0,w_0) \in \rho^{-1}(v_0)$,

$$\bar{\phi}(v_0,r_1,v_1,r_2,\delta r_1) = \check{\phi}(q_0,w_0,r_1,v_1,r_2,\delta r_1).$$

We extend diagram (23) to the following commutative diagram, where $\hat{\mathcal{D}} \subset T(Q/G)$ is the subbundle introduced in Theorem 5.11.

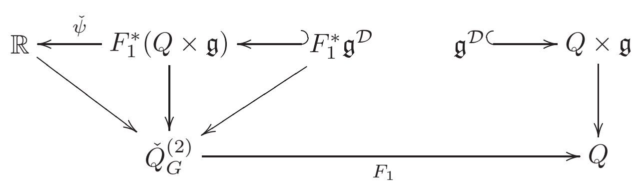


Proposition 6.2. *Condition* (16) *in Theorem 5.11 is equivalent to*

$$\bar{\phi}\mid_{p_2^*(\hat{\mathcal{D}})} = 0. \tag{25}$$

Proof. For v_k and r_k defined as in Theorem 5.11, the explicit form of $\bar{\phi}$ coincides with that of ϕ defined in (15) and the vanishing condition (16) coincides with (25). □

6.2. **Vertical equations.** Consider the commutative diagram



(26)

where

$$\check{\psi}(q_0, w_0, r_1, v_1, r_2, \xi_1) := \big(D_2\check{L}_d(q_0, w_0, r_1)w_0^{-1} - D_2\check{L}_d(q_1, w_1, r_2)w_1^{-1}\big)(\xi_1)$$

for $q_1 := \tilde{F}_1(q_0, w_0, r_1)$ and $w_1 := F_2(q_0, w_0, r_1, v_1, r_2)$.

Notice that $F_1^*(Q \times \mathfrak{g}) = \check{Q}_G^{(2)} \times \mathfrak{g}$. In addition, G acts on $\mathbb{R}$ trivially and on $\check{Q}_G^{(2)} \times \mathfrak{g}$ by $l_g^{\check{Q}_G^{(2)}\times\mathfrak{g}}(q_0, w_0, r_1, v_1, r_2, \xi_1) := (l_g^Q(q_0), l_g^G(w_0), r_1, v_1, r_2, l_g^{TG}(\xi_1))$.

Lemma 6.3. *The map $\check{\psi}$ is a G-equivariant morphism of vector bundles.*

Proof. From the explicit definition, $\check{\psi}$ is a morphism of vector bundles.

In order to check the G-equivariance, for $(q_0, w_0, r_1, v_1, r_2) \in \check{Q}_G^{(2)}$, we let $q_1 := \tilde{F}_1(q_0, w_0, r_1)$ and $w_1 := F_2(q_0, w_0, r_1, v_1, r_2)$. Since

$$\check{\psi}(q_0, w_0, r_1, v_1, r_2, \xi_1) = D_2\check{L}_d(q_0, w_0, r_1)(\xi_1 w_0) - D_2\check{L}_d(q_1, w_1, r_2)(\xi_1 w_1),$$

using the G-equivariance of $\tilde{F}_1$ and F_2, we have

$$\begin{aligned} \check{\psi}(l_g^{\check{Q}_G^{(2)}\times\mathfrak{g}}(q_0, w_0, r_1, v_1, r_2, \xi_1)) =& \check{\psi}(l_g^Q(q_0), l_g^G(w_0), r_1, v_1, r_2, l_g^{TG}(\xi_1)) \\ =& D_2\check{L}_d(l_g^Q(q_0), l_g^G(w_0), r_1)(l_g^{TG}(\xi_1)l_g^G(w_0)) \\ &- D_2\check{L}_d(l_g^Q(q_1), l_g^G(w_1), r_2)(l_g^{TG}(\xi_1)l_g^G(w_1)). \end{aligned} \tag{27}$$

On one hand, recalling the notation of Remark 5.10, we have

$$l_g^{TG}(\xi_1) l_g^G(w_0) = g\xi_1 g^{-1} g w_0 g^{-1} = g\xi_1 w_0 g^{-1} = dl_g^G(w_0)(\xi_1 w_0). \tag{28}$$

On the other, the G-equivariance of $D_2\check{L}_d$ proved in Lemma 5.2 together with (28), applied to (27) lead to the G-equivariance of $\check{\psi}$. □

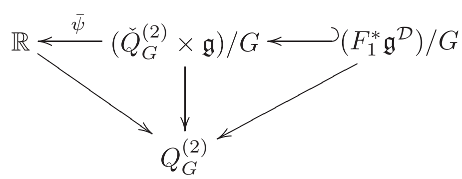
By Lemma 6.3, $\check{\psi}$ defines a morphism of vector bundles $\bar{\psi} : (\check{Q}_G^{(2)} \times \mathfrak{g})/G \to \mathbb{R}$ over $Q_G^{(2)}$. Explicitly, for any $(q_0, w_0) \in \rho^{-1}(v_0)$ and $\xi_1 \in \mathfrak{g}$,

$$\begin{split}\bar{\psi}([(q_0, w_0, r_1, v_1, r_2, \xi_1)]) =& \check{\psi}(q_0, w_0, r_1, v_1, r_2, \xi_1)\\ =& \big(D_2\check{L}_d(q_0, w_0, r_1) w_0^{-1} - D_2\check{L}_d(q_1, w_1, r_2) w_1^{-1}\big)(\xi_1),\end{split} \tag{29}$$

where $q_1 := \tilde{F}_1(q_0, w_0, r_1)$ and $w_1 := F_2(q_0, w_0, r_1, v_1, r_2)$.

A simple computation shows that, because $\mathcal{D} \subset TQ$ is G-invariant, $F_1^*\mathfrak{g}^{\mathcal{D}} \subset \check{Q}_G^{(2)} \times \mathfrak{g}$ is G-invariant. Therefore $(F_1^*\mathfrak{g}^{\mathcal{D}})/G \subset (\check{Q}_G^{(2)} \times \mathfrak{g})/G$ is a vector subbundle.

We collect the different objects in the following commutative diagram, which is the quotient of (part of) (26)



Proposition 6.4. *Condition (18) in Theorem 5.11 is equivalent to*

$$\bar{\psi}\,|_{(F_1^*\mathfrak{g}^{\mathcal{D}})/G} = 0. \tag{30}$$

Proof. For v_k and r_k defined as in Theorem 5.11, the explicit form of $\bar{\psi}$ coincides with that of ψ defined in (17) and the vanishing condition (18) coincides with (30). □

Corollary 6.5. *Any one of the four equivalent conditions of Theorem 5.11 is equivalent to $(v_k, r_{k+1}) \in \hat{\mathcal{D}}_d$ and conditions (25) and (30) are met for all k.*

7. Reconstruction.

Given a discrete curve $q_\cdot$ in Q and its image $(v_\cdot, r_\cdot)$ in $\tilde{G} \times (Q/G)$, Theorem 5.11 establishes an equivalence between $q_\cdot$ being a trajectory of the original system (*i.e.*, a solution of the original dynamics) and $(v_\cdot, r_\cdot)$ being a solution of the reduced dynamics. In this section we study the reconstruction problem, that is, given a curve $(v_\cdot, r_\cdot)$ in $\tilde{G} \times (Q/G)$ that satisfies adequate conditions (constraints and equations of motion), is it possible to find a trajectory $q_\cdot$ of the original system that projects to $(v_\cdot, r_\cdot)$?

Consider the following construction. Given a discrete curve $(v_\cdot, r_\cdot) \in \tilde{G} \times (Q/G)$ and $q_k \in Q$ (one value of k) such that $\pi(q_k) = r_k = p^{Q/G}(v_k)$, if $v_k = \rho(\tilde{q}_k, \tilde{w}_k)$ we define

$$u_k := l^G_{\tau(q_k, \tilde{q}_k)}(\tilde{w}_k) \in G \quad \text{ and } \quad q_{k+1} := \tilde{F}_1(q_k, u_k, r_{k+1}). \tag{31}$$

Since $l^G_{\tau(q_k, l_g^Q(\tilde{q}_k))}(l_g^G(\tilde{w}_k)) = l^G_{\tau(q_k,\tilde{q}_k)g^{-1}}(l_g^G(\tilde{w}_k)) = l^G_{\tau(q_k,\tilde{q}_k)}(\tilde{w}_k)$, the previous construction is independent of the chosen representatives of v_k. Therefore, given $q_0 \in Q$ with $\pi(q_0) = r_0$, applying the construction iteratively defines a unique discrete curve $q_\cdot$ in Q. The following result establishes the properties of $q_\cdot$.

Theorem 7.1. *Consider* $(\bar{q}_0, \bar{q}_1) \in \mathcal{D}_d$ *and* $(v_\cdot, r_\cdot) \in \tilde{G} \times (Q/G)$ *such that* $\pi(\bar{q}_0) = r_0$, $\pi(\bar{q}_1) = r_1$, $v_0 = \rho(\bar{q}_0, \mathcal{A}_d(\bar{q}_0, \bar{q}_1))$. *If* $(v_\cdot, r_\cdot)$ *satisfies* $(v_k, r_{k+1}) \in \hat{\mathcal{D}}_d$ *and conditions* (25) *and* (30) *for all* k, *then the discrete curve* $q_\cdot$ *constructed by* (31) *from* $\bar{q}_0$ *is a trajectory of the system on* Q *whose image by* Υ *is the curve* $(v_\cdot, r_\cdot)$ *and satisfies* $q_0 = \bar{q}_0$ *and* $q_1 = \bar{q}_1$.

Proof. The curve $q_\cdot$ is a lifting of the curve $(v_\cdot, r_\cdot)$ to Q; that is, $r_k = \pi(q_k)$ and $v_k = \rho(q_k, \mathcal{A}_d(q_k, q_{k+1}))$. Indeed, from (31), $\pi(q_k) = \pi(l^Q_{w_{k-1}}(\overline{h_d^{q_{k-1}}}(r_k))) = \pi(\overline{h_d^{q_{k-1}}}(r_k)) = r_k$. On the other hand,

$$\mathcal{A}_d(q_k, q_{k+1}) = \mathcal{A}_d(q_k, l^Q_{u_k}(\overline{h_d^{q_k}}(r_{k+1}))) = u_k \mathcal{A}_d \underbrace{(q_k, \overline{h_d^{q_k}}(r_{k+1}))}_{\in Hor_{\mathcal{A}_d}} = u_k,$$

so that

$$\begin{aligned}\rho(q_k, \mathcal{A}_d(q_k, q_{k+1})) &= \rho(q_k, u_k) = \rho(q_k, l^G_{\tau(q_k, \tilde{q}_k)}(\tilde{w}_k)) \\ &= \rho(l^Q_{\tau(\tilde{q}_k, q_k)}(q_k), \tilde{w}_k) = \rho(\tilde{q}_k, \tilde{w}_k) = v_k,\end{aligned}$$

which completes the argument, that is $\Upsilon(q_k, q_{k+1}) = (v_k, r_{k+1})$ for all k.

The lifted curve satisfies the first initial condition because, by construction, r_0 is lifted to $\bar{q}_0$. Since $v_0 = \rho(\bar{q}_0, \mathcal{A}_d(\bar{q}_0, \bar{q}_1))$, according to (31), $u_0 = \mathcal{A}_d(\bar{q}_0, \bar{q}_1)$ so that r_1 is lifted to $q_1 = l^Q_{\mathcal{A}_d(q_0, q_1)}(\overline{h_d^{q_0}}(r_1))$. Using (8), we conclude that $q_1 = \bar{q}_1$.

Next, we see that $(q_k, q_{k+1}) \in \mathcal{D}_d$ for all k. By the G-invariance of $\mathcal{D}_d$,

$$\begin{aligned}(q_k, q_{k+1}) \in \mathcal{D}_d \ &\Leftrightarrow \tilde{\pi}(q_k, q_{k+1}) \in \mathcal{D}_d/G \ \Leftrightarrow \ \Phi_{\mathcal{A}_d}(\tilde{\pi}(q_k, q_{k+1})) \in \Phi_{\mathcal{A}_d}(\mathcal{D}_d/G) \\ &\Leftrightarrow (v_k, r_{k+1}) \in \hat{\mathcal{D}}_d,\end{aligned}$$

which holds by hypothesis for all k.

The only thing left to do is to check that $q_\cdot$ is a trajectory of the discrete mechanical system. By hypothesis, $(v_\cdot, r_\cdot)$ satisfies conditions (25) and (30). Then, by Corollary 6.5 conditions (16) and (18) hold. But, since the relationship among $q_\cdot$, $v_\cdot$, $r_\cdot$ and $w_\cdot$ is precisely that of the statement of Theorem 5.11 and we proved that condition 4 holds, we conclude that $q_\cdot$ satisfies condition 1 in Theorem 5.11, hence it is a trajectory of the system on Q. □

Example 7.2. The last step to complete our analysis of the system $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ introduced in Example 5.1 is to consider the reconstruction of the evolution of the original system given a trajectory $(r_\cdot, w_\cdot)$ of the reduced system compatible with some initial data $(\bar{q}_0, \bar{q}_1) \in \mathcal{D}_d$.

According to Theorem 7.1 we use (31) to construct the trajectory $q_\cdot$. Since $v_k = (r_k, w_k) = \rho((r_k, 0), w_k)$ and $\tau((x_k, y_k), (r_k, 0)) = y_k$ (remember that $x_k = \pi(x_k, y_k) = r_k$),

$$u_k = l^G_{y_k}(w_k) = w_k,$$

$$q_{k+1} = (x_{k+1}, y_{k+1}) = l^Q_{w_k}(\overline{h_d^{(x_k, y_k)}}(r_{k+1})) = (r_{k+1}, y_k + b(r_{k+1}^2 - x_k^2)/2 + w_k)$$

that expresses $q_\cdot$ in terms of the known data $(r_\cdot, w_\cdot)$. Simplifying we obtain

$$(x_k, y_k) = (r_k, \bar{y}_0 + (r_k^2 - r_0^2)/2) \quad \text{for all} \quad k,$$

with $r_0 = \bar{x}_0$, $r_1 = \bar{x}_1$. Notice that the resulting trajectory is independent of the parameter b chosen to do the reduction, as it should be.

Remark 7.3. The reduction and reconstruction techniques developed so far can be applied in the case where the configuration space is the symmetry group (acting by left multiplication), a problem that has already been studied in [12, 13, 23]. In this context, Theorems 5.11 and 7.1 allow us to re derive Theorem 3 and Corollary 4 of [23].

8. **Nonholonomic discrete momentum.** For discrete or continuous holonomic mechanical systems the presence of continuous symmetries automatically leads to the existence of conserved quantities (momenta), due to Noether's Theorem. In the nonholonomic case, this is no longer true, essentially due to the behavior of constraint forces. Instead, in this case, one obtains an equation that describes the evolution of momenta over the trajectory of the system. Below we discuss the relationship of the discrete momentum evolution equation and the dynamics of a discrete mechanical system with symmetries.

Definition 8.1. Given $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ with symmetry group G, the *nonholonomic discrete momentum map* is the application $J_d : Q \times Q \to (\mathfrak{g}^{\mathcal{D}})^*$ defined by

$$J_d(q_0, q_1)(q_0, \xi) := -D_1 L_d(q_0, q_1)\xi_Q(q_0). \tag{32}$$

for all $(q_0, \xi) \in \mathfrak{g}^{\mathcal{D}}$.

Given any section $\tilde{\xi} \in \Gamma(\mathfrak{g}^{\mathcal{D}})$, define the map $(J_d)_{\tilde{\xi}} : Q \times Q \to \mathbb{R}$ by

$$(J_d)_{\tilde{\xi}}(q_0, q_1) := J_d(q_0, q_1)(q_0, \tilde{\xi}(q_0)).$$

Straightforward computations give the following result.

Lemma 8.2. *For $\xi \in \mathfrak{g}$ and $(q_0, q_1) \in Q \times Q$,*

$$D_1 L_d(q_0, q_1)(\xi_Q(q_0)) = -D_2 L_d(q_0, q_1)(\xi_Q(q_1)). \tag{33}$$

Also, if $(q_0, w_0, r_1) = \tilde{\Phi}_{\mathcal{A}_d}(q_0, q_1) = (q_0, \mathcal{A}_d(q_0, q_1), \pi(q_1))$, then

$$-D_1 L_d(q_0, q_1)(\xi_Q(q_0)) = (D_2 \check{L}_d(q_0, w_0, r_1) w_0^{-1})(\xi). \tag{34}$$

In particular, $J_d(q_0, q_1) = D_2 \check{L}_d(q_0, w_0, r_1) w_0^{-1}$.

Remark 8.3. By definition, when $Q \times Q$ and $\mathfrak{g}^{\mathcal{D}}$ are seen as bundles over Q with respect to the projection on the first variable, J_d is a bundle mapping (that is, it maps fibers to fibers). J. Cortés defines J_d^{nh} in [11, pp. 154-5] in a slightly different way: $J_d^{nh}(q_0, q_1)(q_1, \xi) := D_2 L_d(q_0, q_1)\xi_Q(q_1)$, which is a bundle map when the projection on the second variable is considered in $Q \times Q$. In any case, due to (33), both maps are, essentially, the same. Also $(J_d)_{\tilde{\xi}} = (J_d^{nh})_{\tilde{\xi}}$, for all $\tilde{\xi} \in \Gamma(\mathfrak{g}^{\mathcal{D}})$.

A trajectory of a discrete mechanical system is determined by the Discrete Lagrange–D'Alembert Principle (Definition 3.3). When the variational constraints are decomposed by $\mathcal{D} = \mathcal{S} \oplus \mathcal{H}$, it is possible to decompose all admissible variations into horizontal and vertical variations, in the sense that they belong to $\mathcal{H}$ or $\mathcal{S}$. It is also possible to decompose the variational principle accordingly. The following result, whose proof is obvious, makes a precise statement.

Proposition 8.4. *Let $q_\cdot$ be a discrete curve in Q. Then, the following conditions are equivalent.*

1. *$q_\cdot$ satisfies the variational part of the discrete Lagrange–D'Alembert principle (Definition 3.3).*

2. *$q.$ satisfies*
$$dS_d(q.)(\delta q.^{\mathcal{S}}) = 0 \quad \text{and} \quad dS_d(q.)(\delta q.^{\mathcal{H}}) = 0$$
for all pairs of variations vanishing at the end points $\delta q.^{\mathcal{S}}$ and $\delta q.^{\mathcal{H}}$ in $\mathcal{S}$ and $\mathcal{H}$ respectively.

The next result relates the vertical variational principle, the evolution of the nonholonomic discrete momentum and condition (30).

Theorem 8.5. *In the context of the theorem 5.11, if $q.$ is a discrete curve in Q, let $r_k = \pi(q_k), w_k = \mathcal{A}_d(q_k, q_{k+1})$ and $v_k = \rho(q_k, w_k)$. Then, the following conditions are equivalent.*

1. *$q.$ satisfies the vertical variational principle. That is, $dS_d(q.)(\delta q.^{\mathcal{S}}) = 0$ for all vanishing end point variation $\delta q_k^{\mathcal{S}} \in \mathcal{S}_{q_k}$ for all k.*
2. *Condition (30) is satisfied.*
3. *For all sections $\tilde{\xi} \in \Gamma(\mathfrak{g}^{\mathcal{D}})$,*

$$(J_d)_{\tilde{\xi}}(q_k, q_{k+1}) - (J_d)_{\tilde{\xi}}(q_{k-1}, q_k) = -D_1 L_d(q_{k-1}, q_k)(\tilde{\xi}(q_k) - \tilde{\xi}(q_{k-1}))_Q(q_{k-1}). \quad (35)$$

Proof.
- 1 $\Leftrightarrow$ 2. This is Proposition 6.4 and Lemma 5.12.
- 2 $\Leftrightarrow$ 3. For $\tilde{\xi} \in \Gamma(\mathfrak{g}^{\mathcal{D}})$, using the definition of $(J_d)_{\tilde{\xi}}$, equation (35) is equivalent to
$$-D_1 L_d(q_k, q_{k+1})(\tilde{\xi}(q_k))_Q(q_k) = -D_1 L_d(q_{k-1}, q_k)(\tilde{\xi}(q_k))_Q(q_{k-1}). \quad (36)$$
Using now (34) with $\xi = \tilde{\xi}(q_k)$ and recalling the explicit formula (29) for $\bar{\psi}$ we obtain (30).

□

Remark 8.6. During the proof of Theorem 8.5, we saw that the discrete nonholonomic momentum evolution equation was equivalent to (36), that is simpler to use in practice and, by the same Theorem is equivalent to any of the other conditions in the statement.

Remark 8.7. The discrete nonholonomic evolution equation (35) was first obtained by Cortés and Martínez in [10, Thm. 5.3], where they prove that any solution of the discrete Lagrange–D'Alembert equation satisfies (35). In the context of groupoids, Iglesias et al. obtain the evolution equation of the discrete nonholonomic momentum map in [15, Thm. 3.20]

9. **Reduced equations of motion: trivial bundle case.** In this section we consider the case where $Q := R \times G$, for a manifold R and a Lie group G that acts on Q by $l_g^Q(r, h) := (r, gh)$. In this case, $\pi : Q \to Q/G$ is the trivial principal bundle $p_1 : R \times G \to R$. The goal is to give an explicit description of the reduced system as well as the corresponding equations of motion.

Let $\alpha : \tilde{G} \to R \times G$ be given by $\alpha(\rho((r_0, h_0), w_0)) := (r_0, h_0^{-1} w_0 h_0)$, then α is a diffeomorphism with inverse $\beta(r_0, \vartheta_0) := \rho((r_0, e), \vartheta_0)$. We have the following commutative diagram

$$\begin{array}{ccc} Q \times G \times (Q/G) & \underset{\check{\beta}\times id}{\overset{\check{\alpha}\times id}{\rightleftarrows}} & R \times G \times G \times R \\ \downarrow{\scriptstyle \rho\times id} & & \downarrow{\scriptstyle \rho^t\times id} \\ \tilde{G} \times (Q/G) & \underset{\beta\times id}{\overset{\alpha\times id}{\rightleftarrows}} & R \times G \times R \end{array}$$

where $\rho^t(r_0, h_0, \vartheta_0) := (r_0, h_0^{-1}\vartheta_0 h_0)$, $\check{\alpha}((r_0, h_0), \vartheta_0) := (r_0, h_0, \vartheta_0)$ and $\check{\beta} = \check{\alpha}^{-1}$. In addition, a useful ingredient is the section $s : \tilde{G} \to Q \times G$ given by $s(\rho((r_0, h_0), \vartheta_0)) := ((r_0, e), h_0^{-1}\vartheta_0 h_0)$.

In the current context, by (7), $\mathcal{A}_d$ can be written as $\mathcal{A}_d((r_0, h_0), (r_1, h_1)) = h_1 \mathcal{A}_d^t(r_0, r_1) h_0^{-1}$, where $\mathcal{A}_d^t(r_0, r_1) := \mathcal{A}_d((r_0, e), (r_1, e))$.

Using α the reduced constraint manifold is $\hat{\mathcal{D}}_d^t := (\alpha \times id)(\hat{\mathcal{D}}_d)$, so that the reduced kinematic constraint condition becomes $(r_0, \vartheta_0, r_1) \in \hat{\mathcal{D}}_d^t$ if and only if $\big((r_0, e), (r_1, \vartheta_0 \mathcal{A}_d^t(r_0, r_1)^{-1})\big) \in \mathcal{D}_d$. If $\mathcal{D}_d$ is described by equations $\phi_b(q_0, q_1) = 0$ for all b, the reduced constraint condition becomes $\phi_b((r_0, e), (r_1, \vartheta_0 \mathcal{A}_d^t(r_0, r_1)^{-1})) = 0$ for all b.

We also have the induced lagrangians $\check{L}_d^t := \check{L}_d \circ (\check{\beta} \times id)$ and $\hat{L}_d^t := \hat{L}_d \circ (\beta \times id)$, so that $\hat{L}_d^t(r_0, \vartheta_0, r_1) = \check{L}_d((r_0, e), \vartheta_0, r_1)$.

Next we characterize the second order reduced manifolds $\check{Q}_G^{(2)}$ and $Q_G^{(2)}$, that are needed to set the equations of motion, according to Corollary 6.5. It is clear that

$$\check{Q}_G^{(2)} \simeq R \times G \times G \times R \times G \times R,$$

with $((r_0, h_0), w_0, r_1, \rho((r_1, h_1), w_1), r_2) \mapsto (r_0, h_0, w_0, r_1, h_1^{-1} w_1 h_1, r_2)$, and

$$Q_G^{(2)} \simeq R \times G \times R \times G \times R,$$

with $(\rho((r_0, h_0), w_0), r_1, \rho((r_1, h_1), w_1), r_2) \mapsto (r_0, , h_0^{-1} w_0 h_0, r_1, h_1^{-1} w_1 h_1, r_2)$. Next, we have to characterize the bundles over $Q_G^{(2)}$ where the morphisms $\bar{\phi}$ and $\bar{\psi}$ are defined. Instead, it is easier to notice that the section s provides a diffeomorphism of $Q_G^{(2)}$ with its image, so that we can view $Q_G^{(2)}$ as $R \times \{e\} \times G \times R \times G \times R$. The advantage of this approach is that instead of having to work with quotient bundles we have to work with the restriction of bundles on $\check{Q}_G^{(2)}$ to the image of s.

We start by obtaining trivialized versions of the maps $\tilde{F}_1$, F_1 and F_2. They are

$$\begin{aligned}
\tilde{F}_1^t(r_0, h_0, \vartheta_0, r_1) &:= \tilde{F}_1((r_0, h_0), \vartheta_0, r_1) = (r_1, \vartheta_0 h_0 (\mathcal{A}_d^t(r_0, r_1))^{-1}),\\
F_1^t(r_0, h_0, \vartheta_0, r_1, \vartheta_1, r_2) &:= F_1((r_0, h_0), \vartheta_0, r_1, \rho((r_1, e), \vartheta_1), r_2)\\
&= (r_1, \vartheta_0 h_0 (\mathcal{A}_d^t(r_0, r_1))^{-1}),
\end{aligned}$$

and

$$\begin{aligned}
F_2^t(r_0, h_0, \vartheta_0, r_1, \vartheta_1, r_2) &:= F_2((r_0, h_0), \vartheta_0, r_1, \rho((r_1, e), \vartheta_1), r_2)\\
&= \vartheta_0 h_0 (\mathcal{A}_d^t(r_0, r_1))^{-1}) \vartheta_1 \mathcal{A}_d^t(r_0, r_1) h_0^{-1} \vartheta_0^{-1}.
\end{aligned}$$

Remark 9.1. The case when the symmetry group is abelian has some specially nice features. For one thing, even when $\pi : Q \to Q/G$ is not trivial, there is a diffeomorphism $\alpha : \tilde{G} \to (Q/G) \times G$ that is given by $\alpha(\rho(q_0, w_0)) := (\pi(q_0), w_0)$. When, in addition, $\pi : Q \to Q/G$ is trivial, this diffeomorphism coincides with the one introduced at the beginning of this section.

9.1. **Horizontal equations.** Here we write down the morphism $\bar{\phi} : (Q_G^{(2)} \times p_2^* TR) \to \mathbb{R}$ or, rather, its realization under the identification of $Q_G^{(2)}$ with the image of the section s in $\check{Q}_G^{(2)}$. Let $\check{\phi}^t$ be the pullback of $\check{\phi}$ to $R \times G \times G \times R \times G \times R$. Explicitly,

$$\begin{aligned}
\check{\phi}^t(r_0, h_0, \vartheta_0, r_1, \vartheta_1, r_2, \delta r_1) &:= \check{\phi}((r_0, h_0), \vartheta_0, r_1, \rho((r_1, e), \vartheta_1), r_2, \delta r_1)\\
&= \big(D_1 \check{L}_d((r_1, h_1), w_1, r_2) \circ h^{q_1} + D_3 \check{L}_d((r_0, h_0), \vartheta_0, r_1)\\
&\quad + (\hat{F}_d^-)^t(r_1, \vartheta_1, r_2) + (\hat{F}_d^+)^t(r_0, \vartheta_0, r_1)\big)(\delta r_1),
\end{aligned}$$

where $h_1 := \vartheta_0 h_0 (\mathcal{A}_d^t(r_0, r_1))^{-1}$, $q_1 := (r_1, h_1)$, $w_1 := F_2^t(r_0, h_0, \vartheta_0, r_1, \vartheta_1, r_2)$, and

$$(\hat{F}_d^{\pm})^t(r_1, \vartheta_1, r_2) := \hat{F}_d^{\pm}(\rho((r_1, e), \vartheta_1), r_2).$$

From Definition 5.8, we see that $\hat{F}_d^t : p_1^*TR \oplus p_3^*TR \to \mathbb{R}$ is a morphism of bundles. Explicitly,

$$\hat{F}_d^t(r_0, \vartheta_0, r_1)(\delta r_0, \delta r_1) := D_2\check{L}_d((r_0, e), \vartheta_0, r_1) d\mathcal{A}_d((r_0, e), (r_1, h_1))(h^{(r_0,e)}(\delta r_0), h^{(r_1,h_1)}(\delta r_1)),$$

with h_1 as above. Even more, we can write

$$\begin{aligned} d\mathcal{A}_d((r_0, h_0), (r_1, h_1))(\delta r_0, \delta h_0, \delta r_1, \delta h_1) :=\ & h_1 D_1\mathcal{A}_d^t(r_0, r_1)(\delta r_0) h_0^{-1} \\ & - h_1 \mathcal{A}_d^t(r_0, r_1) h_0^{-1}(\delta h_0) h_0^{-1} + h_1 D_2\mathcal{A}_d^t(r_0, r_1)(\delta r_1) h_0^{-1} + \delta h_1 \mathcal{A}_d^t(r_0, r_1) h_0^{-1}, \end{aligned}$$

and we notice that this expression is written only in terms of $\mathcal{A}_d^t$.

We can also write the horizontal lift explicitly. In order to do so, we view the vector bundle $\mathcal{W} \oplus \mathcal{H} \subset TQ$ as the graph of a bundle map $M : p_1^*TR \to p_2^*TG$ over $Q = R \times G$. Therefore,

$$h^{(r,h)}(\delta r) := (\delta r, M(r, h)(\delta r)) \in T_{(r,h)}Q.$$

Finally, we have

$$\begin{aligned} \bar{\phi}^t(r_0, \vartheta_0, r_1, \vartheta_1, r_2, \delta r_1) &:= \check{\phi}^t(r_0, e, \vartheta_0, r_1, \vartheta_1, r_2, \delta r_1) \\ &= \Big(D_1\hat{L}_d^t(r_1, \vartheta_1, r_2) + D_3\hat{L}_d^t(r_0, \vartheta_0, r_1) \\ &\quad + D_2\hat{L}_d^t(r_1, \vartheta_1, r_2) \circ (\vartheta_1 M(r_1, e) - M(r_1, e)\vartheta_1) \\ &\quad + (\hat{F}_d^-)^t(r_1, \vartheta_1, r_2) + (\hat{F}_d^+)^t(r_0, \vartheta_0, r_1) \Big)(\delta r_1). \end{aligned} \tag{37}$$

On the other hand, we have that the representation of $p_2^*\hat{\mathcal{D}}$ over $R \times G \times R \times G \times R$ is the vector bundle $\hat{\mathcal{D}}^t := p_3^*\hat{\mathcal{D}}$.

Finally, condition (25) becomes

$$\bar{\phi}^t(r_0, \vartheta_0, r_1, \vartheta_1, r_2, \delta r_1) = 0 \tag{38}$$

for all δr_1 such that $h^{(r_1,e)}(\delta r_1) \in \mathcal{D}_{(r_1,e)}$.

Alternatively, if $\langle \omega_1, \ldots, \omega_K \rangle = \mathcal{D}^\circ$, condition (38) becomes

$$\bar{\phi}^t(r_0, \vartheta_0, r_1, \vartheta_1, r_2, \delta r_1) = \sum_{a=1}^{K} \lambda_a \omega^a(r_1, e)(\delta r_1, M(r_1, e)(\delta r_1))$$

for all $\delta r_1 \in T_{r_1}R$ and where $\lambda_a \in \mathbb{R}$ are unknown.

9.2. **Vertical equations.** Here we write down the morphism $\bar{\psi} : (Q_G^{(2)} \times \mathfrak{g})/G \to \mathbb{R}$ or, rather, its realization under the identification of $Q_G^{(2)}$ with the image of the section s in $\check{Q}_G^{(2)}$. Let $\check{\psi}^t$ be the pullback of $\check{\psi}$ to $R \times G \times G \times R \times G \times R$. Explicitly,

$$\begin{aligned} \check{\psi}^t(r_0, h_0, \vartheta_0, r_1, \vartheta_1, r_2, \xi_1) &:= \check{\psi}((r_0, h_0), \vartheta_0, r_1, \rho((r_1, e), \vartheta_1), r_2, \xi_1) \\ &= \big(D_2\check{L}_d((r_0, h_0), \vartheta_0, r_1) w_0^{-1} - D_2\check{L}_d((r_1, h_1), w_1,) w_1^{-1} \big)(\xi_1), \end{aligned}$$

where $h_1 := \vartheta_0 h_0 (\mathcal{A}_d^t(r_0, r_1))^{-1}$ and $w_1 := F_2^t(r_0, h_0, \vartheta_0, r_1, \vartheta_1, r_2)$.

Since, $\bar{\psi}^t := \check{\psi}^t|_{R\times\{e\}\times G\times R\times G\times R}$,

$$\begin{aligned}\bar{\psi}^t(r_0,\vartheta_0,r_1,\vartheta_1,r_2,\xi_1) &= \check{\psi}^t(r_0,e,\vartheta_0,r_1,\vartheta_1,r_2,\xi_1)\\ &= \big(D_2\check{L}_d((r_0,e),\vartheta_0,r_1)\vartheta_0^{-1} - D_2\check{L}_d((r_1,h_1),w_1,r_2)w_1^{-1}\big)(\xi_1) \\ &= \big(D_2\hat{L}_d^t(r_0,\vartheta_0,r_1)\vartheta_0^{-1} - h_1D_2\hat{L}_d^t(r_1,\vartheta_1,r_2)\vartheta_1^{-1}h_1^{-1}\big)(\xi_1),\end{aligned} \tag{39}$$

where

$$h_1 := \vartheta_0(\mathcal{A}_d^t(r_0,r_1))^{-1} \quad \text{and} \quad w_1 := F_2^t(r_0,e,\vartheta_0,r_1,\vartheta_1,r_2). \tag{40}$$

The other required ingredient is the pullback of $F_1^*\mathfrak{g}^{\mathcal{D}}$, that is, $(F_1^t)^*\mathfrak{g}^{\mathcal{D}}$. Then,

$$\begin{aligned}(F_1^t)^*\mathfrak{g}^{\mathcal{D}} &= \big\{(r_0,h_0,\vartheta_0,r_1,\vartheta_1,r_2,\xi_1)\in\check{Q}_G^{(2)}\times\mathfrak{g} : ((r_1,h_1),(0,\xi_1h_1))\in\mathcal{D}_{(r_1,h_1)}\big\}\\ &= \big\{(r_0,h_0,\vartheta_0,r_1,\vartheta_1,r_2,\xi_1)\in\check{Q}_G^{(2)}\times\mathfrak{g} : ((r_1,e),(0,h_1^{-1}\xi_1h_1))\in\mathcal{D}_{(r_1,e)}\big\},\end{aligned}$$

where h_1 is as in (40). Therefore, condition (30), becomes

$$\bar{\psi}^t(r_0,\vartheta_0,r_1,\vartheta_1,r_2,\xi_1) = 0 \tag{41}$$

for all $\xi_1\in\mathfrak{g}$ such that $(0,h_1^{-1}\xi_1h_1)\in\mathfrak{g}^{\mathcal{D}}_{(r_1,e)}$ where h_1 is defined by (40).

Alternatively, if $\langle\omega_1,\ldots,\omega_K\rangle = \mathcal{D}^\circ$, condition (41) becomes

$$\bar{\psi}^t(r_0,\vartheta_0,r_1,\vartheta_1,r_2,\xi_1) = \sum_{a=1}^K \lambda_a\omega^a(r_1,e)(0,h_1^{-1}\xi_1h_1) \quad \text{for all} \quad \xi_1\in\mathfrak{g}, \tag{42}$$

where h_1 is defined by (40) and $\lambda_a\in\mathbb{R}$ are unknown.

9.3. **Reconstruction.** Given initial conditions on $Q\times Q$, the reconstruction process in the current case goes as follows. Let $\bar{q}_0 = (\bar{r}_0,\bar{h}_0)$ and $\bar{q}_1 = (\bar{r}_1,\bar{h}_1)$ with $(\bar{q}_0,\bar{q}_1)\in\mathcal{D}_d$, and a discrete curve $(r_\cdot,\vartheta_\cdot)\in R\times G$. If $(r_\cdot,\vartheta_\cdot)$ satisfies the constraint $\hat{\mathcal{D}}_d^t$, the equations (38) and (41), and $r_0=\bar{r}_0$, $r_1=\bar{r}_1$ and $\vartheta_0 = \bar{h}_0^{-1}\bar{h}_1\mathcal{A}_d^t(\bar{r}_0,\bar{r}_1)$, by Theorem 7.1, the lifted curve $q_\cdot$ constructed from $\bar{q}_0$ using (31) is a trajectory of the original mechanical system on $Q\times Q$ with $q_0=\bar{q}_0$ and $q_1=\bar{q}_1$. In the trivial bundle case, if $q_k=(r_k,h_k)$,

$$q_{k+1} = \big(r_{k+1}, h_k\vartheta_k\mathcal{A}_d^t(r_k,r_{k+1})^{-1}\big) \quad \text{for all } k. \tag{43}$$

9.4. **An example of reduction in the trivial bundle case.** In this section we specialize the previous discussion to the discrete mechanical system $(Q,L_d,\mathcal{D},\mathcal{D}_d)$ where $Q:=\mathbb{R}^2\times S^1\times S^1$ with coordinates (x,y,θ,ϕ) —we consider $S^1:=\mathbb{R}/2\pi\mathbb{Z}$, and identify operations in S^1 with regular addition in $\mathbb{R}^1$—,

$$L_d(q_0,q_1) := \frac{m}{2}\big((q_1^x-q_0^x)^2+(q_1^y-q_0^y)^2\big) + \frac{I}{2}(q_1^\theta-q_0^\theta)^2 + \frac{J}{2}(q_1^\phi-q_0^\phi)^2,$$

$$\mathcal{D}_q := \langle\partial_\phi, \partial_\theta + A\cos(q^\phi)\partial_x + A\sin(q^\phi)\partial_y\rangle = \langle\omega_1(q),\omega_2(q)\rangle^\circ \subset T_qQ,$$

where $\omega_1(q) := dx - r\cos(q^\phi)d\theta$ and $\omega_2(q) := dy - r\sin(q^\phi)d\theta$, and

$$\begin{aligned}\mathcal{D}_d := \{(q_0,q_1)\in Q\times Q : &q_1^x-q_0^x = A(q_1^\theta-q_0^\theta)(\cos((q_0^\phi+q_1^\phi)/2) \text{ and}\\ &q_1^y-q_0^y = A(q_1^\theta-q_0^\theta)(\sin((q_0^\phi+q_1^\phi)/2)\},\end{aligned}$$

for m, A, I and J positive constants. This system can be obtained as a discretization of the classical mechanical system formed by a vertical disk of radius A, mass m with inertia momenta I and J, rolling without slipping on a horizontal plane.

We consider the Lie group $G:=\mathbb{R}^2\times S^1$ acting on Q by

$$l_g^Q(q) := (q^x+g^x, q^y+g^y, q^\theta+g^\theta, q^\phi).$$

The corresponding lifted action is $l_g^{TQ}(q,v) = (l_g^Q(q), v)$. We are in the trivial bundle case analyzed above because $\pi : Q \to Q/G$ is the trivial principal bundle $p_2 : G \times S^1 \to S^1$ with structure group G, which is a symmetry group of the system.

We consider various subbundles of TQ: the vertical bundle $\mathcal{V}_q^G = \langle \partial_x, \partial_y, \partial_\theta \rangle$, the intersection bundle $\mathcal{S}_q = \langle \partial_\theta + A\cos(q^\phi)\partial_x + A\sin(q^\phi)\partial_y \rangle$ and the corresponding complements $\mathcal{H}_q := \langle \partial_\phi \rangle$, $\mathcal{U}_q := \langle \partial_x, \partial_y \rangle$ and $\mathcal{W} := \{0\}$. Following Definition 2.5, the previous bundles induce a nonholonomic connection $\mathcal{A}$ whose horizontal space is $Hor_{\mathcal{A}} := \mathcal{H}$. The corresponding horizontal lift $h^q : T_r S^1 \to \mathcal{H}_q \subset T_q Q$ for $q \in \pi^{-1}(r)$ is $h^q(\dot{r}\partial_r) = \dot{r}\partial_\phi$; notice that, in this case, the bundle map M that describes the horizontal bundle as a graph is identically 0.

By Proposition 4.12, since

$$\mathcal{A}_d(q_0, q_1) = (q_1^x, q_1^y, q_1^\theta)(q_0^x, q_0^y, q_0^\theta)^{-1} = (q_1^x - q_0^x, q_1^y - q_0^y, q_1^\theta - q_0^\theta) \in G$$

satisfies (7), there is a discrete connection with level $\gamma(q) = e$ whose discrete connection 1-form is $\mathcal{A}_d$. Notice that our choice of $\mathcal{A}_d$ has $\mathcal{A}_d^t(r_0, r_1) = e$ for all $r_0, r_1 \in S^1$. The corresponding discrete horizontal lift is $h_d^{q_0}(r_1) = (q_0, (q_0^x, q_0^y, q_0^\theta, r_1))$.

The reduced space in this case is $\tilde{G} \times (Q/G) \simeq S^1 \times G \times S^1$ and the reduced second order manifold is $Q_G^{(2)} \simeq S^1 \times G \times S^1 \times G \times S^1$. The trivialized reduced lagrangian is

$$\hat{L}_d^t(r_0, \vartheta_0, r_1) = \frac{m}{2}\big((\vartheta_0^x)^2 + (\vartheta_0^y)^2\big) + \frac{I}{2}(\vartheta_0^\theta)^2 + \frac{J}{2}(r_1 - r_0)^2.$$

Next we describe the horizontal and vertical equations of the reduced system, but first we notice that since $\mathcal{A}_d$ only depends on the group variables, it turns out that $d\mathcal{A}_d$ annihilates horizontal vectors and the mixed curvature $\mathcal{B}_m$ vanishes. Consequently, there are no reduced discrete forces on the system.

From (37), we derive

$$\bar{\phi}^t(r_{k-1}, \vartheta_{k-1}, r_k, \vartheta_k, r_{k+1}, \delta r_k) = \big(J\big((r_k - r_{k-1}) - (r_{k+1} - r_k)\big)dr|_{r_k}\big)(\delta r_k),$$

and, noticing that $\mathcal{W} = \{0\}$ implies that $\hat{\mathcal{D}}^t = TS^1$, the horizontal equation is

$$(r_k - r_{k-1}) - (r_{k+1} - r_k) = 0. \tag{44}$$

From (39), we obtain

$$\begin{aligned}\bar{\psi}^t(r_{k-1}, \vartheta_{k-1}, r_k, \vartheta_k, r_{k+1}, \xi_k) = \big(&m(\vartheta_{k-1}^x - \vartheta_k^x)dx|_{e^x} + m(\vartheta_{k-1}^y - \vartheta_k^y)dy|_{e^y} \\ &+ I(\vartheta_{k-1}^\theta - \vartheta_k^\theta)d\theta|_{e^\theta}\big)(\xi_k),\end{aligned}$$

where $e = (e^x, e^y, e^\theta)$ is the identity element in G. Since the right hand side of (42) is $\lambda_1(dx|_{e^x} - A\cos(r_k)d\theta|_{e^\theta}) + \lambda_2(dy|_{e^y} - A\sin(r_k)d\theta|_{e^\theta})$, we obtain the following vertical equations

$$\begin{aligned} m(\vartheta_{k-1}^x - \vartheta_k^x) &= \lambda_1, \\ m(\vartheta_{k-1}^y - \vartheta_k^y) &= \lambda_2, \\ I(\vartheta_{k-1}^\theta - \vartheta_k^\theta) &= -\lambda_1 A\cos(r_k) - \lambda_2 A\sin(r_k). \end{aligned}$$

On the other hand, the reduced kinematic constraint equations are

$$\begin{aligned} \vartheta_k^x &= A\vartheta_k^\theta \cos((r_k + r_{k+1})/2), \\ \vartheta_k^y &= A\vartheta_k^\theta \sin((r_k + r_{k+1})/2). \end{aligned} \tag{45}$$

Next, we find the reduced dynamics by solving the reduced equations. From (44)

$$r_k = (r_1 - r_0)k + r_0 \qquad \text{for all} \qquad k \in \mathbb{N} \cup \{0\}. \tag{46}$$

From the vertical equations we obtain

$$I(\vartheta^\theta_{k-1} - \vartheta^\theta_k) = -m(\vartheta^x_{k-1} - \vartheta^x_k)A\cos(r_k) - m(\vartheta^y_{k-1} - \vartheta^y_k)A\sin(r_k).$$

Plugging (45) into this last equation (for $k-1$ and k) and simplifying we get

$$\vartheta^\theta_k = \frac{I + mA^2\cos((r_k - r_{k-1})/2)}{I + mA^2\cos((r_{k+1} - r_k)/2)}\vartheta^\theta_{k-1}.$$

Using the horizontal equation (44) we conclude that $\vartheta^\theta_k = \vartheta^\theta_0$ for all k. Thus,

$$\vartheta^x_k = A\vartheta^\theta_0\cos\left((r_1 - r_0)k + \frac{r_1 + r_0}{2}\right) \quad \text{and} \quad \vartheta^y_k = A\vartheta^\theta_0\sin\left((r_1 - r_0)k + \frac{r_1 + r_0}{2}\right).$$

This last expression completes the description of the reduced system's dynamics.

The reconstruction of the trajectories of the original system in $Q \times Q$ is done as described in Section 9.3. By (43), the reconstructed trajectory $q_\cdot$ is

$$q_{k+1} = \big((q^x_{k+1}, q^y_{k+1}, q^\theta_{k+1}), q^\phi_{k+1}\big) = \big((q^x_k, q^y_k, q^\theta_k)\vartheta_k, r_{k+1}\big), \tag{47}$$

so that $(q^x_{k+1}, q^y_{k+1}, q^\theta_{k+1}) = (q^x_0, q^y_0, q^\theta_0)\prod_{j=0}^k \vartheta_j$ or,

$$q^x_{k+1} = q^x_0 + A(q^\theta_1 - q^\theta_0)\cos((q^\phi_1 - q^\phi_0)/2)\frac{\sin((q^\phi_1 - q^\phi_0)k + q^\phi_0) - \sin(q^\phi_0)}{\sin(q^\phi_1 - q^\phi_0)},$$

$$q^y_{k+1} = q^y_0 + A(q^\theta_1 - q^\theta_0)\cos((q^\phi_1 - q^\phi_0)/2)\frac{\cos(q^\phi_0) - \cos((q^\phi_1 - q^\phi_0)k + r_0)}{\sin(q^\phi_1 - q^\phi_0)},$$

$$q^\theta_{k+1} = (q^\theta_1 - q^\theta_0)k + q^\theta_0.$$

Also, from (47) and (46),

$$q^\phi_{k+1} = (q^\phi_1 - q^\phi_0)(k+1) + q^\phi_0.$$

10. **Reduced equations of motion: Chaplygin case.** In this section we specialize Theorems 5.11 and 7.1 to the case where the original system is a discrete mechanical system with Chaplygin type symmetries. In this case we go beyond the result of equivalence between the discrete mechanical system in Q and a dynamical system in $\tilde{G} \times (Q/G)$ to obtain an equivalence between discrete mechanical systems on Q and Q/G.

Definition 10.1. A symmetry group G of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ is a *Chaplygin type symmetry group* if it satisfies the following conditions.

1. $TQ = \mathcal{V}^G \oplus \mathcal{D}$ and
2. $\mathcal{D}_d$ defines an affine discrete connection $\mathcal{A}_d$ on the principal bundle $\pi : Q \to Q/G$.

Notice that, by definition, the condition $TQ = \mathcal{V}^G \oplus \mathcal{D}$ is equivalent to the fact that the decomposition (2) has the form

$$TQ = \underbrace{\{0\}}_{\mathcal{W}} \oplus \underbrace{\mathcal{V}^G}_{\mathcal{U}} \oplus \underbrace{\{0\}}_{\mathcal{S}} \oplus \underbrace{\mathcal{D}}_{\mathcal{H}}. \tag{48}$$

As in the general case, this decomposition defines a connection $\mathcal{A}$ on $Q \to Q/G$, whose horizontal space is $\mathcal{H} = \mathcal{D}$. Condition 2 in Definition 10.1 requires that, for every (q_0, q_1) there exists a unique $g \in G$ such that $(q_0, l^Q_{g^{-1}}(q_1)) \in \mathcal{D}_d$.

Example 10.2. $G := \mathbb{R}$ is a Chaplygin type symmetry group of the discrete mechanical system $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ defined in Example 5.1. The first condition is clear from the decomposition of TQ that appears in Example 5.6; the second point corresponds to the discrete connection $\mathcal{A}_d^b$ considered in Example 4.17 when $b = 1$.

10.1. **An inclusion.** Let $\mathcal{Y} : (Q/G) \times (Q/G) \to \tilde{G} \times (Q/G)$ be defined by

$$\mathcal{Y}(r_0, r_1) := (\rho(q_0, e), r_1),$$

where $q_0 \in \pi^{-1}(r_0)$ and e is the identity of G.

Lemma 10.3. *The application $\mathcal{Y}$ is well defined. Even more, if s is a local section of $\pi : Q \to Q/G$, then $\mathcal{Y}(r_0, r_1) = (\rho(s(r_0), e), r_1)$.*

Proof. $\mathcal{Y}$ does not depend on the choice of $q_0 \in \pi^{-1}(r_0)$ by the G-invariance of ρ. The expression of $\mathcal{Y}$ in terms of s holds because $s(r_0) \in \pi^{-1}(r_0)$. This formula also shows that $\mathcal{Y}$ is a smooth map. □

Using $\mathcal{Y}$ we transport the existing structure on $\tilde{G} \times (Q/G)$ to $(Q/G) \times (Q/G)$. More precisely, we define the (forced, unconstrained) discrete mechanical system $(Q/G, \check{L}_d, \check{F}_d)$ where $\check{L}_d := \mathcal{Y}^*(\hat{L}_d) = \hat{L}_d \circ \mathcal{Y}$ and the discrete force $\check{F}_d := \mathcal{Y}^*(\hat{F}_d)$. We also define $\check{\mathcal{D}} := \hat{\mathcal{D}}$ as a subbundle of $T(Q/G)$, but notice that, $\check{\mathcal{D}} = T(Q/G)$ in the Chaplygin case, due to (48).

Lemma 10.4. *If $\mathcal{A}$ is a connection on the principal bundle $\pi : Q \to Q/G$ and $q_0 \in \pi^{-1}(r_0)$, then*

$$d\check{L}_d(r_0, r_1)(\delta r_0, \delta r_1) = D_1\check{L}_d(q_0, e, r_1)(h^{q_0}(\delta r_0)) + D_3\check{L}_d(q_0, e, r_1)(\delta r_1). \tag{49}$$

Proof. If s is a local section of $Q \to Q/G$ with $q_0 = s(r_0)$ then $\check{L}_d(r_0, r_1) = \hat{L}_d(\rho(s(r_0), e), r_1) = \check{L}_d(s(r_0), e, r_1)$, so that

$$\begin{aligned} d\check{L}_d(r_0, r_1)(\delta r_0, \delta r_1) =& (d\check{L}_d)(s(r_0), e, r_1)(ds(r_0)(\delta r_0), 0, \delta r_1) \\ =& D_1\check{L}_d(s(r_0), e, r_1)(ds(r_0)(\delta r_0)) + D_3\check{L}_d(s(r_0), e, r_1)(\delta r_1). \end{aligned} \tag{50}$$

Since $\pi \circ s = id_{Q/G}$, we have

$$\delta r_0 = d\pi(s(r_0))ds(r_0)(\delta r_0) = d\pi(s(r_0))Hor_{\mathcal{A}}(ds(r_0)(\delta r_0)),$$

thus $Hor_{\mathcal{A}}(ds(r_0)(\delta r_0)) = h^{s(r_0)}(\delta r_0)$. Also, as $D_1\check{L}_d(s(r_0), e, r_1)$ vanishes on vertical vectors by the G-invariance of $\check{L}_d$,

$$\begin{aligned} D_1\check{L}_d(s(r_0), e, r_1)(ds(r_0)(\delta r_0)) =& D_1\check{L}_d(s(r_0), e, r_1)(Hor_{\mathcal{A}}(ds(r_0)(\delta r_0))) \\ =& D_1\check{L}_d(s(r_0), e, r_1)(h^{s(r_0)}(\delta r_0)) \end{aligned}$$

which, replaced in (50) leads to (49). □

Lemma 10.5. *Let $(v_., r_.)$ be a discrete curve in $\tilde{G} \times (Q/G)$ such that $v_k = \rho(q_k, e)$ for some $q_k \in Q$ and all k. Then*

$$d\hat{S}_d(v_., r_.)(\delta v_., \delta r_.) = d\check{S}_d(r_.)(\delta r_.) + \sum_{k=1}^{N-1} \big(\check{F}_d^-(r_k, r_{k+1}) + \check{F}_d^+(r_{k-1}, r_k)\big)(\delta r_k)$$

for all vanishing end points variations $\delta r_.$ and

$$\delta v_k = d\rho(q_k, e)\big(h^{q_k}(\delta r_k), d\mathcal{A}_d(q_k, q_{k+1})(h^{q_k}(\delta r_k), h^{q_{k+1}}(\delta r_{k+1}))\big)$$

for all k.

Proof. By definition, using the explicit form of δv,

$$d\hat{S}_d(v.,r.)(\delta v.,\delta r.) = \sum_{k=0}^{N-1} \big(D_1\check{L}_d(q_k,e,r_{k+1})(h^{q_k}(\delta r_k)) + D_2\check{L}_d(q_k,e,r_{k+1})(d\mathcal{A}_d(q_k,q_{k+1})(h^{q_k}(\delta r_k),h^{q_{k+1}}(\delta r_{k+1}))) + D_3\check{L}_d(q_k,e,r_{k+1})(\delta r_{k+1})\big).$$

Using Lemma 10.4 and the decomposition $d\mathcal{A}_d = D_1\mathcal{A}_d + D_2\mathcal{A}_d$ we obtain

$$d\hat{S}_d(v.,r.)(\delta v.,\delta r.) = d\check{S}(r.)(\delta r.) + \sum_{k=0}^{N-1} \big(\hat{F}_d^-(q_k,e,r_{k+1})(\delta r_k) + \hat{F}_d^+(q_k,e,r_{k+1})(\delta r_{k+1})\big),$$

and the result follows. □

Lemma 10.6. *Let $(v.,r.)$ be a discrete curve in $\tilde{G}\times(Q/G)$ such that $v_k = \rho(q_k,e)$ for some $q_k\in Q$ and all k. Then $(v.,r.)$ satisfies condition (25) if and only if*

$$D_1\check{L}_d(r_k,r_{k+1}) + D_2\check{L}_d(r_{k-1},r_k) + \check{F}_d^+(r_{k-1},r_k) + \check{F}_d^-(r_k,r_{k+1}) \in (\check{\mathcal{D}}_{r_k})^\circ \quad (51)$$

for all k.

Proof. Considering the form of v_k, the vanishing condition (25) immediately translates, via $\mathcal{Y}$ to the vanishing condition (51). □

10.2. **Reduced dynamics.**

Theorem 10.7. *Let G be a Chaplygin symmetry group of $(Q,L_d,\mathcal{D},\mathcal{D}_d)$, $q.$ be a discrete curve in Q, and $r_k := \pi(q_k)$ be the corresponding curve in Q/G. If $(q_k,q_{k+1})\in\mathcal{D}_d$ for all k, the following statements are equivalent.*

1. *$q.$ satisfies the variational principle $dS_d(q)(\delta q.) = 0$ for all vanishing end points variations $\delta q.$ such that $\delta q_k\in\mathcal{D}_{q_k}$ for all k.*
2. *$q.$ satisfies the Lagrange–D'Alembert equations (3).*
3. *$r.$ satisfies the variational principle*

$$d\check{S}_d(r.)(\delta r.) = -\sum_{k=1}^{N-1}(\check{F}_d^-(r_k,r_{k+1}) + \check{F}_d^+(r_{k-1},r_k))(\delta r_k)$$

for all vanishing end points variations $\delta r.$ with $\delta r_k\in T_{r_k}(Q/G)$ and where

$$\check{S}_d(r.) := \sum_{k=0}^{N-1}\check{L}_d(r_k,r_{k+1}).$$

4. *$r.$ satisfies the following equation, for all k.*

$$D_1\check{L}_d(r_k,r_{k+1}) + D_2\check{L}_d(r_{k-1},r_k) = -(\check{F}_d^+(r_{k-1},r_k) + \check{F}_d^-(r_k,r_{k+1})).$$

Proof. Since $\mathcal{S}=\{0\}$, by Theorem 5.11, we see that each one of the points 1 and 2 of the present result is equivalent to any one of

i. Item 3 in Theorem 5.11 holds for all horizontal variations, ($\xi_k = 0$ for all k).
ii. For all k $(v_k,r_{k+1})\in\hat{\mathcal{D}}_d$ and condition (16) holds.

As $Hor_{\mathcal{A}_d} = \mathcal{D}_d$, $(q_k,q_{k+1})\in\mathcal{D}_d$ if and only if $w_k = \mathcal{A}_d(q_k,q_{k+1}) = e$. Then,

$$(v_k,r_{k+1})\in\hat{\mathcal{D}}_d \Leftrightarrow (\rho(q_k,e),r_{k+1})\in\Phi_{\mathcal{A}_d}(\mathcal{D}_d/G) \Leftrightarrow (q_k,\overline{h_d^{q_k}}(r_{k+1}))\in\mathcal{D}_d,$$

which always holds by definition of $\mathcal{A}_d$. Therefore $\hat{\mathcal{D}}_d = \tilde{G}\times(Q/G)$, so that we can drop the reduced kinematic constraint condition from i and ii.

By Lemma 10.5 the variational principle that appears in i is equivalent to the one in point 3 of the statement. Similarly, as $\breve{\mathcal{D}} = T(Q/G)$, Lemma 10.6 and Proposition 6.2 show that ii is equivalent to the one in point 4 of the statement. □

Remark 10.8. Notice that the reduction of a discrete mechanical system with Chaplygin type symmetry results in an unconstrained discrete mechanical system but with external forces given by $\breve{F}^{\pm}$ in the previous theorem. A similar analysis has been done in the more general groupoid setting in [15].

Example 10.9. As we noted above, $G = \mathbb{R}$ is a Chaplygin type symmetry group of the discrete mechanical system introduced in Example 5.1. We notice that the construction described in this section corresponds to performing the reduction using the discrete connection $\mathcal{A}_d^1$ from Example 4.17. Furthermore, the inclusion $\mathcal{Y}$ from section 10.1 was essentially already present in the analysis of the reduction in Example 5.16 in the form of the section $(Q/G) \times (Q/G) \to \tilde{G}$ given by $s(r_0, r_1) := \rho((r_0, 0), r_1)$.

In any case, the discrete unconstrained mechanical system associated to the reduced system by Theorem 10.7 is

$$\breve{Q} := Q/G = \mathbb{R}, \quad \text{with coordinate} \quad r,$$
$$\breve{L}_d(r_0, r_1) = \frac{m}{2}\big((r_1 - r_0)^2 + (r_1^2 - r_0^2)^2/4\big)$$

with no forces since, by Example 5.6, the mixed curvature vanishes in the case $b = 1$, hence there are no reduced forces. Finally, the discrete Euler–Lagrange equation that determines the evolution of the $(\breve{Q}, \breve{L}_d)$ system is

$$(r_{k+1} - r_k) - (r_k - r_{k-1}) + r_k\big((r_{k+1}^2 - r_k^2) - (r_k^2 - r_{k-1}^2)\big)/2 = 0, \tag{52}$$

that is, precisely, (22).

Last, we adapt Theorem 7.1 to the reconstruction in the present setting.

Theorem 10.10. *Let G be a Chaplygin symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$. Let $(\bar{q}_0, \bar{q}_1) \in \mathcal{D}_d$ and $r_\cdot$ be a discrete trajectory of the forced discrete mechanical system $(Q/G, \breve{L}_d, \breve{\mathcal{D}}, \breve{\mathcal{D}}_d, \breve{F}_d)$ such that $\pi(\bar{q}_j) = r_j$ for $j = 0, 1$. Define the discrete curve $q_\cdot$ in Q inductively by $q_0 = \bar{q}_0$ and $q_{k+1} = \overline{h_d^{q_k}}(r_{k+1})$ for all $k \in \mathbb{N} \cup \{0\}$ (here the horizontal lift is associated to the affine discrete connection whose horizontal space is $\mathcal{D}_d$). Then $q_\cdot$ is a trajectory of the original discrete mechanical system with $q_j = \bar{q}_j$ for $j = 0, 1$.*

Proof. The trajectory $r_\cdot$ defines a trajectory $(v_k, r_{k+1}) = \mathcal{Y}(r_k, r_{k+1})$ in $\tilde{G} \times (Q/G)$. Indeed, by Lemma 10.5 both curves satisfy simultaneously the corresponding variational principles. Also, since $\bar{q}_0 \in \pi^{-1}(r_0)$, $v_0 = \rho(\bar{q}_0, e) = \rho(\bar{q}_0, \mathcal{A}_d(\bar{q}_0, \bar{q}_1))$. In general, $v_k = \rho(\tilde{q}_k, e)$ for some $\tilde{q}_k \in \pi^{-1}(r_k)$.

Then, since $(\bar{q}_0, \bar{q}_1) \in \mathcal{D}_d$, by Theorem 7.1, there is a trajectory $q_\cdot$ in Q such that $\pi(q_k) = r_k$ for all k and $q_0 = \bar{q}_0$, $q_1 = \bar{q}_1$. Furthermore, q_{k+1} satisfies the formula in the statement because, q_{k+1} satisfies (31) with $u_k = e$ for all k. □

Example 10.11. We use Theorem 10.10 to reconstruct the trajectories of the system reduced in Example 10.9. Applying the recursive formula and (9) we have

$$q_{k+1} = (x_{k+1}, y_{k+1}) = \overline{h_d^{q_k}}(r_{k+1}) = (r_{k+1}, y_k + (r_{k+1}^2 - r_k^2)/2),$$

or, simplifying, $(x_k, y_k) = (r_k, y_0 + (r_k^2 - r_0^2)/2)$, agreeing with the result obtained in Example 7.2.

11. **Reduced equations of motion: Horizontal symmetries case.** In this section we specialize theorem 5.11 to the case where the original system is a mechanical system with horizontal symmetries. In the same way as in the Chaplygin case discussed in Section 10, in this case we go beyond the result of equivalence between the mechanical system in Q and a dynamical system in $\tilde{G} \times (Q/G)$ to obtain an equivalence between the discrete mechanical system in Q and another one in Q/G.

Definition 11.1. Let M be a symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$. A closed subgroup $G \subset M$ is said to be a *horizontal symmetry subgroup* for $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ if

$$\mathcal{V}^M(q) \cap \mathcal{D}_q = \mathcal{V}^G(q) \quad \text{for all} \quad q \in Q. \tag{53}$$

From now on we will forget the group M and consider the action of G on the system. It is in this context that we specialize Theorem 5.11. Due to condition (53), we have that $\mathcal{V}^G = \mathcal{S}$, the decomposition (2) of TQ becomes

$$TQ = \mathcal{W} \oplus \underbrace{\{0\}}_{\mathcal{U}} \oplus \mathcal{S} \oplus \mathcal{H}$$

for any complementary subbundles $\mathcal{H}$ of $\mathcal{S}$ in $\mathcal{D}$ and $\mathcal{W}$ of $\mathcal{D}$ in TQ. Fixing one such decomposition we define a connection $\mathcal{A}$ on the principal bundle $\pi : Q \to Q/G$ requiring that $Hor_{\mathcal{A}} = \mathcal{H}$.

In the context of this section the discrete nonholonomic momentum map J_d defined by (32) has some special properties, which are studied next.

Lemma 11.2. *If G is horizontal symmetry subgroup the following statements are true.*

1. $\mathfrak{g}^{\mathcal{D}} = Q \times \mathfrak{g}$ *and* $(\mathfrak{g}^{\mathcal{D}})^* \simeq Q \times \mathfrak{g}^*$.
2. *Composing the nonholonomic momentum map J_d defined in (32) with the projection onto the second variable defines a momentum application $J_d : Q \times Q \to \mathfrak{g}^*$. Explicitly, for $\xi \in \mathfrak{g}$,*

$$J_d(q_0, q_1)\xi := -D_1 L_d(q_0, q_1)\xi_Q(q_0).$$

3. *Any of the equivalent conditions of Theorem 8.5 is equivalent to the condition that J_d is constant on the trajectory $q_\cdot$.*

Proof. Item 1 is a direct consequence of $\mathcal{V}^G \subset \mathcal{D}$, while item 2 is clear from item 1.

Last we check item 3. Let $q_\cdot$ be a discrete curve in Q. Assume that equation (35) holds on $q_\cdot$ for any section $\tilde{\xi}$. Then, since any $\xi \in \mathfrak{g}$ defines a (constant) section, evaluating (35) on this section yields $\big(J_d(q_k, q_{k+1}) - J_d(q_{k-1}, q_k)\big)\xi = 0$. Thus, since $\xi \in \mathfrak{g}$ is arbitrary, J_d is conserved on $q_\cdot$.

Conversely, if J_d is constant on $q_\cdot$, equation (35) holds for constant sections of $\mathfrak{g}^{\mathcal{D}}$. But, it can be readily checked that if (35) holds for a section $\tilde{\xi}$, it also holds for the section $f\tilde{\xi}$, for arbitrary $f : Q \to \mathbb{R}$. Then, since in our setup every section is a linear combination of constant sections with variable coefficients, we conclude that (35) holds for all sections. □

Below we construct an affine discrete connection adapted to the present geometry. Later we use that connection to specialize Theorem 5.11 to the horizontal setting.

11.1. **Affine discrete connection for horizontal symmetries.** We recall two well known results.

Lemma 11.3. *If $\xi \in \mathfrak{g}$, $g \in G$ and $q \in Q$, then $\xi_Q(l_g^Q(q)) = l_g^{TQ}((Ad_{g^{-1}}(\xi))_Q(q))$.*

Lemma 11.4. *If $g \in G$ and $q_0, q_1 \in Q$, then*

$$J_d(l_g^{Q\times Q}(q_0,q_1)) = Ad^*_{g^{-1}}(J_d(q_0,q_1)).$$

When $F : Q \times Q \to \mathbb{R}$ is a smooth map, $D_1F : p_1^*TQ \to \mathbb{R}$ is defined by $D_1F = dF|_{p_1^*TQ}$. Noticing that $p_1^*TQ \simeq TQ \times Q$, we have $D_2(D_1F) : p_2^*TQ \to \mathbb{R}$ (where $p_2 : TQ \times Q \to Q$ is the projection). As $p_2^*TQ \simeq TQ \times TQ$, it is customary to consider $D_2D_1L_d : TQ \times TQ \to \mathbb{R}$. It is easy to check that $D_2D_1L_d$ is bilinear in the tangent vectors.

Definition 11.5. Let $L_d : Q \times Q \to \mathbb{R}$ be a discrete lagrangian. We say that L_d is *regular* at $(q_0, q_1) \in Q\times Q$ if the bilinear mapping $D_2D_1L_d(q_0, q_1) : T_{q_0}Q\times T_{q_1}Q \to \mathbb{R}$ is nondegenerate, *i.e.*, if $X_0 \in T_{q_0}Q$ satisfies $D_2D_1L_d(q_0, q_1)(X_0, X_1) = 0$ for all $X_1 \in T_{q_1}Q$, then $X_0 = 0$. In coordinates, the regularity condition becomes that the matrix $\frac{\partial^2 L_d(q_0,q_1)}{\partial q_0 \partial q_1}$ be invertible.

Definition 11.6. Let G be a symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$. We say that L_d is *G-regular* at $(q_0, q_1) \in Q \times Q$ if the restriction of the bilinear form $D_2D_1L_d(q_0, q_1) : T_{q_0}Q \times T_{q_1}Q \to \mathbb{R}$ to $\mathcal{V}^G(q_0) \times \mathcal{V}^G(q_1)$ is nondegenerate.

The notions of regularity introduced above have already been considered by other authors ([21], [10], [23]). In order to study the relationship between the regularities of a lagrangian and the fact that the discrete momentum values be regular we begin by recalling the following fact.

Lemma 11.7. *Let $\pi : Q \to Q/G$ a principal bundle and $\{v_1, \ldots, v_k\} \subset \mathfrak{g}$ a linearly independent subset. Then, if $q \in Q$, $\{(v_1)_Q(q), \ldots, (v_k)_Q(q)\} \subset T_qQ$ is a linearly independent subset. Furthermore, if the first set is a basis, then the second one is a basis too.*

Proposition 11.8. *Let G be a horizontal symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ with regular L_d, discrete momentum mapping $J_d : Q \times Q \to \mathfrak{g}^*$ and $\mu \in \mathfrak{g}^*$. Then,*

- *μ is a regular value of J_d and, consequently, $\mathcal{J}_\mu := J_d^{-1}(\mu) \subset Q \times Q$ is a submanifold.*
- *if for $q_0 \in Q$ we let $J_d^{q_0} : Q \to \mathfrak{g}^*$ by $J_d^{q_0}(q_1) := J_d(q_0, q_1)$, μ is a regular value of $J_d^{q_0}$ and, consequently, $(J_d^{q_0})^{-1}(\mu) \subset Q$ is a submanifold that, if not empty, has dimension* $\dim Q - \dim G$.

Proof. We have to prove that for all $(q_0, q_1) \in \mathcal{J}_\mu$, the map

$$dJ_d(q_0, q_1) : T_{(q_0,q_1)}(Q \times Q) \to T_\mu\mathfrak{g}^* \simeq \mathfrak{g}^*$$

is onto. Let $\{e_1, \ldots, e_r\}$ be a basis of $\mathfrak{g}$ and $\{e_1^*, \ldots, e_r^*\}$ its dual basis. Hence

$$J_d(q_0, q_1) = \sum_{j=1}^r \phi_j(q_0, q_1)e_j^*,$$

where $\phi_j(q_0, q_1) = J_d(q_0, q_1)(e_j) = -D_1L_d(q_0, q_1)(e_j)_Q(q_0)$. Then,

$$dJ_d(q_0, q_1)(X_0, X_1) = \sum_{j=1}^r (D_1\phi_j(q_0, q_1)(X_0) + D_2\phi_j(q_0, q_1)(X_1))e_j^*.$$

Given $\psi = \sum_{j=1}^r a_je_j^* \in \mathfrak{g}^*$, by the regularity of L_d there is $X_1 \in T_{q_1}Q$ such that

$$D_2\phi_j(q_0, q_1)(X_1) = -D_2D_1L_d(q_0, q_1)((e_j)_Q(q_0), X_1) = a_j$$

for all j. Then,

$$dJ_d(q_0, q_1)(0, X_1) = \sum_{j=1}^{r} D_2\phi_j(q_0, q_1)(X_1)e_j^* = \sum_{j=1}^{r} a_j e_j^* = \psi,$$

so that $dJ_d(q_0, q_1)$ is onto; thus μ is a regular value of J_d and standard results allow us to conclude that $\mathcal{J}_\mu \subset Q \times Q$ is a submanifold.

We see in the previous computation that, for $\psi \in \mathfrak{g}^*$

$$dJ_d^{q_0}(q_1)(X_1) = D_2J_d(q_0, q_1)(X_1) = \psi,$$

so that μ is also a regular value of $J_d^{q_0}$, hence, $(J_d^{q_0})^{-1}(\mu) \subset Q$ is also a submanifold. The dimension of $(J_d^{q_0})^{-1}(\mu)$ can be computed noticing that, being μ a regular value of $J_d^{q_0}$, if $q_1 \in (J_d^{q_0})^{-1}(\mu)$, we have

$$\begin{aligned}\dim((J_d^{q_0})^{-1}(\mu)) &= \dim(\ker(dJ_d^{q_0}(q_1))) = \dim(T_{q_1}Q) - \dim(\operatorname{Im}(dJ_d^{q_0}(q_1))) \\ &= \dim(Q) - \dim(G).\end{aligned}$$

□

Proposition 11.9. *Let G be a horizontal symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ with L_d regular and G-regular. Then, for all $q_0 \in Q$ and $\mu \in \mathfrak{g}^*$, if $q_1 \in l_G^Q(\{q_0\}) \cap (J_d^{q_0})^{-1}(\mu)$, then $T_{q_1}Q = T_{q_1}l_G^Q(\{q_0\}) \oplus T_{q_1}(J_d^{q_0})^{-1}(\mu)$.*

Proof. Assume that $(J_d^{q_0})^{-1}(\mu) \neq \emptyset$ since, otherwise, the statement is valid. As $\dim(J_d^{q_0})^{-1}(\mu) = \dim Q - \dim G$, it suffices to see that $T_{q_1}l_G^Q(\{q_0\}) \cap T_{q_1}(J_d^{q_0})^{-1}(\mu) = \{0\}$ for all q_1 as in the statement. If X_1 is in this last intersection,

$$0 = dJ_d^{q_0}(q_1)(X_1) = \sum_{j=1}^{r} -D_2D_1L_d(q_0, q_1)((e_j)_Q(q_0), X_1)e_j^*,$$

so that $D_2D_1L_d(q_0, q_1)((e_j)_Q(q_0), X_1) = 0$, for all j. By the G-regularity of L_d and being $\{(e_1)_Q(q_0) \ldots, (e_k)_Q(q_0)\}$ a basis of $T_{q_1}l_G^Q(\{q_0\})$, we have that $X_1 = 0$. □

Definition 11.10. Let G be a horizontal symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ with L_d regular and G-regular. Given $\mu \in \mathfrak{g}^*$ we say that G is a group of *μ-good symmetries* if, in addition, for each $q \in Q$ there is a unique $g \in G$ such that $J_d^q(l_g^Q(q)) = \mu$. In this case, we define $\gamma : Q \to G$ by $\gamma(q) := g$.

It is possible to extend the previous notion to systems where there are more than one $g \in G$ with the required property but, in this case, the action must be accompanied by a smooth unique determination of g.

Proposition 11.11. *Let G be a group of μ-good symmetries of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ for some $\mu \in \mathfrak{g}^*$. If μ satisfies $Ad_g^*(\mu) = \mu$ for all $g \in G$, then $\mathcal{J}_\mu \subset Q \times Q$ defines an affine discrete connection in $\pi : Q \to Q/G$ of level γ, given by the μ-goodness of G.*

Proof. By Proposition 11.8, $\mathcal{J}_\mu$ is a submanifold of $Q \times Q$ and, by Lemma 11.4, it is also G-invariant. According to the definition of γ, $J_d(q, l_{\gamma(q)}^Q(q)) = \mu$, so that $\Gamma \subset \mathcal{J}_\mu$. Last, by Proposition 11.8, $\mathcal{J}_\mu(q) \subset Q$ is a submanifold and, using Proposition 11.9, we conclude that $\mathcal{J}_\mu$ defines an affine discrete connection. Notice that γ is G-equivariant by Lemma 11.4 and the condition $Ad_g^*(\mu) = \mu$ for all $g \in G$. □

Remark 11.12. In the context of Proposition 11.11, the condition that $Ad_g^*(\mu) = \mu$ for all $g \in G$ can be avoided by considering the (probably smaller) symmetry group $G_\mu := \{g \in G : Ad_g^*(\mu) = \mu\}$ instead of G. Indeed, G_μ is a Lie group and a symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$. Since G satisfies (53), $\mathcal{V}^G(q) \subset \mathcal{D}_q$ for all q, so that $\mathcal{V}^{G\mu}(q) \subset \mathcal{D}_q$ for all q. Hence, (53) holds if we put G_μ instead of M and G and run the reduction arguments in this setting. This same remark applies to the statements made in what remains of this section.

11.2. **Reduction.** In this section we assume that G is a group of μ-good symmetries of a regular and G-regular system for some $\mu \in \mathfrak{g}^*$. In the previous section we saw that if μ satisfies $Ad_g^*(\mu) = \mu$ for all $g \in G$, we can define an affine discrete connection $\mathcal{A}_d$ such that $Hor_{\mathcal{A}_d} = \mathcal{J}_\mu$. Below, we use this connection to specialize Theorem 5.11 to the context of horizontal symmetries.

As in the Chaplygin case studied in Section 10.1, using $\mathcal{Y}$ we define a forced discrete mechanical system on Q/G with discrete lagrangian $\breve{L}_d(r_0, r_1) := \mathcal{Y}^*(\hat{L}_d) = \hat{L}_d \circ \mathcal{Y}$, variational constraints $\breve{\mathcal{D}} := d\pi(\mathcal{D})$, kinematic constraints $\breve{\mathcal{D}}_d := \mathcal{Y}^{-1}(\hat{\mathcal{D}}_d)$ and forces $\breve{F} := \mathcal{Y}^*(\hat{F})$.

Theorem 11.13. *Let G be a μ-good symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ for some $\mu \in \mathfrak{g}^*$. Assume that $Ad_g^*(\mu) = \mu$ for all $g \in G$. Let $q.$ be a discrete curve in Q and $r_k := \pi(q_k)$ the corresponding discrete curve in Q/G. Then, if $J_d(q_k, q_{k+1}) = \mu$ for all k, the following statements are equivalent.*

1. *$(q_k, q_{k+1}) \in \mathcal{D}_d$ for all k and $q.$ satisfies the variational principle $dS_d(q)(\delta q.) = 0$ for all vanishing end points variation $\delta q.$ such that $\delta q_k \in \mathcal{D}_{q_k}$ for all k.*
2. *$(q_k, q_{k+1}) \in \mathcal{D}_d$ for all k and $q.$ satisfies the discrete Lagrange–D'Alembert equations (3) for all k.*
3. *$r.$ satisfies the variational principle*
$$d\breve{S}_d(r.)(\delta r.) = -\sum_{k=1}^{N-1} (\breve{F}_d^-(r_k, r_{k+1}) + \breve{F}_d^+(r_{k-1}, r_k))(\delta r_k)$$
for all vanishing end points variations such that $\delta r_k \in \breve{\mathcal{D}}_{r_k}$ for all k. In addition, $(r_k, r_{k+1}) \in \breve{\mathcal{D}}_d$ for all k.
4. *$r.$ satisfies (51). In addition, $(r_k, r_{k+1}) \in \breve{\mathcal{D}}_d$ for all k.*

Proof. In the present context J_d is conserved over $q.$. Hence, equation (35) is satisfied and, by Theorem 8.5 the vertical equations in Theorem 5.11 are satisfied. Then, by Theorem 5.11, points 1 or 2 of the present result are equivalent to either one of

i. Item 3 in Theorem 5.11 holds for all horizontal variations, ($\xi_k = 0$ for all k).
ii. For all k $(v_k, r_{k+1}) \in \hat{\mathcal{D}}_d$ and condition (16) holds.

Notice that $w_k = \mathcal{A}_d(q_k, q_{k+1}) = e$ due to the conservation of J_d and the choice of $\mathcal{A}_d$. Then, $(v_k, r_{k+1}) = (\rho(q_k, e), r_{k+1}) = \mathcal{Y}(r_k, r_{k+1}) \in \hat{\mathcal{D}}_d$ if and only if $(r_k, r_{k+1}) \in \mathcal{Y}^{-1}(\hat{\mathcal{D}}_d) = \breve{\mathcal{D}}_d$. Thus the constraint conditions that appear in items i or ii are equivalent to the ones that appear in 3 or 4.

By Lemma 10.5, the variational principle that appears in item i is equivalent to the variational principle in point 3. Similarly, by Lemma 10.6, the equations that appear in item ii are equivalent to the ones that appear in 4. □

The following reconstruction result is the analogue of Theorem 7.1 which is valid in the current setting.

Theorem 11.14. *Let G be a horizontal symmetry group of $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ and $(\bar{q}_0, \bar{q}_1) \in \mathcal{D}_d$. Let $\mu = J_d(\bar{q}_0, \bar{q}_1) \in \mathfrak{g}^*$. Assume that $Ad_g^*(\mu) = \mu$ for all $g \in G$ and that G is μ-good.*

Let $r_\cdot$ *be a discrete trajectory of* $(Q/G, \breve{L}_d, \breve{\mathcal{D}}, \breve{\mathcal{D}}_d, \breve{F}_d)$ *such that* $\pi(\bar{q}_j) = r_j$ *for* $j = 0, 1$*. Define the discrete curve* $q_\cdot$ *in* Q *inductively by* $q_0 = \bar{q}_0$*, and* $q_{k+1} = \overline{h_d^{q_k}}(r_{k+1})$ *for all* k *(here the horizontal lift is associated to the affine discrete connection whose horizontal space is* $\mathcal{J}_\mu$*). Then* $q_\cdot$ *is a trajectory of the original discrete mechanical system that satisfies* $q_0 = \bar{q}_0$ *and* $q_1 = \bar{q}_1$*.*

Proof. See the proof of Theorem 10.10. □

11.3. **An example of reduction of horizontal symmetries.** In this section we apply the analysis developed in the previous sections to a system that exhibits horizontal symmetries. The system is a discretization of the nonholonomic free particle considered by J. Cortés on page 100 of [11]. More precisely, the system $(Q, L_d, \mathcal{D}, \mathcal{D}_d)$ has $Q := \mathbb{R}^3$ and

$$L_d(q_0, q_1) := \frac{m}{2}((q_1^x - q_0^x)^2 + (q_1^y - q_0^y)^2 + (q_1^z - q_0^z)^2),$$

where $q := (q^x, q^y, q^z)$. The constraints are

$$\mathcal{D}_q := \left\{\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z \in T_qQ : \dot{y} = q^x\dot{x}\right\} = \left\langle (\partial_x\big|_q + q^x\partial_y\big|_q), \partial_z\big|_q \right\rangle,$$

$$\mathcal{D}_d := \left\{(q_0, q_1) \in Q \times Q : q_1^y - q_0^y = ((q_1^x)^2 - (q_0^x)^2)/2\right\}.$$

The group $M := \mathbb{R}^2$ acts on Q by $l_g^Q(q) := (q^x, q^y + g^y, q^z + g^z)$, where $g := (g^y, g^z) \in M$. The Lie algebra of M is $\mathfrak{m}$ that we identify with $\mathbb{R}^2$. The corresponding lifted action is $l_g^{TQ}(q, v) = (l_g^Q(q), v)$. Therefore, L_d, $\mathcal{D}$ and $\mathcal{D}_d$ are M-invariant.

The vertical space for the action is $\mathcal{V}_q^M = \langle \partial_y\big|_q, \partial_z\big|_q \rangle \subset T_qQ$, so that

$$\mathcal{S}_q := \mathcal{D}_q \cap \mathcal{V}_q^M = \langle \partial_z|_q \rangle.$$

Thus, the closed subgroup $G := \{0\} \times \mathbb{R} \subset M$, satisfies $\mathcal{D}_q \cap \mathcal{V}_q^M = \mathcal{S}_q = \mathcal{V}_q^G$, for all $q \in Q$. Hence, G is a horizontal symmetry (sub)group of the system. We have $\mathfrak{g} = \{0\} \times \mathbb{R} \subset \mathbb{R}^2 = \mathfrak{m}$.

Clearly, L_d is regular as well as G-regular. Also, being G abelian, $Ad_g^*(\mu) = \mu$ for all $g \in G$ and $\mu \in \mathfrak{g}^*$. A simple computation shows that $J_d(q_0, q_1) = m(q_1^z - q_0^z)1^*$, where 1^* denotes the basis of $\mathfrak{g}^*$ that is dual to $(0, 1)$. Then, if $q \in Q$ and $\mu = \mu^z 1^*$, $J_d^q(l_g^Q(q)) = \mu$ has a unique solution $g := (0, \frac{1}{m}\mu^z) \in G$ and G is a group of μ-good symmetries. Hence Theorem 11.13 applies to identify the reduced system with a forced discrete mechanical system on $Q/G \simeq \mathbb{R}^2$. Below we give an explicit description of this reduced system.

As in the previous sections, we use the affine discrete connection $\mathcal{A}_d$ whose horizontal space is, for a fixed $\mu \in \mathfrak{g}^*$, $\mathcal{J}_\mu = \{(q_0, q_1) \in Q \times Q : m(q_1^z - q_0^z) = \mu^z\}$. Equivalently, $\mathcal{A}_d(q_0, q_1) = (0, q_1^z - q_0^z - \frac{1}{m}\mu^z) \in G$. The discrete horizontal lift of $\mathcal{A}_d$ is $\overline{h_d^q}(r) = (r', r'', \frac{1}{m}\mu^z + q_0^z)$.

As an intermediate step we have to describe the reduced system on $\tilde{G} \times (Q/G)$. Since $\pi : Q \to Q/G$ is a trivial principal bundle with structure group G we apply the description of the reduced system given in Section 9. The resulting system is

$$\begin{cases} \tilde{G} \times (Q/G) \simeq (Q/G) \times G \times (Q/G) = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2, \\ \hat{L}_d^t(r_k, \vartheta_k, r_{k+1}) = \frac{m}{2}\left((r'_{k+1} - r'_k)^2 + (r''_{k+1} - r''_k)^2 + (\frac{1}{m}\mu^z + \vartheta_k)^2\right), \\ \hat{\mathcal{D}}_r^t = \langle \partial_{r'}|_r + r'\partial_{r''}|_r \rangle \subset T_r(Q/G), \\ \hat{\mathcal{D}}_d^t = \{(r_k, \vartheta_k, r_{k+1}) \in (Q/G) \times G \times (Q/G) : r''_{k+1} - r''_k = \left((r'_{k+1})^2 - (r'_k)^2\right)/2\}. \end{cases}$$

In order to complete the description of the reduced system we need to compute the reduced forces $\hat{F}_d$. We fix a splitting

$$T_qQ = \underbrace{\langle \partial_y|_q \rangle}_{=\mathcal{W}_q} \oplus \underbrace{\{0\}}_{=\mathcal{U}_q} \oplus \underbrace{\langle \partial_z|_q \rangle}_{=\mathcal{S}_q} \oplus \underbrace{\langle \partial_x|_q + q^x\partial_y|_q \rangle}_{=\mathcal{H}_q},$$

which determines the nonholonomic connection $\mathcal{A}$. In particular, notice that since $d\mathcal{A}_d(q_0, q_1) = (0, dz|_{q_1} - dz|_{q_0})$ and $Hor_{\mathcal{A}}(\delta q) \in \langle \partial_x\big|_q + q^x\partial_y\big|_q \rangle$ the mixed curvature $\mathcal{B}_m$ vanishes, and so does the reduced force $\hat{F}_d$.

The inclusion $\mathcal{Y}$ is $\mathcal{Y}(r_k, r_{k+1}) := (r_k, e, r_{k+1})$ so that the reduced mechanical system $(\breve{Q}, \breve{L}_d, \breve{\mathcal{D}}, \breve{\mathcal{D}}_d)$ has $\breve{Q} = Q/G$ and

$$\begin{cases} \breve{L}_d(r_k, r_{k+1}) = \frac{m}{2}\big((r'_{k+1} - r'_k)^2 + (r''_{k+1} - r''_k)^2 + (\frac{1}{m}\mu^z)^2\big), \\ \breve{\mathcal{D}}_r = \langle \partial_{r'}\big|_r + r'\partial_{r''}\big|_r \rangle, \\ \breve{\mathcal{D}}_d = \{(r_k, r_{k+1}) \in (Q/G) \times (Q/G) : r''_{k+1} - r''_k = \frac{(r'_{k+1})^2 - (r'_k)^2}{2}\}. \end{cases} \tag{54}$$

The discrete Lagrange–D'Alembert equations (3) for this system are

$$\begin{cases} -(r'_{k+1} - r'_k) + (r'_k - r'_{k-1}) = \lambda_k r'_k, \\ -(r''_{k+1} - r''_k) + (r''_k - r''_{k-1}) = -\lambda_k, \\ r''_{k+1} - r''_k = ((r'_{k+1})^2 - (r'_k)^2)/2. \end{cases}$$

Those equations can be easily solved to determine the evolution of the reduced system, $r_\cdot$. The corresponding trajectory $q_\cdot$ of the original system is obtained recursively, according to Theorem 11.14, by, for a given q_k, defining

$$\begin{aligned} q_{k+1} =& (q^x_{k+1}, q^y_{k+1}, q^z_{k+1}) = \overline{h_d^{q_k}}(r_{k+1}) = (r'_{k+1}, r''_{k+1}, \frac{1}{m}\mu^z + q^z_k) \\ =& (r'_{k+1}, r''_{k+1}, q^z_1 - q^z_0 + q^z_k), \end{aligned}$$

or,

$$(q^x_k, q^y_k, q^z_k) = (r'_k, r''_k, k(q^z_1 - q^z_0) + q^z_0).$$

Interestingly, the discrete mechanical system $(\breve{Q}, \breve{L}_d, \breve{\mathcal{D}}, \breve{\mathcal{D}}_d)$ defined by (54) still has a residual symmetry group. The group $M/G \simeq \mathbb{R}$ acts on Q/G via the action induced by l^Q on Q/G. This is precisely the system and symmetry group whose reduction and reconstruction as a Chaplygin system was discussed in Examples 10.9 and 10.11. Using those results we get the trajectories of the original system

$$(q^x_k, q^y_k, q^z_k) = (r_k, (r_k^2 - r_0^2)/2 + y_0, k(q^z_1 - q^z_0) + q^z_0),$$

where r_k is determined from (52) with $r_0 = q^x_0$ and $r_1 = q^x_1$.

Remark 11.15. The presence of this second reduction step by M/G is an example of reduction by stages for discrete mechanical systems, a topic that will be discussed elsewhere.

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