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E-POINTS IN EXTENSIVE GAMES WITH COMPLETE INFORMATION

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In this paper we characterize a general existence theorem concerning E-points for n-persons extensive games with complete information. We provide a sufficient and necessary condition for the existence of such E-points.

1. Introduction and Formulation

The concept of the equilibrium point of games by Nash (1951) was generalized by Marchi in Marchi (1967) to the concept of E-point.

The existence theorem for the equilibrium points in extensive games with complete information is a matter that goes back to Zermelo, as it is noted in the book by Burger (1959). Some references and examples of general equilibrium points are given in Thomas (1984) and almost an exhaustive study for refinements is given in Van Damme's book (1987).

In this paper we extend the result concerning the equilibrium point in an extensive game with complete information to the general concept of E-points.

The intuitive origin of the idea for the formulation of a sufficient and necessary condition for having E-points in extensive games with complete information was obtained from Marchi (2005). The reader will realize that the necessary and sufficient condition given in (2005) for having interchangeability of equilibrium points in extensive games with complete information is naturally the same condition given in this paper for the existence of the E-points.

As it is a standard matter, we give an n-person extensive game with complete information by a finite rooted tree $\mathrm{G} = \{\mathrm{g}\}$ with the initial root A. The set of players is $\mathrm{N} = \{1, \ldots, \mathrm{n}\}$. The chance player is $\mathrm{i_O}$. The set of nodes in G is partitioned in

$$\mathrm{G} = \bigcup_{i\in N} \mathrm{G_i} \cup \mathrm{G_{i_O}}$$

The ending points are given by $e_1, \dots, e_r$, but indeed we do not need them explicitly. For each player $i \in N$, and any $g \in G_1$, we consider all the edges emanating from g, which are indexed by $\sigma_i(g)$. We write $\sigma_i = \{\sigma_i(g)\}_{g \in G_i}$ as a complete plan to be followed by the player $i \in N$ indicating that if the game reaches the node g he chooses $\sigma_i(g)$ with $g \in G_1$ at that node. σ_i is a complete plan for player $i \in N$ and it is called a strategy. The set of the strategies for player $i \in N$ is written by $\sum_1$.

If $g \in G_{i_O}$ the chance player determines his only strategy at that point by a distribution of probability $p_{i_O}(g, \sigma_{i_O}(g))$ on the set edges emanating from g.

The payoff function in such a game is given in a standard way by means of the expectations considering the corresponding strategies $\sigma_i \in \sum_i i \in N$ and the chance moves p_{i_O}. At the end point we give the payoff $Ai(e)_k$ for each $i \in N$ point and e_k. For this we refer to the books by Burger (1959), Myerson (1951), Osborne & Rubistein (1997), Van Damme (1987), etc. Let A be such expectations, that is to say the payoff functions. We explain below the way that these are computed in extensive games with a chance player and the tree. Given for each player $i \in N$, a non-empty subset $e(i) \subset N$ be a set of players called the set of friend players for player $i \in N$. Then we remind you, as it was introduced by Marchi in (1967), an E-point is a point

$$\overline{\sigma} = (\overline{\sigma}_1, \dots, \overline{\sigma}_n) \in \sum = X_{i \in N} \sum_1$$

such as that for each $i \in N$:

$$A_i(\overline{\sigma}_{e(i)}, \overline{\sigma}_{N-e(j)}) \geq A_i(\sigma_{e(i)}, \overline{\sigma}_{N-e(i)}) \quad \text{for each} \quad \sigma_{e(i)} \in \sum_{e(i)}$$

where the set $\sum_f$ is $X_{j \in N} \sum_j$ for $f \subset N$. We have $\sum_{e(i)} \times \sum_{N-e(i)} = \sum$.

We need some more notation. Given $g \in G_i$ and $\sigma_i(g)$ we write $\eta($ g, σ (g) $)$ as that node $\overline{g} \in G$ being the end point of $\sigma_i(g)$. Given $\sigma_i \in \sum_i$ when $g \in G_i \sigma^g$ is the restriction of σ in the truncation Γ_g. For any $g \in G_i$ consider the notation.

$$A(g)(\sigma(g), \sigma^{\eta(g, \sigma_i(g))}$$

Or simply

$$A_i(\sigma_{i(g)}, \sigma^{\eta(g, \sigma_i(g))})$$

for the expectation or payoff function in the truncation with obvious notation.

We remind you that the expectation functions, which are actually recursively constructed, are given in the following way. Consider $\in$ the chance player i_σ is playing at $g \in G_{i_O}$, then consider $\sigma_{i_O}(g)$ and the truncation $\Gamma\eta(g, \sigma_i(g))$. Let $\overline{e}$ be an end point or terminal in the total game. Let

$$w^{\eta(g, \sigma_i(g))}(\sigma^{\eta(g), \sigma_i(g)})$$

the probability of the realization of $\overline{e}$ in the game $\Gamma\eta(g, \sigma_{i_O}(g))$ with the strategy $(\sigma^{\eta(g, \sigma_i(g)})$ and let w (σ^g) be the probability of the realization of $\overline{e}$ in Γ_g with the strategy σ^g. The evident

$$w(\sigma^g) = \rho_{i_O}(g, \sigma_{i_O}(g)) w^{\eta(g, \sigma_{i_O}(g))}(\sigma^{\eta(g, \sigma_i(g)})$$

which is also held in the case where $\overline{\mathrm{e}}$ cannot be realized with the strategy σ. It follows that if in

$$\mathrm{A_i}(\sigma) = \sum_{\overline{\mathrm{e}}} \mathrm{A_i}(\overline{e})\mathrm{w}_{\overline{e}}(\sigma)$$

The sum over all the end points of Γ_{g} is divided into parts corresponding to the end points of the games $\Gamma_{\eta(\mathrm{g},\sigma_{\mathrm{i_O}}(\rho)}$, then

$$\mathrm{A_i}(\sigma) = \sum_{\overline{\mathrm{e}}} \rho_{\mathrm{i_O}}(\mathrm{g}, \sigma_{\mathrm{i_O}}(\mathrm{g}))\mathrm{A_i}(\sigma^{\eta(\mathrm{g},\sigma_{\mathrm{i_O}}(\mathrm{g}))}))$$

Next we provide the result of this paper.

Theorem. *Let* $\mathrm{I_i(g)} = \{\mathrm{j} \in \mathrm{N/i} \in \mathrm{e(j)}\}$ *for* $\mathrm{g} \in \mathrm{G_i}$, *then under the condition of the existence of* $\overline{\sigma}_{\mathrm{i}}(\mathrm{g})$

$$\mathrm{A_j(g)}(\overline{\sigma}_{\mathrm{i}}(\mathrm{g}),\ \overline{\sigma}^{\eta(\mathrm{g},\sigma_{\mathrm{i}}(\mathrm{g}))} \geq \mathrm{A_j(g)}(\overline{\sigma}_{\mathrm{i}}(\mathrm{g}),\ \overline{\sigma}^{\eta(\mathrm{g},\overline{\sigma}_{\mathrm{i}}(\mathrm{g}))} \tag{1}$$

for each: $\mathrm{j} \in \mathrm{I_i(g)}$ *and* $\mathrm{g} \in \mathrm{G_i}$ *and* $\mathrm{i} \in \mathrm{N}$, *then* $\overline{\sigma}$ *is an E-point. Such a condition is sufficient and necessary.*

Proof. We are going to prove the theorem using the induction principle over the length λ of the game Γ. If $\lambda = 1$ and the root $\mathrm{A} \in \mathrm{G_i}$ $\mathrm{i} \in \mathrm{N}$, for $\mathrm{j} \in \mathrm{I_i(g)}$, by condition **(1)**

$$\mathrm{A_j}(\overline{\sigma}_{\mathrm{e(j)}}, \overline{\sigma}_{\mathrm{N-e(j)}}) = \mathrm{A_j(A)}(\overline{\sigma}_{\mathrm{i}}) \geq \mathrm{A_j(A)}(\sigma_{\mathrm{i}}) = \mathrm{A_j}(\sigma_{\mathrm{e(i)}}, \overline{\sigma}_{\mathrm{N-e(i)}})$$

for each $\sigma_{\mathrm{e(i)}}$.

for $\mathrm{j} \notin \mathrm{I(A)}$

$$\mathrm{A_j}(\overline{\sigma}_{\mathrm{e(j)}}, \overline{\sigma}_{\mathrm{N-e(j)}}) = \mathrm{A_j(A)}(\overline{\sigma}_{\mathrm{i}}) = \mathrm{A_j}(\sigma_{\mathrm{e(j)}}, \overline{\sigma}_{\mathrm{N-e(j)}})$$

for each $\sigma_{\mathrm{e(i)}}$, since $\mathrm{i} \notin \mathrm{e(j)}$ and then $\mathrm{i} \in$ N-e(j) and $\overline{\sigma}_{\mathrm{i}} \sim \overline{\sigma}_{\mathrm{N-e(i)}}$. □

Then the theorem is true for $\lambda = 1$.

Now consider the game Γ having length $\lambda = \lambda_{\mathrm{r}}$. By induction hypothesis we have that the theorem is true for all games Γ' with length $\lambda \leq \lambda_{\mathrm{r}-1}$.Consider the game Γ with root A. If $\mathrm{A} \in \mathrm{G_{i_O}}$, that is to say that at the root the chance player is actually playing. Then, since we have the equality:

$$\mathrm{A_i}(\sigma) =_{\sigma_{\mathrm{i}_\sigma}} \sum\nolimits_{(\mathrm{A})} \rho_{\mathrm{i_O}}(\mathrm{A}, \sigma_{\mathrm{i_O}}(\mathrm{A}))\mathrm{A_i}(\sigma^{\eta(\mathrm{A},\sigma_{\mathrm{i_O}}(\mathrm{A}))})$$

It turns out that if $\overline{\sigma}^{\eta(\mathrm{A},\sigma_{\mathrm{i_O}}(\mathrm{A}))}$ is any E-point in the truncation $\Gamma_{\eta(\mathrm{A},\sigma_{\mathrm{i_O}}(\mathrm{A}))}$ which exist by the induction hypothesis since the length of $\Gamma_{\eta(\mathrm{A},\sigma_{\mathrm{i_O}}(\mathrm{A}))} \leq \lambda_{\mathrm{r}} - 1$. Then we have

$$\begin{aligned}
\mathrm{A_i}(\overline{\sigma}_{\mathrm{e(i)}}, \overline{\sigma}_{\mathrm{N-e(i)}}) &=_{\sigma_{\mathrm{i}_\sigma}} \sum\nolimits_{(\mathrm{A})} \rho_{\mathrm{i_O}}(\mathrm{A}, \sigma_{\mathrm{i_O}}(\mathrm{A}))\mathrm{A_i}(\overline{\sigma}^{\eta(\mathrm{A},\sigma_{\mathrm{i_O}}(\mathrm{A}))}) \\
&=_{\sigma_{\mathrm{i}_\sigma}} \sum\nolimits_{(\mathrm{A})} \rho_{\mathrm{i_O}}(\mathrm{A}, \sigma_{\mathrm{i_O}}(\mathrm{A}))\mathrm{A_i}(\overline{\sigma}_{\mathrm{e(i)}}^{\eta(\mathrm{A},\sigma_{\mathrm{i_O}}(\mathrm{A}))}(\overline{\sigma}_{\mathrm{N-e(i)}}^{\eta(\mathrm{A},\sigma_{\mathrm{i_O}}(\mathrm{A}))}) \\
&\geq_{\sigma_{\mathrm{i}_\sigma}} \sum\nolimits_{(\mathrm{A})} \rho_{\mathrm{i_O}}(\mathrm{A}, \sigma_{\mathrm{i_O}}(\mathrm{A}))\mathrm{A_i}(\sigma_{\mathrm{e(i)}}^{\eta(\mathrm{A},\sigma_{\mathrm{i_O}}(\mathrm{A}))}(\overline{\sigma}_{\mathrm{N-e(i)}}^{\eta(\mathrm{A},\sigma_{\mathrm{i_O}}(\mathrm{A}))}) \\
&= \mathrm{A_i}(\sigma_{\mathrm{e(i)}}, \overline{\sigma}_{\mathrm{N-e(i)}})
\end{aligned}$$

for each i and $\sigma_{e(i)}$. In the inequality we have used the fact that $\sigma^{\eta(A,\sigma_{i_O}^{(A)})}$ is an E-point in the corresponding truncation tree game. Therefore $\overline{\sigma}$ is an E-point for Γ. If $A \in G_i$, then for any player k we have

$$A_k(\overline{\sigma}_{e(k)}, \overline{\sigma}_{N-e(k)}) = A_k(\eta(A, \overline{\sigma}_i(A))(\overline{\sigma}_{e(k)}^{\eta(A,\overline{\sigma}_i(A))}, \overline{\sigma}_{N-e(k)}^{\eta(A,\overline{\sigma}_i(A))})$$

Now choosing in each $\Gamma_{\eta(A,\sigma_i(A))}$ an E-point $\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})$ and $\overline{\sigma}_i$ (A) by condition **(1)** .(We overuse notation, which is clear). Then for any player $k \in I_i$ (A), we have

$$\begin{aligned}
A_k(\overline{\sigma}_{e(k)}, \overline{\sigma}_{N-e(k)}) &= A_k(A)(\overline{\sigma}_i(A)), \overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))} \\
&= A_k(\eta(A, \overline{\sigma}_i(A))(\overline{\sigma}_{e(k)}^{\eta(A,\overline{\sigma}_i(A))}, \overline{\sigma}_{N-e(k)}^{\eta(A,\overline{\sigma}_i(A))}) \\
&\geq A_k(\eta(A, \overline{\sigma}_i(A))(\overline{\sigma}_{e(k)}^{\eta(A,\overline{\sigma}_i(A))}, \overline{\sigma}_{N-e(k)}^{\eta(A,\overline{\sigma}_i(A))}) \\
&\geq A_k(\eta(A, \overline{\sigma}_i(A))(\overline{\sigma}_{e(k)}^{\eta(A,\overline{\sigma}_i(A))}, \overline{\sigma}_{N-e(k)}^{\eta(A,\overline{\sigma}_i(A))}) \\
&= A_k(\sigma_{e(k)}, \sigma_{N-e(k)})
\end{aligned}$$

for each $\sigma_{e(k)} \in \Sigma_{e(k)}$
Finally if $k \notin I_i(A)$

$$\begin{aligned}
A_k(\overline{\sigma}_{e(k)}, \overline{\sigma}_{N-e(k)}) &= A_k(A)(\overline{\sigma}_i(A), \overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))} \\
&= A_k^{\eta(A,\overline{\sigma}_i(A))}(\overline{\sigma}_{e(k)}^{\eta(A,\overline{\sigma}_i(A))}, \overline{\sigma}_{N-e(k)}^{\eta(A,\overline{\sigma}_i(A))}) \\
&\geq A_k^{\eta(A,\overline{\sigma}_i(A))}(\overline{\sigma}_{e(k)}\eta(A, \overline{\sigma}_i(A)), \overline{\sigma}_{N-e(k)}^{\eta(A,\overline{\sigma}_i(A))}) \\
&= A_{k(\sigma_{e(k)}, \overline{\sigma}_{N-e(k)})}
\end{aligned}$$

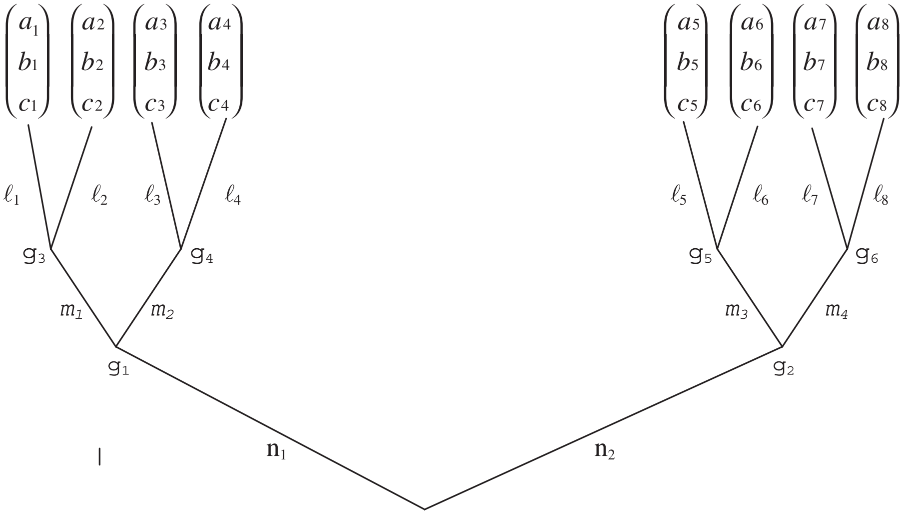
for each ${}_{\sigma e(k)}$. We have used only the fact that $\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))}$ is an E-point in the truncation game $\Gamma_{\eta(A,\sigma_i(A))}$. The last equality is a consequence of $k \notin I(A)$. Therefore the point $\overline{\sigma} \in \Sigma$ is an E-point.

Then the condition (1) is a sufficient condition. And it is easy to see it is also necessary.

2. A Variety of Examples

In this paragraph we are going to present an example which is rather general considering all kinds of variety of games since the payoff functions are given by parameters which, if the condition (1) for the game is satisfied, determines a functional among the parameters of the payoff functions and then, by the theorem, the existence of an E-point.

Indeed we consider the following extensive game with complete information provided by the tree.



With $G_3 = \{A\}$, $G_2 = \{g_1, g_2\}$ $G_1 = \{g_3, g_4, g_5, g_6\}$, $M = \{1,2,3\}$ $G_{i_O} = \emptyset$ where $\emptyset$ indicates the empty set. The structure function is given by

$$\mathrm{e}(1) = \{1, 2\}, \quad \mathrm{e}(2) = \{2, 3\}, \quad \mathrm{e}(3) = \{3, 1\} \tag{2}$$

Then $\sum_3 = \{n_1, n_2\}$, $\sum_2 = \{m_1, m_2\}, \{m_3, m_4\}$ and

$$\sum_1 = \{(l_1, l_2), (l_3, l_4), (l_5, l_6), (l_7, l_8)\} \tag{3}$$

with obvious notation.

The payoff function $\begin{pmatrix} \mathrm{ak} \\ \mathrm{bk} \\ \mathrm{ck} \end{pmatrix}$ determines the payoff of the first player a_k , of the second player b_k and of the third player c_k at the end point k.

Then a sufficient and necessary condition for having

$$\overline{\sigma} = (l_1, l_3, l_5, l_7, m_1, m_3, n_1)$$

as an E-point is given below just applying condition **(1)** to all the nodes.

We have $I_1(g_s) = \{1,2\}$ s = 3, … ,6, $I_2(g_1) = I_2(g_2) = \{2,3\}$
$I_3(A) = \{1,3\}$

We apply condition **(1)** to the different nodes

$a_1 = A_1\ (g_3)\ (l_1) \geq A_1\ (g_3)\ (l_2) = a_2$

$b_1 = A_2\ (g_3)\ (l_1) \geq A_2\ (g_3)\ (l_2) = b_2$

Applying to nodes g_4, $g_5$we have

$a_{3\geq}a_4 \quad a_{5\geq}a_6$ and $a_{7\geq}a_8$

$b_{3\geq}b_4 \quad b_{5\geq}b_6$ and $b_{7\geq}b_8$

In the node g_1, it turns out that

$b_1 = A_2\ (g_1)\ (m_1, l_1) \geq A_2\ (g_1)\ (m_2, l_3) = b_3$

$c_1 = A_3\ (g_1)\ (m_1, l_1) \geq A_3\ (g_1)\ (m_2, l_3) = c_3$

to the node g_2

$b_5 = A_2\ (g_2)\ (m_3, l_5) \geq A_2\ (g_2)\ (m_4, l_7) = b_7$

$c_5 = A_3\ (g_2)\ (m_3, l_5) \geq A_3\ (g_2)\ (m_4, l_7) = c_7$

and finally to A

$a_{1\geq} A_1\ (A)\ (n_1, m_1, l_1) \geq A_1\ (n_2, m_3, l_5) = a_5$

$c_{1\geq} A_3\ (A)\ (n_1, m_1, l_1) \geq A_3\ (n_2, m_3, l_5) = c_5$

Then under the conditions

$$\begin{array}{llll} a_1 \geq a_2 & a_3 \geq a_4 & a_5 \geq a_6 & a_7 \geq a_8 \\ b_1 \geq b_2 & b_3 \geq b_4 & b_5 \geq b_6 & \\ b_1 \geq b_3 & & b_5 \geq b_7 \geq b_8 & \\ c_1 \geq c_3 & & & \\ c_1 \geq c_5 \geq c_7 & & & \end{array}$$

The point $\overline{\sigma}$ is an E-point. Thus the variety for having $\overline{\sigma}$ an E-point is a variety with border but of high dimension. We have chosen a game with $G_{i_O} = \emptyset$ for simplicity.

As a comment we would like to say that it is possible to extend the concept of E-point in an extensive game considering that at each node $g \in G_i$ the structure set e(i) is different, that is to say e(i,g) depends on i and g. It seems clear that generalizing the simple condition **(1)** the existence of a new E-point is then obtained. Perfection might also be studied accordingly; Selten (1975)

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