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Normal Projections in Krein Spaces

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Abstract. Given a complex Krein space $\mathcal{H}$ with fundamental symmetry J, the aim of this note is to characterize the set of J-normal projections

$$\mathcal{Q} = \{Q \in L(\mathcal{H}) : \ Q^2 = Q \text{ and } Q^{\#}Q = QQ^{\#}\}.$$

The ranges of the projections in $\mathcal{Q}$ are exactly those subspaces of $\mathcal{H}$ which are pseudo-regular. For a fixed pseudo-regular subspace $\mathcal{S}$, there are infinitely many J-normal projections onto it, unless $\mathcal{S}$ is regular. Therefore, most of the material herein is devoted to parametrizing the set of J-normal projections onto a fixed pseudo-regular subspace $\mathcal{S}$.

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1. Introduction

It is well-known that a (linear, bounded) projection Q, acting on a Hilbert space $\mathcal{H}$, is normal ($QQ^* = Q^*Q$) if and only if it is selfadjoint ($Q = Q^*$). Therefore, there is a one-to-one correspondence between the closed subspaces of $\mathcal{H}$ and the normal projections acting on $\mathcal{H}$.

On the other hand, if $\mathcal{K}$ is a Krein space with fundamental symmetry J, it is easy to find J-normal projections which are not J-selfadjoint (see Example 1 in Sect. 3). For a fixed Krein space $\mathcal{K}$ with fundamental symmetry J, the purpose of this work is to describe those projections acting on $\mathcal{K}$ which are J-normal, i.e. those $Q = Q^2 \in L(\mathcal{K})$ satisfying

$$QQ^{\#} = Q^{\#}Q,$$

where $Q^{\#}$ is the J-adjoint of Q.

If Q is J-normal, observe that $E = QQ^{\#}$ is a J-selfadjoint projection whose range, hereafter denoted by $R(E)$, is contained in $R(Q)$. Thus, $R(Q)$ contains a regular subspace of $\mathcal{K}$. On the other hand, $P = Q(I - Q^{\#})$ is a projection with $R(P) = R(Q) \cap R(Q)^{[\perp]} = R(Q)^{\circ}$, i.e. $R(P)$ is the isotropic part of $R(Q)$.

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Also, since $EP = PE = 0$ it follows that $Q = E + P$ and

$$R(Q) = R(E)[\dot{+}]R(P) = R(E)[\dot{+}]R(Q)^\circ,$$

that is, $R(Q)$ is a *pseudo-regular* subspace of $\mathcal{K}$, see [9] for the terminology. Conversely, it will be shown that every pseudo-regular subspace of $\mathcal{K}$ admits a J-normal projection onto it. However, it is not hard to prove that a pseudo-regular subspace may admit infinitely many J-normal projections onto it (see Example 2 in Sect. 4).

The importance of pseudo-regular subspaces lies in the fact that they enable to generalize some Pontryagin spaces arguments to general Krein spaces, see [9]. They have also been used as a technical tool for the study of spectral functions (and distributions) for particular classes of operators in Krein spaces [10,11,13,15] and to extend the Beurling–Lax theorem for shifts in indefinite metric spaces [4,5].

Along this work, different characterizations of J-normal projections will be developed. Furthermore, for a fixed pseudo-regular subspace $\mathcal{S}$, we will present a parametrization for the set of J-normal projections onto $\mathcal{S}$.

In the next section, we introduce the basic notations and terminology used in the paper. Section 3 is devoted to describe J-normal projections. In particular, it is shown that every J-normal projection Q admits a unique decomposition $Q = E + P$ where E is J-selfadjoint and P is a J-normal projection with J-neutral range. Then, the main consequences of this decomposition are discussed.

In Sect. 4 it is shown that a (closed) subspace $\mathcal{S}$ is the range of a J-normal projection if and only if it is pseudo-regular, i.e. if $\mathcal{S}+\mathcal{S}^{[\perp]}$ is closed. Then, although there is not a unique J-normal projection onto an arbitrary pseudo-regular subspace $\mathcal{S}$, a formula for a particular J-normal projection onto $\mathcal{S}$ is presented (depending only on the fundamental symmetry J and the orthogonal projections onto $\mathcal{S}$ and $\mathcal{S}^\circ$).

Section 5 deals with J-normal projections onto J-neutral subspaces. It will be shown that there are infinitely many J-normal projections onto a prescribed J-neutral subspace (and their nullspaces can be arbitrarily close). Then, for a fixed J-neutral subspace $\mathcal{N}$, a parametrization for the set of J-normal projections onto $\mathcal{N}$ is presented.

Finally, the aim of Sect. 6 is to present an explicit description of the set of J-normal projections onto a pseudo-regular subspace $\mathcal{S}$. First, it is shown that this set can be decomposed in a disjoint union of decks. Then, considering the projections as block-operator matrices according to an appropriate orthogonal decomposition, each deck is parametrized.

2. Preliminaries

2.1. Notation and Terminology

Along this work $\mathcal{H}$ denotes a complex (separable) Hilbert space. If $\mathcal{K}$ is another Hilbert space then $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$. The group of linear invertible operators acting on $\mathcal{H}$ is denoted by $GL(\mathcal{H})$. Also, $L(\mathcal{H})^+$ denotes the cone of positive semidefinite operators acting on $\mathcal{H}$ and $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap L(\mathcal{H})^+$.

If $T \in L(\mathcal{H}, \mathcal{K})$ then $T^* \in L(\mathcal{K}, \mathcal{H})$ denotes the adjoint operator of T, $R(T)$ stands for its range and $N(T)$ for its nullspace.

Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$, $\mathcal{S} \dot{+} \mathcal{T}$ denotes the direct sum of them. On the other hand, $\mathcal{S} \oplus \mathcal{T}$ stands for their (direct) orthogonal sum and $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$. If $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, there exists a (unique) bounded projection with range $\mathcal{S}$ and nullspace $\mathcal{T}$. Hereafter, it is denoted by $P_{\mathcal{S}//\mathcal{T}}$. If $P_\mathcal{S}$ and $P_\mathcal{T}$ stand for the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, $P_{\mathcal{S}//\mathcal{T}}$ can be represented as:

$$P_{\mathcal{S}//\mathcal{T}} = P_\mathcal{S}(P_\mathcal{S} + P_\mathcal{T})^{-1}, \tag{2.1}$$

see [2, Lemma 3.1].

Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$, the cosine of the *Friedrichs angle* between $\mathcal{S}$ and $\mathcal{T}$ is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y \rangle| : x \in \mathcal{S} \ominus \mathcal{T}, \|x\| = 1, \, y \in \mathcal{T} \ominus \mathcal{S}, \|y\| = 1\}.$$

It is well known that

$$c(\mathcal{S}, \mathcal{T}) < 1 \quad \Leftrightarrow \quad \mathcal{S} + \mathcal{T} \text{ is closed} \quad \Leftrightarrow \quad c(\mathcal{S}^\perp, \mathcal{T}^\perp) < 1.$$

Furthermore, if $P_\mathcal{S}$ and $P_\mathcal{T}$ are the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, then $c(\mathcal{S}, \mathcal{T}) < 1$ if and only if $(I - P_\mathcal{S})P_\mathcal{T}$ has closed range.

On the other hand, the *Dixmier (or minimal) angle* between $\mathcal{S}$ and $\mathcal{T}$ is defined by

$$c_0(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y \rangle| : x \in \mathcal{S}, \|x\| = 1, \, y \in \mathcal{T}, \|y\| = 1\}.$$

It is clear that $c(\mathcal{S}, \mathcal{T}) \le c_0(\mathcal{S}, \mathcal{T})$, and if $\mathcal{S} \cap \mathcal{T} = \{0\}$ then $c(\mathcal{S}, \mathcal{T}) = c_0(\mathcal{S}, \mathcal{T})$.

Remark 2.1. If $P_\mathcal{S}$ and $P_\mathcal{T}$ are the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, then

$$c_0(\mathcal{S}, \mathcal{T}) = \|P_\mathcal{S} P_\mathcal{T}\|.$$

Also, $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$ if and only if $\|P_{\mathcal{S}^\perp} P_{\mathcal{T}^\perp}\| < 1$. See [8] for further details.

2.2. Krein Spaces

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject see [1,6,12].

Given a Krein space $(\mathcal{H}, [\ ,\ ])$ with a *fundamental decomposition* $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$, the direct (orthogonal) sum of the Hilbert spaces $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ is denoted by $(\mathcal{H}, \langle\ ,\ \rangle)$.

Observe that the indefinite metric and the inner product of $\mathcal{H}$ are related by means of a *fundamental symmetry*, i.e. a unitary selfadjoint operator $J \in L(\mathcal{H})$ which satisfies:

$$[\,x, y\,] = \langle\, Jx, y\,\rangle, \qquad x, y \in \mathcal{H}.$$

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the vector space of linear transformations which are bounded with respect to the associated Hilbert spaces $(\mathcal{H}, \langle\ ,\ \rangle_\mathcal{H})$ and $(\mathcal{K}, \langle\ ,\ \rangle_\mathcal{K})$. Given $T \in L(\mathcal{H}, \mathcal{K})$, the J-adjoint operator of T is defined by $T^\# = J_\mathcal{H} T^* J_\mathcal{K}$, where $J_\mathcal{H}$ and $J_\mathcal{K}$ are the fundamental symmetries associated to $\mathcal{H}$ and $\mathcal{K}$, respectively. An operator $T \in L(\mathcal{H})$ is J-selfadjoint if $T = T^{\#}$.

A vector $x \in \mathcal{H}$ is *J-positive* if $[\,x, x\,] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-positive* if every $x \in \mathcal{S}$, $x \neq 0$, is a J-positive vector. J-nonnegative, J-neutral, J-negative and J-nonpositive vectors and subspaces are defined analogously.

Given a subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$, the *J-orthogonal complement* to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [\,x, s\,] = 0, \text{ for every } s \in \mathcal{S}\}.$$

Usually, $\mathcal{S}^{\circ} := \mathcal{S} \cap \mathcal{S}^{[\perp]}$ (the *isotropic part of* $\mathcal{S}$) is a non-trivial subspace. Then, a subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-non-degenerated* if $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$. Otherwise, it is a J-degenerated subspace of $\mathcal{H}$.

Definition. A subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ is a *regular subspace* if it is the range of a J-selfadjoint projection, i.e. if there exists $E \in L(\mathcal{H})$ such that $E = E^2 = E^{\#}$ and $R(E) = \mathcal{S}$.

Given a regular subspace $\mathcal{S}$, observe that $\mathcal{S}^{[\perp]}$ is the nullspace of the J-selfadjoint projection E onto $\mathcal{S}$. Furthermore, if P is the orthogonal projection onto $\mathcal{S}$, the orthogonal projection onto $\mathcal{S}^{[\perp]}$ coincides with $J(I - P)J$. Thus, by (2.1), it follows that

$$E = P(P + I - JPJ)^{-1}, \tag{2.2}$$

see [3] for another formula for E.

Proposition 2.2. [3] *A closed subspace $\mathcal{S}$ is regular if and only if*

$$\|PJ(I - P)\| < 1,$$

or equivalently $(I - P)JPJ(I - P) \leq (1 - \varepsilon)I$ for some $\varepsilon > 0$, where P is the orthogonal projection onto $\mathcal{S}$.

The following result seems to be well known, however its proof is included for the sake of completeness.

Lemma 2.3. *Let $Q \in L(\mathcal{H})$ be a projection acting on a Krein space $\mathcal{H}$ with fundamental symmetry J. Then, the following conditions are equivalent:*

1. $Q^{\#}Q = 0$;
2. *$R(Q)$ is a J-neutral subspace;*
3. *$PJP = 0$, where P is the orthogonal projection onto $R(Q)$;*
4. *the orthogonal projection P onto $R(Q)$ admits the representation (according to the fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$)*

$$P = \frac{1}{2}\begin{pmatrix} V^*V & V^* \\ V & VV^* \end{pmatrix},$$

where $V \in L(\mathcal{H}_+, \mathcal{H}_-)$ is a partial isometry.

Proof. The equivalences 1. $\leftrightarrow$ 2. $\leftrightarrow$ 3. and the implication 4. $\rightarrow$ 1. are easy to check. On the other hand, if $\mathcal{S} = R(Q)$ is a J-neutral subspace of $\mathcal{H}$ then its angular operator $V \in L(\mathcal{H}_+, \mathcal{H}_-)$ is a partial isometry. Therefore

$$\begin{aligned}\mathcal{S} &= \{(x_+, Vx_+) \in \mathcal{H}_+ \oplus \mathcal{H}_- : \ x_+ \in P_+(\mathcal{S}) = N(V)^\perp\} \\ &= \{(V^*Vu, Vu) \in \mathcal{H}_+ \oplus \mathcal{H}_- : \ u \in \mathcal{H}_+\} = R\left(\begin{bmatrix} VV^* \\ V \end{bmatrix}\right),\end{aligned}$$

see [12, Ch. 1, Section 8]. Then, since V is a partial isometry, the operator

$$P = \frac{1}{2}\begin{pmatrix} V^*V & V^* \\ V & VV^* \end{pmatrix},$$

satisfies $P^2 = P = P^*$, i.e. P is the orthogonal projection onto $\mathcal{S}$. □

3. Decompositions of a J-Normal Projection

Every normal projection acting on a Hilbert space is selfadjoint. However, the following example shows that there are J-normal projections acting on a Krein space (i.e. projections that commute with its J-adjoint) which are not J-selfadjoint.

Example 1. Let A be a (possibly unbounded) definitizable operator acting on a Krein space $\mathcal{H}$, i.e. a J-selfadjoint operator A with $\rho(A) \neq \varnothing$ such that there exists a polynomial $p \in \mathbb{C}[\lambda]$ satisfying

$$[\,p(A)x, x\,] \geq 0 \quad \text{for every } x \in \mathrm{dom}(A^k),$$

where k is the degree of the polynomial. Recall that the non-real spectrum of a definitizable operator consists of no more than a finite number of eigenvalues (see [14]).

If σ_1 is a bounded spectral set of A contained in $\mathbb{R}$, let Γ_1 be a Jordan closed rectifiable contour lying in $\rho(A)$ such that σ_1 lies inside Γ_1 and $\sigma(A)\backslash\sigma_1$ lies outside this contour. Then, it is easy to see that the associated Riesz projection

$$P_{\sigma_1} = -\frac{1}{2\pi i}\int_{\Gamma_1}(A - \lambda I)^{-1} d\lambda,$$

is J-selfadjoint.

Analogously, assume that λ_0 is a nonreal (normal) eigenvalue of A and consider the bounded spectral set $\sigma_2 = \{\lambda_0\}$. Then, consider the Riesz projection

$$P_{\sigma_2} = -\frac{1}{2\pi i}\int_{\Gamma_2}(A - \lambda I)^{-1} d\lambda,$$

for an appropriate Jordan closed rectifiable contour Γ_2. Without loss of generality, assume that there exists $\varepsilon > 0$ such that Γ_2 is parametrized by $\gamma(t) = \lambda_0 + \varepsilon e^{2\pi i t}$, $t \in [0, 1]$. Then,

$$P_{\sigma_2} = -\frac{1}{2\pi i}\int_{\Gamma_2}(A - \lambda I)^{-1} d\lambda = -\varepsilon\int_0^1 (A - \lambda_0 - e^{2\pi i t})^{-1} e^{2\pi i t}\, dt.$$

Observe that the J-adjoint of P_{σ_2} can be calculated as follows:

$$\begin{aligned} P_{\sigma_2}^{\#} &= \left(-\int_0^1 (A - \lambda_0 - e^{2\pi it})^{-1} e^{2\pi it}\, dt \right)^{\#} \\ &= -\varepsilon \int_0^1 (A - \overline{\lambda_0} - e^{-2\pi it})^{-1} e^{-2\pi it}\, dt \\ &= -\varepsilon \int_0^1 (A - \overline{\lambda_0} - e^{2\pi it})^{-1} e^{2\pi it}\, dt = P_{\overline{\sigma_2}}, \end{aligned}$$

where $\overline{\sigma_2} = \{\overline{\lambda_0}\}$ is the bounded spectral set symmetric to σ_2 respect to the real line. Hence, $\sigma_2 \cap \overline{\sigma_2} = \varnothing$ and, applying the properties of the Riesz functional calculus, it follows that

$$P_{\sigma_2} P_{\sigma_2}^{\#} = P_{\sigma_2} P_{\overline{\sigma_2}} = 0 \quad (\text{and} \quad P_{\sigma_2}^{\#} P_{\sigma_2} = P_{\overline{\sigma_2}} P_{\sigma_2} = 0),$$

i.e. P_{σ_2} is a J-normal projection, but it is not J-selfadjoint.

Furthermore, $\sigma = \sigma_1 \cup \sigma_2$ is an isolated spectral set for A and, since $\sigma_1 \cap \sigma_2 = \varnothing$, the Riesz projection associated to σ is the J-normal projection $P_\sigma = P_{\sigma_1} + P_{\sigma_2}$.

Following these ideas it can be shown that, for a definitizable operator A, the Riesz projection onto a bounded spectral set is always a J-normal projection. Moreover, it is J-sefadjoint if and only if the bounded spectral set is contained in $\mathbb{R}$.

In what follows, the basic properties of J-normal projections are developed.

Theorem 3.1. *Given a projection $Q \in L(\mathcal{H})$, Q is J-normal if and only if there exist a J-selfadjoint projection $E \in L(\mathcal{H})$ and a projection $P \in L(\mathcal{H})$ satisfying $PP^{\#} = P^{\#}P = 0$ such that*

$$Q = E + P. \tag{3.1}$$

The projections E and P are uniquely determined by Q.

Proof. If $Q \in L(\mathcal{H})$ is a J-normal projection, then $E = QQ^{\#}$ is a J-selfadjoint projection. Notice that $P := Q(I - Q^{\#})$ is also a projection and, since $I - Q$ is also J-normal, it holds that

$$PP^{\#} = Q(I - Q^{\#})(I - Q)Q^{\#} = Q(I - Q)(I - Q^{\#})Q^{\#} = 0.$$

In the same way, $P^{\#}P = 0$.

Conversely, suppose that $Q = E + P$ where E is J-selfadjoint and P is a projection satisfying $PP^{\#} = P^{\#}P = 0$. Since $Q^2 = Q$, it follows that $EP + PE = 0$. Notice that $R(E) \cap R(P) = \{0\}$. In fact, if $x \in R(E) \cap R(P)$ it is easy to see that $0 = (EP + PE)x = 2x$. So, $x = 0$. Therefore, $EP = PE = 0$ (and $EP^{\#} = P^{\#}E = 0$).

Thus, recalling that $PP^{\#} = P^{\#}P = 0$ it follows easily that $QQ^{\#} = Q^{\#}Q = E$, i.e. Q is J-normal. Notice that $P = Q - E = Q(I - Q^{\#})$. The uniqueness of this decomposition follows from the last part of the proof. ☐

If $Q \in L(\mathcal{H})$ is a J-normal projection, notice that the (uniquely) determined projections in the decomposition of Theorem 3.1 are

$$E = QQ^{\#} \quad \text{and} \quad P = Q(I - Q^{\#}). \tag{3.2}$$

Throughout this paper, E and P will be referred as the *regular part* and the *neutral part* of Q, respectively.

Corollary 3.2. *Let $Q \in L(\mathcal{H})$ be a J-normal projection. Then, Q is J-selfadjoint if and only if $R(Q)^\circ$ is trivial.*

Proof. Observe that Q is J-selfadjoint if and only if $Q = QQ^{\#}$, or equivalently, $P = Q(I - Q^{\#}) = 0$. But $R(P) = R(Q) \cap N(Q^{\#}) = R(Q)^\circ$. So, $P = 0$ if and only if $R(Q)^\circ = \{0\}$. $\square$

Corollary 3.3. *Given a projection $Q \in L(\mathcal{H})$, Q is J-normal if and only if*

$$Q = GH,$$

where $G \in L(\mathcal{H})$ is a J-selfadjoint projection and $H \in L(\mathcal{H})$ is a J-normal projection with J-neutral kernel contained in $R(G)$. Furthermore, this factorization is unique and the projections G and H commute.

Proof. If Q is J-normal, then $G = I-(I-Q)(I-Q)^{\#}$ and $H = I-(I-Q)Q^{\#}$ satisfy the desired properties.

Conversely, if $Q = GH$ for a pair of projections G and H satisfying the assumptions, notice that $(I-G)(I-H) = 0$, or equivalently, $I+GH = G+H$. Thus,

$$I - Q = I - GH = (I - G) + (I - H),$$

$I-G$ is J-selfadjoint and $I-H$ satisfies $(I-H)(I-H)^{\#} = (I-H)^{\#}(I-H) = 0$. Then, by Theorem 3.1, Q is J-normal.

The uniqueness of the factorization and the commutativity of G and H also follow from the above theorem. $\square$

Corollary 3.4. *If $Q \in L(\mathcal{H})$ is a J-normal projection and $Q = E + P$ is the decomposition given by Theorem 3.1, then there exists a unique J-selfadjoint projection $F \in L(\mathcal{H})$ such that*

$$I - Q = F + P^{\#}. \tag{3.3}$$

Moreover, $EF = 0$.

Proof. Applying Theorem 3.1 to $I - Q$ it follows that its J-selfadjoint part is $F = (I - Q)(I - Q)^{\#}$ and

$$(I - Q) - F = (I - Q) - (I - Q)(I - Q)^{\#} = (I - Q)Q^{\#} = P^{\#}.$$

Furthermore, $E = QQ^{\#} = Q^{\#}Q$ and then it is obvious that $EF = 0$. $\square$

Lemma 3.5. *Let $Q \in L(\mathcal{H})$ be a J-normal projection and consider the neutral part $P \in L(\mathcal{H})$ of Q. Then,*

$$R(P) = R(Q)^\circ \quad \text{and} \quad R(P^{\#}) = N(Q)^\circ. \tag{3.4}$$

Therefore, $R(Q)^\circ$ and $N(Q)^\circ$ have the same dimension and codimension.

Proof. Indeed, if Q is J-normal then $P = Q(I - Q^{\#}) = (I - Q^{\#})Q$ and

$$R(P) = R(Q) \cap N(Q^{\#}) = R(Q) \cap R(Q)^{[\perp]} = R(Q)^{\circ}.$$

The assertion on $R(P^{\#})$ follows analogously. Finally, notice that

$$\begin{aligned}\dim R(Q)^{\circ} &= \dim R(P) = \dim N(P)^{\perp} = \dim R(P^{*}) = \dim R(P^{\#})\\ &= \dim N(Q)^{\circ},\end{aligned}$$

and $\operatorname{codim} R(Q)^{\circ} = \dim N(P) = \dim R(P)^{\perp} = \dim N(P^{*}) = \dim N(P^{\#}) = \operatorname{codim} N(Q)^{\circ}$. □

Remark 3.6. Let $Q \in L(\mathcal{H})$ be a J-normal projection with decompositions $Q = E+P$ and $I-Q = F+P^{\#}$. From the J-normality of Q and the formulas

$$E = QQ^{\#}, \quad P = Q(I - Q^{\#}), \quad F = (I - Q)(I - Q)^{\#} \quad \text{and} \quad PE = PF = 0,$$

the following facts are easily deduced:

1. $R(E) = R(Q) \cap R(Q^{\#})$ and $R(F) = N(Q) \cap N(Q^{\#})$. Moreover,
$$R(Q) = R(E)\,[\dot{+}]\,R(P) \quad \text{and} \quad N(Q) = R(F)\,[\dot{+}]\,R(P^{\#}).$$
2. Also, since $PP^{\#} = P^{\#}P = 0$, observe that $P + P^{\#}$ is a J-selfadjoint projection with range $R(Q)^{\circ} \dot{+} N(Q)^{\circ}$. Therefore, $R(Q)^{\circ} \dot{+} N(Q)^{\circ}$ is regular.
3. Finally, by the items above, notice that
$$\mathcal{H} = R(Q) \dot{+} N(Q) = (R(E)[\dot{+}]\,R(P)) \;\dot{+}\; (R(F)[\dot{+}]\,R(P^{\#})).$$
Then, if Q is J-normal, $\mathcal{H}$ can be decomposed as
$$\mathcal{H} = R(Q) \cap R(Q^{\#})\;[\dot{+}]\;(R(Q)^{\circ} \dot{+} N(Q)^{\circ})\;[\dot{+}]\;N(Q) \cap N(Q^{\#}). \tag{3.5}$$

In fact, (3.5) is equivalent to the J-normality of Q.

Proposition 3.7. *Let $Q \in L(\mathcal{H})$ be a projection. Then, Q is J-normal if and only if*

$$\mathcal{H} = R(Q) \cap R(Q^{\#}) \dot{+} R(Q) \cap N(Q^{\#}) \dot{+} N(Q) \cap R(Q^{\#}) \dot{+} N(Q) \cap N(Q^{\#}).$$

Proof. If Q is J-normal, the decomposition follows from item 3. in the above remark. Conversely, suppose that the above equation holds. Given $x \in \mathcal{H}$ there exist (unique) $x_1 \in R(Q) \cap R(Q^{\#})$, $x_2 \in R(Q) \cap N(Q^{\#})$, $x_3 \in N(Q) \cap R(Q^{\#})$ and $x_4 \in N(Q) \cap N(Q^{\#})$ such that $x = x_1 + x_2 + x_3 + x_4$. Then,

$$Q^{\#}Qx = Q^{\#}(x_1 + x_2) = x_1 = Q(x_1 + x_3) = QQ^{\#}x.$$

Therefore, $Q^{\#}Qx = QQ^{\#}x$ for every $x \in \mathcal{H}$, i.e. Q is J-normal. □

4. The Range of a *J*-Normal Projection

The aim of this section is to characterize the ranges of the family of J-normal projections acting on a Krein space. The main result in this direction addresses the fact that a (closed) subspace is the range of a J-normal projection if and only if it is a pseudo-regular subspace. Thus, the first paragraphs are devoted to recall the definition of pseudo-regularity and to state some well known equivalent conditions. Throughout this section, $\mathcal{H}$ denotes a Krein space with fundamental symmetry J.

Definition. A closed subspace $\mathcal{S}$ of $\mathcal{H}$ is called *pseudo-regular* if the algebraic sum $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed.

The following proposition compiles several conditions which are equivalent to pseudo-regularity. These facts are well known but they are scattered throughout the literature and different research papers, e.g. see [5,9,12,13].

Proposition 4.1. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and consider its Gramian operator $G_{\mathcal{S}} = P_{\mathcal{S}} J|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$. Then, the following conditions are equivalent:*

1. *$\mathcal{S}$ is pseudo-regular.*
2. *$(\mathcal{S}^\circ)^{[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$.*
3. *There exists a regular subspace $\mathcal{M}$ such that $\mathcal{S} = \mathcal{S}^\circ [\dot{+}]\, \mathcal{M}$.*
4. *If $\mathcal{S} = \mathcal{T} \dot{+} \mathcal{S}^\circ$, then $\mathcal{T}$ is regular.*
5. *There exists a regular subspace $\mathcal{N} \supseteq \mathcal{S}$ such that $\mathcal{S}^\circ = \mathcal{N} \cap \mathcal{S}^{[\perp]}$.*
6. *$\mathcal{S}/\mathcal{S}^\circ$ is a Krein space.*
7. *0 is an isolated point of $\sigma(G_{\mathcal{S}})$.*

Proposition 4.2. (T. Ando) *Given a (closed) subspace $\mathcal{S}$ of $\mathcal{H}$, consider its isotropic part $\mathcal{S}^\circ$. Let P and P_0 denote the orthogonal projections onto $\mathcal{S}$ and $\mathcal{S}^\circ$, respectively. Then, $\mathcal{S}$ is pseudo-regular if and only if*

$$\|(P - P_0)J(I - P)\| < 1.$$

Proof. Observe that $J(I - P)J$ is the orthogonal projection onto $\mathcal{S}^{[\perp]}$. By definition, $\mathcal{S}$ is pseudo-regular if

$$\mathcal{S} + \mathcal{S}^{[\perp]} \quad \text{is closed.}$$

But $\mathcal{S}+\mathcal{S}^{[\perp]}$ is closed if and only if $c(\mathcal{S}, \mathcal{S}^{[\perp]}) < 1$. Also, notice that $c(\mathcal{S}, \mathcal{S}^{[\perp]}) = c_0(\mathcal{S} \ominus \mathcal{S}^\circ, \mathcal{S}^{[\perp]}) = \|(P - P_0)J(I - P)J\|$ (see the Preliminaries). Hence, $\mathcal{S}$ is pseudo-regular if and only if

$$\|(P - P_0)J(I - P)\| < 1. \qquad \square$$

Theorem 4.3. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, $\mathcal{S}$ is the range of a J-normal projection if and only if $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$.*

Proof. If $\mathcal{S}$ is the range of a J-normal projection Q then, by Remark 3.6, $\mathcal{S} = R(E)[\dot{+}]\mathcal{S}^\circ$ where $E = QQ^\#$. Furthermore, $R(E)$ is regular because E is a J-selfadjoint projection. Thus, $\mathcal{S}$ is a pseudo-regular subspace.

Conversely, suppose that $\mathcal{S}$ is a pseudo-regular subspace and let P be the orthogonal projection onto the isotropic subspace $\mathcal{S}^\circ$. Since $R(P)$ is J-neutral, it follows by Lemma 2.3 that $PJP = 0$. Then, $PP^\# = P^\# P = 0$.

Consider the subspace $\mathcal{T} = \mathcal{S} \ominus \mathcal{S}^\circ$. Since $\mathcal{S} = \mathcal{T}[\dot{+}]\mathcal{S}^\circ$, Proposition 4.1 assures that $\mathcal{T}$ is a regular subspace of $\mathcal{H}$. Thus, there is a (unique) J-selfadjoint projection E with $R(E) = \mathcal{T}$.

Furthermore, $PE = EP = 0$ because $\mathcal{T} \subset (\mathcal{S}^\circ)^\perp$ and $\mathcal{S}^\circ \subset \mathcal{S}^{[\perp]} \subset \mathcal{T}^{[\perp]}$. Then $Q = E + P$ is also a projection with

$$R(Q) = R(E) + R(P) = \mathcal{T} \dot{+} \mathcal{S}^{\circ} = \mathcal{S}.$$

Finally, the J-normality of Q follows from Theorem 3.1. $\square$

Recall that if $\kappa = \min\{\dim \mathcal{H}_+, \dim \mathcal{H}_-\} < \infty$, the Krein space with fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$ is called a *Pontryagin space* and is denoted by Π_κ. In a Pontryagin space Π_κ, a closed subspace $\mathcal{S}$ is regular if and only if it is J-non-degenerated (see e.g. [12]). Thus, every J-non-degenerated subspace of Π_κ admits a (unique) J-selfadjoint projection onto it. Furthermore,

Corollary 4.4. *If Π_κ is a Pontryagin space, then every closed subspace $\mathcal{S}$ of Π_κ admits a J-normal projection onto it.*

Proof. Since $\mathcal{S}^{\circ}$ is a closed subspace of $\mathcal{S}$, $\mathcal{S}$ can be written as

$$\mathcal{S} = \mathcal{S}^{\circ} \oplus (\mathcal{S} \ominus \mathcal{S}^{\circ}).$$

Furthermore, $\mathcal{T} := \mathcal{S} \ominus \mathcal{S}^{\circ}$ is J-orthogonal to $\mathcal{S}^{\circ}$. Hence, $\mathcal{S} = \mathcal{S}^{\circ}[\dot{+}]\mathcal{T}$. It is easy to see that $\mathcal{T}$ is a J-non-degenerated subspace of $\mathcal{H}$ and therefore, $\mathcal{T}$ is regular because Π_κ is a Pontryagin space. Thus, $\mathcal{S}$ is the direct sum of its isotropic part and a regular subspace and, by Theorem 4.3, $\mathcal{S}$ is the range of a J-normal projection. $\square$

The last paragraphs of this section are devoted to discussing the non-uniqueness of J-normal projections associated to a pseudo-regular subspace. First of all, observe the following example.

Example 2. Consider the Minkowski space $(\mathbb{C}^3, [\ ,\ ])$, i.e. $\mathbb{C}^3$ endowed with the indefinite inner product given by $[x, y] = x_1\overline{y_1} + x_2\overline{y_2} - x_3\overline{y_3}$, where $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbb{C}^3$,

Fix $\mathcal{S}$ by $\mathcal{S} = \operatorname{span}\{(1,0,0), (0,1,1)\}$. Given a vector $v = (x, y, z) \in \mathbb{C}^3 \setminus \mathcal{S}$, let Q_v be the projection onto $\mathcal{S}$ along the subspace spanned by v. According to the canonical basis of $\mathbb{C}^3$, its matrix representation is

$$Q_v = \frac{1}{z-y}\begin{pmatrix} z-y & x & -x \\ 0 & z & -y \\ 0 & z & -y \end{pmatrix}.$$

A few calculations show that

$$Q_v^{\#} = \frac{1}{\overline{z-y}}\begin{pmatrix} \overline{z-y} & 0 & 0 \\ \overline{x} & \overline{z} & -\overline{z} \\ \overline{x} & \overline{y} & -\overline{y} \end{pmatrix}.$$

Then, it is easy to see that

$$Q_v^{\#}Q_v = \frac{1}{|z-y|^2}\begin{pmatrix} |z-y|^2 & x\overline{(z-y)} & -x\overline{(z-y)} \\ \overline{x}(z-y) & |x|^2 & -|x|^2 \\ \overline{x}(z-y) & |x|^2 & -|x|^2 \end{pmatrix} \quad \text{and}$$

$$Q_vQ_v^{\#} = \frac{1}{|z-y|^2}\begin{pmatrix} |z-y|^2 & x\overline{(z-y)} & -x\overline{(z-y)} \\ \overline{x}(z-y) & |z|^2-|y|^2 & -|z|^2+|y|^2 \\ \overline{x}(z-y) & |z|^2-|y|^2 & -|z|^2+|y|^2 \end{pmatrix}.$$

Therefore, Q_v is a J-normal projection onto $\mathcal{S}$ if and only if $|z|^2 = |x|^2 + |y|^2$.

The above example also shows that, for a fixed projection $Q \in L(\mathcal{H})$, the idempotency of the J-selfadjoint operators $QQ^{\#}$ and $Q^{\#}Q$ is not a sufficient condition for the J-normality of Q. In fact, notice that $Q_v^{\#}Q_v$ and $Q_vQ_v^{\#}$ are projections for every $v \in \mathbb{C}^3 \backslash \mathcal{S}$, even if $|z|^2 \neq |x|^2 + |y|^2$.

Although there is not a unique J-normal projection onto a fixed arbitrary pseudo-regular subspace $\mathcal{S}$, it is possible to present a particular J-normal projection onto $\mathcal{S}$ in terms of the orthogonal projections onto $\mathcal{S}$ and $\mathcal{S}^\circ$. Observe that this particular J-normal projection onto $\mathcal{S}$ is the one discussed in Theorem 4.3.

Corollary 4.5. *Given a (closed) pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, let P and P_0 denote the orthogonal projections onto $\mathcal{S}$ and $\mathcal{S}^\circ$, respectively. Then,*

$$Q = (P - P_0)(P - P_0 + I - J(P - P_0)J)^{-1} + P_0, \tag{4.1}$$

is a J-normal projection onto $\mathcal{S}$.

Proof. Since $\mathcal{S} \ominus \mathcal{S}^\circ$ is a regular subspace of $\mathcal{H}$, the J-selfadjoint projection E onto $\mathcal{S} \ominus \mathcal{S}^\circ$ can be written as

$$E = (P - P_0)(P - P_0 + I - J(P - P_0)J)^{-1},$$

see (2.2). Furthermore, by Theorem 3.1, $Q = E + P_0 = (P - P_0)(P - P_0 + I - J(P - P_0)J)^{-1} + P_0$ is a J-normal projection onto $\mathcal{S}$. $\square$

5. *J*-Normal Projections with *J*-Neutral Range

From now on, every subspace considered is assumed to be closed.

As it was shown in the previous section, a pseudo-regular subspace may admit infinitely many J-normal projections onto it. In order to provide a parametrization of the set of J-normal projections onto a prescribed pseudo-regular subspace, consider the simplest case first, i.e. a J-neutral subspace. This section is devoted to studying J-normal projections onto J-neutral subspaces, i.e. those projections $P \in L(\mathcal{H})$ satisfying $PP^{\#} = P^{\#}P = 0$.

It is obvious that every J-neutral subspace $\mathcal{N}$ of a Krein space $\mathcal{H}$ is a pseudo-regular one, since $\mathcal{N} = \mathcal{N}^\circ$. In particular,

Lemma 5.1. *If $\mathcal{N}$ is a J-neutral subspace then the orthogonal projection $P := P_{\mathcal{N}} \in L(\mathcal{H})$ is J-normal. Furthermore, $PP^{\#} = P^{\#}P = 0$.*

Proof. By Lemma 2.3, the assumption on $\mathcal{N}$ is equivalent to $PJP = 0$. Thus,

$$PP^{\#} = PJP \cdot J = 0 \quad \text{and} \quad P^{\#}P = J \cdot PJP = 0.$$

$\square$

Proposition 5.2. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be (closed) J-neutral subspaces of $\mathcal{H}$ such that $\mathcal{N}_1 \cap \mathcal{N}_2 = \{0\}$. Then, the following conditions are equivalent:*

1. *there exists a J-normal projection $P \in L(\mathcal{H})$ such that $R(P) = \mathcal{N}_1$ and $R(P^{\#}) = \mathcal{N}_2$;*
2. *$\mathcal{N}_1 + \mathcal{N}_2$ is regular;*
3. *$\mathcal{N}_1 \dot{+} \mathcal{N}_2^{[\perp]} = \mathcal{H}$.*

Proof. *1.* $\Rightarrow$ *2.* follows from item 2. of Remark 3.6.

2. $\Rightarrow$ *3.*: Suppose that $\mathcal{M} = \mathcal{N}_1 + \mathcal{N}_2$ is regular. Then, $\mathcal{M}^{[\perp]} = \mathcal{N}_1^{[\perp]} \cap \mathcal{N}_2^{[\perp]}$ is also regular and

$$\mathcal{H} = \mathcal{M} \dotplus \mathcal{M}^{[\perp]} = \mathcal{N}_1 \dotplus (\mathcal{N}_2 \dotplus \mathcal{N}_1^{[\perp]} \cap \mathcal{N}_2^{[\perp]}) \subseteq \mathcal{N}_1 + \mathcal{N}_2^{[\perp]},$$

because $\mathcal{N}_2$ is J-neutral. Analogously, $\mathcal{H} = \mathcal{N}_1^{[\perp]} + \mathcal{N}_2$ and $\mathcal{N}_1 \cap \mathcal{N}_2^{[\perp]} = (\mathcal{N}_1^{[\perp]} + \mathcal{N}_2)^{[\perp]} = \{0\}$. Thus, $\mathcal{H} = \mathcal{N}_1 \dotplus \mathcal{N}_2^{[\perp]}$.

3. $\Rightarrow$ *1.*: If $\mathcal{N}_1 \dotplus \mathcal{N}_2^{[\perp]} = \mathcal{H}$, consider the projection $P := P_{\mathcal{N}_1 // \mathcal{N}_2^{[\perp]}}$. Then, $P^{\#} = P_{\mathcal{N}_2 // \mathcal{N}_1^{[\perp]}}$ and it is easy to see that $PP^{\#} = P^{\#}P = 0$. Therefore, P is a J-normal projection with $R(P) = \mathcal{N}_1$ and $R(P^{\#}) = \mathcal{N}_2$. □

As a consequence of the above proposition, if P is a J-normal projection onto a J-neutral subspace, the subspaces $R(P)$ and $R(P^{\#})$ are *skewly linked* (see [12, Def. 1.29]). Moreover, in a Pontryagin space Π_κ, a pair of J-neutral subspaces $\mathcal{N}_1, \mathcal{N}_2$ of Π_κ is skewly linked if and only if there exists a J-normal projection $P \in L(\mathcal{H})$ such that $R(P) = \mathcal{N}_1$ and $R(P^{\#}) = \mathcal{N}_2$.

Remark 5.3. If $\mathcal{N}$ is a J-neutral subspace then $\mathcal{N} + J(\mathcal{N})$ is regular. In fact, by Lemma 5.1, the orthogonal projection P onto $\mathcal{N}$ is a J-normal projection and $R(P^{\#}) = J(\mathcal{N})$. So, by the above proposition, $\mathcal{N} + J(\mathcal{N})$ is regular.

Proposition 5.4. *Let $Q \in L(\mathcal{H})$ be a projection such that $R(Q)^\circ + N(Q)^\circ$ is regular. Then, there exist projections $E, P \in L(\mathcal{H})$ such that $PP^{\#} = P^{\#}P = 0$ and*

$$Q = E + P.$$

Proof. By Proposition 5.2, $\mathcal{H}$ can be decomposed as $\mathcal{H} = R(Q)^\circ \dotplus (N(Q)^\circ)^{[\perp]}$ and $P = P_{R(Q)^\circ // (N(Q)^\circ)^{[\perp]}}$ is J-normal. Since $R(P) \subseteq R(Q)$, it follows that $QP = P$. Also, PQ is a projection and $R(PQ) = R(P)$. Furthermore,

$$\begin{aligned} N(PQ) &= N(Q) + R(Q) \cap N(P) = N(Q) + R(Q) \cap (N(Q)^\circ)^{[\perp]} \\ &\subseteq (N(Q)^\circ)^{[\perp]} = N(P). \end{aligned}$$

Thus, $PQ = P$ and $E := Q - P$ is a projection because of

$$E^2 = Q - QP - PQ + P = Q - P - P + P = Q - P = E.$$

Notice that $PE = EP = 0$ and therefore $Q = E + P$. □

Following the notation of the above proof, observe that $E = Q - P = Q(I - P) = (I - P)Q$. Hence, $R(E) = R(Q) \cap N(P) = R(Q) \cap (N(Q)^\circ)^{[\perp]}$ and $N(E) = R(P) + N(Q) = R(Q)^\circ + N(Q)$. Therefore,

$$E = P_{R(Q)\cap(N(Q)^\circ)^{[\perp]} // R(Q)^\circ + N(Q)}.$$

Thus, the following is a sufficient condition to guarantee that the decomposition of the above proposition is the same as in Theorem 3.1.

Corollary 5.5. *Let $Q \in L(\mathcal{H})$ be a projection such that $R(Q)^\circ + N(Q)^\circ$ is regular. Then, the following conditions are equivalent:*

1. *Q is J-normal;*
2. *$R(Q) \cap (N(Q)^\circ)^{[\perp]} \subseteq R(Q) \cap R(Q^\#)$;*
3. *$N(Q) \cap (R(Q)^\circ)^{[\perp]} \subseteq N(Q) \cap N(Q^\#)$.*

Proof. If Q is J-normal, then $N(Q)$ is a pseudo-regular subspace. So,

$$(N(Q)^\circ)^{[\perp]} = N(Q) + N(Q)^{[\perp]} = N(Q) + R(Q^\#).$$

Then, if $x \in R(Q) \cap (N(Q)^\circ)^{[\perp]}$, there exist $u \in N(Q)$ and $v \in \mathcal{H}$ such that $x = u + Q^\# v$. Hence,

$$x = Qx = Q(u + Q^\# v) = QQ^\# v,$$

i.e. $x \in R(Q) \cap R(Q^\#)$. Thus, $R(Q) \cap (N(Q)^\circ)^{[\perp]} \subseteq R(Q) \cap R(Q^\#)$.

Conversely, suppose that $R(Q) \cap (N(Q)^\circ)^{[\perp]} \subseteq R(Q) \cap R(Q^\#)$. Then, consider the decomposition $Q = E + P$ given by Proposition 5.4, where $E, P \in L(\mathcal{H})$ are projections and $PP^\# = P^\# P = 0$. Observe that

$$R(E) = R(Q) \cap (N(Q)^\circ)^{[\perp]} = R(Q) \cap R(Q^\#),$$

because $N(Q)^\circ \subseteq N(Q) = R(Q^\#)^{[\perp]}$. Also,

$$R(E^\#) = N(E)^{[\perp]} = N(Q)^{[\perp]} \cap (R(Q)^\circ)^{[\perp]} \supseteq R(Q^\#) \cap R(Q) = R(E).$$

Thus, $E^\# E = E$ and, by Theorem 3.1, Q is J-normal.

Finally, notice that the equivalence 1. $\leftrightarrow$ 3. follows considering $I - Q$ instead of Q. □

The following result shows that, for a fixed J-neutral subspace, there are infinitely many J-normal projections onto it. Furthermore, the nullspaces of these projections can be arbitrarily close.

Proposition 5.6. (T. Ando) *Suppose that a (non-trivial) projection $P \in L(\mathcal{H})$ satisfies $PP^\# = P^\# P = 0$. Then, there exists a one-parameter family of (different) J-normal projections $P_\varepsilon \in L(\mathcal{H})$ onto $R(P)$ (for $0 < \varepsilon < \varepsilon_0$) such that*

$$\|P_\varepsilon - P\| \to 0 \quad \textit{as } \varepsilon \to 0.$$

Proof. Let P_R (resp. P_N) be the orthogonal projection onto $R(P)$ (resp. $N(P)$). Then, the ranges of these projections are J-neutral subspaces and, by Lemma 2.3, there is a partial isometry $V \in L(\mathcal{H}_+, \mathcal{H}_-)$ such that

$$I - P_N = \frac{1}{2}\begin{pmatrix} V^*V & V^* \\ V & VV^* \end{pmatrix}.$$

Since $e^{i\varepsilon}V$ is also a partial isometry (for every $\varepsilon > 0$), there is an orthogonal projection Q_ε such that

$$I - Q_\varepsilon = \frac{1}{2}\begin{pmatrix} V^*V & e^{-i\varepsilon}V^* \\ e^{i\varepsilon}V & VV^* \end{pmatrix},$$

so that $(I - Q_\varepsilon)J(I - Q_\varepsilon) = 0$. It is clear that $\|P_N - Q_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

Since $\|P_R P_N\| < 1$ and $\|(I - P_R)(I - P_N)\| < 1$, there exists $\varepsilon_0 > 0$ such that

$$\|P_R Q_\varepsilon\| < 1 \quad \text{and} \quad \|(I - P_R)(I - Q_\varepsilon)\| < 1 \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

Hence, there is a projection $P_\varepsilon \in L(\mathcal{H})$ with $R(P_\varepsilon) = R(P)$ and $N(P_\varepsilon) = R(Q_\varepsilon)$, see Remark 2.1. Then, by Lemma 2.3, $P_\varepsilon P_\varepsilon^{\#} = P_\varepsilon^{\#} P_\varepsilon = 0$. Finally, P_ε can be represented as:

$$P_\varepsilon = P_R(P_R + Q_\varepsilon)^{-1},$$

see (2.1). So, $P_\varepsilon \neq P$ for every $0 < \varepsilon \leq \varepsilon_0$, and $\|P_\varepsilon - P\| \to 0$ as $\varepsilon \to 0$. $\square$

Corollary 5.7. *Suppose that a (non-trivial) projection $P \in L(\mathcal{H})$, satisfies $PP^{\#} = P^{\#}P = 0$. Then, there exists a one-parameter family of (different) J-normal projections $P_\varepsilon \in L(\mathcal{H})$ onto $R(P)$ (for $0 < \varepsilon < \varepsilon_0$) such that*

$$c(N(P), N(P_\varepsilon)) \longrightarrow 1 \quad \text{as } \varepsilon \to 0.$$

Proof. Consider the projections P_ε obtained in Proposition 5.6. Following the notations in the proof above, $N(P) = R(P_N)$ and $N(P_\varepsilon) = R(Q_\varepsilon)$. Then,

$$c(N(P), N(P_\varepsilon)) = c(R(P_N), R(Q_\varepsilon)) = c(R(I - P_N), R(I - Q_\varepsilon)),$$

because P_N and Q_ε are orthogonal projections. By Remark 2.1,

$$\begin{aligned}
&c(R(I - P_N), R(I - Q_\varepsilon))^2 \\
&\quad = \|(I - Q_\varepsilon)(I - P_N)\|^2 = \|(I - Q_\varepsilon)(I - P_N)(I - Q_\varepsilon)\| \\
&\quad = \frac{|(1 + e^{i\varepsilon})(1 + e^{-i\varepsilon})|}{4} \left\| \frac{1}{2} \begin{pmatrix} V^*V & \frac{1+e^{-i\varepsilon}}{1+e^{i\varepsilon}} V^* \\ \frac{1+e^{i\varepsilon}}{1+e^{-i\varepsilon}} V & VV^* \end{pmatrix} \right\| \\
&\quad = \frac{|(1 + e^{i\varepsilon})(1 + e^{-i\varepsilon})|}{4} = \frac{1 + cos(\varepsilon)}{2} = \cos^2(\tfrac{\varepsilon}{2}).
\end{aligned}$$

Therefore, $c(N(P), N(P_\varepsilon)) = \cos(\frac{\varepsilon}{2}) \longrightarrow 1$ as $\varepsilon \to 0$. $\square$

5.1. *J*-Normal Projections with Prescribed *J*-Neutral Range

Let $\mathcal{N}$ be a J-neutral subspace of a Krein space $\mathcal{H}$ with fundamental symmetry J. Along these paragraphs, a parametrization for the set of J-normal projections onto $\mathcal{N}$ is presented. These results are generalized to an arbitrary pseudo-regular subspace in Sect. 6.

According to the orthogonal decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$, the symmetry J can be written as a block-operator-matrix

$$J = \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{N}^\perp \end{matrix} \tag{5.1}$$

where $a \in L(\mathcal{N}^\perp, \mathcal{N})$ and $b = b^* \in L(\mathcal{N}^\perp)$ satisfy

$$aa^* = I_{\mathcal{N}}, \quad ab = 0 \quad \text{and} \quad a^*a + b^2 = I_{\mathcal{N}^\perp}. \tag{5.2}$$

Since $a \in L(\mathcal{N}^\perp, \mathcal{N})$ is a coisometry, it follows that $a^* \in L(\mathcal{N}, \mathcal{N}^\perp)$ is a partial isometry with final space:

$$R(a^*a) = R(a^*) = J(\mathcal{N}).$$

Thus, $a^*a \in L(\mathcal{N}^\perp)$ is the orthogonal projection onto $J(\mathcal{N})$.

On the other hand, if P is a projection with range $\mathcal{N}$ then P can be written as a block-operator-matrix

$$P = \begin{pmatrix} I & x \\ 0 & 0 \end{pmatrix},$$

with $x \in L(\mathcal{N}^{\perp}, \mathcal{N})$. Furthermore, P satisfies $PP^{\#} = 0$ if and only if

$$0 = \begin{pmatrix} I & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix} \begin{pmatrix} I & 0 \\ x^* & 0 \end{pmatrix} = \begin{pmatrix} ax^* + xa^* + xbx^* & 0 \\ 0 & 0 \end{pmatrix},$$

or equivalently, $x \in L(\mathcal{N}^{\perp}, \mathcal{N})$ is a solution of the equation

$$ax^* + xa^* + xbx^* = 0. \tag{5.3}$$

Thus, in order to describe the set of J-normal projections onto the J-neutral subspace $\mathcal{N}$, the above equation has to be solved. The following result provides a parametrization for the set of solutions of (5.3).

Lemma 5.8. *Let $\mathcal{N}$ be a J-neutral subspace of $\mathcal{H}$. Then, $x \in L(\mathcal{N}^{\perp}, \mathcal{N})$ is a solution of (5.3) if and only if there exist operators $A \in L(\mathcal{N})$ and $B \in L(\mathcal{N}^{\perp}, \mathcal{N})$ such that A is antihermitian, $J(\mathcal{N}) \subseteq N(B)$ and*

$$x = (A - \tfrac{1}{2}BbB^*)a + B.$$

Proof. Recall that the operators a and b considered in (5.3) satisfy the conditions in (5.2). First, suppose that $x \in L(\mathcal{N}^{\perp}, \mathcal{N})$ is a solution of (5.3). Since $a^*a + b^2 = I_{\mathcal{N}^{\perp}}$, x can be written as $x = x_1 + x_2$, where $x_1 = xa^*a$ and $x_2 = xb^2$.

Observe that $x_2a^* = x_1b = 0$. Thus, $0 = ax^* + xa^* + xbx^* = ax_1^* + x_1a^* + x_2bx_2^*$. In other words,

$$2\,\mathrm{Re}(x_1a^*) = ax_1^* + x_1a^* = -x_2bx_2^*.$$

So, the antihermitian operator $A = i\,\mathrm{Im}(x_1a^*) \in L(\mathcal{N})$ satisfies

$$x_1 = x_1a^*a = (A - \tfrac{1}{2}x_2bx_2^*)a.$$

Then, considering $B = x_2 = x(I_{\mathcal{N}^{\perp}} - a^*a) \in L(\mathcal{N}^{\perp}, \mathcal{N})$ it follows that $J(\mathcal{N}) \subseteq N(B)$ and

$$x = (A - \tfrac{1}{2}BbB^*)a + B.$$

Conversely, given an antihermitian operator $A \in L(\mathcal{N})$ and $B \in L(\mathcal{N}^{\perp}, \mathcal{N})$ such that $J(\mathcal{N}) \subseteq N(B)$, consider

$$x := (A - \tfrac{1}{2}BbB^*)a + B.$$

Then, it is easy to see that $xa^* = A - \frac{1}{2}BbB^*$ and $xbx^* = BbB^*$. Therefore,

$$xa^* + ax^* + xbx^* = (A - \tfrac{1}{2}BbB^*) + (-A - \tfrac{1}{2}BbB^*) + BbB^* = 0,$$

i.e. $x \in L(\mathcal{N}^{\perp}, \mathcal{N})$ is a solution of (5.3). □

Proposition 5.9. *Let $\mathcal{N}$ be a J-neutral subspace of $\mathcal{H}$. Then, $P \in L(\mathcal{H})$ is a J-normal projection onto $\mathcal{N}$ if and only if there exist $A = -A^* \in L(\mathcal{N})$ and $B \in L(\mathcal{N}^{\perp}, \mathcal{N})$ with $J(\mathcal{N}) \subseteq N(B)$ such that*

$$P = \begin{pmatrix} I & (A - \frac{1}{2}BbB^*)a + B \\ 0 & 0 \end{pmatrix},$$

according to the orthogonal decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^{\perp}$.

6. A Parametrization for the Set of J-Normal Projections

Let $\mathcal{S}$ be a pseudo-regular subspace of a Krein space $\mathcal{H}$ with fundamental symmetry J, and denote

$$\mathcal{Q}_{\mathcal{S}} = \{Q \in L(\mathcal{H}) : \ Q^2 = Q, \ QQ^{\#} = Q^{\#}Q \text{ and } R(Q) = \mathcal{S}\}.$$

The aim of this section is to present an explicit parametrization of $\mathcal{Q}_{\mathcal{S}}$. First, notice that there are as many projections in $\mathcal{Q}_{\mathcal{S}}$ as in $\mathcal{Q}_{\mathcal{S}^\circ}$.

Lemma 6.1. *Suppose that $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$. If P is a J-normal projection onto $\mathcal{S}^\circ$ then there is a unique J-normal projection Q onto $\mathcal{S}$ such that P is the neutral part of Q, i.e. $P = Q(I-Q)^{\#}$.*

Proof. Suppose that $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$ and consider $\mathcal{T} = \mathcal{S} \cap N(P)$. Since P is a projection onto $\mathcal{S}^\circ \subseteq \mathcal{S}$, given $s \in \mathcal{S}$, $(I-P)s \in \mathcal{S} + \mathcal{S}^\circ = \mathcal{S}$. So that, $(I-P)s \in \mathcal{S} \cap N(P)$. Therefore,

$$\mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{T}.$$

Then, by Proposition 4.1, $\mathcal{T}$ is a regular subspace of $\mathcal{H}$. Let E be the J-selfadjoint projection onto $\mathcal{T}$.

Notice that $EP = 0$ because $\mathcal{S}^\circ \subseteq \mathcal{S}^{[\perp]} \subseteq \mathcal{T}^{[\perp]}$. On the other hand, $R(E) = \mathcal{T} \subseteq N(P)$. So, $PE = 0$ and, since E is J-selfadjoint, the following commutativity relations have been established:

$$EP = PE = 0 \quad \text{and} \quad EP^{\#} = P^{\#}E = 0.$$

Now, define $Q = E + P$. Then, by Theorem 3.1, Q is a J-normal projection and $P = Q - E = Q - QQ^{\#} = Q(I - Q^{\#})$.

Finally, suppose that there is another J-normal projection $Q' \in L(\mathcal{H})$ onto $\mathcal{S}$ such that $P = Q'(I-Q')^{\#}$. Then, $E' = Q' - P = Q'(Q')^{\#}$ is a J-selfadjoint projection onto a subspace of $\mathcal{S}$. Notice that $R(E') \subseteq N(P)$ because $PE' = 0$. Hence, $R(E') \subseteq \mathcal{T}$. But,

$$R(E') \dot{+} \mathcal{S}^\circ = \mathcal{S} = \mathcal{T} \dot{+} \mathcal{S}^\circ.$$

Thus, $R(E') = \mathcal{T}$ and, by the uniqueness of the J-selfadjoint projection onto a regular subspace, $E' = E$. □

Theorem 6.2. *Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$ with isotropic part $\mathcal{S}^\circ$, there is a (continuous) bijection between $\mathcal{Q}_{\mathcal{S}}$ and $\mathcal{Q}_{\mathcal{S}^\circ}$.*

Proof. For a fixed pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, let $\Phi : \mathcal{Q}_{\mathcal{S}} \to \mathcal{Q}_{\mathcal{S}^\circ}$ be defined by

$$\Phi(Q) = Q(I - Q^{\#}).$$

It follows by the above lemma that Φ is bijective, because for every $P \in \mathcal{Q}_{\mathcal{S}^\circ}$ there exists a unique $Q \in \mathcal{Q}_{\mathcal{S}}$ such that $\Phi(Q) = P$. □

Corollary 6.3. *Let $\mathcal{S}$ be a pseudo-regular subspace of a Krein space $\mathcal{H}$ with fundamental symmetry J. Then, there is a unique J-normal projection Q onto $\mathcal{S}$ if and only if $\mathcal{S}^\circ = \{0\}$. Moreover, in this case Q is J-selfadjoint.*

Proof. If $\mathcal{S}^\circ = \{0\}$ then $\mathcal{S}$ is a regular subspace and there exists a (unique) J-selfadjoint projection onto $\mathcal{S}$. Moreover, if Q is a J-normal projection onto $\mathcal{S}$ then, by Theorem 3.1, $Q = E + P$ where E is J-selfadjoint and P is a projection onto $\mathcal{S}^\circ = \{0\}$. Thus, $P = 0$ and $Q = E$.

On the other hand, if $\mathcal{S}^\circ \neq \{0\}$ then, as a consequence of Theorem 6.2 and Proposition 5.6, there are infinitely many J-normal projections onto $\mathcal{S}$. □

By Proposition 4.1, for a fixed pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, if $\mathcal{S}^\circ$ is the isotropic part of $\mathcal{S}$ and $\mathcal{M}$ is a subspace of $\mathcal{S}$ such that $\mathcal{S} = \mathcal{S}^\circ[\dot{+}]\mathcal{M}$ (i.e. $\mathcal{M}$ is a complement of $\mathcal{S}^\circ$ in $\mathcal{S}$), then $\mathcal{M}$ is a regular subspace of $\mathcal{H}$. Hence, consider

$$\mathcal{Q}_{\mathcal{S},\mathcal{M}} = \{Q \in \mathcal{Q}_{\mathcal{S}} : \ QQ^{\#} = E_{\mathcal{M}}\},$$

where $E_{\mathcal{M}}$ stands for the J-selfadjoint projection onto $\mathcal{M}$.

Notice that $\mathcal{Q}_{\mathcal{S}}$ can be written as the disjoint union of the family $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$, as $\mathcal{M}$ varies on the complements of $\mathcal{S}^\circ$ in $\mathcal{S}$:

Lemma 6.4. *If $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$, then*

$$\mathcal{Q}_{\mathcal{S}} = \dot{\bigcup}_{\{\mathcal{M}:\ \mathcal{S}=\mathcal{S}^\circ[\dot{+}]\mathcal{M}\}} \mathcal{Q}_{\mathcal{S},\mathcal{M}}, \tag{6.1}$$

where $\dot{\cup}$ denotes a disjoint union.

Proof. It is obvious that $\mathcal{Q}_{\mathcal{S}} = \bigcup_{\{\mathcal{M}:\ \mathcal{S}=\mathcal{S}^\circ[\dot{+}]\mathcal{M}\}} \mathcal{Q}_{\mathcal{S},\mathcal{M}}$. Suppose that $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}_1} \cap \mathcal{Q}_{\mathcal{S},\mathcal{M}_2}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are regular subspaces of $\mathcal{H}$. Then,

$$E_{\mathcal{M}_1} = QQ^{\#} = E_{\mathcal{M}_2},$$

or equivalently, $\mathcal{M}_1 = \mathcal{M}_2$. Hence, $\mathcal{Q}_{\mathcal{S},\mathcal{M}_1} = \mathcal{Q}_{\mathcal{S},\mathcal{M}_2}$. □

6.1. Parametrizing the Deck $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$ for a Pseudo-Regular Subspace $\mathcal{S}$

The following paragraphs are devoted to studying those J-normal projections onto $\mathcal{S}$ which have a fixed regular part. Along this section operators are treated as block-operator matrices according to the orthogonal decomposition

$$\mathcal{H} = \mathcal{S}^\circ \oplus (\mathcal{S} \ominus \mathcal{S}^\circ) \oplus \mathcal{S}^\perp,$$

and $P_{\mathcal{S}^\perp}$, $P_{\mathcal{S}^\circ}$ and $P_{\mathcal{S}\ominus\mathcal{S}^\circ}$ denote the orthogonal projections onto $\mathcal{S}^\perp$, $\mathcal{S}^\circ$ and $\mathcal{S} \ominus \mathcal{S}^\circ$, respectively.

If $\mathcal{M}$ is a regular subspace of $\mathcal{H}$ such that $\mathcal{S} = \mathcal{S}^\circ[\dot{+}]\mathcal{M}$, it is necessary to describe the fundamental symmetry J and the J-selfadjoint projection $E_{\mathcal{M}}$ onto $\mathcal{M}$ as block-operator matrices.

Lemma 6.5. *If $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$, then J is represented as the block-operator matrix*

$$J = \begin{array}{l} \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a^* & c^* & d \end{pmatrix} \begin{array}{l} \mathcal{S}^\circ \\ \mathcal{S}\ominus\mathcal{S}^\circ \\ \mathcal{S}^\perp \end{array} \end{array}, \tag{6.2}$$

where $a \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$*,* $b = b^* \in GL(\mathcal{S} \ominus \mathcal{S}^\circ)$*,* $c \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$ *and* $d = d^* \in L(\mathcal{S}^\perp)$ *satisfy the following equations:*

$$\begin{cases} aa^* = I_{\mathcal{S}^\circ} \\ b^2 + cc^* = I_{\mathcal{S}\ominus\mathcal{S}^\circ} \\ a^*a + c^*c + d^2 = I_{\mathcal{S}^\perp} \\ bc + cd = ad = ac^* = 0 \end{cases}. \tag{6.3}$$

Proof. Notice that $P_{\mathcal{S}^\circ} J P_{\mathcal{S}^\circ} = 0$ because $\mathcal{S}^\circ$ is J-neutral. Also, $P_{\mathcal{S}^\circ} J P_{\mathcal{S}\ominus\mathcal{S}^\circ} = 0$ because $\mathcal{S} \ominus \mathcal{S}^\circ \subseteq \mathcal{S}$ and $\mathcal{S}^\circ \subseteq \mathcal{S}^{[\perp]}$. Then,

$$J = \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a^* & c^* & d \end{pmatrix}.$$

On the other hand, the system of equations (6.3) follows from $J^2 = I$.

By Proposition 4.1, $\mathcal{S} \ominus \mathcal{S}^\circ$ is a regular subspace of $\mathcal{H}$. Furthermore, the regularity of $\mathcal{S} \ominus \mathcal{S}^\circ$ is equivalent to the range inclusion

$$R(c) \subseteq R(b),$$

see [7, Prop. 3.3]. Then, the second equation in (6.3) implies that $\mathcal{S} \ominus \mathcal{S}^\circ \subseteq R(b)$. Hence, b is an invertible selfadjoint operator in $L(\mathcal{S} \ominus \mathcal{S}^\circ)$. □

Remark 6.6. Observe that the operator $a \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ appearing in the above lemma is a coisometry. Then, $a^* \in L(\mathcal{S}^\circ, \mathcal{S}^\perp)$ is a partial isometry with final space $J(\mathcal{S}^\circ)$.

Indeed, by the block-operator matrix representation of J given in (6.2), it is easy to see that $R(a^*) = J(\mathcal{S}^\circ)$. Hence,

$$R(a^*a) = R(a^*) = J(\mathcal{S}^\circ). \tag{6.4}$$

Thus, $a^*a \in L(\mathcal{S}^\perp)$ is the orthogonal projection onto $J(\mathcal{S}^\circ)$.

The following lemma presents a block-matrix representation for the J-selfadjoint projection $E_{\mathcal{M}}$ onto a particular complement $\mathcal{M}$ of $\mathcal{S}^\circ$ in $\mathcal{S}$. This is a technical tool necessary to parametrize the deck $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$.

Lemma 6.7. *Given a pseudo-regular subspace* $\mathcal{S}$ *of* $\mathcal{H}$*, let* $\mathcal{M}$ *be a complement of* $\mathcal{S}^\circ$ *in* $\mathcal{S}$*. Then, the* J*-selfadjoint projection onto* $\mathcal{M}$ *is*

$$E_{\mathcal{M}} = \begin{pmatrix} 0 & ar^*b & ar^*(c+br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.5}$$

where $r = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E_{\mathcal{M}} P_{J(\mathcal{S}^\circ)}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$*.*

Proof. Suppose that $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$. Then, by Proposition 4.1, $\mathcal{M}$ is regular.

Denote by $E_{\mathcal{M}}$ the J-selfadjoint projection onto $\mathcal{M}$. Since $R(E_{\mathcal{M}}) = \mathcal{M} \subseteq \mathcal{S}$ it follows that $P_{\mathcal{S}^\perp} E_{\mathcal{M}} = 0$, so that the third row in the matrix representation of $E_{\mathcal{M}}$ is zero. Also, since $\mathcal{S}^\circ \subseteq \mathcal{S}^{[\perp]} \subseteq \mathcal{M}^{[\perp]} = N(E_{\mathcal{M}})$, it follows that $E_{\mathcal{M}}P_{\mathcal{S}^\circ} = 0$. So that the first column is also zero. Therefore,

$$E_{\mathcal{M}} = \begin{pmatrix} 0 & u & v \\ 0 & p & q \\ 0 & 0 & 0 \end{pmatrix},$$

where $u \in L(\mathcal{S}\ominus\mathcal{S}^\circ, \mathcal{S}^\circ)$, $v \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$, $p \in L(\mathcal{S}\ominus\mathcal{S}^\circ)$ and $q \in L(\mathcal{S}^\perp, \mathcal{S}\ominus\mathcal{S}^\circ)$ satisfy

$$\begin{cases} up = u \\ uq = v \\ p^2 = p \\ pq = q \end{cases}.$$

Thus, $p = P_{\mathcal{S}\ominus\mathcal{S}^\circ}E_{\mathcal{M}}|_{\mathcal{S}\ominus\mathcal{S}^\circ}$ is a projection with

$$R(p) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}E_{\mathcal{M}}(\mathcal{S}\ominus\mathcal{S}^\circ) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}E_{\mathcal{M}}(\mathcal{S}) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}(\mathcal{M}) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}(\mathcal{S}) = \mathcal{S}\ominus\mathcal{S}^\circ,$$

because $\mathcal{S}^\circ \subseteq N(P_{\mathcal{S}\ominus\mathcal{S}^\circ}) \cap N(E_{\mathcal{M}})$. Hence, $p = I_{\mathcal{S}\ominus\mathcal{S}^\circ}$.

Furthermore, $E_{\mathcal{M}}$ is J-selfadjoint if and only if

$$JE_{\mathcal{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & bq \\ 0 & a^*u + c^* & (a^*u + c^*)q \end{pmatrix}$$

is selfadjoint, or equivalently, if

$$a^*u + c^* = q^*b. \tag{6.6}$$

By (6.3), $aa^* = I_{\mathcal{S}^\circ}$ and $ac^* = 0$. Thus, multiplying on the left by a, it follows that $u = aq^*b$. Thus,

$$E_{\mathcal{M}} = \begin{pmatrix} 0 & aq^*b & aq^*bq \\ 0 & I & q \\ 0 & 0 & 0 \end{pmatrix},$$

where $q = P_{\mathcal{S}\ominus\mathcal{S}^\circ}E_{\mathcal{M}}|_{\mathcal{S}^\perp}$. Replacing u is (6.6), notice that q satisfies $a^*aq^*b + c^* = q^*b$, or equivalently,

$$q = q(a^*a) + b^{-1}c.$$

Therefore, if $r = q(a^*a)$ then $aq^*b = a(c^*b^{-1} + r^*)b = ar^*b$, and (6.5) follows. □

Finally, a block-matrix representation of a projection $Q \in L(\mathcal{H})$ onto $\mathcal{S}$ is needed. Since $R(Q) = \mathcal{S}$, observe that $P_{\mathcal{S}^\circ}QP_{\mathcal{S}^\circ} = P_{\mathcal{S}^\circ}$, $P_{\mathcal{S}\ominus\mathcal{S}^\circ}QP_{\mathcal{S}\ominus\mathcal{S}^\circ} = P_{\mathcal{S}\ominus\mathcal{S}^\circ}$ and

$$P_{\mathcal{S}^\circ}QP_{\mathcal{S}\ominus\mathcal{S}^\circ} = P_{\mathcal{S}\ominus\mathcal{S}^\circ}QP_{\mathcal{S}^\circ} = 0.$$

Then, Q is represented as the block-operator matrix

$$Q = \begin{pmatrix} I & 0 & x \\ 0 & I & y \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.7}$$

where $x = P_{\mathcal{S}^\circ}Q|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ and $y = P_{\mathcal{S}\ominus\mathcal{S}^\circ}Q|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}\ominus\mathcal{S}^\circ)$.

Furthermore, if $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}}$ then, by Theorem 3.1, $P = Q - E_{\mathcal{M}}$ is a projection onto $\mathcal{S}^\circ$ such that $PP^\# = P^\# P = 0$. Moreover, by (6.5), P has the form

$$P = Q - E_{\mathcal{M}} = \begin{pmatrix} I & -ar^*b & x - ar^*(c+br) \\ 0 & 0 & y - b^{-1}c - r \\ 0 & 0 & 0 \end{pmatrix}.$$

But, $R(P) = \mathcal{S}^\circ$ if and only if

$$y = b^{-1}c + r.$$

Also, $PP^\# = 0$ if and only if $PJP^* = 0$, or equivalently,

$$\begin{pmatrix} I & -ar^*b & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a^* & c^* & d \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ -bra^* & 0 & 0 \\ z^* & 0 & 0 \end{pmatrix} = 0,$$

where $z = x - ar^*(c+br)$. But the above equation is equivalent to

$$z(I - r^*bc)^*a^* + a(I - r^*bc)z^* + zdz^* + ar^*b^3ra^* = 0. \tag{6.8}$$

The following lemma is devoted to describe the solutions of (6.8), where a, b, c, d and r are the operators appearing in (6.2) and in (6.5).

Lemma 6.8. *An operator $z \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ is a solution of (6.8) if and only if there exist $A = -A^* \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ with $J(\mathcal{S}^\circ) \subseteq N(B)$ such that*

$$z = (A + \mathrm{Re}(Bc^*bra^*) - \tfrac{1}{2}(BdB^* + ar^*b^3ra^*))a + B.$$

Proof. Let $z \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ be a solution of (6.8) and consider the operators

$$z_1 = z(a^*a) \quad \text{and} \quad z_2 = z(I_{\mathcal{S}^\perp} - a^*a).$$

Notice that $z_1(I - r^*bc)^*a^* + a(I - r^*bc)z_1^* = z_1a^* + az_1^* = 2\,\mathrm{Re}(z_1a^*)$ because $ac^* = ca^* = 0$. Also,

$$z_2(I - r^*bc)^*a^* + a(I - r^*bc)z_2^* = -z_2c^*bra^* - ar^*bcz_2^* = -2\,\mathrm{Re}(z_2c^*bra^*),$$

because $z_2a^* = az_2^* = 0$. On the other hand, since $ad = da^* = 0$ it is easy to see that

$$zdz^* = (z_1 + z_2)d(z_1 + z_2)^* = z_2dz_2^*.$$

Therefore, (6.8) is equivalent to

$$2\,\mathrm{Re}(z_1a^*) = 2\,\mathrm{Re}(z_2c^*bra^*) - z_2dz_2^* - ar^*b^3ra^*. \tag{6.9}$$

Then, considering the antihermitian operator $A = i\,\mathrm{Im}(z_1a^*) \in L(\mathcal{S}^\circ)$, it follows that

$$\begin{aligned} z_1 &= (z_1a^*)a = (i\,\mathrm{Im}(z_1a^*) + \mathrm{Re}(z_1a^*))a \\ &= (A + \mathrm{Re}(z_2c^*bra^*) - \tfrac{1}{2}(z_2dz_2^* + ar^*b^3ra^*))a. \end{aligned}$$

Hence, $B = z_2 \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ satisfies $J(\mathcal{S}^\circ) \subseteq N(B)$ and

$$z = z_1 + z_2 = (A + \mathrm{Re}(Bc^*bra^*) - \tfrac{1}{2}(BdB^* + ar^*b^3ra^*))a + B.$$

Conversely, given an antihermitian operator $A \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ such that $N(b)^\perp \subseteq N(d)$, consider

$$z_{A,B} := (A + \mathrm{Re}(Bc^*bra^*) - \tfrac{1}{2}(BdB^* + ar^*b^3ra^*))a + B.$$

Then, it is easy to see that $z_{A,B} \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ is a solution of (6.8). □

Finally, it is possible to parametrize the deck $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$ as follows:

Theorem 6.9. *Let $Q \in L(\mathcal{H})$ be a projection onto a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$. Suppose that $\mathcal{M}$ is a regular subspace of $\mathcal{H}$ such that $\mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{M}$. Then, $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}}$ if and only if*

$$Q = \begin{pmatrix} I & 0 & (A + \mathrm{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c+br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix},$$

where $r = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E_{\mathcal{M}}(a^*a) \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$, $A = -A^* \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ is such that $J(\mathcal{S}^\circ) \subseteq N(B)$.

Proof. Suppose that $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}}$, i.e. $Q \in L(\mathcal{H})$ is a J-normal projection onto $\mathcal{S}$ satisfying $QQ^\# = Q^\# Q = E_{\mathcal{M}}$. Then, $P = Q - E_{\mathcal{M}}$ is a projection onto $\mathcal{S}^\circ$ such that $PP^\# = P^\# P = 0$. Hence, if Q is written as in (6.7) it follows that $y = b^{-1}c$.

Then, by the discussion above,

$$P = \begin{pmatrix} I & -ar^*b & x - ar^*(c+br) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $z = x - ar^*(c+br)$ is a solution of (6.8). Thus, by Proposition 6.8, there exist $A = -A^* \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ with $J(\mathcal{S}^\circ) \subseteq N(B)$ such that

$$P = \begin{pmatrix} I & -ar^*b & (A + \mathrm{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$Q = \begin{pmatrix} I & 0 & (A + \mathrm{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c+br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix}.$$

The converse follows immediately. □

Given a pseudo regular subspace $\mathcal{S}$ of $\mathcal{H}$, denote by $\mathcal{C}(\mathcal{S}^\circ)$ the set of complements of $\mathcal{S}^\circ$ in $\mathcal{S}$. Recall that, by Lemma 6.4, the set of J-normal projections onto $\mathcal{S}$ is decomposed as

$$\mathcal{Q}_{\mathcal{S}} = \dot{\bigcup}_{\mathcal{M}\in\mathcal{C}(\mathcal{S}^\circ)} \mathcal{Q}_{\mathcal{S},\mathcal{M}}.$$

Furthermore, for a fixed $\mathcal{M} \in \mathcal{C}(\mathcal{S}^\circ)$, Theorem 6.9 states that the deck $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$ is parametrized by the bijection $\Psi_{\mathcal{M}} : \mathcal{AH}(\mathcal{S}^\circ) \times \mathcal{N}_\circ \to \mathcal{Q}_{\mathcal{S},\mathcal{M}}$ given by

$$\Psi_{\mathcal{M}}(A,B)=\begin{pmatrix} I & 0 & (A+\mathrm{Re}(Bc^*bra^*)-\frac{1}{2}(BdB^*+ar^*b^3ra^*))a+B+ar^*(c+br) \\ 0 & I & b^{-1}c+r \\ 0 & 0 & 0 \end{pmatrix},$$

where $\mathcal{AH}(\mathcal{S}^\circ)$ stands for the real vector space of antihermitian operators acting on $\mathcal{S}^\circ$ and $\mathcal{N}_\circ$ is the set composed by those operators $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ such that $J(\mathcal{S}^\circ) \subseteq N(B)$.

Therefore, the set $\mathcal{Q}_{\mathcal{S}}$ of J-normal projections onto $\mathcal{S}$ is parametrized as follows:

Theorem 6.10. *Let $\mathcal{S}$ be a pseudo-regular subspace of $\mathcal{H}$. Then, the function $\Psi : \mathcal{RC}(\mathcal{S}^\circ) \times \mathcal{AH}(\mathcal{S}^\circ) \times \mathcal{N}_\circ \to \mathcal{Q}_{\mathcal{S}}$ defined by*

$$\Psi(\mathcal{M},A,B)=\begin{pmatrix} I & 0 & (A+\mathrm{Re}(Bc^*bra^*)-\frac{1}{2}(BdB^*+ar^*b^3ra^*))a+B+ar^*(c+br) \\ 0 & I & b^{-1}c+r \\ 0 & 0 & 0 \end{pmatrix},$$

is one-to one.

Observe that in the expression defining Ψ appears the operator

$$r = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E_{\mathcal{M}} P_{J(\mathcal{S}^\circ)}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}\ominus\mathcal{S}^\circ),$$

given in Lemma 6.7, where $P_{\mathcal{S}\ominus\mathcal{S}^\circ}$ and $P_{J(\mathcal{S}^\circ)}$ are the orthogonal projections onto $\mathcal{S}\ominus\mathcal{S}^\circ$ and $J(\mathcal{S}^\circ)$, respectively, and $E_{\mathcal{M}}$ is the J-selfadjoint projection onto $\mathcal{M}$.

6.2. An Interesting Particular Deck: $\mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$

Let $\mathcal{S}$ be a fixed pseudo-regular subspace of a Krein space $\mathcal{H}$ with fundamental symmetry J. These paragraphs are devoted to describe the set $\mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$, i.e. the family of J-normal projections $Q \in L(\mathcal{H})$ onto $\mathcal{S}$ such that $QQ^\#$ is the J-selfadjoint projection onto the (regular) subspace $\mathcal{S}\ominus\mathcal{S}^\circ$. In this particular deck there is a minimal norm projection, see Remark 6.12.

First of all, since $\mathcal{S}\ominus\mathcal{S}^\circ$ is a complement of $\mathcal{S}^\circ$ in $\mathcal{S}$, it follows by Lemma 6.7 that the J-selfadjoint projection onto $\mathcal{S}\ominus\mathcal{S}^\circ$ (hereafter denoted by E) is the block-operator matrix given by (6.5), where

$$r = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E P_{J(\mathcal{S}^\circ)}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}\ominus\mathcal{S}^\circ).$$

But, $J(\mathcal{S}^\circ) \subseteq J(\mathcal{S}^\circ)+\mathcal{S}^{[\perp]} = J(\mathcal{S}^\circ+\mathcal{S}^\perp) = J((\mathcal{S}\ominus\mathcal{S}^\circ)^\perp) = N(E)$. Therefore, $r=0$ and the block-operator matrix representation of E is

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & b^{-1}c \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, as a consequence of Theorem 6.9, $\mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$ is parametrized as:

Proposition 6.11. *Let $\mathcal{S}$ be a pseudo-regular subspace of a Krein space $\mathcal{H}$ with fundamental symmetry J. A projection Q onto $\mathcal{S}$ satisfies $QQ^\# = Q^\#Q = E$ if and only if*

$$Q = \begin{pmatrix} I & 0 & (A - \frac{1}{2}BdB^*)a + B \\ 0 & I & b^{-1}c \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.10}$$

where a, b, c and d are the operators appearing in (6.2)*, $A = -A^* \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ is such that $J(\mathcal{S}^\circ) \subseteq N(B)$.*

Remark 6.12. In this particular case it is possible to estimate

$$\min\{\|Q\| : \ Q \in \mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}\}.$$

Indeed, if P_0 is the orthogonal projection onto $\mathcal{S}^\circ$ and E stands for the J-selfadjoint projection onto $\mathcal{S}\ominus\mathcal{S}^\circ$, then $Q_0 = E+P_0 \in \mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$. Furthermore,

$$\|Q_0\|^2 = \|Q_0Q_0^*\| = \|EE^* + P_0\| = \max\{\|EE^*\|, \|P_0\|\} = \|EE^*\| = \|E\|^2,$$

because $R(EE^*) = \mathcal{S} \ominus \mathcal{S}^\circ$ is orthogonal to $R(P_0) = \mathcal{S}^\circ$. Therefore, $\|Q_0\| = \|E\|$.

On the other hand, if $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$ then there exists a (unique) $P = P^2 \in L(\mathcal{H})$ such that $PP^\# = P^\# P = 0$ and $Q = E + P$.

Consider a sequence $\{x_n\}_{n\geq 1}$ in the unit ball of $\mathcal{H}$ such that $\|Ex_n\| \to \|E\|$ as $n \to \infty$. Then,

$$\|Q\|^2 \geq \|Qx_n\|^2 = \|Ex_n\|^2 + \|Px_n\|^2 \geq \|Ex_n\|^2 \to \|E\|^2 = \|Q_0\|^2.$$

Hence, $\|Q_0\| = \min\{\|Q\| : \ Q \in \mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}\}$.

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