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Further refinements of the Heinz inequality

Rupinderjit Kaur [a], Mohammad Sal Moslehian [b,*], Mandeep Singh [a], Cristian Conde [c]

[a] Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal 148106, Punjab, India
[b] Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran
[c] Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J. M. Gutierrez 1150, (B1613GSX) Los Polvorines and Instituto Argentino de Matemática "Alberto P. Calderón", Saavedra 15 3 piso, (C1083ACA) Buenos Aires, Argentina

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ABSTRACT

The celebrated Heinz inequality asserts that $2|||A^{1/2}XB^{1/2}||| \leqslant |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| \leqslant |||AX + XB|||$ for $X \in \mathbb{B}(\mathscr{H})$, $A, B \in \mathbb{B}(\mathscr{H})_+$, every unitarily invariant norm $||| \cdot |||$ and $\nu \in [0, 1]$. In this paper, we present several improvement of the Heinz inequality by using the convexity of the function $F(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$, some integration techniques and various refinements of the Hermite–Hadamard inequality. In the setting of matrices we prove that

$$\begin{aligned}
&\left|\left|\left|A^{\frac{\alpha+\beta}{2}}XB^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}}XB^{\frac{\alpha+\beta}{2}}\right|\right|\right| \\
&\quad \leqslant \frac{1}{|\beta-\alpha|}\left|\left|\left|\int_{\alpha}^{\beta}\left(A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\right)d\nu\right|\right|\right| \\
&\quad \leqslant \frac{1}{2}\left|\left|\left|A^{\alpha}XB^{1-\alpha} + A^{1-\alpha}XB^{\alpha} + A^{\beta}XB^{1-\beta} + A^{1-\beta}XB^{\beta}\right|\right|\right|,
\end{aligned}$$

for real numbers α, β.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a complex separable Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the full matrix algebra

* Corresponding author. Tel.: +98 9151140894; fax: +98 511 8644594.
E-mail addresses: rupinder_grewal_86@yahoo.co.in (R. Kaur), moslehian@um.ac.ir, moslehian@member.ams.org, moslehian@yahoo.com (M.S. Moslehian), msrawla@yahoo.com (M. Singh), cconde@ungs.edu.ar (C. Conde).

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$\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field. The cone of positive operators is denoted by $\mathbb{B}(\mathscr{H})_+$. A unitarily invariant norm $|||\cdot|||$ is defined on a norm ideal $\mathfrak{J}_{|||\cdot|||}$ of $\mathbb{B}(\mathscr{H})$ associated with it and has the property $|||UXV||| = |||X|||$, where U and V are unitaries and $X \in \mathfrak{J}_{|||.|||}$. Whenever we write $|||X|||$, we mean that $X \in \mathfrak{J}_{|||.|||}$. The operator norm on $\mathbb{B}(\mathscr{H})$ is denoted by $\|\cdot\|$.

The arithmetic–geometric mean inequality for two positive real numbers a, b is $\sqrt{ab} \leqslant (a+b)/2$, which has been generalized in the context of bounded linear operators as follows. For $A, B \in \mathbb{B}(\mathscr{H})_+$ and an unitarily invariant norm $|||\cdot|||$ it holds that

$$2|||A^{1/2}XB^{1/2}||| \leqslant |||AX + XB|||.$$

For $0 \leqslant \nu \leqslant 1$ and two nonnegative real numbers a and b, the *Heinz mean* is defined as

$$H_\nu(a, b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}.$$

The function H_ν is symmetric about the point $\nu = \frac{1}{2}$. Note that $H_0(a, b) = H_1(a, b) = \frac{a+b}{2}$, $H_{1/2}(a, b) = \sqrt{ab}$ and

$$H_{1/2}(a, b) \leqslant H_\nu(a, b) \leqslant H_0(a, b) \tag{1.1}$$

for $0 \leqslant \nu \leqslant 1$, i.e., the Heinz means interpolate between the geometric mean and the arithmetic mean. The generalization of (1.1) in $B(\mathscr{H})$ asserts that for operators A, B, X such that $A, B \in \mathbb{B}(\mathscr{H})_+$, every unitarily invariant norm $|||\cdot|||$ and $\nu \in [0, 1]$ the following double inequality due to Bhatia and Davis [3] holds

$$2|||A^{1/2}XB^{1/2}||| \leqslant |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \leqslant |||AX + XB|||. \tag{1.2}$$

Indeed, it has been proved that $F(\nu) = |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$ is a convex function of ν on $[0, 1]$ with symmetry about $\nu = 1/2$, which attains its minimum there at and its maximum at $\nu = 0$ and $\nu = 1$.

The second part of the previous inequality is one of the most essential inequalities in the operator theory, which is called *the Heinz inequality*; see [10]. The proof given by Heinz [11] is based on the complex analysis and is somewhat complicated. In [18], McIntosh showed that the Heinz inequality is a consequence of the following inequality

$$\|A^*AX + XBB^*\| \geqslant 2\,\|AXB\|\ ,$$

where $A, B, X \in \mathbb{B}(\mathscr{H})$. In the literature, the above inequality is called the *arithmetic–geometric mean inequality*. Fujii et al. [9] proved that the Heinz inequality is equivalent to several other norm inequalities such as the *Corach–Porta–Recht inequality* $\|AXA^{-1} + A^{-1}XA\| \geqslant 2\|X\|$, where A is a selfadjoint invertible operator and X is a selfadjoint operator; see also [6]. Audenaert [2] gave a singular value inequality for Heinz means by showing that if $A, B \in \mathcal{M}_n$ are positive semidefinite and $0 \leqslant \nu \leqslant 1$, then $s_j(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu) \leqslant s_j(A+B)$ for $j = 1, \ldots, n$, where s_j denotes the jth singular value. Also, Yamazaki [22] used the classical Heinz inequality $\|AXB\|^r\|X\|^{1-r} \geq \|A^rXB^r\|$ $(A, B, X \in \mathbb{B}(\mathscr{H}), A \geqslant 0, B \geqslant 0, r \in [0, 1])$ to characterize the chaotic order relation and to study isometric Aluthge transformations.

For a detailed study of these and associated norm inequalities along with their history of origin, refinements and applications, one may refer to [3–5,12–15].

It should be noticed that $F(1/2) \leqslant F(\nu) \leqslant \frac{F(0)+F(1)}{2}$ provides a refinement to the Jensen inequality $F(1/2) \leqslant \frac{F(0)+F(1)}{2}$ for the function F. Therefore it seems quite reasonable to obtain a new refinement of (1.2) by utilizing a refinement of Jensen's inequality. This idea was recently applied by Kittaneh [17] in virtue of the Hermite–Hadamard inequality (2.1).

One of the purposes of the present article is to obtain some new refinements of (1.2), from different refinements of inequality (2.1). We also aim to give a unified study and further refinements to the recent works for matrices.

2. The Hermite–Hadamard inequality and its refinements

For a convex function f, the double inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{2.1}$$

is known as the *Hermite–Hadamard* (H-H) inequality. This inequality was first published by Hermite in 1883 in an elementary journal and independently proved in 1893 by Hadamard. It gives us an estimation of the mean value of the convex function f; see [18,19].

There is an extensive amount of literature devoted to this simple and nice result, which has many applications in the theory of special means from which we would like to refer the reader to [20]. Interestingly, each of two sides of the H-H inequality characterizes convex functions. More precisely, if J is an interval and $f : J \to \mathbb{R}$ is a continuous function, whose restriction to every compact subinterval $[a, b]$ verifies the first inequality of (2.1) then f is convex. The same works when the first inequality is replaced by the second one.

Applying the H-H inequality, one can obtain the well-known geometric–logarithmic–arithmetic inequality

$$H_{1/2}(a, b) \leqslant L(a, b) \leqslant H_0(a, b),$$

where $L(a, b) = \int_0^1 a^t b^{1-t} dt$. An operator version of this has been proved by Hiai and Kosaki [13], which says that for $A, B \in \mathbb{B}(\mathscr{H})_+$,

$$|||A^{1/2}XB^{1/2}||| \leqslant \left|\left|\left|\int_0^1 A^\nu XB^{1-\nu} d\nu\right|\right|\right| \leqslant \frac{1}{2}|||AX + XB|||,$$

which is another refinement of the arithmetic–geometric operator inequality.

Throughout this paper we will use the following notations: For $a, b \in \mathbb{R}$ and $t \in [0, 1]$,

$$m_f(a, b) = \frac{1}{b-a}\int_a^b f(x)dx,$$

and

$$[a, b]_t = (1-t)a + tb.$$

If f is an integrable function on $[a, b]$ then

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 f(ta + (1-t)b)dt = \int_0^1 f(tb + (1-t)a)dt,$$

and if f is convex on $[a, b]$ we get

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 F_{(a,b)}(t)dt,$$

where $F_{(a,b)}(t) = \frac{1}{2}\left(f\left(a + \frac{t(b-a)}{2}\right) + f\left(b - \frac{t(b-a)}{2}\right)\right)$; see [1, Theorem 1.2].

In this section we collect various refinements of the H-H inequality for convex functions.

Theorem 2.1 [7,21]. *If $f : [a, b] \to \mathbb{R}$ is a convex function and H_t, G_t are defined on $[0, 1]$ by*

$$H_t(a, b) = \frac{1}{b-a}\int_a^b f\left(\left[\frac{a+b}{2}, x\right]_t\right) dx,$$

and

$$G_t(a, b) = \frac{1}{2(b-a)}\int_a^b \left[f([x, a]_t) + f([x, b]_t)\right]dx,$$

then H_t and G_t are convex, increasing and

$$f\left(\frac{a+b}{2}\right)=H_0(a,b)\leqslant H_t(a,b)\leqslant H_1(a,b)=m_f(a,b), \tag{2.2}$$

$$m_f(a,b)=G_0(a,b)\leqslant G_t(a,b)\leqslant G_1(a,b)=\frac{f(a)+f(b)}{2} \tag{2.3}$$

for all $t\in[0,1]$. Furthermore,

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leqslant\frac{2}{b-a}\int_{\frac{(3a+b)}{4}}^{\frac{(a+3b)}{4}}f(x)dx\leqslant\int_0^1 H_t(a,b)dt\\&\leqslant\frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+m_f(a,b)\right)\leqslant m_f(a,b)\end{aligned}$$

and

$$\begin{aligned}\frac{2}{b-a}\int_{\frac{(3a+b)}{4}}^{\frac{(a+3b)}{4}}f(x)dx&\leqslant\frac{1}{2}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\leqslant\int_0^1 G_t(a,b)dt\\&\leqslant\frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right)\leqslant\frac{f(a)+f(b)}{2}.\end{aligned} \tag{2.4}$$

Remark 2.2. (1) From (2.4) we get that

$$m_f(a,b)\leqslant\frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right)\leqslant\frac{f(a)+f(b)}{2},$$

which is the well-known Bullen's inequality; see [20, p. 140]. As an immediate consequence, from the previous inequality, we note that the first inequality is stronger than the second one in (2.1), i.e.

$$m_f(a,b)-f\left(\frac{a+b}{2}\right)\leqslant\frac{f(a)+f(b)}{2}-m_f(a,b).$$

(2) We note some properties of H_t and G_t useful in the next sections. For $\mu\in[0,1]$ we get
  (a) $H_t(\mu,1-\mu)=\frac{1}{1-2\mu}\int_\mu^{1-\mu}f\left(\left[\frac{1}{2},x\right]_t\right)dx=\frac{1}{2\mu-1}\int_{1-\mu}^{\mu}f\left(\left[\frac{1}{2},x\right]_t\right)dx=H_t(1-\mu,\mu)$.
  (b) $G_t(\mu,1-\mu)=\frac{1}{2(1-2\mu)}\int_\mu^{1-\mu}[f([x,\mu]_t)+f([x,1-\mu]_t)]dx=G_t(1-\mu,\mu)$.

Recently, the following result was proved:

Theorem 2.3 [21]. *If f is a convex function defined on open interval J, $a,b\in J$ with $a<b$ and the mapping T_t is defined by*

$$T_t(a,b)=\frac{1}{2}\left(f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right),$$

then T_t is convex and increasing on $[0,1]$ and

$$f\left(\frac{a+b}{2}\right)\leqslant T_\eta(a,b)\leqslant T_\xi(a,b)\leqslant T_\lambda(a,b)\leqslant\frac{f(a)+f(b)}{2},$$

for all $\eta\in(0,\xi)$, $\lambda\in(\xi,1)$, where $T_\xi(a,b)=m_f(a,b)$.

In [8], the author asked whether for a convex function f on an interval J there exist real numbers l, L such that

$$f\left(\frac{a+b}{2}\right) \leqslant l \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant L \leqslant \frac{f(a)+f(b)}{2}.$$

An affirmative answer to this question is given as follows.

Theorem 2.4 [8]. *Assume that $f : [a, b] \to \mathbb{R}$ is a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leqslant l(\lambda) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant L(\lambda) \leqslant \frac{f(a)+f(b)}{2} \tag{2.5}$$

for all $\lambda \in [0, 1]$, where

$$l(\lambda) = \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$L(\lambda) = \frac{1}{2}\big(f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda)f(b)\big).$$

Remark 2.5. Applying inequality (2.5) for $\lambda = \frac{1}{2}$ we get

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right) \leqslant m_f(a,b)$$
$$\leqslant \frac{1}{2}\left(f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right) \leqslant \frac{f(a)+f(b)}{2}.$$

This result has been obtained by Akkouchi in [1].

3. Refinements of the Heinz inequality for operators

In this section we use the convexity of $F(\nu) = |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$; $\nu \in [0, 1]$ and the different refinements of inequality (2.1) described in the previous section.

Theorem 3.1. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$. Then for any $t, \mu \in [0, 1]$ and any unitarily invariant norm $|||\cdot|||$,*

$$\begin{aligned} 2|||A^{1/2}XB^{1/2}||| &\leqslant \frac{1}{1-2\mu}\int_\mu^{1-\mu} F([1/2, x]_t)dx \\ &\leqslant \frac{1}{1-2\mu}\int_\mu^{1-\mu} |||A^x XB^{1-x} + A^{1-x}XB^x|||dx \\ &\leqslant \frac{1}{2(1-2\mu)}\int_\mu^{1-\mu} [F([x,\mu]_t) + F([x, 1-\mu]_t)]dx \\ &\leqslant |||A^\mu XB^{1-\mu} + A^{1-\mu}XB^\mu|||. \end{aligned}$$

Proof. For $\mu \neq \frac{1}{2}$ the inequalities follows by applying inequalities (2.2) and (2.3) on the interval $[\mu, 1-\mu]$ if $0 \leqslant \mu < \frac{1}{2}$ or $[1-\mu, \mu]$ if $\frac{1}{2} < \mu \leqslant 1$. Finally

$$\lim_{\mu \to \frac{1}{2}} \frac{1}{2(1-2\mu)}\int_\mu^{1-\mu} \big(F([x,\mu]_t) + F([x, 1-\mu]_t)\big)\, dx = 2|||A^{1/2}XB^{1/2}|||$$

completes the proof. □

Applying Theorem 2.1 to the function F on the interval $\left[\mu, \frac{1}{2}\right]$ or $\left[\frac{1}{2}, \mu\right]$ for $\mu \in [0, 1]$ we obtain the following refinement of [17, Theorem 2 and Corollary 1].

Theorem 3.2. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$. Then for every $\mu \in [0, 1]$ and every unitarily invariant norm $|||\cdot|||$,*

$$\begin{aligned}
2|||A^{1/2}XB^{1/2}||| &\leqslant |||A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}||| \\
&\leqslant \frac{4}{1-2\mu}\int_{\frac{(6\mu+1)}{8}}^{\frac{(2\mu+3)}{8}} |||A^xXB^{1-x} + A^{1-x}XB^x|||dx \leqslant \int_0^1 H_t(1/2, \mu)dt \\
&\leqslant \frac{1}{2}|||A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}||| + \frac{1}{1-2\mu}\int_\mu^{1/2} F(x)dx \\
&\leqslant \frac{2}{1-2\mu}\int_\mu^{1/2} |||A^xXB^{1-x} + A^{1-x}XB^x|||dx = G_0(1/2, \mu) \leqslant \int_0^1 G_t(1/2, \mu)dt \\
&\leqslant \frac{1}{2}\left(|||A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}||| + |||A^\mu XB^{1-\mu} + A^{1-\mu}XB^\mu||| + F(1/2)\right) \\
&\leqslant \frac{1}{2}|||A^\mu XB^{1-\mu} + A^{1-\mu}XB^\mu||| + |||A^{1/2}XB^{1/2}||| \\
&\leqslant |||A^\mu XB^{1-\mu} + A^{1-\mu}XB^\mu|||.
\end{aligned}$$

Now, we have the following refinement of the first part of the the Heinz inequality via certain sequences.

Theorem 3.3. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$ and for $n_0 \geqslant 0$,*

$$x_n(F, a, b) = \frac{1}{2^n}\sum_{i=1}^{2^n} F\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{2^n}\right),$$

$$y_n(F, a, b) = \frac{1}{2^n}\left(\frac{F(a)+F(b)}{2} + \sum_{i=1}^{2^n-1} F\left([a, b]_{\frac{i}{2^n}}\right)\right).$$

Then

(1) For $\mu \in [0, 1/2]$ and for every unitarily invariant norm $|||\cdot|||$,

$$\begin{aligned}
2|||A^{1/2}XB^{1/2}||| &= x_0(F, \mu, 1-\mu) \leqslant \cdots \leqslant x_n(F, \mu, 1-\mu) \\
&\leqslant \frac{1}{1-2\mu}\int_\mu^{1-\mu} |||A^xXB^{1-x} + A^{1-x}XB^x|||dx \\
&\leqslant y_n(F, \mu, 1-\mu) \leqslant \cdots \leqslant y_0(F, \mu, 1-\mu) = F(\mu)
\end{aligned}$$

(2) For $\mu \in [1/2, 1]$ and for every unitarily invariant norm $|||\cdot|||$,

$$\begin{aligned}
2|||A^{1/2}XB^{1/2}||| &= x_0(F, 1-\mu, \mu) \leqslant \cdots \leqslant x_n(F, 1-\mu, \mu) \\
&\leqslant \frac{1}{2\mu-1}\int_{1-\mu}^{\mu} |||A^xXB^{1-x} + A^{1-x}XB^x|||dx \\
&\leqslant y_n(F, 1-\mu, \mu) \leqslant \cdots \leqslant y_0(F, 1-\mu, \mu) = F(\mu)
\end{aligned}$$

Applying the Theorem 2.4, we obtain the following refinement.

Theorem 3.4. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$ and $\alpha, \beta \in [0,1]$ and $|||\cdot|||$ be a unitarily invariant norm. Then*

$$F\left(\frac{\alpha+\beta}{2}\right) \leqslant l(\lambda) \leqslant \frac{1}{b-a}\int_a^b F(x)dx \leqslant L(\lambda) \leqslant \frac{F(\alpha)+F(\beta)}{2}$$

for all $\lambda \in [0,1]$, where

$$l(\lambda) = \lambda F\left(\frac{\lambda\beta+(2-\lambda)\alpha}{2}\right) + (1-\lambda)F\left(\frac{(1+\lambda)\beta+(1-\lambda)\alpha}{2}\right)$$

and

$$L(\lambda) = \frac{1}{2}\big(F(\lambda\beta+(1-\lambda)\alpha) + \lambda F(\alpha) + (1-\lambda)F(\beta)\big).$$

Finally, using the refinement presented in Theorem 2.3 we get the following statement.

Theorem 3.5. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$. For $a, b \in (0,1)$ with $a < b$ let T_t be the mapping defined in $[0,1]$ by*

$$T_t(a,b) = \frac{1}{2}\left(F\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + F\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right).$$

Then, there exists $\xi \in (0,1)$ such that for any $\mu \in (0,1)$ and any unitary invariant norm $|||\cdot|||$,

$$\begin{aligned} 2|||A^{1/2}XB^{1/2}||| &\leqslant T_\eta(\mu, 1-\mu) \leqslant T_\xi(\mu, 1-\mu) = \frac{1}{1-2\mu}\int_\mu^{1-\mu} F(x)dx \\ &\leqslant T_\lambda(\mu, 1-\mu) \leqslant |||A^\mu XB^{1-\mu} + A^{1-\mu}XB^\mu|||, \end{aligned}$$

where $\eta \in [0,\xi]$ and $\lambda \in [\xi, 1]$.

From the generalization of the H-H inequality due to Vasić and Lacković, we get

Theorem 3.6. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$ and let p, q be positive numbers and $0 \leqslant \alpha < \beta \leqslant 1$. Then the double inequality*

$$F\left(\frac{p\alpha+q\beta}{p+q}\right) \leqslant \frac{1}{2y}\int_{c-y}^{c+y} F(t)dt \leqslant \frac{pF(\alpha)+qF(\beta)}{p+q}$$

holds for $c = \frac{p\alpha+q\beta}{p+q}$, $y > 0$ if and only if $y \leqslant \frac{\beta-\alpha}{p+q}\min\{p,q\}$.

4. Refinement of the Heinz inequality for matrices

In what follows, the capital letters $A, B, X, \ldots$ denote arbitrary elements of $\mathcal{M}_n$. By $\mathbb{P}_n$ we denote the set of positive definite matrices. The Schur product of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in M_n is the entrywise product and denoted by $A \circ B$. We shall state the following preliminary result, which is needed to prove our main results.

If $X = [x_{ij}]$ is positive semidefinite, then for any matrix Y, we have

$$|||X \circ Y||| \leqslant |||Y||| \max_i x_{ii} \tag{4.1}$$

for every unitarily invariant norm $|||\cdot|||$. For a proof of this, the reader may be referred to [11].

Theorem 4.1. *Let $A, B \in \mathbb{P}_n$ and $X \in M_n$. Then for any real numbers α, β and any unitarily invariant norm $|||\cdot|||$,*

$$\left|\left|\left|A^{\frac{\alpha+\beta}{2}}XB^{1-\frac{\alpha+\beta}{2}}+A^{1-\frac{\alpha+\beta}{2}}XB^{\frac{\alpha+\beta}{2}}\right|\right|\right| \leqslant \frac{1}{|\beta-\alpha|}\left|\left|\left|\int_\alpha^\beta \left(A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right)d\nu\right|\right|\right|$$
$$\leqslant \frac{1}{2}\left|\left|\left|A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^\alpha+A^\beta XB^{1-\beta}+A^{1-\beta}XB^\beta\right|\right|\right|. \tag{4.2}$$

Proof. Without loss of generality assume that $\alpha < \beta$. We shall first prove the result for the case $A = B$. Since the norms considered here are unitarily invariant, so we can assume that A is diagonal, i.e. $A = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Note that

$$A^{\frac{\alpha+\beta}{2}}XA^{1-\frac{\alpha+\beta}{2}}+A^{1-\frac{\alpha+\beta}{2}}XA^{\frac{\alpha+\beta}{2}} = Y\circ\left(\int_\alpha^\beta \left(A^\nu XA^{1-\nu}+A^{1-\nu}XA^\nu\right)d\nu\right),$$

where Y is a Hermitian matrix. If $X = [x_{ij}]$ and $Y = [y_{ij}]$, then

$$\left[\lambda_i^{\frac{\alpha+\beta}{2}}x_{ij}\lambda_j^{1-\frac{\alpha+\beta}{2}}+\lambda_i^{1-\frac{\alpha+\beta}{2}}x_{ij}\lambda_j^{\frac{\alpha+\beta}{2}}\right] = \left[y_{ij}\int_\alpha^\beta\left(\lambda_i^\nu x_{ij}\lambda_j^{1-\nu}+\lambda_i^{1-\nu}x_{ij}\lambda_j^\nu\right)d\nu\right],$$

whence

$$\begin{aligned} y_{ij} &= \frac{\lambda_i^{\frac{\alpha+\beta}{2}}\lambda_j^{1-\frac{\alpha+\beta}{2}}+\lambda_i^{1-\frac{\alpha+\beta}{2}}\lambda_j^{\frac{\alpha+\beta}{2}}}{\int_\alpha^\beta\left(\exp\left(\log(\lambda_i)\nu+\log(\lambda_j)(1-\nu)\right)+\exp\left(\log(\lambda_i)(1-\nu)+\log(\lambda_j)\nu\right)\right)d\nu} \\ &= \frac{\lambda_i^{\frac{\beta-\alpha}{2}}\left(\lambda_i^\alpha\lambda_j^{1-\beta}+\lambda_i^{1-\beta}\lambda_j^\alpha\right)\lambda_j^{\frac{\beta-\alpha}{2}}(\log\lambda_i-\log\lambda_j)}{\lambda_i^\beta\lambda_j^{1-\beta}-\lambda_i^{1-\beta}\lambda_j^\beta-\lambda_i^\alpha\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^\alpha} \\ &= \frac{\lambda_i^{\frac{\beta-\alpha}{2}}(\log\lambda_i-\log\lambda_j)\lambda_j^{\frac{\beta-\alpha}{2}}}{\lambda_i^{\beta-\alpha}-\lambda_j^{\beta-\alpha}}, \quad \text{for } i\neq j \end{aligned}$$

and $y_{ii} = \frac{1}{\beta-\alpha} > 0$. By (4.1), it is enough to show that the matrix Y is positive semidefinite, or equivalently the matrix

$$y'_{ij} = \begin{cases} \frac{\log\lambda_i-\log\lambda_j}{\lambda_i^{\beta-\alpha}-\lambda_j^{\beta-\alpha}} & \text{if } i\neq j \\ \frac{1}{(\beta-\alpha)\lambda_i^{\beta-\alpha}} & \text{if } i=j \end{cases}$$

is positive semidefinite. On taking $\lambda_i^{\beta-\alpha} = s_i$, we get

$$(\beta-\alpha)y'_{ij} = \begin{cases} \frac{\log s_i-\log s_j}{s_i-s_j} & \text{if } i\neq j \\ \frac{1}{s_i} & \text{if } i=j, \end{cases}$$

which is a positive semidefinite matrix, since the matrix on the right hand side is the Löwner matrix corresponding to the matrix monotone function $\log x$; see [4, Theorem 5.3.3]. This proves the first inequality in (4.2) for the case $A = B$.

The second inequality will follow on the same lines. We indeed have

$$\int_\alpha^\beta\left(A^\nu XA^{1-\nu}+A^{1-\nu}XA^\nu\right)d\nu = Z\circ\left(A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^\alpha+A^\beta XB^{1-\beta}+A^{1-\beta}XB^\beta\right),$$

where Z is the Hermitian matrix with entries

$$z_{ij} = \begin{cases} \frac{\lambda_i^{\beta-\alpha} - \lambda_j^{\beta-\alpha}}{(\log \lambda_i - \log \lambda_j)(\lambda_i^{\beta-\alpha} + \lambda_j^{\beta-\alpha})} & \text{if } i \neq j \\ \frac{(\beta-\alpha)}{2} & \text{if } i = j\,. \end{cases}$$

On taking $\lambda_i^{\beta-\alpha} = e^{t_i}$ we conclude that Z is positive semidefinite if and only if so is the following matrix

$$\frac{2}{\beta-\alpha} z'_{ij} = \begin{cases} \frac{\tanh((t_i-t_j)/2)}{(t_i-t_j)/2} & \text{if } i \neq j \\ 1 & \text{if } i = j\,. \end{cases}$$

The right hand side matrix is positive semidefinite since the function $f(x) = \frac{\tanh x}{x}$ is positive definite; see [4, Example 5.2.11]. This proves the second inequality in (4.2) for the case $A = B$.

The general case follows on replacing A by $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and X by $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$. □

The first corollary provides some variants of [17, Theorems 2 and 3]. It should be noticed that

$$\lim_{\mu \to 1/2} \left(\frac{2}{|1-2\mu|} \left|\left|\left| \int_\mu^{1/2} (A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu) d\nu \right|\right|\right| \right) = 2 \left|\left|\left| A^{1/2} X B^{1/2} \right|\right|\right|$$

and

$$\lim_{\mu \to 0} \left(\frac{1}{|\mu|} \left|\left|\left| \int_0^\mu (A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu) d\nu \right|\right|\right| \right) = |||AX + XB|||\,.$$

Corollary 4.2. *Let $A, B \in \mathbb{P}_n$, $X \in M_n$, μ be a real number and $|||\cdot|||$ be any unitarily invariant norm. Then*

$$\left|\left|\left| A^{\frac{2\mu+1}{4}} X B^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}} X B^{\frac{2\mu+1}{4}} \right|\right|\right| \leqslant \frac{2}{|1-2\mu|} \left|\left|\left| \int_\mu^{1/2} (A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu) d\nu \right|\right|\right|$$

$$\leqslant \frac{1}{2} \left|\left|\left| A^\mu X B^{1-\mu} + A^{1-\mu} X B^\mu + 2A^{1/2} X B^{1/2} \right|\right|\right|\,,$$

$$\left|\left|\left| A^{\frac{\mu}{2}} X B^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}} X B^{\frac{\mu}{2}} \right|\right|\right| \leqslant \frac{1}{|\mu|} \left|\left|\left| \int_0^\mu (A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu) d\nu \right|\right|\right|$$

$$\leqslant \frac{1}{2} \left|\left|\left| AX + XB + A^\mu X B^{1-\mu} + A^{1-\mu} X B^\mu \right|\right|\right|\,.$$

The following consequence provides a matrix analogue of (1.1).

Corollary 4.3. *Let $A, B \in \mathbb{P}_n$ and $X \in M_n$. Then for any $0 \leqslant \alpha < \beta \leqslant 1$ with $\alpha + \beta \leqslant 2$ and any unitarily invariant norm $|||\cdot|||$,*

$$2|||A^{1/2} X B^{1/2}||| \leqslant \left|\left|\left| A^{\frac{\alpha+\beta}{2}} X B^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} X B^{\frac{\alpha+\beta}{2}} \right|\right|\right|$$

$$\leqslant \frac{1}{|\beta-\alpha|} \left|\left|\left| \int_\alpha^\beta \left(A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu \right) d\nu \right|\right|\right|$$

$$\leqslant \frac{1}{2}\left|\left|\left|A^{\alpha}XB^{1-\alpha}+A^{1-\alpha}XB^{\alpha}+A^{\beta}XB^{1-\beta}+A^{1-\beta}XB^{\beta}\right|\right|\right|$$
$$\leqslant \frac{1}{2}\left|\left|\left|A^{\alpha}XB^{1-\alpha}+A^{1-\alpha}XB^{\alpha}\right|\right|\right|+\frac{1}{2}\left|\left|\left|A^{\beta}XB^{1-\beta}+A^{1-\beta}XB^{\beta}\right|\right|\right|$$
$$\leqslant |||AX+XB|||.$$

Proof. Applying the triangle inequality, the properties of the function $f(\nu) = |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||$ and Theorem 4.1 we get the required inequalities. □

It is shown in [17, Corollary 3] that

$$|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}||| \leqslant 4r_0|||A^{1/2}XB^{1/2}|||+(1-2r_0)|||AX+XB|||. \tag{4.3}$$

A natural generalization of (4.3) would be

$$|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}||| \leqslant |||4r_0A^{1/2}XB^{1/2}+(1-2r_0)(AX+XB)|||$$

for $0 \leqslant \nu \leqslant 1$ and $r_0 = \min\{\nu, 1-\nu\}$ with $A, B \in \mathbb{P}_n$ and $X \in M_n$, which in fact is not true, in general. The following counterexample justifies this:

Take $X = \begin{bmatrix} 52.39 & 38.71 & 12.36 \\ 32.86 & 35.38 & 64.82 \\ 91.79 & 99.45 & 66.10 \end{bmatrix}$, $A = \begin{bmatrix} 92.315 & 87.791 & 71.090 \\ 87.791 & 120.130 & 83.340 \\ 71.090 & 83.340 & 103.610 \end{bmatrix}$,

$B = \begin{bmatrix} 118.482 & 23.249 & 112.676 \\ 23.249 & 10.343 & 38.224 \\ 112.676 & 38.224 & 156.551 \end{bmatrix}$ and $\nu = 0.4680$. Then $\mathrm{tr}|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}| = 78135.5$, while $\mathrm{tr}|4r_0A^{1/2}XB^{1/2}+(1-2r_0)(AX+XB)| = 78125.4$.

We shall, however, present another result, which is a possible generalization of (4.3).

Theorem 4.4. *Let $A, B \in \mathbb{P}_n$ and $X \in M_n$. Then for $\nu \in [0,1]$ and for every unitarily invariant norm $|||\cdot|||$,*

$$|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}||| \leqslant |||4r_1(\nu)A^{1/2}XB^{1/2}+(1-2r_1(\nu))(AX+XB)|||, \tag{4.4}$$

where $r_1(\nu) = \min\{\nu, \left|\frac{1}{2}-\nu\right|, 1-\nu\}$.

Proof. First, we consider the case $\nu \in [0, 1/2]$. Notice that by some simple algebraic or geometrical arguments, we may conclude that $0 \leqslant r_1 \leqslant 1/4$. Again, by following a similar way as in Theorem 4.1, we can write the matrix

$$A^{\nu}XA^{1-\nu}+A^{1-\nu}XA^{\nu} = W \circ (4r_1A^{1/2}XA^{1/2}+(1-2r_1)(AX+XA)),$$

where W is a Hermitian matrix with entries

$$w_{ij} = \begin{cases} \frac{\lambda_i^{\nu}(\lambda_i^{1-2\nu}+\lambda_j^{1-2\nu})\lambda_j^{\nu}}{4r_1\lambda_i^{1/2}\lambda_j^{1/2}+(1-2r_1)(\lambda_i+\lambda_j)} & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Now, observe that $0 \leqslant \frac{4r_1}{1-2r_1} \leqslant 2$ and $0 \leqslant 1-2\nu \leqslant 1$, so the matrix W is positive semidefinite; see [5, Theorem 5.2, p. 225]. On repeating the same argument as in Theorem 4.1, the required inequality (4.4) follows.

Finally, if $\nu \in [\frac{1}{2}, 1]$ let $\mu = 1 - \nu \in [0, \frac{1}{2}]$, then by the previous case we have

$$\begin{aligned}|||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| &= |||A^{1-\mu}XB^{\mu} + A^{\mu}XB^{1-\mu}||| \\ &\leqslant |||4r_1(\mu)A^{\frac{1}{2}}XB^{\frac{1}{2}} + (1 - 2r_1(\mu))(AX + XB)|||,\end{aligned}$$

where $r_1(\mu) = \min\left\{\mu, \left|\frac{1}{2} - \mu\right|, 1 - \mu\right\} = r_1(\nu)$. □

From the previous theorem, we deduce a new refinement of the Heinz inequality for matrices.

Corollary 4.5. *Let $A, B \in \mathbb{P}_n$ and $X \in M_n$. Then for $\nu \in [0, 1]$ and for every unitarily invariant norm $|||\cdot|||$,*

$$\begin{aligned}|||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| &\leqslant |||4r_1(\nu)A^{1/2}XB^{1/2} + (1 - 2r_1(\nu))(AX + XB)||| \\ &\leqslant 4r_1(\nu)|||A^{1/2}XB^{1/2}||| + (1 - 2r_1(\nu))|||AX + XB||| \\ &\leqslant 2(2r_1(\nu) - 1)|||A^{1/2}XB^{1/2}||| + 2(1 - r_1(\nu))|||AX + XB||| \\ &\leqslant |||AX + XB|||,\end{aligned}$$

where $r_1(\nu) = \min\{\nu, \left|\frac{1}{2} - \nu\right|, 1 - \nu\}$.

As a direct consequence of Theorem 4.4, we obtain the following refinement of an inequality (see [6]).

Corollary 4.6. *Let $A, B \in \mathbb{P}_n$, $X \in M_n$, $r \in \left[\frac{1}{2}, \frac{3}{2}\right]$ and $t \in (-2, 2]$. Then for every unitarily invariant norm $|||\cdot|||$,*

$$\begin{aligned}|||A^{r}XB^{2-r} + A^{2-r}XB^{r}||| &\leqslant |||4sAXB + (1 - 2s)(A^{3/2}XB^{1/2} + A^{1/2}XB^{3/2})||| \\ &\leqslant 4s|||AXB||| + (1 - 2s)|||A^{3/2}XB^{1/2} + A^{1/2}XB^{3/2}||| \\ &\leqslant 4s|||AXB||| + (1 - 2s)\frac{2}{t+2}|||A^2X + tAXB + XB^2||| \\ &\leqslant 2(2s - 1)|||AXB||| + \frac{4(1 - s)}{t+2}|||A^2X + tAXB + XB^2||| \\ &\leqslant \frac{2}{t+2}|||A^2X + tAXB + XB^2|||\end{aligned}$$

in which $s = \min\left\{r - \frac{1}{2}, |1 - r|, \frac{3}{2} - r\right\}$.

Proof. Let $Y = A^{1/2}XB^{1/2} \in M_n$ and $\nu = r - \frac{1}{2} \in [0, 1]$. It follows from Theorem 4.4 that

$$\begin{aligned}|||A^{r}XB^{2-r} + A^{2-r}XB^{r}||| &= |||A^{r}A^{-1/2}YB^{-1/2}B^{2-r} + A^{2-r}A^{-1/2}YB^{-1/2}B^{r}||| \\ &= |||A^{\nu}YB^{1-\nu} + A^{1-\nu}YB^{1-\nu}||| \\ &\leqslant |||4r_1(\nu)A^{1/2}YB^{1/2} + (1 - 2r_1(\nu))(AY + YB)||| \\ &= |||4r_1(\nu)AXB + (1 - 2r_1(\nu))(A^{3/2}XB^{1/2} + A^{1/2}XB^{3/2})|||,\end{aligned}$$

where $r_1(\nu) = \min\left\{\nu, \left|\frac{1}{2} - \nu\right|, 1 - \nu\right\}$. Let $s = r_1\left(r - \frac{1}{2}\right)$. Applying the triangle inequality and Zhan's inequality, we obtain

$$\begin{aligned}|||A^{r}XB^{2-r} + A^{2-r}XB^{r}||| &\leqslant |||4sAXB + (1 - 2s)(A^{3/2}XB^{1/2} + A^{1/2}XB^{3/2})||| \\ &\leqslant 4s|||AXB||| + (1 - 2s)|||A^{3/2}XB^{1/2} + A^{1/2}XB^{3/2}|||\end{aligned}$$

$$
\begin{aligned}
&\leqslant 4s|||AXB||| + \frac{2(1-2s)}{t+2}|||A^2X + tAXB + XB^2||| \\
&\leqslant 2(2s-1)|||AXB||| + \frac{4(1-s)}{t+2}|||A^2X + tAXB + XB^2||| \\
&\leqslant \frac{2}{t+2}|||A^2X + tAXB + XB^2|||. \quad \square
\end{aligned}
$$

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