THE HOMOLOGY OF FREE RACKS AND QUANDLES

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Abstract. We prove that rack (resp. quandle) homology of the free rack (resp. free quandle) is trivial.

Introduction

A rack is a set X equipped with a bijective, self-right-distributive binary operation, while a quandle is a rack which satisfies an idempotency condition. The earliest work on racks is due to Conway and Wraith (which used the name wracks for this concept) and appears in unpublished correspondence between the authors. Their work is inspirated by the conjugacy operation in a group and so focuses in the special case of quandles, but they also were aware of the general notion. Some of the first published works about racks are [K], where they are called crystals, [B], where they are called automorphic sets and [F-R], in which the name rack (a modification of wrack) was introduced. The algebraic structure of quandle was introduced independently by Joyce and Matveev ([J] and [M]) in order to obtain invariants of knotted circles, i.e. invariants of embeddings of the circle in three space.

Rack homology and cohomology with coefficients in an abelian group was introduced by Fenn, Rourke and Sanderson in [F-R-S]. The quandle homology and cohomology theory was developed in [C-J-K-L-S] in order to define invariants of classical knots and knotted surfaces. We refer to [C] for the history of quandles and applications of quandle (co)homology. A suitable property for a (co)homology theory is that it vanishes in positive degree on free objects. Professor J. L. Loday asked the first author of the present paper if this happen in the theory of racks and quandles. In this note we give a positive answer to this question. In order to carry out this task first we give an explicit description of the free rack on a set X, and then we prove that the rack cohomology of the free rack on X vanishes for positive degree, and it is free of rank the cardinality of X in degree zero. Finally we prove the analogous statement for free quandles and quandle homology.

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Note. The first author remember and thanks heartily his last visit to Strasbourg, by generous invitation of Jean Louis Loday, and the large list of comments and remarks about rack and quandle homology that he received from him. At that time, in analogy with the relation between groups and Lie algebras, J.L. Loday looked at rack homology as a candidate to replace group homology, if one also replace Lie algebras with Leibnitz algebras. The question of triviality of the homology of free racks was part of natural conditions for this analogy to work fine. After writing this mauscript, prof. Jean Louis Loday passed away unexpectedly, we dedicate this work to his memory.

1. The free rack on X

In this section we recall the definition of rack and give a construction of the free rack on a set X.

Definition 1.1. A non empty set R together with a binary operation $\triangleleft\colon R \times R \to R$ is called a *rack* if the following two axioms are satisfied

(1) for any $a \in R$, the map $-\triangleleft a\colon R \to R$, $b \mapsto b \triangleleft a$ is bijective, and
(2) for any $a, b, c \in R$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

A rack $(Q, \triangleleft)$ is a *quandle* if $a \triangleleft a = a$ for all $a \in Q$.

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Definition 1.2. The *free rack* on a non empty set X is a rack $R(X)$ together with a map $\jmath\colon X \to R(X)$, such that for any other map $f\colon X \to Y$, from X into a rack Y, there exists a unique morphism of racks $\widetilde{f}\colon R(X) \to Y$, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \jmath}\big\downarrow & \nearrow_{\widetilde{f}} & \\ R(X) & & \end{array}$$

commutes.

It is clear that if the free rack on X exists, then it is unique up to rack isomorphism.

Proposition 1.3. *Let X be a non empty set and let $F(X)$ be the free group on X. The free rack $R(X)$ on X always exists, and it is the set $X \times F(X)$ endowed with the rack operation*

$$(x,\alpha) \triangleleft (y,\beta) := (x, \alpha\beta^{-1}y\beta) \quad \textit{for } x,y \in X,\ \alpha,\beta \in F(X).$$

The map $\jmath\colon X \to R(X)$ is given by $\jmath(x) = (x,1)$.

Before the proof it is convenient to recall the notion of a G-crossed set.

Definition 1.4. Let G be a group. A *G-crossed set* is a right G-set X endowed with a map $\partial\colon X \to G$ such that

$$\partial(x \cdot g) = g^{-1}\partial(x)g \quad \text{for all } g \in G,\ x \in X.$$

The following result is straightforward.

Lemma 1.5. *If $\partial\colon X \to G$ is a G-crossed set, then*

1. *The binary operation $\triangleleft\colon X \times X \to X$ given by $x \triangleleft y := x \cdot \partial(y)$ defines a rack structure on X.*
2. *Let $\operatorname{Conj}(G)$ be the underlying set of G viewed as a rack via conjugation, namely via $g \triangleleft h := h^{-1}gh$ for all $g,h \in G$. Then $\partial\colon (X,\triangleleft) \to \operatorname{Conj}(G)$ is a rack homomorphism.*

Proof of Proposition 1.3. It is clear that $R(X) := X \times F(X)$ is a right $F(X)$-set via $(x,\alpha)\cdot\beta := (x,\alpha\beta)$. It is easy to check that the map

$$\begin{array}{rcl} R(X) & \xrightarrow{\ \partial\ } & F(X) \\ (x,\alpha) & \longmapsto & \alpha^{-1}x\alpha \end{array}$$

gives an $F(X)$-crossed set structure on $R(X)$. Hence $R(X)$ is a rack via

$$(x,\alpha) \triangleleft (y,\beta) := (x,\alpha) \cdot \partial(y,\beta) = (x,\alpha) \cdot \beta^{-1}y\beta = (x,\alpha\beta^{-1}y\beta).$$

Let us consider a function $f\colon X \to Y$, where Y is a rack. We will see that there exists a unique morphism of racks $\widetilde{f}\colon R(X) \to Y$ such that $\widetilde{f}(x,1) = f(x)$ for all $x \in X$. To begin with, observe that if $x,z \in X$ and $\alpha \in F(X)$, then

$$(x,\alpha) \triangleleft (z,1) = (x,\alpha z),$$

and consequently

$$(x,\alpha z^{-1}) = (x,\alpha) \triangleleft^{-1} (z,1).$$

We conclude that if $\alpha = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ with $x_i \in X$ and $\epsilon_i = \pm 1$, then

$$(x,\alpha) = ((\cdots(((x,1) \triangleleft^{\epsilon_1} (x_1,1)) \triangleleft^{\epsilon_2} (x_2,1)) \cdots) \triangleleft^{\epsilon_{n-1}} (x_{n-1},1)) \triangleleft^{\epsilon_n} (x_n,1).$$

In this way we get that if there exists such $\widetilde{f}$, then necessarily

$$\widetilde{f}(x,\alpha) = ((\dots((f(x) \triangleleft^{\epsilon_1} f(x_1)) \triangleleft^{\epsilon_2} f(x_2)) \triangleleft \cdots) \triangleleft^{\epsilon_{n-1}} f(x_{n-1})) \triangleleft^{\epsilon_n} f(x_n). \tag{1.1}$$

So, if $\widetilde{f}$ exists, it is unique. The idea is to define $\widetilde{f}$ by the above formula in such a way that will be clear that $\widetilde{f}$ is well defined and to prove that it is a morphism of racks, namely, that

$$\widetilde{f}\big((x,\alpha) \triangleleft (y,\beta)\big) = \widetilde{f}(x,\alpha) \triangleleft \widetilde{f}(y,\beta). \tag{1.2}$$

In order to carry out this task we define a map $\widetilde{f}\colon X \times M(X,X^{-1}) \to Y$, where $M(X,X^{-1})$ is the free monoid in X and the "inverses" X^{-1}, by the formula (1.1). One will have for instance

$$\begin{aligned} \widetilde{f}(x,\alpha yy^{-1}) &= [[(\dots(f(x) \triangleleft^{\epsilon_1} f(x_1)) \triangleleft \cdots) \triangleleft^{\epsilon_n} f(x_n)] \triangleleft f(y)] \triangleleft^{-1} f(y) \\ &= ((\widetilde{f}(x,\alpha)) \triangleleft f(y)) \triangleleft^{-1} f(y) \\ &= \widetilde{f}(x,\alpha), \end{aligned}$$

and so, $\widetilde{f}$ is well-defined for α in the free group.

We will prove that $\widetilde{f}\colon X \times F(X) \to Y$ satisfies equality (1.2) by induction on the length $\mathrm{lh}(\beta)$, of β as reduced word. By definition of $\widetilde{f}$, this equality is fulfilled when $\beta = 1$. Let us suppose that equality (1.2) is satisfied when $\mathrm{lh}(\beta) < n$ and that $\mathrm{lh}(\beta) = n$. Write $\beta = \beta' z^{\epsilon}$ with $\mathrm{lh}(\beta') = n-1$, $z \in X$ and $\epsilon = \pm 1$. If $\epsilon = 1$, then

$$\begin{aligned}\widetilde{f}\big((x,\alpha) \triangleleft (y,\beta)\big) &= \widetilde{f}\big((x,\alpha) \triangleleft \big((y,\beta') \triangleleft (z,1)\big)\big)\\ &= \widetilde{f}\big(\big(\big((x,\alpha) \triangleleft^{-1} (z,1)\big) \triangleleft (y,\beta')\big) \triangleleft (z,1)\big)\\ &= \big(\big(\widetilde{f}(x,\alpha) \triangleleft^{-1} \widetilde{f}(z,1)\big) \triangleleft \widetilde{f}(y,\beta')\big) \triangleleft \widetilde{f}(z,1)\\ &= \widetilde{f}(x,\alpha) \triangleleft \big(\widetilde{f}(y,\beta') \triangleleft \widetilde{f}(z,1)\big)\\ &= \widetilde{f}(x,\alpha) \triangleleft \widetilde{f}(y,\beta),\end{aligned}$$

and if $\epsilon = -1$, then

$$\begin{aligned}\widetilde{f}\big((x,\alpha) \triangleleft (y,\beta)\big) \triangleleft \widetilde{f}(z,1) &= \widetilde{f}\big(\big((x,\alpha) \triangleleft (y,\beta)\big) \triangleleft (z,1)\big)\\ &= \widetilde{f}\big(\big((x,\alpha) \triangleleft \big((y,\beta') \triangleleft^{-1} (z,1)\big)\big) \triangleleft (z,1)\big)\\ &= \widetilde{f}\big(\big((x,\alpha) \triangleleft (z,1)\big) \triangleleft (y,\beta')\big)\\ &= \big(\widetilde{f}(x,\alpha) \triangleleft \widetilde{f}(z,1)\big) \triangleleft \widetilde{f}(y,\beta')\\ &= \big(\widetilde{f}(x,\alpha) \triangleleft \big(\widetilde{f}(y,\beta') \triangleleft^{-1} \widetilde{f}(z,1)\big)\big) \triangleleft \widetilde{f}(z,1),\end{aligned}$$

which is equivalent to

$$\widetilde{f}\big((x,\alpha) \triangleleft (y,\beta)\big) = \widetilde{f}(x,\alpha) \triangleleft \big(\widetilde{f}(y,\beta') \triangleleft^{-1} \widetilde{f}(z,1)\big) = \widetilde{f}(x,\alpha) \triangleleft \widetilde{f}(y,\beta).$$

This complete the inductive step and finishes the proof. □

Example 1.6. When $X = \{x\}$, then $F(X) \cong (\mathbb{Z},+)$ and so $R(X) = X \times F(X) \simeq \{x\} \times \mathbb{Z}$ (set-theoretical bijection). Under this identification, the rack operation is

$$(x,n) \triangleleft (x,m) = (x, n-m+1+m) = (x, n+1).$$

Namely, one may identify $R(X)$ with $\mathbb{Z}$ endowed with the operation $n \triangleleft m := n+1$.

Corollary 1.7. *Let $f\colon X \to Y$ be a morphism of racks. Then f is a (categorical) monomorphism if and only if f is injective (set theoretically).*

Proof. It is clear that if $f\colon X \to Y$ is injective, then it is a categorical monomorphism. For the converse, if f is not injective then there exist $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Let $R(x)$ be the free rack on the set $\{x\}$ and let $g_i\colon R(x) \to X$ $(i = 1,2)$ be the morphisms determined by $g_i(x,0) = x_i$. Since $f \circ g_1 = f \circ g_2$, we conclude that f is not a categorical monomorphism. □

2. The free quandle on X

Recall that a quandle is a rack $(Q, \triangleleft)$ satisfying the additional property $a \triangleleft a = a$ for all $a \in Q$. In this section we give a construction of the free quandle.

Definition 2.1. The free quandle on a set X is a quandle $Q(X)$ together with a function $\jmath\colon X \to Q(X)$, such that for every function $f\colon X \to Y$, where Y is a quandle, there exists a unique morphism of quandles $\widetilde{f}\colon Q(X) \to Y$, such that $\widetilde{f} \circ \jmath = f$.

It was proved in [J] that the free quandle $Q(X)$ on X may be described as the union of the conjugacy classes of the elements of X inside $F(X)$ with the quandle operation given by $\alpha \triangleleft \beta := \beta^{-1}\alpha\beta$. This construction is very concrete, but one needs to prove that it has the universal property.

As an opposite point of view, an alternative way for construct the free quandle is to consider the free rack $R(X)$, and the equivalence relation $\sim$ on $R(X)$ generated by $(x,\alpha) \sim (x,\alpha) \triangleleft (x,\alpha)$. If $\sim$ is compatible with the rack structure of $R(X)$, then the map

$$\begin{aligned} X &\xrightarrow{\ \jmath\ } Q'(X)\ ,\\ x &\longmapsto [x,1] \end{aligned}$$

where $Q'(X)$ is the quotient $R(X)/\sim$ and $[x,1]$ is the class of $(x,1)$ in $Q'(X)$, will clearly satisfy the required universal property.

In order to check that $\sim$ is compatible with the rack operation, we need to see that

a) if $(x,\alpha)\sim(x',\alpha')$ and $(y,\beta)\sim(y',\beta')\Rightarrow(x,\alpha)\triangleleft(y,\beta)\sim(x',\alpha')\triangleleft(y',\beta')$,

b) if $(x,\alpha)\triangleleft(y,\beta)\sim(x',\alpha')\triangleleft(y,\beta)$, then $(x,\alpha)\sim(x',\alpha')$.

To accomplish this it is convenient first to note that

$$(x,\alpha)\sim(y,\beta)\iff x=y \text{ and } \beta=x^k\alpha \text{ for some } k\in\mathbb{Z}.$$

Condition a) follows from

$$(x,x^k\alpha)\triangleleft(y,y^\ell\beta)=(x,x^k\alpha(y^\ell\beta)^{-1}yy^\ell\beta)=(x,x^k\alpha\beta^{-1}y\beta)\sim(x,\alpha\beta^{-1}y\beta)=(x,\alpha)\triangleleft(y,\beta),$$

while condition b) is true because

$$(x,\alpha\beta^{-1}y\beta)=(x,\alpha)\triangleleft(y,\beta)\sim(x',\alpha')\triangleleft(y,\beta)=(x',\alpha'\beta^{-1}y\beta)$$

if and only if $x=x'$ and there exists $k\in\mathbb{Z}$ such that

$$\alpha'\beta^{-1}y\beta=x^k\alpha\beta^{-1}y\beta,$$

and so $\alpha'=x^k\alpha$. We conclude that $\triangleleft$ is well-defined in $Q'(X)$.

It is possible to check that $Q(X)$ is free using that $Q'(X)$ is. For this recall that $R(X)$ is an $F(X)$-crossed set via the map $\partial\colon R(X)\to F(X)$ defined by $\partial(x,\alpha)=\alpha^{-1}x\alpha$. By Lemma 1.5, we know that

$$\partial\colon R(X)\to \mathrm{Conj}(F(X))$$

is a rack homomorphism, whose image is precisely the union of conjugacy classes of X. In this way one gets a morphism of racks $R(X)\to Q(X)$. Since $Q(X)$ is a quandle, it factorizes through $Q'(X)$. In order to prove that the induced surjective a map

$$\overline{\partial}\colon Q'(X)\to Q(X)$$

is an isomorphism, it suffices to prove that it is injective, or equivalently, that

$$\partial(x,\alpha)=\partial(y,\beta)\quad\text{if and only if}\quad x=y \text{ and } \alpha=x^k\beta \text{ for some } k\in\mathbb{Z}.$$

But notice that if $\alpha^{-1}x\alpha=\beta^{-1}y\beta$ in $F(X)$, then the equality is also true in its abelianization

$$F(X)_{ab}:=\frac{F(X)}{[F(X),F(X)]}\cong\mathbb{Z}[X],$$

where $\mathbb{Z}[X]$ denotes the free abelian group with basis X, and so necessarily $x=y$. Hence

$$\beta\alpha^{-1}x=x\beta\alpha^{-1}.$$

But an element $\gamma\in F(X)$ commutes with a generator x if and only if in a reduced expression of γ there is no letters different from x, namely, if and only if $\gamma=x^k$ for some $k\in\mathbb{Z}$. We conclude $\beta\alpha^{-1}=x^k$ for some $k\in\mathbb{Z}$, and so $\beta=x^k\alpha$ as desired.

3. Rack homology and the homology of the free rack

The *homology with integer coefficients* of a rack $(R,\triangleleft)$ is defined as the homology of the complex

$$C(R):=\ C_0(R)\xleftarrow{\partial}C_1(R)\xleftarrow{\partial}C_2(R)\xleftarrow{\partial}C_3(R)\xleftarrow{\partial}C_4(R)\xleftarrow{\partial}\cdots,$$

with objects $C_n(R):=\mathbb{Z}[R^{n+1}]$ and boundary maps $\partial\colon C_n(R)\to C_{n-1}(R)$ given by

$$\partial(r_0,\dots,r_n):=\partial_t(r_0,\dots,r_n)-\partial_\triangleleft(r_0,\dots,r_n),$$

where $\partial_\star:=\sum_{i=0}^n(-1)^i\partial_\star^i$ for $\star=t$ or $\triangleleft$, with

$$\partial_t^i(r_0,\dots,r_n):=(r_0,\dots,r_{i-1},r_{i+1},\dots,r_n),$$
$$\partial_\triangleleft^i(r_0,\dots,r_n):=(r_0\triangleleft r_i,\dots,r_{i-1}\triangleleft r_i,r_{i+1},\dots,r_n).$$

For example, $\partial(r_0,r_1)=-r_0+r_0\triangleleft r_1$.

Given an abelian group G and a rack R, by applying the functors $\_\otimes G$ and $\mathrm{Hom}_\mathbb{Z}(-,G)$ to $C(R)$ we obtain the *chain* and *cochain complexes of* R *with coefficients en* G. By definition, the *homology* $H_*^R(R,G)$, *of* R *with coefficients in* G, is the homology of the first one, and the *cohomology* $H_R^*(R,G)$, *of* R *with coefficients in* G, is the cohomology of the second one.

Consider the free rack $R(X)$ on a set X. There is an augmentation map $\mu\colon\mathbb{Z}[R(X)]\to\mathbb{Z}[X]$ given by $\mu(x,\alpha):=x$. The main result of this work is the following:

Theorem 3.1. *Let X be a non empty set. The complex*

$$C^{\mathrm{Au}}(R(X)) := \; 0 \longleftarrow \mathbb{Z}[X] \xleftarrow{\mu} C_0(Y) \xleftarrow{\partial} C_1(Y) \xleftarrow{\partial} C_2(Y) \xleftarrow{\partial} \cdots, \tag{3.3}$$

where $Y := R(X)$, is contractile.

Corollary 3.2. *For each abelian group G,*

$$H_n^R(R(X),G) = \begin{cases} G^{(|X|)} & \text{for } n=0,\\ 0 & \text{otherwise,}\end{cases} \qquad \text{and} \qquad H_R^n(R(X),G) = \begin{cases} G^{|X|} & \text{for } n=0,\\ 0 & \text{otherwise,}\end{cases}$$

where $G^{(|X|)}$ and $G^{|X|}$ denotes the direct sum of $|X|$ copies of G and the direct product of $|X|$ copies of G, respectively.

Proof. It follows from the fact that the complexes $C^{\mathrm{Au}}(R(X))\otimes G$ and $\mathrm{Hom}_{\mathbb{Z}}(C^{\mathrm{Au}}(R(X)),G)$ are contractile. □

Given a set X we let $\widetilde{Y}$ denote the direct product $X \times S(X^{\pm})$, where

- $X^{\pm} = X \coprod X^{-}$ with $X^{-} = \{x^{-1} : x \in X\}$,
- $S(X^{\pm})$ is the free monoid in $X^{\pm}$.

For each $(\mathbf{x},\boldsymbol{\alpha}) := \big((x_0,\alpha_0),\dots,(x_i,\alpha_i)\big) \in \widetilde{Y}^{i+1}$ and $\beta \in S(X^{\pm})$, we write

$$(\mathbf{x},\boldsymbol{\alpha})\beta := \big((x_0,\alpha_0\beta),\dots,(x_i,\alpha_i\beta)\big).$$

For $n \ge 0$, we recursively define a linear map $\widetilde{s}_n\colon \mathbb{Z}[\widetilde{Y}^{n+1}] \to \mathbb{Z}[\widetilde{Y}^{n+2}]$, by

$$\begin{aligned}
&\widetilde{s}_n\big((\mathbf{x},\boldsymbol{\alpha}),(x_n,1)\big) := 0,\\
&\widetilde{s}_n\big((\mathbf{x},\boldsymbol{\alpha}),(x_n,\alpha_n x)\big) := \widetilde{s}_n\big((\mathbf{x},\boldsymbol{\alpha})x^{-1},(x_n,\alpha_n)\big) + (-1)^n\big((\mathbf{x},\boldsymbol{\alpha})x^{-1},(x_n,\alpha_n),(x,1)\big),\\
&\widetilde{s}_n\big((\mathbf{x},\boldsymbol{\alpha}),(x_n,\alpha_n x^{-1})\big) := \widetilde{s}_n\big((\mathbf{x},\boldsymbol{\alpha})x,(x_n,\alpha_n)\big) - (-1)^n\big((\mathbf{x},\boldsymbol{\alpha}),(x_n,\alpha_n x^{-1}),(x,1)\big),
\end{aligned}$$

where $(\mathbf{x},\boldsymbol{\alpha}) := \big((x_0,\alpha_0),\dots,(x_{n-1},\alpha_{n-1})\big)$ for $n>0$. Since

$$\widetilde{s}_n\big((\mathbf{x},\boldsymbol{\alpha}),(x_n,\alpha_n x x^{-1})\big) = \widetilde{s}_n\big((\mathbf{x},\boldsymbol{\alpha}),(x_n,\alpha_n x^{-1}x)\big) = \widetilde{s}_n\big((\mathbf{x},\boldsymbol{\alpha}),(x_n,\alpha_n)\big),$$

the map $\widetilde{s}_n$ induces a linear map

$$s_n\colon C_n(Y) \to C_{n+1}(Y) \qquad (n \ge 0).$$

From the definition of s_n it follows immediately that

$$s_n\big((\mathbf{x},\boldsymbol{\alpha}),(x_n,\alpha_n x)\big) = s_n\big((\mathbf{x},\boldsymbol{\alpha})x^{-1},(x_n,\alpha_n)\big) + (-1)^n\big((\mathbf{x},\boldsymbol{\alpha})x^{-1},(x_n,\alpha_n),(x,1)\big),$$

for each $(\mathbf{x},\boldsymbol{\alpha}) := \big((x_0,\alpha_0),\dots,(x_{n-1},\alpha_{n-1})\big) \in R(X)^n$, $(x_n,\alpha_n) \in R(X)$ and $x \in X$.

For a sake of brevity, given β and γ in the free group $F(X)$ on X, we write β^{γ} instead of $\gamma^{-1}\beta\gamma$, and given an tuple $(\mathbf{x},\boldsymbol{\alpha}) := \big((x_0,\alpha_0),\dots,(x_i,\alpha_i)\big)$ of elements of the free rack $R(X)$, we write

$$(\mathbf{x},\boldsymbol{\alpha})\beta := \big((x_0,\alpha_0\beta),\dots,(x_i,\alpha_i\beta)\big).$$

Lemma 3.3. *Let us consider*

$$A := \big((x_0,\alpha_0),\dots,(x_n,\alpha_n)\big) \in R(X)^{n+1}, \quad D := (x_{n+1},\alpha_{n+1}) \in R(X) \quad \text{and} \quad x \in X.$$

The following equalities are fulfilled for each $\star \in \{t,\triangleleft\}$:

$$\begin{aligned}
s_n \circ \partial_\star^i\big(A,Dx\big) &= s_n \circ \partial_\star^i\big(Ax^{-1},D\big) + (-1)^n\partial_\star^i\big(Ax^{-1},D,(x,1)\big) && (0 \le i \le n+1),\\
s_n \circ \partial_\star^i\big(A,Dx^{-1}\big) &= s_n \circ \partial_\star^i\big(Ax,D\big) - (-1)^n\partial_\star^i\big(A,Dx^{-1},(x,1)\big) && (0 \le i \le n+1),\\
\partial_\star^i \circ s_{n+1}\big(A,Dx\big) &= \partial_\star^i \circ s_{n+1}\big(Ax^{-1},D\big) - (-1)^n\partial_\star^i\big(Ax^{-1},D,(x,1)\big) && (0 \le i \le n+2)
\end{aligned}$$

and

$$\partial_\star^i \circ s_{n+1}\big(A,Dx^{-1}\big) = \partial_\star^i \circ s_{n+1}\big(Ax,D\big) + (-1)^n\partial_\star^i\big(A,Dx^{-1},(x,1)\big) \qquad (0 \le i \le n+2).$$

Proof. For $0 \le i \le n$ we write

- $A_i := \big((x_0,\alpha_0),\dots,(x_{i-1},\alpha_{i-1})\big)$,
- $B_i := \big((x_0,\alpha_0),\dots,\widehat{(x_i,\alpha_i)},\dots,(x_n,\alpha_n)\big)$,
- $C_i := \big((x_{i+1},\alpha_{i+1}),\dots,(x_n,\alpha_n)\big)$,

- $u_i := x_i^{\alpha_i} x^{-1}$.

Notice that for each $i \le n$,

$$\begin{aligned}
s_n \circ \partial_t^i\big(A, Dx\big) &= s_n\big(B_i, Dx\big)\\
&= s_n\big(B_i x^{-1}, D\big) + (-1)^n\big(B_i x^{-1}, D, (x,1)\big)\\
&= s_n \circ \partial_t^i\big(Ax^{-1}, D\big) + (-1)^n \partial_t^i\big(Ax^{-1}, D, (x,1)\big),\\
s_n \circ \partial_\triangleleft^i\big(A, Dx\big) &= s_n\big(A_i x_i^{\alpha_i}, C_i, Dx\big)\\
&= s_n\big(A_i u_i, C_i x^{-1}, D\big) + (-1)^n\big(A_i u_i, C_i x^{-1}, D, (x,1)\big)\\
&= s_n \circ \partial_\triangleleft^i\big(Ax^{-1}, D\big) + (-1)^n \partial_\triangleleft^i\big(Ax^{-1}, D, (x,1)\big),\\
\partial_t^i \circ s_{n+1}\big(A, Dx\big) &= \partial_t^i \circ s_{n+1}\big(Ax^{-1}, D\big) - (-1)^n \partial_t^i\big(Ax^{-1}, D, (x,1)\big)
\end{aligned}$$

and

$$\partial_\triangleleft^i \circ s_{n+1}\big(A, Dx\big) = \partial_\triangleleft^i \circ s_{n+1}\big(Ax^{-1}, D\big) - (-1)^n \partial_\triangleleft^i\big(Ax^{-1}, D, (x,1)\big).$$

Furthermore

$$\begin{aligned}
s_n \circ \partial_t^{n+1}\big(A, Dx\big) &= s_n(A)\\
&= s_n \circ \partial_t^{n+1}\big(Ax^{-1}, D\big) + s_n(A) - s_n\big(Ax^{-1}\big)\\
&= s_n \circ \partial_t^{n+1}\big(Ax^{-1}, D\big) + (-1)^n \partial_t^{n+1}\big(Ax^{-1}, D, (x,1)\big),\\
s_n \circ \partial_\triangleleft^{n+1}\big(A, Dx\big) &= s_n\big(Ax_{n+1}^{\alpha_{n+1}x}\big)\\
&= s_n \circ \partial_\triangleleft^{n+1}\big(Ax^{-1}, D\big) + s_n\big(Ax_{n+1}^{\alpha_{n+1}x}\big) - s_n\big(Ax^{-1}x_{n+1}^{\alpha_{n+1}}\big)\\
&= s_n \circ \partial_\triangleleft^{n+1}\big(Ax^{-1}, D\big) + (-1)^n \partial_\triangleleft^{n+1}\big(Ax^{-1}, D, (x,1)\big),\\
\partial_t^{n+1} \circ s_{n+1}\big(A, Dx\big) &= \partial_t^{n+1} \circ s_{n+1}\big(Ax^{-1}, D\big) - (-1)^n \partial_t^{n+1}\big(Ax^{-1}, D, (x,1)\big),\\
\partial_\triangleleft^{n+1} \circ s_{n+1}\big(A, Dx\big) &= \partial_\triangleleft^{n+1} \circ s_{n+1}\big(Ax^{-1}, D\big) - (-1)^n \partial_\triangleleft^{n+1}\big(Ax^{-1}, D, (x,1)\big)\\
\partial_t^{n+2} \circ s_{n+1}\big(A, Dx\big) &= \partial_t^{n+2} \circ s_{n+1}\big(Ax^{-1}, D\big) - (-1)^n \partial_t^{n+2}\big(Ax^{-1}, D, (x,1)\big),\\
\partial_\triangleleft^{n+2} \circ s_{n+1}\big(A, Dx\big) &= \partial_\triangleleft^{n+2} \circ s_{n+1}\big(Ax^{-1}, D\big) - (-1)^n \partial_\triangleleft^{n+2}\big(Ax^{-1}, D, (x,1)\big).
\end{aligned}$$

This finishes the proof of the first and third equality in the statement. The other equalities follows from these ones substituting Ax for A and Dx^{-1} for D. □

Proof of Teorem 3.1. Recall that we have maps

$$s_n \colon C_n(Y) \longrightarrow C_{n+1}(Y) \qquad \text{for } n \ge 0,$$

We now define $s_{-1} \colon \mathbb{Z}[X] \to C_0(Y)$ by $s_{-1}(x) := (x, 1)$. We claim that $(s_n)_{n\ge -1}$ is a contracting homotopy of $C(Y)$. Note that

$$\begin{aligned}
&\mu \circ s_{-1}(x) = \mu(x,1) = x,\\
&s_{-1} \circ \mu(x,\alpha) = (x,1),\\
&\partial \circ s_0(x,1) = 0 = (x,1) - s_{-1} \circ \mu(x,1).
\end{aligned}$$

for all $x \in X$. Assume that

$$(s_{-1} \circ \mu + \partial \circ s_0)(x_0, \alpha) = (x_0, \alpha).$$

Then

$$\begin{aligned}
\partial \circ s_0(x_0, \alpha x) &= \partial \circ s_0(x_0, \alpha) + \partial\big((x_0,\alpha),(x,1)\big)\\
&= (x_0,\alpha) - (x_0,1) + \partial\big((x_0,\alpha),(x,1)\big)\\
&= (x_0,\alpha) - (x_0,1) - (x_0,\alpha) + (x_0,\alpha x)\\
&= (x_0,\alpha x) - s_{-1} \circ \mu(x_0,\alpha x)
\end{aligned}$$

and

$$\begin{aligned}\partial\circ s_0(x_0,\alpha x^{-1}) &= \partial\circ s_0(x_0,\alpha) - \partial\big((x_0,\alpha x^{-1})(x,1)\big)\\ &= (x_0,\alpha)-(x_0,1)-\partial\big((x_0,\alpha x^{-1}),(x,1)\big)\\ &= (x_0,\alpha)-(x_0,1)+(x_0,\alpha x^{-1})-(x_0,\alpha)\\ &= (x_0,\alpha x^{-1}) - s_{-1}\circ\mu(x_0,\alpha x^{-1}).\end{aligned}$$

From the fact that $s_k(\cdots,(x_{n+1},1))=0$ we have

$$\begin{aligned}s_n\circ\partial\big((\mathbf{x},\boldsymbol{\alpha}),(x_{n+1},1)\big) &= (-1)^{n+1}s_n(\mathbf{x},\boldsymbol{\alpha}) + (-1)^n s_n\big((\mathbf{x},\boldsymbol{\alpha})x_{n+1}\big)\\ &= \big((\mathbf{x},\boldsymbol{\alpha}),(x_{n+1},1)\big) - \partial\circ s_{n+1}\big((\mathbf{x},\boldsymbol{\alpha}),(x_{n+1},1)\big).\end{aligned}$$

Lastly, suppose that

$$s_n\circ\partial\big((\mathbf{x},\boldsymbol{\alpha}),(x_{n+1},\alpha_{n+1})\big) = \big((\mathbf{x},\boldsymbol{\alpha}),(x_{n+1},\alpha_{n+1})\big) - \partial\circ s_{n+1}\big((\mathbf{x},\boldsymbol{\alpha}),(x_{n+1},\alpha_{n+1})\big),$$

for all $(\mathbf{x},\boldsymbol{\alpha}) := \big((x_0,\alpha_0),\dots,(x_n,\alpha_n)\big)$. Write $A := (\mathbf{x},\boldsymbol{\alpha})$ and $D := (x_{n+1},\alpha_{n+1})$. By the inductive hypothesis and Lemma 3.3

$$\begin{aligned}s_n\circ\partial\big(A,Dx\big) &= s_n\circ\partial\big(Ax^{-1},D\big) + (-1)^n\sum_{i=1}^{n+1}(-1)^i\big(\partial_t^i-\partial_\triangleleft^i\big)\big(Ax^{-1},D,(x,1)\big)\\ &= \big(Ax^{-1},D\big) - \partial\circ s_{n+1}\big(Ax^{-1},D\big) + (-1)^n\partial\big(Ax^{-1},D,(x,1)\big)\\ &\quad + (\partial_\triangleleft^{n+2}-\partial_t^{n+2}\big)\big(Ax^{-1},D,(x,1)\big)\\ &= \big(Ax^{-1},D\big) - \partial\circ s_{n+1}\big(A,Dx\big) + (\partial_\triangleleft^{n+2}-\partial_t^{n+2}\big)\big(Ax^{-1},D,(x,1)\big)\\ &= \big(A,Dx\big) - \partial\circ s_{n+1}\big(A,Dx\big)\end{aligned}$$

and

$$\begin{aligned}s_n\circ\partial\big(A,Dx^{-1}\big) &= s_n\circ\partial\big(Ax,D\big) - (-1)^n\sum_{i=1}^{n+1}\big(\partial_t^i-\partial_\triangleleft^i\big)\big(A,Dx^{-1},(x,1)\big)\\ &= \big(Ax,D\big) - \partial\circ s_{n+1}\big(Ax,D\big) - (-1)^n\partial\big(A,Dx^{-1},(x,1)\big)\\ &\quad + \big(\partial_t^{n+2}-\partial_\triangleleft^{n+2}\big)\big(A,Dx^{-1},(x,1)\big)\\ &= \big(A,Dx^{-1}\big) - \partial\circ s_{n+1}\big(A,Dx^{-1}\big).\end{aligned}$$

This finishes the proof. □

4. Quandle homology and the homology of the free quandle

Given a subset S of an abelian group G we let $\langle S\rangle$ denote the subgroup of G generated by S. It is easy to see that if Q is a quandle, then the complex $C(Q)$ has a degeneration subcomplex

$$C^{\mathrm{D}}(Q) :=\ C_0^{\mathrm{D}}(Q) \xleftarrow{\partial} C_1(Q) \xleftarrow{\partial} C_2^{\mathrm{D}}(Q) \xleftarrow{\partial} C_3^{\mathrm{D}}(Q) \xleftarrow{\partial} C_4^{\mathrm{D}}(Q) \xleftarrow{\partial} \cdots,$$

where, for each $n\ge 0$,

$$C_n^{\mathrm{D}}(Q) := \langle\{(r_0,\dots,r_{i-1},r,r,r_{i+2},\dots,r_n) : r_0,\dots,r_{i-1},r,r,r_{i+2},\dots,r_n\in Q\}\rangle.$$

The appropriate complex for computing quandle homology and cohomology are defined by

$$C_\bullet^{\mathrm{Q}}(Q) := C_\bullet(Q)/C_\bullet^{\mathrm{D}}(Q),\ \ C_{\mathrm{Q}}^\bullet(Q,k) := \mathrm{Hom}(C_\bullet^{\mathrm{Q}}(Q),k).$$

For instance, given an abelian group k and a rack Q, if $f\colon Q\times Q\to k$ is a map, then it is easy to check that the operation

$$(k,q)\triangleleft_f (k',q') := (k+f(q,q'),q\triangleleft q')$$

gives a rack structure on $k\times Q$ if and only if f is a 2-cocycle. If Q is a quandle, the above operation is not a quandle in general, but it is so if and only if f is a quandle cocycle.

The quandle version of Theorem 3.1 is the following:

Theorem 4.1. *Let X be a non empty set. The complex*

$$C^{\mathrm{Au,Q}}(Q(X)) :=\ 0 \longleftarrow \mathbb{Z}[X] \xleftarrow{\mu} C_0^{\mathrm{Q}}(Z) \xleftarrow{\partial} C_1^{\mathrm{Q}}(Z) \xleftarrow{\partial} C_2^{\mathrm{Q}}(Z) \xleftarrow{\partial} \cdots, \tag{4.4}$$

where $Z := Q(X)$ and $\mu(\alpha^{-1}x\alpha) := x$, is contractile.

Proof. As in the proof of Theorem 3.1, let $Y := R(X)$. We consider the canonical projection

$$\wp_n \colon C_n(Y) \to C_n(Z) \to C_n^{\mathrm{Q}}(Z).$$

It is easy to see that the kernel of $\wp_n$ is the subgroup $\mathfrak{I}_n$ of $C_n(Y)$ generated by the elements

$$A - \big(A_i, (x_i, x_i\alpha_i), C_i\big) \quad \text{and} \quad \big(A_i, (x_i, \alpha_i), (x_i, \alpha_i), C_{i+1}\big),$$

where $0 \le i \le n$,

- $A := \big((x_0, \alpha_0), \dots, (x_n, \alpha_n)\big)$,
- $A_i := \big((x_0, \alpha_0), \dots, (x_{i-1}, \alpha_{i-1})\big)$,
- $C_i := \big((x_{i+1}, \alpha_{i+1}), \dots, (x_n, \alpha_n)\big)$.

We claim that $s_n(\mathfrak{I}_n) \subseteq \mathfrak{I}_{n+1}$. By the recursive definition of s_n, in order to verify this it suffices to prove that the following facts hold for $0 \le i \le n$:

(1) If

$$s_n(A_n, (x_n, \alpha_n)) - s_n\big(A_i, (x_i, x_i\alpha_i), C_i\big) \in \mathfrak{I}_{n+1},$$

for all tuple $A_n := \big((x_0, \alpha_0), \dots, (x_{n-1}, \alpha_{n-1})\big)$, then

$$s_n(A_n, (x_n, \alpha_n x)) - s_n\big(A_i, (x_i, x_i\alpha_i), C_i', (x_n, \alpha_n x)\big) \in \mathfrak{I}_{n+1},$$

and

$$s_n(A_n, (x_n, \alpha_n x^{-1})) - s_n\big(A_i, (x_i, x_i\alpha_i), C_i', (x_n, \alpha_n x^{-1})\big) \in \mathfrak{I}_{n+1},$$

where $C_i' := \big((x_{i+1}, \alpha_{i+1}), \dots, (x_{n-1}, \alpha_{n-1})\big)$.

(2) If

$$s_n\big(A_i, (x_i, \alpha_i), (x_i, \alpha_i), C_{i+1}\big) \in \mathfrak{I}_{n+1},$$

for all tuple $A_n := \big((x_0, \alpha_0), \dots, (x_{n-1}, \alpha_{n-1})\big)$, then

$$s_n\big(A_i, (x_i, \alpha_i), (x_i, \alpha_i), C_{i+1}', (x_n, \alpha_n x)\big) \in \mathfrak{I}_{n+1},$$

and

$$s_n\big(A_i, (x_i, \alpha_i), (x_i, \alpha_i), C_{i+1}', (x_n, \alpha_n x^{-1})\big) \in \mathfrak{I}_{n+1},$$

where $C_{i+1}' := \big((x_{i+2}, \alpha_{i+2}), \dots, (x_{n-1}, \alpha_{n-1})\big)$.

It is convenient to consider separately the cases $i < n$ and $i = n$. To illustrate the method of proof we will consider the first half of the first case with $i < n$, and we let the other ones to the reader. To abbreviate we write

$$L_n := s_n(A_n, (x_n, \alpha_n x)) - s_n\big(A_i, (x_i, x_i\alpha_i), C_i', (x_n, \alpha_n x)\big).$$

By definition

$$\begin{aligned} L_n &= s_n\big(A_n x^{-1}, (x_n, \alpha_n)\big) - s_n\big(A_i x^{-1}, (x_i, x_i\alpha_i x^{-1}), C_i' x^{-1}, (x_n, \alpha_n)\big) \\ &\quad + (-1)^n\big(A_n x^{-1}, (x_n, \alpha_n), (x, 1)\big) - (-1)^n\big(A_i x^{-1}, (x_i, x_i\alpha_i x^{-1}), C_i' x^{-1}, (x_n, \alpha_n), (x, 1)\big). \end{aligned}$$

By the inductive hypothesis we know that

$$s_n\big(A_n x^{-1}, (x_n, \alpha_n)\big) - s_n\big(A_i x^{-1}, (x_i, x_i\alpha_i x^{-1}), C_i' x^{-1}, (x_n, \alpha_n)\big) \in \mathfrak{I}_{n+1},$$

and it is evident that

$$\big(A_n x^{-1}, (x_n, \alpha_n), (x, 1)\big) - \big(A_i x^{-1}, (x_i, x_i\alpha_i x^{-1}), C_i' x^{-1}, (x_n, \alpha_n), (x, 1)\big) \in \mathfrak{I}_{n+1}.$$

Consequently, s_n induces a morphism of groups $\overline{s}_n \colon C_n^{\mathrm{Q}}(Z) \to C_{n+1}^{\mathrm{Q}}(Z)$. It is clear that the family $(\overline{s}_n)_{n \ge -1}$, where $\overline{s}_{-1} \colon \mathbb{Z}X \to C_0^{\mathrm{Q}}(X)$ is the canonical inclusion, is a contractile homotopy of $C^{\mathrm{Q}}(Z)$. □

Corollary 4.2. *For each abelian group G,*

$$H_n^Q(Q(X), G) = \begin{cases} G^{(|X|)} & \text{for } n = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad H_Q^n(Q(X), G) = \begin{cases} G^{|X|} & \text{for } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It follows from the fact that the complexes $C^{\mathrm{Au},Q}(Q(X)) \otimes G$ and $\mathrm{Hom}_{\mathbb{Z}}(C^{\mathrm{Au},Q}(Q(X)), G)$ are contractile. □

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