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E-POINTS FOR DIAGONAL GAMES I

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In this paper we study and compute E-points in an explicit way for diagonal games of three, four, five and n-players.

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1. Introduction

In the non-cooperative n-person games there is a strong *cooperative* component which is the concept of E-points introduced by Marchi (1967). At the present there is not much done in this subject in the literature.

Here in this paper we are going to study and compute explicitly E-points for some diagonal n-person games. In our knowledge they are the first non trivial calculations of such points, which rise a somewhat new subject relating game theory, with algebraic geometric. Other calculations of E-points for three person games are due to Ruth Martínez (1989).

2. Preliminaries

Consider an n-person non-cooperative game

$$\Gamma = \{\Sigma_1, \ldots, \Sigma_n;\ \mathrm{A}_1, \ldots, \mathrm{A}_n\}$$

where Σ_i is the strategy set of player $i \in \mathrm{N} = \{1, \ldots, n\}$ and A_i its payoff function in pure strategies. The cardinality of $\Sigma_i : |\Sigma_i|$ is finite. Then the mixed extension is given by

$$\widetilde{\Gamma} = \{\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_n;\ \mathrm{E}_1, \ldots, \mathrm{E}_n\}$$

where the ith mixed strategy set is

$$\widetilde{\Sigma}_i = \left\{\mathrm{x}_i : \Sigma_i \to \mathrm{R} : \sum_{\sigma_i \in \Sigma_i} \mathrm{x}_i(\sigma_i) = 1\ \mathrm{x}_i(\sigma_i) \geq 0\ \forall \sigma_i \in \Sigma_i\right\}.$$

and E_i is the standard expectation.

We recall that a structure function for it, is a function

$$\mathrm{d} : \mathrm{N} \to 2^{\mathrm{N}}$$

such that $\mathrm{d}(i) \ni i$. The set $\mathrm{d}(i)$ is the set of friend players for $i \in \mathrm{N}$.

Given $(\widetilde{\Gamma}, \mathrm{d})$ we say that a point

$$\mathrm{x} = (\mathrm{x}_{\mathrm{d}(i)} x_{\mathrm{N}-\mathrm{d}(i)}) = (\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \mathrm{X}_{\mathrm{N}} = \mathrm{X}$$

(where clearly $\mathrm{x}_{\mathrm{d}(i)} \in \mathrm{X}_{\mathrm{d}(i)} = \mathrm{X}_{j \in \mathrm{d}(i)} \widetilde{\Sigma}_j$) is an E-point if

$$\mathrm{E}_i(\bar{\mathrm{x}}_{\mathrm{d}(i)}, \bar{\mathrm{x}}_{\mathrm{N}-\mathrm{d}(i)}) \geq \mathrm{E}_i(\mathrm{x}_{\mathrm{d}(i)}, \bar{\mathrm{x}}_{\mathrm{N}-\mathrm{d}(i)}) \quad \forall i \ \forall \mathrm{x}_{\mathrm{d}(i)} \in \overline{\mathrm{X}}_{\mathrm{d}(i)}.$$

Given a point x_j let $\mathrm{S}(\mathrm{x}_j)$ be the support of x_j. Then we have the following:

Lemma 1.1. *Given* $(\widetilde{\Gamma}, \mathrm{d})$, *a point* $\bar{\mathrm{x}} \in \mathrm{X}$ *is an E-point if and only if it satisfies the system*

$$\begin{aligned} &\lambda_i - \mathrm{E}_i(\sigma_{\mathrm{d}(i)}, \bar{\mathrm{x}}_{\mathrm{N}-\mathrm{d}(i)}) = 0 \quad \forall \sigma_{\mathrm{d}(i)} \in \prod_{j \in \mathrm{d}(i)} \mathrm{S}(\bar{\mathrm{x}}_j) \\ &\lambda_i - \mathrm{E}_i(\sigma_{\mathrm{d}(i)}, \bar{\mathrm{x}}_{\mathrm{N}-\mathrm{d}(i)}) \geq 0 \quad \forall \sigma_{\mathrm{d}(i)} \notin \prod_{j \in \mathrm{d}(i)} \mathrm{S}(\bar{\mathrm{x}}_j) \, \forall i \\ &\sum_{\sigma_i \in \Sigma_i} \mathrm{x}_i(\sigma_i) = 1 \quad \forall i \\ &\mathrm{x}_i(\sigma_i) \geq 0 \quad \forall i \forall \sigma_i \in \Sigma_i. \end{aligned} \tag{1}$$

Proof. Consider an E-point $\bar{\mathrm{x}} \in \mathrm{X}$ and define $\lambda_i = \mathrm{E}_i(\bar{\mathrm{x}})$. Then for any pure strategy pair $\sigma_{\mathrm{d}(i)} \in \Sigma_{\mathrm{d}(i)} = \mathrm{X}_{j \in \mathrm{d}(i)} \Sigma_j$ we have

$$\lambda_i - \mathrm{E}_i(\bar{\sigma}_{\mathrm{d}(i)}, \bar{\mathrm{x}}_{\mathrm{N}-\mathrm{d}(i)}) \geq 0,$$

and the second part of (1) is fulfilled.

Now suppose that there exists an $\bar{i} \in N$ and a $\bar{\sigma}_{d(\bar{i})} \in \prod_{j \in d(\bar{i})} S(\bar{x}_j)$ such that

$$\lambda_{\bar{i}} - E_{\bar{i}}(\bar{\sigma}_{d(\bar{i})}, \bar{x}_{N-d(\bar{i})}) > 0.$$

On the other hand we have

$$\begin{aligned} \lambda_{\bar{i}} = E_{\bar{i}}(\bar{x}) &= \sum_{\sigma_{d(\bar{i})} \in \Sigma_{d(\bar{i})}} E_{\bar{i}}(\sigma_{d(\bar{i})}, \bar{x}_{N-d(\bar{i})}) \bar{x}_{d(\bar{i})}(\sigma_{d(\bar{i})}) \\ &= \sum_{\sigma_{d(\bar{i})} \neq \bar{\sigma}_{d(\bar{i})}} E_{\bar{i}}(\sigma_{d(\bar{i})}, \bar{x}_{N-d(\bar{i})}) \bar{x}_{d(\bar{i})}(\sigma_{d(\bar{i})}) \\ &\quad + E_{\bar{i}}(\bar{\sigma}_{d(\bar{i})}, \bar{x}_{N-d(\bar{i})}) \bar{x}_{d(\bar{i})}(\bar{\sigma}_{d(\bar{i})}) < \lambda_{\bar{i}} \end{aligned}$$

since $\bar{\sigma}_{d(\bar{i})} \in \prod_{j \in d(\bar{i})} S(\bar{x}_j)$.

This implies a contradiction and the first part of (1) is proved.

Conversely, if for each $i \in \mathrm{N}$ and $\sigma_{\mathrm{d}(i)} \in \prod_{j \in \mathrm{S}_{\mathrm{d}(i)}} \mathrm{S}(\bar{\mathrm{x}}_j)$

$$\lambda_i - \mathrm{E}_i(\bar{\sigma}_{\mathrm{d}(i)}, \bar{\mathrm{x}}_{\mathrm{N}-\mathrm{d}(i)}) = 0$$

Then since $\mathrm{S}(\bar{\mathrm{x}}) = \prod_j \mathrm{S}_j(\bar{\mathrm{x}}_j)$ we have

$$\lambda_i = \mathrm{E}_i(\bar{\mathrm{x}})$$

and for the second part of (1)

$$\lambda_i \geq \mathrm{E}_i(\sigma_{\mathrm{d}(i)}, \bar{\mathrm{x}}_{\mathrm{N}-\mathrm{d}(i)})$$

which implies immediately

$$\mathrm{E}_i(\bar{\mathrm{x}}_{\mathrm{d}(i)}, \bar{\mathrm{x}}_{\mathrm{N}-\mathrm{d}(i)}) \geq \mathrm{E}_i(\mathrm{x}_{\mathrm{d}(i)}, \bar{\mathrm{x}}_{\mathrm{N}-\mathrm{d}(i)}) \quad \forall i\ \forall\ \mathrm{x}_{\mathrm{d}(i)}.$$

This says that the point $\bar{\mathrm{x}}$ is an *E*-point. □

This simple characterization of *E*-points will permit us to compute explicitly *E*-points for diagonal games which are completely mixed. We recall that an *E*-point $\bar{\mathrm{x}}$ is completely mixed if and only if $\mathrm{X}_{j\in\mathrm{N}}\,\mathrm{S}(\bar{\mathrm{x}}_j) = \Sigma$. This notion for zero sum two person games goes back to Kaplansky (1945). Completely mixed *E*-points are characterized only by using the first part of (1).

3. Examples for Three Players and General

Now we only consider n-person games such that $\left|\Sigma_i\right| = \mathrm{n}_i$ satisfying $\mathrm{n}_i = \mathrm{m}$ for each i. For this game consider the function

$$\delta(\sigma_{j_1}, \ldots, \sigma_{j_k}) = \delta(\sigma_{j_1}, \sigma_{j_2})\delta(\sigma_{j_2}, \sigma_{j_3}) \cdots \delta(\sigma_{j_k}, \sigma_{j_1})$$

where δ stand for the Kronecker's delta.

Take as an example of (Γ, d) the given game Γ with the structure function $\mathrm{d}(i) = \{i, i+1\}$ with $\mathrm{A}_i(\sigma_1, \ldots, \sigma_n) = \delta(1, 2, \ldots, i-1, i+2, \ldots, n)\mathrm{a}_i(\sigma)$. Then for them a completely mixed strategy by Lemma 1.1 is characterized by the system

$$\lambda_i - \mathrm{a}_i(\sigma) \prod_{j \notin \{i, i+1\}} \mathrm{x}_j(\sigma) = 0 \quad \sigma = \sigma_i = \sigma_j \qquad i \neq j$$

when the payoff functions are given by

$$\mathrm{A}_i(\sigma_1, \ldots, \sigma_n) = \delta(\sigma_i, \sigma_{i+2})\delta(\sigma_{i+2}, \sigma_{i+3}) \cdots \delta(\sigma_{i-1}, \sigma_i) \qquad \mathrm{a}_i(\sigma_i)\mathrm{a}_i(\sigma_i) > 0.$$

We call it diagonal.

In this case $\mathrm{M} = \{1, 2, 3\}$ we have the following equations

$$\begin{aligned} &\lambda_1 - \mathrm{a}_1(\sigma)\mathrm{x}_3(\sigma) = 0 \\ &\lambda_2 - \mathrm{a}_2(\sigma)\mathrm{x}_1(\sigma) = 0 \qquad \forall \sigma \in \Sigma = \Sigma_i \\ &\lambda_3 - \mathrm{a}_3(\sigma)\mathrm{x}_2(\sigma) = 0 \end{aligned} \tag{2}$$

The solution of this system is

$$\mathrm{x}_i(\sigma) = \frac{1}{\mathrm{a}_{i+1}(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_{i+1}(\sigma)}}$$

and

$$\lambda_i(\sigma) = \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_{i+1}(\sigma)}} \quad \text{with } i+1 \bmod 3.$$

Now we present some simple examples for three person games. Consider first the case $\mathrm{d}(1) = \{1\}, \mathrm{d}(2) = \{2\}$ and $\mathrm{d}(3) = \{3, 1\}$, with payoff functions

$$\begin{aligned} \mathrm{A}_1(\sigma_1, \sigma_2, \sigma_3) &= \delta(1,2,3)\mathrm{a}_1(\sigma_1) \\ \mathrm{A}_2(\sigma_1, \sigma_2, \sigma_3) &= \delta(1,2,3)\mathrm{a}_2(\sigma_2) \\ \mathrm{A}_3(\sigma_1, \sigma_2, \sigma_3) &= \delta(2,3)\mathrm{a}_3(\sigma_3) \end{aligned}$$

Then the system (1) for this diagonal game is given by

$$\begin{aligned} \lambda_1 &= \mathrm{a}_1(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma) \\ \lambda_2 &= \mathrm{a}_2(\sigma)\mathrm{x}_1(\sigma)\mathrm{x}_3(\sigma) \qquad \forall\sigma \\ \lambda_3 &= \mathrm{a}_3(\sigma)\mathrm{x}_2(\sigma) \end{aligned} \tag{3}$$

From the last equality we easily obtain

$$\mathrm{x}_2(\sigma) = \frac{1}{\mathrm{a}_3(\sigma)} \frac{1}{\Sigma_\sigma 1/\mathrm{a}_3(\sigma)}$$

and

$$\lambda_3 = \frac{1}{\Sigma_\sigma 1/\mathrm{a}_3(\sigma)}$$

Replacing it in the first Eq. (3) then

$$\lambda_1 = \frac{1}{\Sigma_\sigma \mathrm{a}_1(\sigma)\mathrm{a}_3(\sigma)} \frac{1}{\Sigma_\sigma 1/\mathrm{a}_3(\sigma)}$$

and

$$\mathrm{x}_3 = \frac{1}{\Sigma_\sigma \mathrm{a}_1(\sigma)\mathrm{a}_3(\sigma)} \mathrm{a}_1(\sigma)\mathrm{a}_3(\sigma)$$

Finally working in the second Eq. (3) it follows

$$\lambda_2 = \frac{1}{\Sigma_\sigma \mathrm{a}_1(\sigma)\mathrm{a}_3(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_1(\sigma)\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)}}$$

and

$$\mathrm{x}_1(\sigma) = \frac{1}{\mathrm{a}_1(\sigma)\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_1(\sigma)\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)}}.$$

Again this is the unique E-point completely mixed in the game (Γ, d).

In a similar way it is possible to obtain the solution of the game with $\mathrm{N} = \{1, 2, 3\}$, $\mathrm{d}(1) = \{1\}, \mathrm{d}(2) = \{1, 2\}$ and $\mathrm{d}(3) = \{3, 1\}$ having the equations given by Eq. (1) as

$$\begin{aligned} \lambda_1 - \mathrm{a}_1(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma) &= 0 \\ \lambda_2 - \mathrm{a}_2(\sigma)\mathrm{x}_3(\sigma) &= 0 \qquad \forall\sigma \\ \lambda_3 - \mathrm{a}_3(\sigma)\mathrm{x}_2(\sigma) &= 0 \end{aligned} \tag{4}$$

4. Examples of Four Players

In this paragraph we will present and solve some four person non-cooperative games with different structure functions. We end this section with a simple game having difficulties in solving it, and therefore we just point them out.

First consider the diagonal four person game with $\mathrm{d}(1) = \{1\}$, $\mathrm{d}(2) = \{1, 2\}$, $\mathrm{d}(3) = \{1, 2, 3\}$ and $\mathrm{d}(4) = \{4\}$. The payoff functions are of the form

$$\begin{aligned} \mathrm{A}_1(\sigma_1, \ldots, \sigma_4) &= \delta(1, \ldots, 4)\mathrm{a}_1(\sigma_1) \\ \mathrm{A}_2(\sigma_1, \ldots, \sigma_4) &= \delta(2, 3, 4)\mathrm{a}_2(\sigma_2) \\ \mathrm{A}_3(\sigma_1, \ldots, \sigma_4) &= \delta(3, 4)\mathrm{a}_3(\sigma_3) \\ \mathrm{A}_4(\sigma_1, \ldots, \sigma_4) &= \delta(1, \ldots, 4)\mathrm{a}_4(\sigma_4) \end{aligned}$$

Then in order to find completely mixed E-points of (Γ, d) the system (1) becomes

$$\left.\begin{aligned} \lambda_1 - \mathrm{a}_1(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma)\mathrm{x}_4(\sigma) &= 0 \\ \lambda_2 - \mathrm{a}_2(\sigma)\mathrm{x}_3(\sigma)\mathrm{x}_4(\sigma) &= 0 \\ \lambda_3 - \mathrm{a}_3(\sigma)\mathrm{x}_4(\sigma) &= 0 \\ \lambda_4 - \mathrm{a}_4(\sigma)\mathrm{x}_1(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma) &= 0 \end{aligned}\right. \quad \forall\sigma \tag{4}$$

From the third equation one easily derive

$$\mathrm{x}_4(\sigma) = \frac{1}{\mathrm{a}_3(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_3(\sigma)}}$$

and

$$\lambda_3 = \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_3(\sigma)}}.$$

Replacing $\mathrm{x}_4(\sigma)$ in the second equation of (4) it turns out

$$\mathrm{x}_3(\sigma) = \frac{1}{\mathrm{a}_2(\sigma)\mathrm{x}_4(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_2(\sigma)\mathrm{x}_4(\sigma)}}$$

and

$$\lambda_2 = \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_2(\sigma)\mathrm{x}_4(\sigma)}}$$

Following in this way using the remaining two equations of (4) it is easy to get the following expressions

$$\mathrm{x}_2(\sigma) = \frac{1}{\mathrm{a}_1(\sigma)\mathrm{x}_3(\sigma)\mathrm{x}_4(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_1(\sigma)\mathrm{x}_3(\sigma)\mathrm{x}_4(\sigma)}}$$

$$\lambda_1 = \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_1(\sigma)\mathrm{x}_3(\sigma)\mathrm{x}_4(\sigma)}}$$

$$\mathrm{x}_4(\sigma) = \frac{1}{\mathrm{a}_4(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_4(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma)}}$$

$$\lambda_4 = \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_4(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma)}}$$

As a second example consider the four person diagonal game with $\mathrm{d}(1) = \{1\}$, $\mathrm{d}(2) = \{2\}$, $\mathrm{d}(3) = \{3, 4\}$ and $\mathrm{d}(4) = \{2, 4\}$. The payoff functions are of the form

$$\begin{aligned} \mathrm{A}_i(\sigma_1, \ldots, \sigma_4) &= \delta(1, \ldots, 4)\mathrm{a}_1(\sigma_1) \qquad i = 1, 2 \\ \mathrm{A}_3(\sigma_1, \ldots, \sigma_4) &= \delta(1, 2, 3)\mathrm{a}_3(\sigma_3) \\ \mathrm{A}_4(\sigma_1, \ldots, \sigma_4) &= \delta(1, 3, 4)\mathrm{a}_4(\sigma_4) \end{aligned}$$

Then the system (1) is now for this game (Γ, d)

$$\left.\begin{aligned} \lambda_1 - \mathrm{a}_1(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma)\mathrm{x}_4(\sigma) &= 0 \\ \lambda_2 - \mathrm{a}_2(\sigma)\mathrm{x}_1(\sigma)\mathrm{x}_3(\sigma)\mathrm{x}_4(\sigma) &= 0 \\ \lambda_3 - \mathrm{a}_3(\sigma)\mathrm{x}_1(\sigma)\mathrm{x}_2(\sigma) &= 0 \\ \lambda_4 - \mathrm{a}_4(\sigma)\mathrm{x}_1(\sigma)\mathrm{x}_3(\sigma) &= 0 \end{aligned}\right. \quad \forall\sigma \tag{5}$$

From the second and the fourth equations of (5) we have

$$\mathrm{x}_4(\sigma) = \frac{\lambda_2}{\lambda_4}\frac{\mathrm{a}_4(\sigma)}{\mathrm{a}_2(\sigma)}$$

and therefore

$$\lambda_4 = \Sigma_\sigma \frac{\mathrm{a}_4(\sigma)}{\mathrm{a}_2(\sigma)}\lambda_2$$

On the other hand from the first and the second, we obtain

$$\mathrm{x}_2(\sigma) = \frac{\lambda_1}{\lambda_2}\frac{\mathrm{a}_2(\sigma)}{\mathrm{a}_1(\sigma)}\mathrm{x}_1(\sigma)$$

Replacing this in the third one, it easily obtained

$$\frac{\lambda_1}{\lambda_2\lambda_3} = \frac{1}{\left(\Sigma_\sigma \sqrt{\frac{\mathrm{a}_1(\sigma)}{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)}}\right)^2}$$

From the third and fourth equations we have

$$\mathrm{x}_2(\sigma) = \frac{\lambda_3}{\lambda_4}\frac{\mathrm{a}_4(\sigma)}{\mathrm{a}_3(\sigma)}\mathrm{x}_3(\sigma)$$

Replacing this in the first equation, we obtain

$$\mathrm{x}_3^2(\sigma) = \frac{\lambda_1\lambda_4^2}{\lambda_1\lambda_3}\frac{\mathrm{a}_2(\sigma)\lambda_3(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_4^2(\sigma)}$$

and operating, we get

$$\lambda_4 = \frac{1}{\Sigma_\sigma \frac{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_4^2(\sigma)}} \sum_\sigma \sqrt{\frac{\mathrm{a}_1(\sigma)}{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)}}$$

$$\lambda_2 = \frac{1}{\Sigma_\sigma \frac{\mathrm{a}_4(\sigma)}{\mathrm{a}_2(\sigma)}}\lambda_4$$

$$\mathrm{x}_4(\sigma) = \frac{1}{\Sigma_\sigma \frac{\mathrm{a}_4(\sigma)}{\mathrm{a}_2(\sigma)}}\frac{\mathrm{a}_4(\sigma)}{\mathrm{a}_2(\sigma)}$$

Therefore replacing x_4 in the first equation of (5) and forgetting the second one we obtain the system

$$\begin{aligned}\mu_1 &= \mathrm{b}_1(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma)\\ \mu_2 &= \mathrm{b}_2(\sigma)\mathrm{x}_1(\sigma)\mathrm{x}_3(\sigma)\\ \mu_3 &= \mathrm{b}_3(\sigma)\mathrm{x}_1(\sigma)\mathrm{x}_2(\sigma)\end{aligned} \tag{6}$$

where $\mathrm{b}_1(\sigma) = \mathrm{a}_1(\sigma)\mathrm{x}_4(\sigma)$, $\mathrm{b}_2(\sigma) = \mathrm{a}_4(\sigma)$ and $\mathrm{b}_3(\sigma) = \mathrm{a}_3(\sigma)$ where $\mu_2 = \lambda_4$ is known and given above. This system (6) is a particular case of the three person diagonal game given in Marchi (1990). The solution can be obtained using the formulas given there. Since $\mu_2 = \lambda_4$ is known we have the existence of one solution under the condition

$$1 \Bigg/ \sum_\sigma \sqrt{\frac{\mathrm{a}_1(\sigma)}{\mathrm{a}_2(\sigma)}} \cdot \sum_\sigma \sqrt{\frac{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_4^2(\sigma)}} = \frac{1}{\sum_\sigma \frac{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_4^2(\sigma)}} \sum_\sigma \sqrt{\frac{\mathrm{a}_1(\sigma)}{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)}}$$

It is of interest to see that the family of functions $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3$ and a_4 satisfying this equation is non empty. Consider for example

$$\mathrm{a}_1(\sigma) = \theta_1\mathrm{a}(\sigma), \quad \mathrm{a}_2(\sigma) = \theta_2\mathrm{a}(\sigma), \quad \mathrm{a}_3(\sigma) = \theta_3\mathrm{a}(\sigma), \quad \mathrm{a}_4(\sigma) = \theta_4\mathrm{a}(\sigma)$$

for an arbitrary $\mathrm{a}(\sigma) > 0$ and given $\theta_1, \theta_2, \theta_3$ and $\theta_4 > 0$. Then the previous condition is transformed to

$$\theta_2^5\theta_3^4 = \theta_4^6\theta_1^3$$

Clearly the family of this kind of functions is non-empty.

5. Examples of Five Players

As a first example of five players that we mention but we do not present here explicitly since we leave it for the general case of n player is the case with $\mathrm{d}(i) = \{i, i+1, i+2\}$ which generalizes (2) in the case of five players. The reader at this point is referred to the next section.

Another example which can be studied accordingly is the generalization of (3) and (4) but for reasons which go beyond the scope of this paper we just mention them.

The generalization of (4) in the case of n players is presented in the next section.

Now in this paragraph we present explicitly the following five person diagonal game (Γ, d) where $\mathrm{d}(i) = \{i, i+1, i+2\}$ and the payoff function has the shape

$$\mathrm{A}_i(\sigma_1, \ldots, \sigma_5) = \delta(i, i+3, i+4)\mathrm{a}_i(\sigma_i) \mod 5.$$

and the system (1) for this game is

$$\begin{aligned}\lambda_1 - \mathrm{a}_1(\sigma)\mathrm{x}_4(\sigma)\mathrm{x}_5(\sigma) &= 0\\ \lambda_2 - \mathrm{a}_2(\sigma)\mathrm{x}_5(\sigma)\mathrm{x}_1(\sigma) &= 0\\ \lambda_3 - \mathrm{a}_3(\sigma)\mathrm{x}_1(\sigma)\mathrm{x}_2(\sigma) &= 0 \qquad \forall\sigma\\ \lambda_4 - \mathrm{a}_4(\sigma)\mathrm{x}_2(\sigma)\mathrm{x}_3(\sigma) &= 0\\ \lambda_5 - \mathrm{a}_5(\sigma)\mathrm{x}_3(\sigma)\mathrm{x}_4(\sigma) &= 0\end{aligned} \tag{7}$$

From them, we immediately get

$$\frac{\lambda_i}{\mathrm{a}_i(\sigma)\mathrm{x}_{i+1}(\sigma)} = \frac{\lambda_{i+1}}{\mathrm{a}_{i+1}(\sigma)\mathrm{x}_{i+4}(\sigma)} \quad \mathrm{mod}\, 5$$

Therefore

$$\mathrm{x}_2(\sigma) = \frac{\lambda_4\lambda_1}{\lambda_5\lambda_2}\frac{\mathrm{a}_5(\sigma)\mathrm{a}_2(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_1(\sigma)}\mathrm{x}_1(\sigma)$$

$$\mathrm{x}_3(\sigma) = \frac{\lambda_4}{\lambda_3}\frac{\mathrm{a}_3(\sigma)}{\mathrm{a}_4(\sigma)}\mathrm{x}_1(\sigma)$$

$$\mathrm{x}_4(\sigma) = \frac{\lambda_1}{\lambda_2}\frac{\mathrm{a}_2(\sigma)}{\mathrm{a}_1(\sigma)}\mathrm{x}_1(\sigma)$$

$$\mathrm{x}_5(\sigma) = \frac{\lambda_4\lambda_1}{\lambda_3\lambda_5}\frac{\mathrm{a}_3(\sigma)\mathrm{a}_5(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_4(\sigma)}\mathrm{x}_1(\sigma)$$

Replacing x_4 and x_5 in the first equation of (7) we get

$$\mathrm{x}_1^2(\sigma) = \frac{\mathrm{a}_1(\sigma)\mathrm{a}_4(\sigma)}{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)\mathrm{a}_5(\sigma)}\frac{\lambda_2\lambda_3\lambda_5}{\lambda_1\lambda_4}$$

For symmetry it is possible to derive

$$\mathrm{x}_2^2(\sigma) = \frac{\mathrm{a}_2(\sigma)\mathrm{a}_5(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_3(\sigma)\mathrm{a}_4(\sigma)}\frac{\lambda_1\lambda_3\lambda_4}{\lambda_2\lambda_5}$$

$$\mathrm{x}_3^2(\sigma) = \frac{\lambda_2\lambda_4\lambda_5}{\lambda_1\lambda_3}\frac{\mathrm{a}_1(\sigma)\mathrm{a}_3(\sigma)}{\mathrm{a}_2(\sigma)\mathrm{a}_4(\sigma)\mathrm{a}_5(\sigma)}$$

$$\mathrm{x}_4^2(\sigma) = \frac{\lambda_3\lambda_5\lambda_1}{\lambda_2\lambda_4}\frac{\mathrm{a}_2(\sigma)\mathrm{a}_4(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_3(\sigma)\mathrm{a}_5(\sigma)}$$

and

$$\mathrm{x}_5^2(\sigma) = \frac{\lambda_1\lambda_2\lambda_4}{\lambda_3\lambda_5}\frac{\mathrm{a}_3(\sigma)\mathrm{a}_5(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_2(\sigma)\mathrm{a}_4(\sigma)}$$

From the expression of $\mathrm{x}_1^2(\sigma)$ we obtain:

$$\sqrt{\frac{\lambda_2\lambda_3\lambda_5}{\lambda_1\lambda_4}}\sum_\sigma\sqrt{\frac{\mathrm{a}_1(\sigma)\mathrm{a}_4(\sigma)}{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)\mathrm{a}_5(\sigma)}} = 1$$

or

$$\frac{\lambda_2\lambda_3\lambda_5}{\lambda_1\lambda_4} = \mathrm{b}_1 = \frac{1}{\left(\Sigma_\sigma\sqrt{\frac{\mathrm{a}_1(\sigma)\mathrm{a}_4(\sigma)}{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)\mathrm{a}_5(\sigma)}}\right)^2}$$

and by symmetry with the remaining terms

$$\frac{\lambda_1\lambda_3\lambda_4}{\lambda_2\lambda_5} = \mathrm{b}_2$$

$$\frac{\lambda_2\lambda_4\lambda_5}{\lambda_1\lambda_3} = \mathrm{b}_3$$

$$\frac{\lambda_3\lambda_4\lambda_1}{\lambda_2\lambda_4} = \mathrm{b}_4$$

$$\frac{\lambda_4\lambda_1\lambda_2}{\lambda_3\lambda_5} = \mathrm{b}_5$$

where the b_i are defined correspondingly. Now multiplying consecutively these last equalities it follows:

$$\begin{aligned} \mathrm{b}_1\mathrm{b}_2 &= \lambda_3^2 \\ \mathrm{b}_2\mathrm{b}_3 &= \lambda_2^2 \\ \mathrm{b}_3\mathrm{b}_4 &= \lambda_5^2 \\ \mathrm{b}_4\mathrm{b}_5 &= \lambda_1^2 \\ \mathrm{b}_5\mathrm{b}_1 &= \lambda_2^2 \end{aligned}$$

from where are obtained the values of λ_i and therefore those of x_i. We just give the explicit expression of

$$\lambda_3 = \frac{1}{\Sigma_\sigma \sqrt{\frac{\mathrm{a}_1(\sigma)\mathrm{a}_4(\sigma)}{\mathrm{a}_2(\sigma)\mathrm{a}_3(\sigma)\mathrm{a}_5(\sigma)}}} \frac{1}{\Sigma_\sigma \sqrt{\frac{\mathrm{a}_2(\sigma)\mathrm{a}_5(\sigma)}{\mathrm{a}_1(\sigma)\mathrm{a}_3(\sigma)\mathrm{a}_4(\sigma)}}}$$

Thus we have solved (7). Computing explicitly the only one E-point completely mixed.

6. Examples of n-Person Diagonal Game

In this section we present generalization of some examples given in the previous paragraphs.

First consider the generalization to n-person game of the three person game given in (2).

In the general case when $\mathrm{d}(i) = \{i, i+1, \ldots, i-2\}$, where the payoff functions are of the form

$$\mathrm{A}_i(\sigma_1, \ldots, \sigma_n) = \delta(i, i-1)\mathrm{a}_i(\sigma_i).$$

Then the system (1) for completely mixed points requires the solution of the system

$$\lambda_i - \mathrm{a}_i(\sigma)\mathrm{x}_{i-1}(\sigma) = 0 \quad \forall\sigma.$$

The solution of it is given by

$$\lambda_i(\sigma) = \frac{1}{\Sigma_\sigma 1/\mathrm{a}_i(\sigma)}$$

$$\mathrm{x}_{i-1}(\sigma) = \frac{1}{\mathrm{a}_i(\sigma)} \frac{1}{\Sigma_\sigma 1/\mathrm{a}_i(\sigma)} \quad i \bmod n.$$

Such a point is an E-point for (Γ, d).

Indeed we point out that the point $(\bar{\sigma}, \bar{\sigma}, \ldots, \bar{\sigma})$ is an E-point. Besides there are exactly

$$\sum_{i=1}^{n} \binom{n}{i} = 2^{\mathrm{n}} - 1$$

E-points which might be computed accordingly. Anyway there is only one E-point completely mixed. The phenomenon appears on all the games studied here and for this reason we do not repeat the same argument.

As a second example, we consider the generalization of the game given in (4). That is to say $\mathrm{N} = \{1, \ldots, n\}, n \geq 4, d(i) = \{1, \ldots, i\}$ for $i \neq n$ and $\mathrm{d}(n) = \{n\}$, where the payoff functions are of the form

$$\mathrm{A}_i(\sigma_1, \ldots, \sigma_n) = \delta(i, i+1, \ldots, n)\mathrm{a}_i(\sigma).$$

Then the system of equations of (1) gives rise to

$$\begin{aligned} \lambda_i - \mathrm{a}_i(\sigma) \prod_{j=i+1}^{n} \mathrm{x}_j(\sigma) = 0 \qquad \forall\sigma \\ \lambda_n - \mathrm{a}_n(\sigma) \prod_{j=1}^{n-1} \mathrm{x}_j(\sigma) = 0 \qquad \forall\sigma \end{aligned} \tag{8}$$

From the $n-1$'s equation we get

$$\mathrm{x}_n(\sigma) = \frac{1}{\mathrm{a}_{n-1}(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_{n-1}(\sigma)}}$$

$$\lambda_{n-1}(\sigma) = \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_{n-1}(\sigma)}}$$

and recursively

$$\mathrm{x}_i(\sigma) = \frac{1}{\mathrm{a}_{i-1}(\sigma) \prod_{j=i+1}^{n} x_j(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_{i-1}(\sigma) \prod_{j=1}^{n} x_j(\sigma)}}$$

$$\lambda_i = \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_i(\sigma) \prod_{j=i+1}^{n} \mathrm{x}_j(\sigma)}}$$

and finally

$$\mathrm{x}_i(\sigma) = \frac{1}{\mathrm{a}_n(\sigma) \prod_{j=i+1}^{n-1} \mathrm{x}_j(\sigma)} \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_n(\sigma) \prod_{j=i+1}^{n-1} \mathrm{x}_j(\sigma)}}$$

$$\lambda_n = \frac{1}{\Sigma_\sigma \frac{1}{\mathrm{a}_n(\sigma) \prod_{j=2}^{n-1} \mathrm{x}_j(\sigma)}}$$

In this way we have obtained the unique completely mixed E-points of the game (Γ, d).

Finally in this section we present a generalization of the game given in the Sec. 4 for five players whose equations are expressed in (7).

Now consider the game (Γ, d) with $\mathrm{N} = \{1, \ldots, 2k-1\}$ and $\mathrm{d}(i) = \{i, i+1, \ldots, 2k-1, 1, 2, \ldots, i-3\}$ where the payoff functions have the form

$$\mathrm{A}_i(\sigma_1, \ldots, \sigma_n) = \delta(i-1, i, i+1)\mathrm{a}_i(\sigma_1).$$

The system (1) for this game after a suitable change of variable is equivalent to the system

$$\lambda_i - \mathrm{a}_i(\sigma)\mathrm{x}_i(\sigma)\mathrm{x}_{i+1}(\sigma) = 0 \mod 2k-1 \quad \forall i = 1, \ldots, 2k-1 \quad \forall \sigma \tag{9}$$

Now from here we get

$$x_{i+2}(\sigma) = \frac{\lambda_{i+1}}{\lambda_i}\frac{\mathrm{a}_i(\sigma)}{\mathrm{a}_{i+1}(\sigma)}\mathrm{x}_i(\sigma)$$

and recursively

$$\mathrm{x}_{i+2}(\sigma) = \prod_{\pi=0}^{\pi} \frac{\lambda_i - 2\mathrm{u} + 1}{\lambda_i - 2\mathrm{u}} \frac{\mathrm{a}_i - 2\mathrm{u}(\sigma)}{\mathrm{a}_{i-2\mathrm{u}+1}(\sigma)}\mathrm{x}_{i-2\mathrm{u}}(\sigma)$$

for $3 \leq i+2 \leq 2k-1$ and $i - 2\bar{\mathrm{u}} \geq 1$.

The first and final equations provide

$$\mathrm{x}_2(\sigma) = \frac{\lambda_1}{\lambda_{2\mathrm{k}-1}} \frac{\mathrm{a}_{2\mathrm{k}-1}(\sigma)}{\mathrm{a}_1(\sigma)}\mathrm{x}_{2\mathrm{k}-1}(\sigma)$$

and

$$\mathrm{x}_1(\sigma) = \frac{\lambda_{2\mathrm{k}-1}}{\lambda_{2\mathrm{k}-2}} \frac{\mathrm{a}_{2\mathrm{k}-1}(\sigma)}{\mathrm{a}_{2\mathrm{s}-2\mathrm{n}}(\sigma)}\mathrm{x}_{2\mathrm{k}-2}(\sigma).$$

Consider $2\mathrm{s}+1 = i+2$, or $i = 2\mathrm{s}-1$ and $\bar{\mathrm{u}} = \mathrm{s}-1$, then the above expression gives us

$$\mathrm{x}_{2\mathrm{s}+1}(\sigma) = \prod_{\mathrm{u}=0}^{\mathrm{s}-1} \frac{\lambda_{2\mathrm{s}-2\mathrm{u}}}{\lambda_{2\mathrm{s}-2\mathrm{u}-1}} \frac{\mathrm{a}_{2\mathrm{s}-2\mathrm{u}-1}(\sigma)}{\mathrm{a}_{2\mathrm{s}-2\mathrm{u}}(\sigma)}\mathrm{x}_1(\sigma)$$

On the other hand we obtain

$$\begin{aligned}\mathrm{x}_1(\sigma) &= \frac{\lambda_{2\mathrm{k}-1}}{\lambda_{2\mathrm{k}-2}} \frac{\mathrm{a}_{2\mathrm{k}-2}(\sigma)}{\mathrm{a}_{2\mathrm{k}-1}(\sigma)}\mathrm{x}_{2\mathrm{k}-2}(\sigma) \\ &= \frac{\lambda_{2\mathrm{k}-1}}{\lambda_{2\mathrm{k}-2}} \frac{\mathrm{a}_{2\mathrm{k}-2}(\sigma)}{\mathrm{a}_{2\mathrm{k}-1}(\sigma)} \cdot \prod_{\mathrm{u}=0}^{\mathrm{k}-\mathrm{s}-2} \frac{\lambda_{2\mathrm{k}-2\mathrm{u}-3}}{\lambda_{2\mathrm{k}-2\mathrm{u}-4}} \frac{\mathrm{a}_{2\mathrm{k}-2\mathrm{u}-4}(\sigma)}{\mathrm{a}_{2\mathrm{k}-2\mathrm{u}-3}(\sigma)}\mathrm{x}_{2\mathrm{s}}(\sigma)\end{aligned}$$

where we have used $i+2 = 2\mathrm{k}-2$ and $\bar{\mathrm{u}} = \mathrm{k}-\mathrm{s}-2$. Replacing $\mathrm{x}_{2\mathrm{s}+1}$ in the 2s equality with the insertion of $\mathrm{x}_1(\sigma)$ given above in terms of $\mathrm{x}_{2\mathrm{s}}(\sigma)$ we get

$$\begin{aligned}&\lambda_{2\mathrm{s}} - \mathrm{a}_{2\mathrm{s}} \prod_{\mathrm{u}=0}^{\mathrm{s}-1} \frac{\lambda_{2\mathrm{s}-2\mathrm{u}}}{\lambda_{2\mathrm{s}-2\mathrm{u}-1}} \frac{\mathrm{a}_{2\mathrm{s}-2\mathrm{u}-1}(\sigma)}{\mathrm{a}_{2\mathrm{s}-2\mathrm{u}-1}(\sigma)} \prod_{\mathrm{u}=0}^{\mathrm{k}-\mathrm{s}-2} \frac{\lambda_{2\mathrm{k}-2\mathrm{u}-3}}{\lambda_{2\mathrm{k}-2\mathrm{u}-4}} \frac{\mathrm{a}_{2\mathrm{k}-2\mathrm{u}-4}(\sigma)}{\mathrm{a}_{2\mathrm{k}-2\mathrm{u}-3}(\sigma)} \\ &\quad \cdot \frac{\lambda_{2\mathrm{k}-1}\mathrm{a}_{2\mathrm{k}-2}(\sigma)}{\lambda_{2\mathrm{k}-2}\mathrm{a}_{\mathrm{ak}-1}(\sigma)}\mathrm{x}_{2\mathrm{s}}^2 = 0\end{aligned}$$

or simplifying

$$x_{2s}^2(\sigma) = \frac{\lambda_{2s}}{a_{2s}(\sigma)} \prod_{u=0}^{s-1} \frac{\lambda_{2s-2u-1}}{\lambda_{2s-2u}} \frac{a_{2s-2u}(\sigma)}{a_{2s-2u-1}(\sigma)} \prod_{u=0}^{k-s-1} \frac{\lambda_{2k-2u-2}}{\lambda_{2k-2u-1}} \frac{a_{2k-2u-1}(\sigma)}{a_{2k-2u-2}(\sigma)}$$

In a similar way it is possible to derive

$$x_{2k-1}(\sigma) = \prod_{u=0}^{k-s-2} \frac{\lambda_{2k-3-2u+1}}{\lambda_{2k-3-2u}} \frac{a_{2k-3-2u}(\sigma)}{a_{2k-2-2u}(\sigma)} x_{2s+1}(\sigma)$$

and then

$$\lambda_{2s+1} - a_{2s+1}(\sigma) \prod_{u=0}^{s-1} \frac{\lambda_{2s-2u+1}}{\lambda_{2s-2u}} \frac{a_{2s-2u}(\sigma)}{a_{2s-2u+1}(\sigma)} \prod_{u=0}^{k-s-2} \frac{\lambda_{2k-2-2u}}{\lambda_{2k-3-2u}} \frac{a_{2k-3-2u}(\sigma)}{a_{2k-2-2u}(\sigma)} \frac{\lambda_1}{\lambda_{2k-1}}$$

$$\cdot \frac{a_{2k-1}(\sigma)}{a_1(\sigma)} x_{2s+1}^2(\sigma) = 0$$

or

$$x_{2s+1}^2(\sigma) = \frac{\lambda_{2s+1}}{a_{2s+1}(\sigma)} \prod_{u=0}^{s-1} \frac{\lambda_{2s-2u}}{\lambda_{2s-2u+1}} \frac{a_{2s-2u+1}(\sigma)}{a_{2s-2u-1}(\sigma)}$$

$$\times \prod_{u=0}^{k-s-2} \frac{\lambda_{2k-3-2u}}{\lambda_{2k-2-2u}} \frac{a_{2k-2-2u}(\sigma)}{a_{2k-3-2u}(\sigma)} \frac{\lambda_{2k-1} a_1(\sigma)}{\lambda_1 a_{2k-1}(\sigma)}$$

Now computing $\Sigma_\sigma x_{2s}(\sigma) = 1$ we get

$$\lambda_{2s} \prod_{u=0}^{s-1} \frac{\lambda_{2s-1-2u}}{\lambda_{2s-2u}} \prod_{u=0}^{k-s-1} \frac{\lambda_{2k-2u-2}}{\lambda_{2k-2u-1}} = b_{2s}$$

where

$$b_{2s} = 1 \Bigg/ \left[\sum_\sigma \left(\frac{1}{a_{2s}(\sigma)} \prod_{u=0}^{s-1} \frac{a_{2s-2u}(\sigma)}{a_{2s-2u-1}(\sigma)} \prod_{u=0}^{k-s-1} \frac{a_{2k-2u-1}(\sigma)}{a_{2k-2u-2}(\sigma)} \right)^{1/2} \right]^2$$

Similarly for $2s+1$:

$$\lambda_{2s+1} \prod_{u=0}^{s-1} \frac{\lambda_{2s-2u}}{\lambda_{2s-2u+1}} \prod_{u=0}^{k-s-2} \frac{\lambda_{2k-2u-3}}{\lambda_{2k-2u-2}} \frac{\lambda_{2k-1}}{\lambda_1} = b_{2s+1}$$

where

$$b_{2s+1} = 1 \Bigg/ \left[\sum_\sigma \left(\frac{1}{a_{2s+1}(\sigma)} \prod_{u=0}^{s-1} \frac{a_{2s-2u+1}(\sigma)}{a_{2s-2u}(\sigma)} \prod_{u=0}^{k-s-2} \frac{a_{2k-3-2u}(\sigma)}{a_{2k-2-2u}(\sigma)} \right)^{1/2} \right]^2$$

Now for any s consider $b_{2s} b_{2s+1}$ which turns out to be

$$b_{2s} b_{2s+1} = \lambda_{2s}^2$$

and then

$$\lambda_{2s} = \sqrt{b_{2s} b_{2s+1}}.$$

Similarly

$$\lambda_{2s+1} = \sqrt{b_{2s+1} \lambda_{2s+2}}$$

and in this way the problem is solved. We just have computed the only one completely mixed E-point of the problem (Γ, d).

7. Final Remarks

We would like to point out that in the previous examples introduced and studied in this paper, which in some ways appear to be elemental, they show the relationship of game theory and algebraic geometry. The study of the last example is more difficult. If the reader would infer that all the problems are simple, the answer is not. Consider the four person game (Γ, d) with $d(i) = \{i, i+1\} \bmod 4$, with the payoffs

$$A_i(\sigma_1 - \sigma_4) = a_i(\sigma_i)\delta(i, i+2, i+3) \bmod 4.$$

Then the system (1) for this game is

$$\begin{aligned} \lambda_1 - a_1(\sigma)x_3(\sigma)x_4(\sigma) &= 0 \\ \lambda_2 - a_2(\sigma)x_4(\sigma)x_1(\sigma) &= 0 \\ \lambda_3 - a_3(\sigma)x_1(\sigma)x_2(\sigma) &= 0 \\ \lambda_4 - a_4(\sigma)x_2(\sigma)x_3(\sigma) &= 0 \end{aligned} \qquad \forall\sigma$$

and are very difficult for the general case with arbitrary functions a's.

Many other examples of diagonal games can be given in order to compute E-points but one has to be very careful with the group-combinatorial structure of the equations.

These are the first examples of computations of E-points which in our opinion open a new way of possible applications in real situations of competition and cooperation and the way to compute in more general context strategic solutions of games.

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