# Regular elements and Kolmogorov translation in residuated lattices 

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#### Abstract

In this article, we study in detail the regular elements of a bounded, commutative and integral residuated lattice. We introduce the notion of a regular variety and explore its relationship with the Kolmogorov negative translation. In addition, we investigate the corresponding notions in the axiomatic extensions of the Full Lambek Calculus with exchange and weakening.


## 1. Introduction and purposes

The variety of residuated lattices is the equivalent algebraic semantics, in the sense of Blok-Pigozzi [1], of the Full Lambek Calculus FL (see for example [8] and the references given there). In this article, we will focus our attention on the subvariety of bounded commutative and integral residuated lattices, that is, the equivalent algebraic semantics of $\mathbf{F} \mathbf{L}_{\text {ew }}$, the calculus that results from $\mathbf{F L}$ by adding two structural rules: exchange and weakening. For brevity, we shall refer to the members of this subvariety simply as residuated lattices.

Given a residuated lattice $\boldsymbol{A}$, we can define an (involutive) residuated lattice structure $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ on the set of its regular (involutive) elements $\operatorname{Reg}(\boldsymbol{A})$. In general, there is no direct relation between $\boldsymbol{A}$ and $\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A})$; for instance, $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ need not be either a subalgebra or a homomorphic image of $\boldsymbol{A}$ (see [5] and [8, Chapter 8]). The latter case has been studied in [5], where it is shown that the condition " $\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A})$ is a homomorphic image of $\boldsymbol{A}$ " is equationally definable, and defines Glivenko's variety. This gives, via algebraization, the axiomatic extension of $\mathbf{F} \mathbf{L}_{\text {ew }}$ admitting a generalization of Glivenko's theorem, which was stated originally by V. Glivenko in [9] to give an interpretation of the classical propositional logic into the intuitionistic propositional logic.

[^0]In this article, we will investigate in more detail the relationship between these algebras. In addition, we will study the relationship between a given variety $\mathbb{V}$ of residuated lattices and the class $R(\mathbb{V})=\{\boldsymbol{\operatorname { R e g }}(\boldsymbol{A}): \boldsymbol{A} \in \mathbb{V}\}$. In general, the latter is neither a variety nor is it contained in the former, as we will illustrate with special examples. However, the condition $R(\mathbb{V}) \subseteq \mathbb{V}$ is used implicitly in the proof of [8, Theorem 8.43 , p. 373, line 11]. Thus, the examples we give in this article contradict that theorem, which motivated us to pursue a deeper study of the relation between a variety $\mathbb{V}$ and the class $R(\mathbb{V})$. We will give necessary and sufficient conditions for $R(\mathbb{V})$ to be a variety and for it to be contained in $\mathbb{V}$. We will also show that most of the wellknown varieties of residuated lattices fulfill these conditions. A tool that will be useful to study the class $R(\mathbb{V})$ will be the Kolmogorov negative translation, which was originally introduced to give another interpretation of the classical propositional calculus into the intuitionistic logic (see [10]). This translation is a transformation on the terms in the language of residuated lattices that allows us to relate the equations valid in $\boldsymbol{A}$ to those valid in $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$. Taking into account the correspondence between subvarieties of residuated lattices and axiomatic extensions of the calculus $\mathbf{F L}_{\text {ew }}$, we will also study the logical counterpart of the Kolmogorov translation and the regular varieties.

The paper is organized as follows. In the next section, we recall some basic definitions and properties about residuated lattices that will be used throughout the article. In Section 3, we study the regular elements of a residuated lattice, we define the class $R(\mathbb{V})$ and explore its relation to $\mathbb{V}$. In the following section, we introduce the notion of regular variety and we develop the connections between them and the Kolmogorov translation. In Section 5, we show that the variety of distributive residuated lattices is not regular and we give a construction that shows that every involutive residuated lattice is the algebra of regular elements of a distributive residuated lattice. In Section 6, we discuss briefly the lattice of regular varieties. In the final section, we translate the algebraic results obtained so far into the axiomatic extensions of $\mathbf{F L} \mathbf{L e w}_{\text {. }}$.

We assume familiarity with residuated lattices; for general results, see [8] and the references given there. The material from universal algebra required for this article can be found in [3] and [2].

## 2. Preliminaries

Throughout this paper, $\mathbb{R L}$ will denote the class of all bounded commutative and integral residuated latticed-ordered monoids (residuated lattices for short), that is, the class of algebras $\boldsymbol{A}=\langle A ; \wedge, \vee, *, \rightarrow, 0,1\rangle$ in the algebraic language $\{\wedge, \vee, *, \rightarrow, 0,1\}$ of type $(2,2,2,2,0,0)$ such that $\langle A ; \wedge, \vee, 0,1\rangle$ is a bounded lattice, $\langle A ; *, 1\rangle$ is a commutative monoid and the following residuation condition holds

$$
x * y \leqslant z \text { if and only if } x \leqslant y \rightarrow z
$$

where $x, y, z$ denote arbitrary elements of $A$ and $\leqslant$ is the order given by the lattice structure.

It is well known that the class $\mathbb{R} \mathbb{L}$ is equationally definable (see for example [8]), and so, it is a variety; that is, it is closed under homomorphic images, subalgebras, and the formation of direct products.

In the next lemma, we list, for further references, some well-known and easily provable consequences of the previous definition that will be used throughout this paper.

Lemma 2.1. The following properties hold in any residuated lattice $\boldsymbol{A}$, for any $a, b, c$ in $A$ :
(a) $a \leqslant b$ if and only if $a \rightarrow b=1$,
(b) $1 \rightarrow a=a$,
(c) $(a \rightarrow b) \rightarrow((b \rightarrow c) \rightarrow(a \rightarrow c))=1$,
(d) $(a * b) \rightarrow c=a \rightarrow(b \rightarrow c)$,
(e) $(a \vee b) \rightarrow c=(a \rightarrow c) \wedge(b \rightarrow c)$,
(f) $a=b$ if and only if $(a \rightarrow b) *(b \rightarrow a)=1$.

If we consider the unary term $\neg x:=x \rightarrow 0$, then residuated lattices in which the equation $\neg \neg x \approx x$ (or equivalently, $\neg \neg x \rightarrow x \approx 1$ ) holds are called involutive residuated lattices. For any subvariety $\mathbb{V}$ of $\mathbb{R} \mathbb{L}$, $\mathbb{I V}$ will denote the variety of all its involutive members, that is, $\mathbb{I V}=\mathbb{V} \cap \mathbb{R} \mathbb{R}$.

In the next lemma, we collect some well-known elementary properties involving $\neg$ that will be constantly used throughout the article (see [6] and [5] for details).

Lemma 2.2. Given a residuated lattice $\boldsymbol{A}$ and $a, b$ arbitrary elements in $A$, we have
(a) if $a \leqslant b$, then $\neg b \leqslant \neg a$,
(b) $a \leqslant \neg \neg a$,
(c) $\neg \neg \neg a=\neg a$,
(d) $\neg \neg(a \rightarrow \neg b)=a \rightarrow \neg b$ and $\neg \neg(\neg a \wedge \neg b)=\neg a \wedge \neg b$,
(e) $\neg(a \vee b)=\neg(\neg \neg a \vee \neg \neg b)$.

For a residuated lattice $\boldsymbol{A}$, we consider $\mathcal{F}(\boldsymbol{A})$ the family of its implicative filters ( $i$-filters for short), that is, $f \in \mathcal{F}(\boldsymbol{A})$ if and only if $f$ is a subset of $A$ such that

- $1 \in f$ and
- $a, a \rightarrow b \in f$ implies $b \in f$;
or equivalently,
- $f$ is non-empty,
- for any $a, b \in f, a * b \in f$,
- for any $a, b \in A, a \in f$ and $a \leqslant b$ imply $b \in f$.

Moreover, $f$ is called proper provided that $f \neq A$, or equivalently, $0 \notin f$. Then the correspondence $\theta \mapsto 1 / \theta$, where $1 / \theta$ denotes the class of 1 modulo $\theta$, gives
an order isomorphism from the family of all congruence relations on $\boldsymbol{A}$ onto $\mathcal{F}(\boldsymbol{A})$, both ordered by inclusion; its inverse is

$$
f \mapsto \theta(f)=\{(a, b):(a \rightarrow b) *(b \rightarrow a) \in f\}
$$

As a consequence of this isomorphism, we write $a / f$ instead of $a / \theta(f)$ to denote the class of the element $a$ modulo the congruence $\theta(f)$.

It is easy to see that for any non-empty $Y \subseteq A$,

$$
F^{\boldsymbol{A}}(Y)=\left\{a \in A: a \geqslant b_{1} * \cdots * b_{n}, \text { for some } n \geqslant 1 \text { and } b_{1}, \ldots, b_{n} \in Y\right\}
$$

is the least i-filter containing $Y$.
The variety $\mathbb{R L}$ satisfies the congruence extension property, or equivalently, the $i$-filter extension property; that is, if $\boldsymbol{B}$ is a subalgebra of a residuated lattice $\boldsymbol{A}$, then for each i-filter $g$ of $\boldsymbol{B}$ there is an i-filter $f$ of $\boldsymbol{A}$ such that $g=f \cap B$.

In this paper, several varieties of residuated lattices are considered. Here we list some of them and fix the corresponding notation.

Residuated lattices satisfying the equation $x * y \approx x \wedge y$ will be called Heyting algebras. They are also called bounded Brouwerian algebras, and they form a subvariety of $\mathbb{R} \mathbb{L}$ denoted by $\mathbb{H}$.

The smallest non trivial variety of residuated lattices, relative to inclusion, is the class $\mathbb{B}$ of Boolean algebras, which is the subvariety of $\mathbb{R L}$ determined by the equation $x \vee \neg x \approx 1$. In any Boolean algebra, the complement of $x$ is given by $\neg x$ and the equation $x * y \approx x \wedge y$ holds. Hence, any Boolean algebra is an involutive Heyting algebra and, in fact, $\mathbb{B}=\mathbb{H} \cap \mathbb{R} \mathbb{R}$.

Let $\mathbb{P R} \mathbb{R}$ denote the variety of pseudocomplemented residuated lattices, i.e., the subvariety of $\mathbb{R L}$ determined by the equation $x \wedge \neg x \approx 0$. It is easy to see that any Heyting algebra is pseudocomplemented, hence $\mathbb{H} \subseteq \mathbb{P} \mathbb{R} \mathbb{L}$. Moreover, $\mathbb{B}=\mathbb{P} \mathbb{R} \mathbb{L} \cap \mathbb{R} \mathbb{R}$.

We write $x^{0}=1$ and $x^{n}$ for the $n$th $*$-power of $n$. The power notation has precedence over the operator $\neg$, that is, $\neg x^{n}$ stands for $\neg\left(x^{n}\right)$. For each $n>0$, we represent by $\mathbb{E}_{n}$ the subvariety of $\mathbb{R} \mathbb{L}$ determined by the equation $x^{n+1} \approx x^{n}$.

In [11] (see also [8, Chapter 11]), it is shown that for any variety $\mathbb{V}$ of residuated lattices, the following conditions are equivalent:

- $\mathbb{V}$ is semisimple, i.e., all its members are semisimple;
- $\mathbb{V}$ is a discriminator variety;
- there is a positive integer $n$ such that $\mathbb{V}$ satisfies the equation

$$
\begin{equation*}
x \vee \neg x^{n} \approx 1 \tag{2.1}
\end{equation*}
$$

The subvariety of $\mathbb{R} \mathbb{L}$ determined by equation (2.1) is usually denoted by $\mathbb{E M}_{n}$. Moreover, $\mathbb{E M}_{n} \subseteq \mathbb{E}_{n}$.

We denote by $\mathbb{G}$ the variety of Glivenko residuated lattices, i.e., the subvariety of $\mathbb{R} \mathbb{L}$ determined by the equation:

$$
\begin{equation*}
\neg \neg(\neg \neg x \rightarrow x) \approx 1 \tag{2.2}
\end{equation*}
$$

Well-known examples of Glivenko varieties are the varieties of BL-algebras, given by continuous $t$-norms, Heyting algebras, and involutive residuated lattices.

The following remark is important for understanding how we can axiomatize the subvarieties of $\mathbb{R} \mathbb{L}$.

Remark 2.3. From $(f)$ of Lemma 2.1, one deduces that any variety $\mathbb{V}$ of residuated lattices is determined by the axioms for residuated lattices plus a set of equations of the form $t \approx 1$, where $t$ is an $\mathbb{R} \mathbb{L}$-term. More specifically, for any $\boldsymbol{A} \in \mathbb{R} \mathbb{L}$ :

- $\boldsymbol{A} \in \mathbb{V}$ if and only if $\boldsymbol{A} \models t \approx 1$ for any term $t$ such that $\mathbb{V} \models t \approx 1$.


## 3. Algebra of regular elements

Given a residuated lattice $\boldsymbol{A}$, we consider the set of its regular elements, $\operatorname{Reg}(\boldsymbol{A})=\{\neg \neg a: a \in A\}$. For any $\odot \in\{\wedge, \vee, *, \rightarrow\}$, we consider the term operation $x \odot_{r} y:=\neg \neg(x \odot y)$. Then $\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A})=\left\langle\operatorname{Reg}(\boldsymbol{A}) ; \wedge_{r}, \vee_{r}, *_{r}, \rightarrow_{r}, 0,1\right\rangle$ is an involutive residuated lattice, i.e., $\operatorname{Reg}(\boldsymbol{A}) \in \mathbb{R} \mathbb{L}$ (see [5, 13, 4], for example). Observe that by $(d)$ of Lemma 2.2, for any $a, b \in \operatorname{Reg}(\boldsymbol{A})$, we have $a \rightarrow_{r} b=a \rightarrow b$ and $a \wedge_{r} b=a \wedge b$. However, $\vee_{r}$ and $*_{r}$ are different, in general, from $\vee$ and $*$, respectively, and so $\boldsymbol{\operatorname { R e g }} \boldsymbol{( A )}$ may not be a subalgebra of $\boldsymbol{A}$. Nevertheless, in some cases $\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A})$ can be obtained as a homomorphic image of $\boldsymbol{A}$, because from the results given in [5] (see also [8]), we deduce the following:

Lemma 3.1. For every residuated lattice $\boldsymbol{A}$, the following are equivalent:
(1) the map $x \mapsto \neg \neg x$ defines a homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$;
(2) $\boldsymbol{A} \in \mathbb{G}$, i.e., equation (2.2) holds in $\boldsymbol{A}$.

Moreover, $\operatorname{Reg}(\boldsymbol{A})$ contains the set of boolean (or complemented) elements of $\boldsymbol{A}, B(\boldsymbol{A})=\{a \in A: a \vee \neg a=1\}$, which is the universe of $\boldsymbol{B}(\boldsymbol{A})$, a subalgebra both of $\boldsymbol{A}$ and of $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$.

Remark 3.2. In general, $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ is not equal to $\boldsymbol{B}(\boldsymbol{A})$. However, $\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A})$ is a Boolean algebra if and only if $\boldsymbol{A}$ is pseudocomplemented, and $\boldsymbol{\operatorname { R e }} \boldsymbol{g}(\boldsymbol{A})=$ $\boldsymbol{B}(\boldsymbol{A})$ if and only if $\boldsymbol{A}$ is Stonean, i.e., the equation $\neg x \vee \neg \neg x \approx 1$ holds in $\boldsymbol{A}$ (see for example [4]).

In a straightforward way, any homomorphism between two residuated lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ induces a homomorphism between $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ and $\boldsymbol{\operatorname { R e g }} \boldsymbol{\operatorname { B }} \boldsymbol{B})$, as the following lemma shows.

Lemma 3.3. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be residuated lattices and let $h$ be a homomorphism from $\boldsymbol{A}$ into $\boldsymbol{B}$. Then $\operatorname{Reg}(h)$, the restriction of $h$ to $\operatorname{Reg}(\boldsymbol{A})$, gives a homomorphism from $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ into $\boldsymbol{\operatorname { R e g }}(\boldsymbol{B})$. Moreover, if $h$ is onto, then $\operatorname{Reg}(h)$ is also onto.

Proof. Let $a, b \in \operatorname{Reg}(\boldsymbol{A})$. Since $h(a), h(b) \in \operatorname{Reg}(\boldsymbol{B}) \cap h[A]$, we have that $\operatorname{Reg}(h)[\operatorname{Reg}(\boldsymbol{A})]=h[\operatorname{Reg}(\boldsymbol{A})] \subseteq \operatorname{Reg}(h(\boldsymbol{A}))$. Also, if $\odot \in\{\wedge, \vee, *, \rightarrow\}$, then

$$
h\left(a \odot_{r} b\right)=h(\neg \neg(a \odot b))=\neg \neg(h(a) \odot h(b))=h(a) \odot_{r} h(b) .
$$

Hence, $\operatorname{Reg}(h)$ is a homomorphism from $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ into $\boldsymbol{\operatorname { R e g }}(\boldsymbol{B})$. In addition, if $h$ is onto, then $h(\boldsymbol{A})=\boldsymbol{B}$, and $h[\operatorname{Reg}(\boldsymbol{A})]=\operatorname{Reg}(\boldsymbol{B})$. Indeed, if $h(a) \in \operatorname{Reg}(\boldsymbol{B})$, then $h(a)=\neg \neg h(a)=h(\neg \neg a) \in h[\operatorname{Reg}(\boldsymbol{A})]$. Thus, $\operatorname{Reg}(h)$ is onto.

There is also a close connection between the i-filters of a residuated lattice $\boldsymbol{A}$ and those of $\boldsymbol{\operatorname { R e g }} \boldsymbol{\operatorname { A }} \boldsymbol{A}$.

Lemma 3.4. Let $\boldsymbol{A}$ be a residuated lattice. Then for any $i$-filter $f$ of $\boldsymbol{A}$, $f \cap \operatorname{Reg}(\boldsymbol{A})$ is an $i$-filter of $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$. Conversely, for each $i$-filter $g$ of $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$, there is an $i$-filter $f$ of $\boldsymbol{A}$ such that $g=f \cap \operatorname{Reg}(\boldsymbol{A})$.

Proof. Is is easy to check that $f \in \mathcal{F}(\boldsymbol{A})$ implies $f \cap \operatorname{Reg}(\boldsymbol{A}) \in \mathcal{F}(\boldsymbol{\operatorname { R e g }}(\boldsymbol{A}))$. To show the converse take $f=F^{\boldsymbol{A}}(g)$. Then it is also easy to see that $g=$ $F^{\boldsymbol{A}}(g) \cap \operatorname{Reg}(\boldsymbol{A})$.

For $\boldsymbol{A} \in \mathbb{R} \mathbb{L}$, an element $a \in A$ is dense if $\neg \neg a=1$. The set of dense elements of a residuated lattice $\boldsymbol{A}$ is denoted by $D(\boldsymbol{A})$ and forms an i-filter.

Lemma 3.5. Let $\boldsymbol{A}$ be a non trivial residuated lattice.
(a) If $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ is directly indecomposable, then $\boldsymbol{A}$ is also directly indecomposable.
(b) If $\boldsymbol{A}$ is subdirectly irreducible and $D(\boldsymbol{A})=\{1\}$, then $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ is subdirectly irreducible.
(c) $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ is simple if and only if $D(\boldsymbol{A})$ is the maximum proper $i$-filter of $\boldsymbol{A}$.

Proof. (a): In [12, Proposition 1.5], it is shown that $\boldsymbol{A}$ is directly indecomposable if and only if $B(\boldsymbol{A})=\{0,1\}$. We have already noted that $B(\boldsymbol{A}) \subseteq$ $B(\boldsymbol{\operatorname { e r }} \boldsymbol{g}(\boldsymbol{A}))$, so (a) follows immediately.
(b): Assume $\boldsymbol{A}$ is subdirectly irreducible. Let $m=\min \mathcal{F}(\boldsymbol{A}) \backslash\{\{1\}\}$, the monolith of $\boldsymbol{A}$. If $a \in m \backslash\{1\}$, then since $D(\boldsymbol{A})=\{1\}$, we have that $\neg \neg a \in(m \cap \operatorname{Reg}(\boldsymbol{A})) \backslash\{1\}$, and so $m \cap \operatorname{Reg}(\boldsymbol{A}) \neq\{1\}$. Let $g \neq\{1\}$ be an i-filter of $\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A})$; then there is an i-filter $f$ of $\boldsymbol{A}$ such that $g=f \cap \operatorname{Reg}(\boldsymbol{A})$. Since $f \neq\{1\}$, we have $m \subseteq f$, and so $m \cap \operatorname{Reg}(\boldsymbol{A}) \subseteq f \cap \operatorname{Reg}(\boldsymbol{A})=g$. Thus, $m \cap \operatorname{Reg}(\boldsymbol{A})$ is the monolith of $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$. That completes the proof of $(b)$.
(c): Observe that since $0 \notin D(\boldsymbol{A}), D(\boldsymbol{A})$ is a proper i-filter of $\boldsymbol{A}$. Furthermore, recall that a residuated lattice $\mathbf{A}$ is simple if and only if for each $a \in A \backslash\{1\}$, there exists a positive integer $n$ such that $a^{n}=0$. Assume now that $\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A})$ is simple, and let $g$ be an i-filter of $\boldsymbol{A}$. If $g \nsubseteq D(\boldsymbol{A})$, then there is $a \in g$ such that $\neg \neg a \neq 1$, and so $a^{n} \leqslant \neg \neg(\neg \neg a)^{n}=0$ for some $n>0$; hence, $0 \in g$ and $g=A$. Thus, every i-filter $g \neq A$ is contained in $D(\boldsymbol{A})$. Conversely, assume that $D(\boldsymbol{A})$ is the maximum proper i-filter of $\boldsymbol{A}$. Let $a \in \operatorname{Reg}(\boldsymbol{A}) \backslash\{1\}$.

Since $a \notin D(\boldsymbol{A})$, we have that $0 \in F^{\boldsymbol{A}}(a)=A$. Hence, there is $n>0$ such that $a^{n}=0$, whence $\neg \neg a^{n}=0$. Thus, $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ is simple.

Given a class $\mathbb{K}$ of residuated lattices, let $R(\mathbb{K})=\{\boldsymbol{\operatorname { R e g }}(\boldsymbol{A}): \boldsymbol{A} \in \mathbb{K}\}$. As usual, $S, H, P$, and $V$ stand for the operators for subalgebras, homomorphic images, direct products, and generated variety, respectively.

Lemma 3.6. Let $\mathbb{K}$ be a class of residuated lattices. Then the following properties hold:
(a) $R(\mathbb{K}) \subseteq \mathbb{K}$ if and only if $R(\mathbb{K})=\mathbb{K} \cap \mathbb{R} \mathbb{L}$;
(b) if $O \in\{H, S, P, V\}$, then $R O(\mathbb{K}) \subseteq O R(\mathbb{K})$;
(c) $R V(\mathbb{K}) \subseteq V(\mathbb{K})$ if and only if $R(\mathbb{K}) \subseteq V(\mathbb{K})$.

Proof. We prove only (b), since (a) is trivial and (c) follows immediately from (b). Since $\boldsymbol{A} \subseteq \boldsymbol{B}$ implies $\boldsymbol{\operatorname { R e }} \boldsymbol{g}(\boldsymbol{A}) \subseteq \boldsymbol{\operatorname { R e }} \boldsymbol{g}(\boldsymbol{B})$, we have the case $O=S$. From $\boldsymbol{\operatorname { R e }} \boldsymbol{g}\left(\prod \boldsymbol{A}_{i}\right)=\prod \boldsymbol{\operatorname { R e g }}\left(\boldsymbol{A}_{i}\right)$, we deduce the property for $O=P$. The case $O=H$ is a consequence of Lemma 3.3. Finally, the corresponding property for $O=V$ is now straightforward.

In what follows, given a variety $\mathbb{V}$, we denote by $\mathbb{V}_{\text {si }}$ the class of subdirectly irreducible algebras belonging to $\mathbb{V}$.

Lemma 3.7. The following properties hold true for each variety $\mathbb{V} \subseteq \mathbb{R} \mathbb{L}$.
(a) $R(\mathbb{V}) \subseteq \mathbb{V}$ if and only if $R\left(\mathbb{V}_{\mathrm{si}}\right) \subseteq \mathbb{V}$.
(b) $R(\mathbb{V})$ is closed under homomorphic images and direct products, that is, $H P R(\mathbb{V}) \subseteq R(\mathbb{V})$.
(c) $S R(\mathbb{V})=V R(\mathbb{V})$.

Proof. (a): This follows from Lemma 3.6 by taking $\mathbb{K}=\mathbb{V}_{\text {si }}$.
(b): To see that $H R(\mathbb{V}) \subseteq R(\mathbb{V})$, let $\boldsymbol{A} \in \mathbb{V}$ and $g \in \mathcal{F}(\boldsymbol{\operatorname { R e }} \boldsymbol{g}(\boldsymbol{A}))$. We show that $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A}) / g \in I R(\mathbb{V})=R(\mathbb{V})$. Let $\pi: \boldsymbol{A} \rightarrow \boldsymbol{A} / F^{\boldsymbol{A}}(g)$ be the natural map. By Lemma 3.3, $\operatorname{Reg}(\pi): \boldsymbol{\operatorname { R e g }}(\boldsymbol{A}) \rightarrow \boldsymbol{\operatorname { L e g }}\left(\boldsymbol{A} / F^{\boldsymbol{A}}(g)\right)$ is a homomorphism. From the proof of Lemma 3.4, we know that $F^{\boldsymbol{A}}(g) \cap \operatorname{Reg}(\boldsymbol{A})=g$. Hence, by the homomorphism theorem, $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A}) / g$ is isomorphic to $\boldsymbol{\operatorname { R e g }}\left(\boldsymbol{A} / F^{\boldsymbol{A}}(g)\right)$. The fact that $R(\mathbb{V})$ is closed under products follows easily since $\boldsymbol{\operatorname { R e } \boldsymbol { g }}\left(\prod_{i \in I} \boldsymbol{A}_{i}\right)$ $=\prod_{i \in I} \boldsymbol{R e} \boldsymbol{g}\left(\boldsymbol{A}_{i}\right)$ for any family $\left\{\boldsymbol{A}_{i}\right\}_{i \in I}$ in $\mathbb{R} \mathbb{L}$.
(c): Note that by the congruence extension property, $S H R(\mathbb{V})=H S R(\mathbb{V})$. Hence, $V R(\mathbb{V})=H S P R(\mathbb{V}) \subseteq H S R(\mathbb{V})=S H R(\mathbb{V}) \subseteq S R(\mathbb{V})$.

As shown in the previous lemma, $R(\mathbb{V})$ is closed under $H$ and $P$, but in general, it is not closed under $S$. We give an example of this in the following theorem.

Theorem 3.8. There is a variety $\mathbb{V}$ of residuated lattices such that $R(\mathbb{V})$ is not a variety.
Proof. Consider $\boldsymbol{A}=\langle\{0,1,2,3,4,5,6,7,8\} ; \wedge, \vee, *, \rightarrow, 0,1\rangle$, the residuated lattice whose lattice order is given by the diagram depicted in Figure 1 and whose operations $*$ and $\rightarrow$ are given by the tables in Figure 2.


Figure 1. The Hasse diagram of $\boldsymbol{A}$

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 0 | 2 | 0 | 5 | 5 | 0 | 5 | 5 | 5 |
| 3 | 0 | 3 | 5 | 0 | 0 | 0 | 5 | 5 | 0 |
| 4 | 0 | 4 | 5 | 0 | 5 | 0 | 5 | 5 | 5 |
| 5 | 0 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 6 | 5 | 5 | 5 | 0 | 5 | 5 | 5 |
| 7 | 0 | 7 | 5 | 5 | 5 | 0 | 5 | 5 | 5 |
| 8 | 0 | 8 | 5 | 0 | 5 | 0 | 5 | 5 | 0 |


| $\rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 2 | 1 | 1 | 6 | 6 | 6 | 1 | 1 | 6 |
| 3 | 4 | 1 | 6 | 1 | 1 | 6 | 1 | 1 | 1 |
| 4 | 3 | 1 | 6 | 6 | 1 | 6 | 1 | 6 | 6 |
| 5 | 6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 5 | 1 | 6 | 6 | 6 | 6 | 1 | 6 | 6 |
| 7 | 5 | 1 | 6 | 6 | 6 | 6 | 1 | 1 | 6 |
| 8 | 8 | 1 | 6 | 6 | 1 | 6 | 1 | 1 | 1 |

Figure 2. The operations $*$ and $\rightarrow$ of $\boldsymbol{A}$

First note that from the table for $*$, it is clear that for all $a \in A, a \neq 1$ implies $a^{3}=0$, and so $\boldsymbol{A} \models x \vee \neg x^{3} \approx 1$. Thus, $V(\boldsymbol{A})$ satisfies $x \vee \neg x^{3} \approx 1$ and, consequently, it is a finitely generated discriminator variety. We claim that $R V(\boldsymbol{A})$ is not a variety.

From the table for $\rightarrow$, we infer that $\operatorname{Reg}(\boldsymbol{A})=A \backslash\{7\}$. Moreover, it is easy to check that $B=A \backslash\{7,8\}$ is the universe of a subalgebra of $\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A})$, which we denote by $\boldsymbol{B}$. We claim that $\boldsymbol{B} \notin R V(\boldsymbol{A})$. Indeed, suppose that $\boldsymbol{B}=\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{C})$ for some $\boldsymbol{C} \in V(\boldsymbol{A})$. We can assume that $\boldsymbol{C}$ is generated by $B$, for otherwise we would consider the subalgebra of $C$ generated by $B$. Thus, since $V(\boldsymbol{A})$ is locally finite and is finitely generated, $\boldsymbol{C}$ is finite. Besides, since $\boldsymbol{B}$ is directly indecomposable, so too is $\boldsymbol{C}$ by Lemma 3.5 . Therefore,
since $V(\boldsymbol{A})$ is a discriminator variety, $\boldsymbol{C}$ is simple and thus belongs to $I S(\boldsymbol{A})$, whence $\boldsymbol{B}=\boldsymbol{\operatorname { R e g }}(\boldsymbol{C}) \in \operatorname{RIS}(\boldsymbol{A})$, which is easily seen not to be true. This shows that $R V(\boldsymbol{A})$ is not closed under $S$ and hence is not a variety.

Observe that if $R(\mathbb{V})$ is not a variety, then $\mathbb{I V} \varsubsetneqq R(\mathbb{V})$, and then from the above theorem and item $(a)$ of Lemma 3.6, we deduce the next result.

Corollary 3.9. There is a variety $\mathbb{V}$ of residuated lattices such that $R(\mathbb{V}) \nsubseteq \mathbb{V}$.
Our next task is to characterize, by means of free algebras, the varieties $\mathbb{V}$ of residuated lattices such that $R(\mathbb{V})$ is a variety.

For a set $X, \boldsymbol{F}_{\mathbb{V}}(\bar{X})$ denotes the $|X|$-free algebra in the variety $\mathbb{V}$, with set of free generators $\bar{X}=\{\bar{x}: x \in X\}$, and $\boldsymbol{S} \boldsymbol{g}_{\mathbb{V}}^{r}(\neg \neg \bar{X})$ denotes the subalgebra of $\boldsymbol{\operatorname { R e }} \boldsymbol{g}\left(\boldsymbol{F}_{\mathbb{V}}(\bar{X})\right)$ generated by the set $\neg \neg \bar{X}=\{\neg \neg \bar{x}: x \in X\}$. In particular, if $X=\left\{x_{n}: n \in \omega\right\}$ is denumerable, $\boldsymbol{F}_{\mathbb{V}}(\omega)$ stands for the $\omega$-free algebra in $\mathbb{V}$; in this case, we write $\boldsymbol{S} \boldsymbol{g}_{\mathbb{V}}^{r}(\neg \neg \omega)$ in place of $\boldsymbol{S} \boldsymbol{g}_{\mathbb{V}}^{r}(\neg \neg \bar{X})$.

Lemma 3.10. Let $\mathbb{V}$ be a variety of residuated lattices. Then for every set $X$, $\boldsymbol{S g}_{\mathbb{V}}^{r}(\neg \neg \bar{X})$ is the $|X|$-free algebra in $S R(\mathbb{V})$.

Proof. It is clear that $\boldsymbol{S} \boldsymbol{g}_{\mathbb{V}}^{r}(\neg \neg \bar{X}) \in S R(\mathbb{V})$. Moreover, the map $\bar{x} \mapsto \neg \neg \bar{x}$ is a bijection from $\bar{X}$ onto $\neg \neg \bar{X}$, and so $|X|=|\neg \neg \bar{X}|$.

Let $\boldsymbol{A} \in S R(\mathbb{V})$, and let $h: \neg \neg \bar{X} \rightarrow A$ be a map. Take $\boldsymbol{B} \in \mathbb{V}$ such that $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{\operatorname { R e }} \boldsymbol{g}(\boldsymbol{B})$, and consider $\bar{h}: \boldsymbol{F}_{\mathbb{V}}(\bar{X}) \rightarrow \boldsymbol{B}$, the homomorphism such that $\bar{h}(\bar{x})=h(\neg \neg \bar{x})$. Then it is easy to see that $\operatorname{Reg}(\bar{h}) \upharpoonright_{S g_{V}^{r}(\neg \neg \bar{X})}$ is a homomorphism from $\boldsymbol{S} \boldsymbol{g}_{\mathbb{V}}^{r}(\neg \neg \bar{X})$ into $\boldsymbol{B}$ that extends $h$.

Theorem 3.11. $R(\mathbb{V})$ is a variety if and only if $\boldsymbol{S}_{\boldsymbol{g}_{\mathbb{V}}}^{r}(\neg \neg \bar{X}) \in R(\mathbb{V})$ for every set $X$.

Proof. The direct implication is trivial. To see the converse, assume that $\boldsymbol{S} \boldsymbol{g}_{\mathbb{V}}^{r}(\neg \neg \bar{X}) \in R(\mathbb{V})$ for every set $X$. Then by Lemma 3.10, $\boldsymbol{F}_{V R(\mathbb{V})}(\bar{X}) \in R(\mathbb{V})$ for every set $X$. Since any algebra in $V R(\mathbb{V})$ is a homomorphic image of $\boldsymbol{F}_{V R(\mathbb{V})}(\bar{X})$ for some $X$, Lemma 3.7 implies $V R(\mathbb{V}) \subseteq H R(\mathbb{V}) \subseteq R(\mathbb{V})$. Hence, $R(\mathbb{V})=V R(\mathbb{V})$ is a variety.

Observe that from Theorems 3.8 and 3.11 , we can deduce the following.
Corollary 3.12. There is a variety $\mathbb{V}$ of residuated lattices with $\boldsymbol{S g}_{\mathbb{V}}^{r}(\neg \neg \bar{X})$ $\neq \boldsymbol{\operatorname { R e g }}\left(\boldsymbol{F}_{\mathbb{V}}(\bar{X})\right)$ for some set $X$. That is, in general, $\boldsymbol{\operatorname { R e g }}\left(\boldsymbol{F}_{\mathbb{V}}(\bar{X})\right)$ is not generated by $\neg \neg \bar{X}$.

## 4. Kolmogorov translation and regular varieties

In what follows, by a term we understand a $\{\wedge, \vee, *, \rightarrow, 0,1\}$-term. Given a term $t$ we write $t\left(x_{1}, \ldots, x_{n}\right)$ to indicate that the variables appearing in $t$ are in $\left\{x_{1}, \ldots, x_{n}\right\}$. We will denote by $T(X)$ the set of all terms whose variables belong to $X$. If $\sigma \subseteq\{\wedge, \vee, *, \rightarrow\}$, then $T_{\sigma}(X)$ will denote the set
of all $(\{0,1\} \cup \sigma)$-terms with variables in $X$. Observe that $T_{\sigma}(X) \subseteq T(X)$. Moreover, if $t\left(x_{1}, \ldots, x_{n}\right) \in T_{\sigma}(X)$ and $\boldsymbol{A}$ is a residuated lattice, then for any $a_{1}, \ldots, a_{n} \in A, t^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right)$ represents the interpretation of $t$ on $\boldsymbol{A}$ given by the assignment $\left\{x_{i} \mapsto a_{i}\right\}_{1 \leqslant i \leqslant n}$.

Given a term $t \in T(X)$, its Kolmogorov translation $\widetilde{t}$ is defined recursively on the complexity of $t$ as follows

- $\widetilde{t}=\neg \neg t$, if $t \in X \cup\{0,1\}$,
- for any $\odot \in\{\wedge, \vee, *, \rightarrow\}$, if $t=t_{1} \odot t_{2}$, then $\widetilde{t}=\neg \neg\left(\widetilde{t}_{1} \odot \tilde{t}_{2}\right)$.

The Kolmogorov translation of a term satisfies a key property that we state in the following lemma.

Lemma 4.1. Let $t\left(x_{1}, \ldots, x_{n}\right)$ be a term. Then for any residuated lattice $\boldsymbol{A}$ and any $a_{1}, \ldots, a_{n} \in A$, we have

$$
\begin{equation*}
t_{\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right)=\widetilde{t}^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right) \tag{4.1}
\end{equation*}
$$

Proof. The proof is by induction on the complexity of $t$. The claim is trivial for $t=x_{i}$ with $1 \leqslant i \leqslant n$, and for $t \in\{0,1\}$.

If $t=t_{1} \odot t_{2}$, with $\odot \in\{\wedge, \vee, *, \rightarrow\}$, then

$$
\begin{aligned}
& t^{\boldsymbol{R e g}(\boldsymbol{A})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right) \\
= & t_{1}^{\boldsymbol{R e g}(\boldsymbol{A})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right) \odot_{r} t_{2}^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right) \\
= & \neg \neg\left(t_{1}^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right) \odot t_{2}^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right)\right) \\
= & \neg \neg\left(\widetilde{t_{1}^{\boldsymbol{A}}}\left(a_{1}, \ldots, a_{n}\right) \odot \widetilde{t}_{2}^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right)\right)=\widetilde{t}^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right) .
\end{aligned}
$$

The following easy consequence of the above lemma will be crucial to understand the classes $R(\mathbb{V})$.

Corollary 4.2. For any residuated lattice $\boldsymbol{A}$ and any terms $t, s$, we have $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A}) \models t \approx s$ if and only if $\boldsymbol{A} \models \widetilde{t} \approx \widetilde{s}$.

Moreover, we have the following (see [5, Corollary 4.5]):
Lemma 4.3. Given a residuated lattice $\boldsymbol{A}$, the following are equivalent:
(1) $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{A}$,
(2) $\boldsymbol{A} \vDash \widetilde{t}\left(x_{1}, \ldots, x_{n}\right) \approx t\left(\neg \neg x_{1}, \ldots, \neg \neg x_{n}\right)$ for any term $t\left(x_{1}, \ldots, x_{n}\right)$,
(3) $\boldsymbol{A} \mid=\neg \neg(\neg \neg x \vee \neg \neg y) \approx \neg \neg x \vee \neg \neg y$ and $\boldsymbol{A} \mid=\neg \neg(\neg \neg x * \neg \neg y) \approx \neg \neg x * \neg \neg y$.

Corollary 4.4. For any variety $\mathbb{V}$ of residuated lattices, the following are equivalent:
(1) $\boldsymbol{R e g}\left(\boldsymbol{F}_{\mathbb{V}}(\bar{X})\right)$ is a subalgebra of $\boldsymbol{F}_{\mathbb{V}}(\bar{X})$ for every set $X$,
(2) $\mathbb{V} \models \widetilde{t}\left(x_{1}, \ldots, x_{n}\right) \approx t\left(\neg \neg x_{1}, \ldots, \neg \neg x_{n}\right)$ for any term $t\left(x_{1}, \ldots, x_{n}\right)$,
(3) $\mathbb{V} \models \neg \neg(\neg \neg x \vee \neg \neg y) \approx \neg \neg x \vee \neg \neg y$ and $\mathbb{V} \models \neg \neg(\neg \neg x * \neg \neg y) \approx \neg \neg x * \neg \neg y$.

Observe that by (4.1), the set

$$
\left\{\tilde{t}^{\boldsymbol{F}_{\mathrm{v}}(\bar{X})}\left(\bar{x}_{0}, \ldots, \bar{x}_{n}\right): t \in T(X), x_{i} \in X, i \leqslant n \in \omega\right\}
$$

is the universe of $\boldsymbol{S} \boldsymbol{g}_{\mathbb{V}}^{r}(\neg \neg \bar{X})$. In general, the correspondence induced by $t \mapsto \widetilde{t}$ on $\boldsymbol{F}_{\mathbb{V}}(\bar{X})$ does not give a mapping. Indeed, if $\mathbb{V}$ is such that $R(\mathbb{V}) \nsubseteq \mathbb{V}$, then there are $s, t \in T(X)$ such that $\mathbb{V} \models s \approx t$ and $R(\mathbb{V}) \not \vDash s \approx t$; by Corollary $4.2, \mathbb{V} \not \vDash \widetilde{s} \approx \widetilde{t}$. In others words, we have $s^{\boldsymbol{F}_{\mathrm{V}}(\omega)}=t^{\boldsymbol{F}(\omega)}$ but $\widetilde{s}^{\boldsymbol{F}_{\mathrm{V}}(\omega)} \neq \widetilde{t}^{\boldsymbol{F}_{\mathrm{V}}(\omega)}$, always interpreting the variables on free generators.

Theorem 4.5. For any variety $\mathbb{V}$ of residuated lattices, the following properties are equivalent:
(1) $R(\mathbb{V}) \subseteq \mathbb{V}$,
(2) for any set $X$, the correspondence $\sim^{\boldsymbol{F}_{\mathrm{V}}(\bar{X})}: t^{\boldsymbol{F}_{\mathrm{V}}(\bar{X})} \mapsto \widetilde{t}^{\boldsymbol{F}_{\mathrm{V}}(\bar{X})}$ gives a homomorphism from $\boldsymbol{F}_{\mathbb{V}}(\bar{X})$ onto $\boldsymbol{S g}_{\mathbb{V}}^{r}(\neg \neg \bar{X})$,
(3) $\sim^{\boldsymbol{F}_{\mathbb{V}}}(\omega)$ gives a mapping from $\boldsymbol{F}_{\mathbb{V}}(\omega)$ into $\boldsymbol{R e} \boldsymbol{g}\left(\boldsymbol{F}_{\mathbb{V}}(\omega)\right)$.

Proof. (1) implies (2): By Lemmas 3.6 and $3.7, R(\mathbb{V}) \subseteq \mathbb{V}$ is equivalent to $\mathbb{I V}=R(\mathbb{V})=S R(\mathbb{V})$; hence by Lemma 3.10, $\boldsymbol{F}_{\mathbb{I}}(\bar{X}) \cong \boldsymbol{S}_{\mathbb{V}}^{r}(\neg \neg \bar{X}) \in \mathbb{V}$. Then the map $\bar{x} \mapsto \neg \neg \bar{x}$ extends to a homomorphism $h$ from $\boldsymbol{F}_{\mathbb{V}}(\bar{X})$ onto $\boldsymbol{S} \boldsymbol{g}_{\mathbb{V}}^{r}(\neg \neg \bar{X}) \in \mathbb{V}$. For any $t\left(x_{1}, \ldots, x_{n}\right) \in T(X)$, by Lemma 4.1, we have
$h\left(t^{\boldsymbol{F}_{\mathrm{V}}(\bar{X})}\left(\bar{x}_{1}, \ldots, \bar{x}_{n}\right)\right)=t^{\boldsymbol{R e g}\left(\boldsymbol{F}_{\mathrm{V}}(\bar{X})\right)}\left(\neg \neg \bar{x}_{1}, \ldots, \neg \neg \bar{x}_{n}\right)=\tilde{t}^{\boldsymbol{F}_{\mathrm{V}}(\bar{X})}\left(\bar{x}_{1}, \ldots, \bar{x}_{n}\right)$,
so (2) holds true.
(2) implies (3): This is trivial because $S g^{r}(\neg \neg \omega) \subseteq \operatorname{Reg}\left(\boldsymbol{F}_{\mathbb{V}}(\omega)\right)$.
(3) implies (1): Let $t, s$ be terms. Since every term depends only on a finite number of variables, we can assume, without loss of generality, that $t=t\left(x_{0}, \ldots, x_{n}\right), s=s\left(x_{0}, \ldots, x_{n}\right) \in T\left(\left\{x_{n}: n \in \omega\right\}\right)$. Suppose $\mathbb{V} \models t \approx s$. Then $t^{\boldsymbol{F}_{\mathrm{V}}(\omega)}\left(\bar{x}_{0}, \ldots, \bar{x}_{n}\right)=s^{\boldsymbol{F}_{\mathrm{V}}(\omega)}\left(\bar{x}_{0}, \ldots, \bar{x}_{n}\right)$; consequently,

$$
\begin{aligned}
\tilde{t}^{\boldsymbol{\boldsymbol { F } _ { \mathrm { V } }}(\omega)}\left(\bar{x}_{0}, \ldots, \bar{x}_{n}\right) & =\widetilde{s}^{\boldsymbol{F}_{\mathrm{V}}(\omega)}\left(\bar{x}_{0}, \ldots, \bar{x}_{n}\right), \\
t^{\boldsymbol{R e g}\left(\boldsymbol{F}_{\mathrm{V}}(\bar{X})\right)}\left(\neg \neg \bar{x}_{0}, \ldots, \neg \neg \bar{x}_{n}\right) & =s^{\boldsymbol{\operatorname { R e g }}\left(\boldsymbol{F}_{\mathrm{V}}(\bar{X})\right)}\left(\neg \neg \bar{x}_{0}, \ldots, \neg \neg \bar{x}_{n}\right), \\
t^{\boldsymbol{S} \boldsymbol{g}_{\mathrm{V}}^{r}(\neg \neg \bar{X})}\left(\neg \neg \bar{x}_{0}, \ldots, \neg \neg \bar{x}_{n}\right) & =s^{\boldsymbol{S} \boldsymbol{g}_{\mathrm{V}}^{r}(\neg \neg \bar{X})}\left(\neg \neg \bar{x}_{0}, \ldots, \neg \neg \bar{x}_{n}\right),
\end{aligned}
$$

and, by Lemma 3.10, $S R(\mathbb{V}) \models t \approx s$.
Corollary 4.6. If $\mathbb{V}$ is a subvariety of $\mathbb{R} \mathbb{L}$, then $R(\mathbb{V}) \subseteq \mathbb{V}$ if and only if $\boldsymbol{F}_{S R(\mathbb{V})}(\omega)$ is a homomorphic image of $\boldsymbol{F}_{\mathbb{V}}(\omega)$.

Proof. The result is a consequence of Theorem 4.5 above and the facts that $\boldsymbol{F}_{S R(\mathbb{V})}(\omega) \cong \boldsymbol{S}_{\mathbb{V}}^{r}(\neg \neg \omega)$ and $V\left(\boldsymbol{F}_{\mathbb{V}}(\omega)\right)=\mathbb{V}$.

We say that a variety $\mathbb{V}$ of residuated lattices is regular provided that it satisfies the condition $R(\mathbb{V}) \subseteq \mathbb{V}$. By Lemma 3.6, if $\mathbb{V}$ is a regular variety, then $R(\mathbb{V})=\mathbb{I} \mathbb{V}$, and so $R(\mathbb{V})$ is a variety.

It is clear that $\mathbb{R} \mathbb{L}$ is regular, but there are plenty of examples of regular subvarieties of $\mathbb{R} \mathbb{L}$. For instance, any subvariety $\mathbb{V}$ of $\mathbb{R L}$ such that $\mathbb{V} \supseteq \mathbb{R} \mathbb{R}$ is trivially regular.

On the other hand, Property (a) of Lemma 3.7 provides a way to show that some well-known subvarieties of $\mathbb{R L}$ are regular. More precisely, to show that $\mathbb{V}$ is regular, it suffices to show that $R\left(\mathbb{V}_{\text {si }}\right) \subseteq \mathbb{V}$. For example:

- For all $n \geqslant 1$, the subdirectly irreducible members of the variety $\mathbb{E M}_{n}$ are simple. Using this and Lemma 3.5, we see that $R\left(\left(\mathbb{E M}_{n}\right)_{\mathrm{si}}\right) \subseteq\left(\mathbb{E M}_{n}\right)_{\mathrm{si}}$. Hence, $\mathbb{E M}_{n}$ is regular. Observe that the variety considered in Theorem 3.8 is a non regular subvariety of $\mathbb{E M}_{3}$.
- MTLL, the variety generated by totally ordered residuated lattices, is a regular variety. Indeed, since all the algebras in $\mathbb{M T L}_{\text {si }}$ are totally ordered, so are the algebras in $R\left(\mathbb{M T L}_{\text {si }}\right)$, and thus $R\left(\mathbb{M T L}_{\mathrm{si}}\right) \subseteq \mathbb{M T L}$.
- Let $\mathbb{W N M}$ be the subvariety of MTL given by the equation

$$
\neg(x * y) \vee((x \wedge y) \rightarrow(x * y)) \approx 1
$$

The algebras in $\mathbb{W N M}$ are called weak nilpotent minimum algebras. This variety is also a regular variety. Indeed, given $\boldsymbol{A} \in \mathbb{W N M}_{\text {si }}$, we know that $\boldsymbol{A}$ is totally ordered. In particular, since 1 is $\vee$-irreducible in $\boldsymbol{A}$, for each $a, b \in \operatorname{Reg}(\boldsymbol{A})$ such that $\neg\left(a *_{r} b\right)<1$, we have that $\neg(a * b) \neq 1$, and so $a \wedge_{r} b=a \wedge b \leqslant a * b \leqslant a *_{r} b$. Hence, $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A}) \in \mathbb{W} N M$.
We will now give another source of regular varieties. Consider $\sigma_{\text {ex }}=\{\vee, *\}$, $\sigma_{\text {in }}=\{\wedge, \rightarrow\}$, and take $\widehat{T}(X)$ to be the set of terms $t\left(t_{1}, \ldots, t_{n}\right)$ such that $t\left(x_{1}, \ldots, x_{n}\right) \in T_{\sigma_{\text {ex }}}(X)$ for $n \geqslant 1$, and $t_{1}, \ldots, t_{n} \in T_{\sigma_{\text {in }}}(X)$.

Lemma 4.7. The following properties hold:
(a) if $t \in T_{\sigma_{\text {ex }}}\left(x_{1}, \ldots, x_{n}\right)$, then $\mathbb{R} \mathbb{L} \models \widetilde{t}\left(x_{1}, \ldots, x_{n}\right) \approx \neg \neg t\left(x_{1}, \ldots, x_{n}\right)$,
(b) if $t \in T_{\sigma_{\text {in }}}\left(x_{1}, \ldots, x_{n}\right)$, then $\mathbb{R} \mathbb{L} \models \widetilde{t}\left(x_{1}, \ldots, x_{n}\right) \approx t\left(\neg \neg x_{1}, \ldots, \neg \neg x_{n}\right)$,
(c) if $t \in \widehat{T}\left(x_{1}, \ldots, x_{n}\right)$, then $\mathbb{R} \mathbb{L} \models \widetilde{t}\left(x_{1}, \ldots, x_{n}\right) \approx \neg \neg t\left(\neg \neg x_{1}, \ldots, \neg \neg x_{n}\right)$.

Proof. (a): This is proved by induction on the complexity of the term $t$ using

$$
\begin{aligned}
& \mathbb{R} \mathbb{L} \models \neg \neg(\neg \neg x \vee \neg \neg y) \approx \neg \neg(x \vee y), \\
& \mathbb{R} \mathbb{L} \models \neg \neg(\neg \neg x * \neg \neg y) \approx \neg \neg(x * y) .
\end{aligned}
$$

(b): This is proved by induction on the complexity of the term $t$ using

$$
\begin{aligned}
& \mathbb{R} \mathbb{L} \models \neg \neg(\neg \neg x \wedge \neg \neg y) \approx \neg \neg x \wedge \neg \neg y, \\
& \mathbb{R} \mathbb{L} \models \neg \neg(\neg \neg x \rightarrow \neg \neg y) \approx \neg \neg x \rightarrow \neg \neg y .
\end{aligned}
$$

(c): An easy induction on the complexity of terms shows that for any terms $t, t_{1}, \ldots, t_{n}$, if $\alpha=t\left(t_{1}, \ldots, t_{n}\right)$, then $\mathbb{R} \mathbb{L} \models \widetilde{\alpha} \approx \widetilde{t}\left(\widetilde{t}_{1}, \ldots, \widetilde{t}_{n}\right)$. Therefore, (c) follows by combining (a) and (b).

The importance of the terms in $\widehat{T}(X)$ lies in the following property.
Lemma 4.8. For any residuated lattice $\boldsymbol{A}$ and any $t, s \in \widehat{T}(X)$, we have

$$
\boldsymbol{A} \models t \approx s \text { implies } \boldsymbol{\operatorname { R e }} \boldsymbol{g}(\boldsymbol{A}) \models t \approx s
$$

Proof. Since $\boldsymbol{A} \models t\left(x_{1}, \ldots, x_{n}\right) \approx s\left(x_{1}, \ldots, x_{n}\right)$, we immediately see that

$$
\boldsymbol{A} \models \neg \neg t\left(\neg \neg x_{1}, \ldots, \neg \neg x_{n}\right) \approx \neg \neg s\left(\neg \neg x_{1}, \ldots, \neg \neg x_{n}\right) .
$$

Hence, by Lemma 4.7 above, $\boldsymbol{A} \models \tilde{t} \approx \widetilde{s}$. By Corollary 4.2, we get that $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A}) \models t \approx s$.

Corollary 4.9. If $\mathbb{V}$ is a variety of residuated lattices admitting an equational basis relative to $\mathbb{R} \mathbb{L}$ whose terms are in $\widehat{T}(X)$, then $\mathbb{V}$ is regular.

As a consequence of Corollary 4.9, we obtain another proof that the variety $\mathbb{M T L}$ is regular. Indeed, it is known that $\mathbb{M T L}$ may be characterized within $\mathbb{R L}$ by the equation $(x \rightarrow y) \vee(y \rightarrow x) \approx 1$; hence, the previous corollary applies. Another example of a regular variety is the variety $\mathbb{E}_{n}$ of residuated lattices given by the equation $x^{n} \approx x^{n+1}$. Yet another example is the variety $\mathbb{B L}$ of BL-algebras, axiomatized within $\mathbb{M T L}$ by the axiom $x \wedge y \approx x *(x \rightarrow y)$.

A large class of regular varieties is the class of Glivenko varieties, that is, the varieties contained in $\mathbb{G}$.

By Lemma 3.1, for every algebra $\boldsymbol{A} \in \mathbb{G}, \boldsymbol{\operatorname { R e g }}(\boldsymbol{A}) \in H(\boldsymbol{A})$, so for any subvariety $\mathbb{V}$ of $\mathbb{G}$, we have $R(\mathbb{V}) \subseteq \mathbb{V}$, and so we have the following result.

Theorem 4.10. Every Glivenko variety is regular.
The following theorem gives some characterizations for Glivenko varieties in terms of regular elements and the Kolmogorov translation.

Theorem 4.11. If $\mathbb{V}$ is a subvariety of $\mathbb{R} \mathbb{L}$, the following conditions are equivalent:
(1) $\boldsymbol{S} \boldsymbol{g}_{\mathbb{V}}^{r}(\neg \neg \bar{X})=\boldsymbol{R e} \boldsymbol{g}\left(\boldsymbol{F}_{\mathbb{V}}(\bar{X})\right)$, for any set $X$
(2) $\boldsymbol{S}_{\boldsymbol{g}}^{\mathbb{V}} r(\neg \neg\{\bar{x}\})=\boldsymbol{R e} \boldsymbol{g}\left(\boldsymbol{F}_{\mathbb{V}}(\{\bar{x}\})\right)$,
(3) $\mathbb{V}$ is a Glivenko variety,
(4) for any term $t\left(x_{1}, \ldots, x_{n}\right), \mathbb{V} \models \widetilde{t}\left(x_{1}, \ldots, x_{n}\right)=\neg \neg t\left(x_{1}, \ldots, x_{n}\right)$.

Proof. (1) implies (2): This is trivial.
(2) implies (3): If (2) holds, then since $\neg \neg(\neg \neg \bar{x} \rightarrow \bar{x})$ is regular, there is a unary term $t(x)$ such that $\neg \neg(\neg \neg \bar{x} \rightarrow \bar{x})=t^{\boldsymbol{\operatorname { R e g }}\left(\boldsymbol{F}_{\mathrm{V}}(\{\bar{x}\})\right)}(\neg \neg \bar{x})$, and so $\neg \neg(\neg \neg \bar{x} \rightarrow \bar{x})=\widetilde{t}^{\boldsymbol{F}_{\mathrm{V}}(\{x\})}(\bar{x})$. Since the last equation relates elements in a free algebra, it follows that $\mathbb{V} \models \neg \neg(\neg \neg x \rightarrow x) \approx \widetilde{t}(x)$. Thus,

$$
\tilde{t}^{\boldsymbol{F} \mathfrak{V}(\{x\})}(\neg \neg \bar{x})=\neg \neg(\neg \neg \bar{x} \rightarrow \neg \neg \bar{x})=1 .
$$

Note however that $\widetilde{t}_{\boldsymbol{F}_{\mathfrak{V}}(\{x\})}(\neg \neg \bar{x})=\widetilde{t}^{\boldsymbol{F}_{\mathrm{V}}}(\{x\})(\bar{x})$, as the variables in $\widetilde{t}(x)$ are all preceded by a double negation. Putting together the last three equations, we obtain $\neg \neg(\neg \neg \bar{x} \rightarrow \bar{x})=1$, that is, Glivenko's equation holds in $\mathbb{V}$.
(3) implies (4): Let $t\left(x_{1}, \ldots, x_{n}\right)$ be a term. If $\mathbb{V}$ satisfies Glivenko's equation, then since it is a regular variety, $\neg \neg$ and $\sim$ both define a homomorphism from $\boldsymbol{F}_{\mathbb{V}}\left(\bar{x}_{1}, \ldots, \bar{x}_{n}\right)$ onto $\boldsymbol{\operatorname { R e }} \boldsymbol{g}\left(\boldsymbol{F}_{\mathbb{V}}\left(\bar{x}_{1}, \ldots, \bar{x}_{n}\right)\right)$ such that $\bar{x}_{i} \mapsto \neg \neg \bar{x}_{i}=\widetilde{\bar{x}}_{i}$ for $1 \leqslant i \leqslant n$. Since $\left\{\bar{x}_{1}, \ldots, \bar{x}_{n}\right\}$ is the set of free generators of $\boldsymbol{F}_{\mathbb{V}}\left(\left\{\bar{x}_{1}, \ldots, \bar{x}_{n}\right\}\right)$, we have

$$
\begin{aligned}
& (\neg \neg t)^{\boldsymbol{F}_{\mathrm{V}}\left(\left\{\bar{x}_{1}, \ldots, \bar{x}_{n}\right\}\right)}\left(\bar{x}_{1}, \ldots, \bar{x}_{n}\right)=\neg \neg\left(t^{\boldsymbol{F} \mathrm{V}\left(\left\{\bar{x}_{1}, \ldots, \bar{x}_{n}\right\}\right)}\left(\bar{x}_{1}, \ldots, \bar{x}_{n}\right)\right) \\
= & t^{\boldsymbol{R e g}\left(\boldsymbol{F}_{\mathrm{V}}\left(\left\{\bar{x}_{1}, \ldots, \bar{x}_{n}\right\}\right)\right)}\left(\neg \neg \bar{x}_{1}, \ldots, \neg \neg \bar{x}_{n}\right)=\widetilde{t}^{\boldsymbol{F}_{\mathrm{V}}\left(\left\{\bar{x}_{1}, \ldots, \bar{x}_{n}\right\}\right)}\left(\bar{x}_{1}, \ldots, \bar{x}_{n}\right),
\end{aligned}
$$

and so $\mathbb{V} \models \neg \neg t \approx \tilde{t}$.
(4) implies (1): This follows from the definition of $\boldsymbol{R e} \boldsymbol{g}$ and Lemma 4.1.

## 5. Distributive residuated lattices

In this section, we show that the variety $\mathbb{D} \mathbb{R L}$ of all distributive residuated lattices is not regular, that is, $R(\mathbb{D} \mathbb{R} \mathbb{L}) \nsubseteq \mathbb{D} \mathbb{R} \mathbb{L}$. Nonetheless, $R(\mathbb{D} \mathbb{R} \mathbb{L})$ is a variety, and, in fact, we will see that $R(\mathbb{D} \mathbb{R L})$ is the variety $\mathbb{I R L}$ of all involutive residuated lattices.
$\mathbb{D} \mathbb{R} \mathbb{L}$ is the subvariety of $\mathbb{R} \mathbb{L}$ given by $x \wedge(y \vee z) \approx(x \wedge y) \vee(x \wedge z)$. Observe that the term $x \wedge(y \vee z)$ is not in $\widehat{T}(\{x, y, z\})$.

To prove that $R(\mathbb{D} \mathbb{R} \mathbb{L})$ is, in fact, the whole variety $\mathbb{R} \mathbb{L}$, we will see that for any $\boldsymbol{A} \in \mathbb{R} \mathbb{R} L$, we can build $\boldsymbol{B} \in \mathbb{D} \mathbb{R L}$ such that $\operatorname{Reg}(\boldsymbol{B})=\boldsymbol{A}$.

Let $\boldsymbol{A}$ be a residuated lattice. For any $X \subseteq A$, we consider

$$
(X]=\{a \in A: a \leqslant x \text { for some } x \in X\} .
$$

We also write $(x]$ instead of $(\{x\}]$. Given $X \subseteq A$, we say that $X$ is decreasing if $y \in X$ whenever $y \leqslant x$ and $x \in X$, that is, $X$ is decreasing if and only if $X=(X]$. If $\operatorname{Dec}(\boldsymbol{A})$ denotes the family of non-empty decreasing subsets of $\boldsymbol{A}$, then $\langle\operatorname{Dec}(\boldsymbol{A}) ; \cap, \cup,\{0\}, A\rangle$ is a complete bounded distributive lattice.

In $\operatorname{Dec}(\boldsymbol{A})$, we define the operation: $X * Y=(\{x * y: x \in X, y \in Y\}]$. It is straightforward to see that this operation is associative, commutative, and has $A$ as identity element. Therefore, $\langle\operatorname{Dec}(\boldsymbol{A}) ; *, A\rangle$ is a commutative monoid. Moreover, for any $X, Y_{i} \in \operatorname{Dec}(\boldsymbol{A})$, we have $X * \bigcup_{i \in I} Y_{i}=\bigcup_{i \in I}\left(X * Y_{i}\right)$. Hence, for any $X, Y \in \operatorname{Dec}(\boldsymbol{A})$, the family $\{Z \in \operatorname{Dec}(\boldsymbol{A}): X * Z \subseteq Y\}$ is closed under arbitrary unions, so it has maximum, which we denote by $X \rightarrow Y$. Moreover, one can easily show that $X \rightarrow Y=\{z \in A: x * z \in Y$ for all $x \in X\}$. Then the following residuation property holds:

$$
X * Z \subseteq Y \Longleftrightarrow Z \subseteq X \rightarrow Y
$$

Therefore, $\boldsymbol{\operatorname { D e c }}(\boldsymbol{A})=\langle\operatorname{Dec}(\boldsymbol{A}) ; \cap, \cup, *, \rightarrow,\{0\}, A\rangle$ is a distributive residuated lattice.

Observe that $x \mapsto \alpha(x)=(x]$ gives a one to one mapping $\alpha: A \rightarrow \operatorname{Dec}(\boldsymbol{A})$. This map preserves almost all operations, indeed:

- $(x] \cap(y]=(x \wedge y]$,
- $(x] *(y]=(x * y]$,
- $(x] \rightarrow(y]=(x \rightarrow y]$,
- $(0]=\{0\}$,
- $(1]=A$.

In the following, given a subset $Y$ of $A$, we denote by $\operatorname{lb}(Y)$ the set of lower bounds of $Y$ and by $\operatorname{rub}(Y)$ the set of upper bounds of $Y$ that belong to $\operatorname{Reg}(\boldsymbol{A})$. Note that

$$
\begin{aligned}
\neg X=X \rightarrow\{0\} & =\{z \in A: x * z=0 \text { for all } x \in X\} \\
& =\{z \in A: z \leqslant \neg x \text { for all } x \in X\}=\operatorname{lb}(\{\neg x: x \in X\})
\end{aligned}
$$

We claim that

$$
\begin{equation*}
\neg \neg X=\operatorname{lb}(\operatorname{rub}(X)) . \tag{5.1}
\end{equation*}
$$

To prove this claim, it is enough to show that $\{\neg y: y \in \neg X\}=\operatorname{rub}(X)$. Indeed, if $y \in \neg X$, then $y \leqslant \neg x$ for any $x \in X$, and so $\neg y \geqslant \neg \neg x \geqslant x$ for any $x \in X$. Thus, $\neg y \in \operatorname{rub}(X)$. Conversely, suppose that $y=\neg \neg y \geqslant x$ for every $x \in X$. Then $\neg y \leqslant \neg x$ for every $x \in X$, which means that $\neg y \in \neg X$. Thus, $y=\neg \neg y$ with $\neg y \in \neg X$, as was to be proved.

Observe that if $a_{0}, \ldots, a_{n-1} \in A$ for $n>0$, and $a \in \operatorname{Reg}(\boldsymbol{A})$, then $a \geqslant a_{i}$ for every $i<n$ if and only if $a \geqslant \bigvee_{i<n} a_{i}$ if and only if $a \geqslant \neg \neg \bigvee_{i<n} a_{i}$. Hence, for any $n>0$ and any $a_{0}, \ldots, a_{n-1} \in A$, we have that

$$
\operatorname{rub}\left(\bigcup_{i<n}\left(a_{i}\right]\right)=\operatorname{rub}\left(\left\{a_{0}, \ldots, a_{n-1}\right\}\right)=\operatorname{rub}\left(\left\{\neg \neg \bigvee_{i<n} a_{i}\right\}\right)
$$

Then by (5.1), we have $\neg \neg \bigcup_{i<n}\left(a_{i}\right]=\left(\neg \neg \bigvee_{i<n} a_{i}\right]$. In particular, if $\boldsymbol{A} \in \mathbb{R} \mathbb{R} \mathbb{L}$, then for any $a_{0}, \ldots, a_{n-1} \in A$ for $n>0$, we have

$$
\begin{equation*}
\neg \neg \bigcup_{i<n}\left(a_{i}\right]=\left(\bigvee_{i<n} a_{i}\right] \tag{5.2}
\end{equation*}
$$

Now we are ready to prove the main result of this section.
Theorem 5.1. $R(\mathbb{D} \mathbb{R L})=\mathbb{I R} \mathbb{L}$.
Proof. Let $\boldsymbol{A} \in \mathbb{R} \mathbb{R} \mathbb{L}$. Consider the distributive residuated lattice $\operatorname{Dec}(\boldsymbol{A})$ defined above. Let $\alpha[A]$ be the image of the map $\alpha: x \mapsto(x]$, that is, $\alpha[A]=$ $\{(a]: a \in A\}$. Let $\boldsymbol{C}$ be the subalgebra of $\boldsymbol{\operatorname { D e c }}(\boldsymbol{A})$ generated by $\alpha[A]$. We claim that $C$ is given by $C=\left\{\bigcup_{i<n}\left(a_{i}\right]: a_{i} \in A, n>0\right\}$.

To show this, it is enough to verify that $C$ is a subuniverse of $\boldsymbol{D e c}(\boldsymbol{A})$. Indeed, $\{0\}$ and $A$ belong to $C$ and, by definition, $C$ is closed under $\cup$. Moreover, for any $a_{0}, \ldots, a_{n-1}, b_{0}, \ldots, b_{m-1} \in A$ with $n, m>0$, we have

$$
\begin{aligned}
& \left(\bigcup_{i<n}\left(a_{i}\right]\right) \cap\left(\bigcup_{j<m}\left(b_{j}\right]\right)=\bigcup_{i<n, j<m}\left(\left(a_{i}\right] \cap\left(b_{j}\right]\right)=\bigcup_{i<n, j<m}\left(a_{i} \wedge b_{j}\right] \\
& \left(\bigcup_{i<n}\left(a_{i}\right]\right) *\left(\bigcup_{j<m}\left(b_{j}\right]\right)=\bigcup_{i<n, j<m}\left(\left(a_{i}\right] *\left(b_{j}\right]\right)=\bigcup_{i<n, j<m}\left(a_{i} * b_{j}\right] .
\end{aligned}
$$

In order to show that $C$ is closed under $\rightarrow$, we need to prove first that $\left(a_{i}\right] \rightarrow \bigcup_{j<m}\left(b_{j}\right]=\bigcup_{j<m}\left(a_{i} \rightarrow b_{j}\right]$. Indeed,

$$
\begin{aligned}
& \left(a_{i}\right] \rightarrow \bigcup_{j<m}\left(b_{j}\right]=\left\{z \in A: a_{i} * z \in \bigcup_{j<m}\left(b_{j}\right]\right\}=\bigcup_{j<m}\left\{z \in A: a_{i} * z \in\left(b_{j}\right]\right\} \\
& =\bigcup_{j<m}\left\{z \in A: a_{i} * z \leqslant b_{j}\right\}=\bigcup_{j<m}\left\{z \in A: z \leqslant a_{i} \rightarrow b_{j}\right\}=\bigcup_{j<m}\left(a_{i} \rightarrow b_{j}\right]
\end{aligned}
$$

Thus, $\left(\bigcup_{i<n}\left(a_{i}\right]\right) \rightarrow\left(\bigcup_{j<m}\left(b_{j}\right]\right)=\bigcap_{i<n}\left(\left(a_{i}\right] \rightarrow \bigcup_{j<m}\left(b_{j}\right]\right)=\bigcap_{i<n} \bigcup_{j<m}\left(a_{i} \rightarrow b_{j}\right]$.
This completes the proof of our claim about $C$. Next, we show that $\alpha[A]=\operatorname{Reg}(\boldsymbol{C})$. By (5.2), if $a \in A$, then $\neg \neg(a]=(a]$, hence $\alpha[A] \subseteq \operatorname{Reg}(\boldsymbol{C})$. Moreover, if $a_{0}, \ldots, a_{n-1} \in A$ for $n>0$, are such that $\bigcup_{i<n}\left(a_{i}\right] \in \operatorname{Reg}(\boldsymbol{C})$, then by (5.2), we have that $\bigcup_{i<n}\left(a_{i}\right]=\neg \neg \bigcup_{i<n}\left(a_{i}\right]=\left(\bigvee_{i<n} a_{i}\right] \in \alpha[A]$.

Therefore, the map $\alpha: A \rightarrow \operatorname{Reg}(\boldsymbol{C})$ is one to one and onto. In fact, $\alpha$ is an isomorphism between $\boldsymbol{A}$ and $\boldsymbol{\operatorname { R e g }} \boldsymbol{\boldsymbol { C }} \boldsymbol{C})$ because for every $a, b \in A$, we have

- $\alpha(0)=(0]=\{0\}$ and $\alpha(1)=(1]=A$,
- $\alpha(a \wedge b)=(a \wedge b]=(a] \cap(b]$,
- $\alpha(a \vee b)=(a \vee b]=\neg \neg((a] \cup(b])=(a] \cup_{r}(b]$,
- $\alpha(a * b)=(a * b]=\neg \neg(a * b]=\neg \neg((a] *(b])=(a] *_{r}(b]$,
- $\alpha(a \rightarrow b)=(a \rightarrow b]=(a] \rightarrow(b]$.

Finally, since $R(\mathbb{D} \mathbb{R} \mathbb{L})$ is closed under isomorphic images, we conclude that $\boldsymbol{A} \in R(\mathbb{D} \mathbb{R L})$.

Corollary 5.2. For each $\boldsymbol{A} \in \mathbb{R} \mathbb{L}$, there is $\boldsymbol{B} \in \mathbb{D} \mathbb{R} \mathbb{L}$ such that, up to isomorphism, $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})=\boldsymbol{\operatorname { R e g }}(\boldsymbol{B})$.

Observe that $R(\mathbb{D} \mathbb{R} \mathbb{L}) \nsubseteq \mathbb{D} \mathbb{R L}$ implies that it is not possible to give an axiomatization of $\mathbb{D} \mathbb{R} \mathbb{L}$ with equations obtained from terms in $\widehat{T}(X)$.

Remark 5.3. If $\boldsymbol{A}$ and $\boldsymbol{C}$ are as in the proof of Theorem 5.1, we have the following:

- $\boldsymbol{C} \models \neg \neg(\neg \neg x * \neg \neg y) \approx \neg \neg x * \neg \neg y$, that is, $\operatorname{Reg}(\boldsymbol{C})$ is closed under $*$.
- $\boldsymbol{C}$ satisfies Glivenko's identity if and only if $\boldsymbol{A} \in \mathbb{M} T \mathbb{L}$. Indeed, it may be easily checked that for any $a_{0}, \ldots, a_{n-1} \in A$ for $n>0$, the following relation holds:

$$
\neg \neg\left(\neg \neg \bigcup_{i<n}\left(a_{i}\right] \rightarrow \bigcup_{i<n}\left(a_{i}\right]\right)=\left(\bigvee_{j<n} \bigwedge_{i<n}\left(a_{i} \rightarrow a_{j}\right)\right] .
$$

- $\boldsymbol{C}$ is pseudocomplemented if and only if $\boldsymbol{A}$ is pseudocomplemented, and hence a Boolean algebra (see Remark 3.2). This follows from the fact that for every $a_{0}, \ldots, a_{n-1} \in A$ with $n>0$,

$$
\neg\left(\bigcup_{i<n}\left(a_{i}\right] \wedge \neg \bigcup_{i<n}\left(a_{i}\right]\right)=\left(\bigwedge_{i<n} \neg\left(a_{i} \wedge \bigwedge_{j<n} \neg a_{j}\right)\right] .
$$

## 6. Lattices of regular varieties

Since $\mathbb{R L}$ is a congruence distributive variety, its non-trivial subvarieties, ordered by inclusion, constitute a complete distributive lattice $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{R L})$ whose least element is the variety $\mathbb{B}$ of Boolean algebras and whose greatest element is the whole class $\mathbb{R L}$. Moreover, the collection of all non-trivial subvarieties of $\mathbb{I R} \mathbb{L}$ also form a complete distributive lattice $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{I} \mathbb{R} \mathbb{L})$. In fact, $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{T} \mathbb{R} \mathbb{L})$ is a complete sublattice of $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{R} \mathbb{L})$, because for any family $\left(\mathbb{V}_{i}\right)_{i \in I}$ of varieties in $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{\mathbb { R }} \mathbb{L})$ and any $\mathbb{W}$ in $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{R} \mathbb{L})$ with $\bigcup_{i \in I} \mathbb{V}_{i} \subseteq \mathbb{W}$, we have $\bigcup_{i \in I} \mathbb{V}_{i} \subseteq \mathbb{I} \mathbb{W}$.

Using (1) of Lemma 3.7, it is easy to check that for any family of non-trivial regular subvarieties $\left(\mathbb{V}_{i}\right)_{i \in I}$ of $\mathbb{R} \mathbb{L}$, we have

$$
R\left(\bigcap_{i \in I} \mathbb{V}_{i}\right) \subseteq \bigcap_{i \in I} \mathbb{V}_{i} \text { and } R\left(\bigvee_{i \in I} \mathbb{V}_{i}\right) \subseteq \bigvee_{i \in I} \mathbb{V}_{i}
$$

Hence, non-trivial regular subvarieties of $\mathbb{R} \mathbb{L}$, ordered by inclusion, form a complete sublattice of $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{R} \mathbb{L})$, denoted by $\boldsymbol{L}^{\boldsymbol{r} \boldsymbol{v}}(\mathbb{R} \mathbb{L})$.

If $\mathbb{V}$ is a subvariety of $\mathbb{R} \mathbb{L}$, we define $\widetilde{\mathbb{V}}=\{\boldsymbol{A} \in \mathbb{R} \mathbb{L}: \boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A}) \in \mathbb{V}\}$. Thus, we have the following:

Lemma 6.1. If $\mathbb{V}$ is a variety of involutive residuated lattices, then:
(a) $\boldsymbol{A} \in \widetilde{\mathbb{V}}$ if and only if $\boldsymbol{A} \models \widetilde{s} \approx \widetilde{t}$ for each equation $s \approx t$ valid in $\mathbb{V}$, and consequently, $\widetilde{\mathbb{V}}$ is a variety;
(b) $R(\widetilde{\mathbb{V}})=\mathbb{V}$.

Proof. (a): This follows from Corollary 4.2.
(b): If $\boldsymbol{A} \in \mathbb{V}$, then $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})=\boldsymbol{A}$. Thus, $\boldsymbol{A} \in \widetilde{\mathbb{V}}$, and so $\boldsymbol{A} \in R(\widetilde{\mathbb{V}})$. The converse is trivial.

Recall that the variety $\mathbb{B}$ of Boolean algebras is the variety of residuated lattices given by the equation $x \vee \neg x \approx 1$. Thus, for each $\boldsymbol{A} \in \widetilde{\mathbb{B}}$, we have $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A}) \models x \vee \neg x \approx 1$, so $\boldsymbol{A} \models \neg \neg(\neg \neg x \vee \neg x) \approx 1$. We have $\boldsymbol{A} \models x \wedge \neg x \approx 0$ since $\mathbb{R L} \vDash x \wedge \neg x \leqslant \neg(\neg \neg x \vee \neg x)$; hence, $\boldsymbol{A}$ is a pseudocomplemented residuated lattice. Taking into account Remark 3.2, we conclude that $\widetilde{\mathbb{B}}=\mathbb{P R} \mathbb{L}$ (cf. [4, Theorem 1.3]).

Theorem 6.2. If $\mathbb{V}$ is a subvariety of $\mathbb{R} \mathbb{L}$ and $\mathbb{W}$ is a subvariety of $\mathbb{R} \mathbb{L}$, then the following are equivalent:
(1) $\mathbb{W}$ is regular and $\mathbb{I W}=\mathbb{V}$,
(2) $\mathbb{V} \subseteq \mathbb{W} \subseteq \widetilde{\mathbb{V}}$.

Proof. If $\mathbb{W}$ is regular and $\mathbb{I W}=\mathbb{V}$, then $R(\mathbb{W})=\mathbb{V}$, and by definition, $\mathbb{W} \subseteq \widetilde{\mathbb{V}}$. Conversely, if $\mathbb{V} \subseteq \mathbb{W} \subseteq \widetilde{\mathbb{V}}$, then $R(\mathbb{V}) \subseteq R(\mathbb{W}) \subseteq R(\widetilde{\mathbb{V}})$. Since $R(\mathbb{V})=R(\widetilde{\mathbb{V}})=\mathbb{V}$, we have that $R(\mathbb{W})=\mathbb{V} \subseteq \mathbb{W}$. Thus, $\mathbb{W}$ is regular and $\mathbb{I W}=R(\mathbb{W})=\mathbb{V}$.

Observe that, given a non-trivial variety $\mathbb{V} \in \mathbb{R} \mathbb{R} \mathbb{L}$,

$$
[\mathbb{V}, \widetilde{\mathbb{V}}]=\left\{\mathbb{W} \in \boldsymbol{L}^{\boldsymbol{v}}(\mathbb{R} \mathbb{L}): \mathbb{V} \subseteq \mathbb{W} \subseteq \widetilde{\mathbb{V}}\right\}=\left\{\mathbb{W} \in \boldsymbol{L}^{\boldsymbol{r} \boldsymbol{v}}(\mathbb{R} \mathbb{L}): R(\mathbb{W})=\mathbb{V}\right\}
$$

that is, $[\mathbb{V}, \widetilde{\mathbb{V}}]$ is the family of all regular varieties $\mathbb{W}$ such that $R(\mathbb{W})=\mathbb{V}$. Then $[\mathbb{V}, \widetilde{\mathbb{V}}]$, ordered by inclusion, is a complete distributive sublattice of $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{R} \mathbb{L})$.

Consider $\widetilde{\boldsymbol{L}^{\boldsymbol{v}}}(\mathbb{R} \mathbb{L})=\left\{\widetilde{\mathbb{V}}: \mathbb{V} \in \boldsymbol{L}^{\boldsymbol{v}}(\mathbb{\mathbb { R }} \mathbb{L})\right\}$. It follows from the above that the correspondence $\mathbb{V} \mapsto \widetilde{\mathbb{V}}$ defines an order isomorphism from $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{\mathbb { R }} \mathbb{L})$ onto $\widetilde{\boldsymbol{L}^{\boldsymbol{v}}}(\mathbb{R} \mathbb{R} \mathbb{L})$, both ordered by inclusion. Therefore, $\widetilde{\boldsymbol{L}^{\boldsymbol{v}}}(\mathbb{\mathbb { R } \mathbb { L } )}$ is a complete distributive lattice. Note also that by item (a) of Lemma 6.1, we have that a variety $\mathbb{W}$ of residuated lattices belongs to $\widetilde{\boldsymbol{L}^{\boldsymbol{v}}}(\mathbb{I} \mathbb{R} \mathbb{L})$ if and only if $\mathbb{W}$ can be axiomatized by means of equations of the form $\widetilde{t} \approx \widetilde{s}$. Moreover, $\widetilde{\boldsymbol{L}^{\boldsymbol{v}}}(\mathbb{R} \mathbb{R})$ is cofinal in $\boldsymbol{L}^{\boldsymbol{v}}(\mathbb{R} \mathbb{L})$, because for any variety $\mathbb{V}$ of residuated lattices, $\mathbb{V} \subseteq \widetilde{V R(\mathbb{V})} \in \widetilde{\boldsymbol{L}^{\boldsymbol{v}}}(\mathbb{R} \mathbb{R} \mathbb{L})$.

Let $\mathbb{V}$ be a subvariety of $\mathbb{I R} \mathbb{L}$, and let $\mathbb{W}$ be a variety of residuated lattices. If $\mathbb{W}$ is a Glivenko variety such that $\mathbb{I W}=\mathbb{V}$, then since $\mathbb{W}$ is regular, we have
that $\mathbb{V} \subseteq \mathbb{W} \subseteq \widetilde{\mathbb{V}}$, and so $\mathbb{W} \subseteq \mathbb{G} \cap \widetilde{\mathbb{V}}$. Conversely, if $\mathbb{V} \subseteq \mathbb{W} \subseteq \mathbb{G} \cap \widetilde{\mathbb{V}}$, then $\mathbb{I W}=\mathbb{V}$, and $\mathbb{W}$ is a Glivenko variety. Hence we have the following:

Corollary 6.3. If $\mathbb{V}$ is a subvariety of $\mathbb{R} \mathbb{R}$ and $\mathbb{W}$ is a subvariety of $\mathbb{R L}$, then the following are equivalent:
(1) $\mathbb{W}$ is Glivenko and $\mathbb{I W}=\mathbb{V}$,
(2) $\mathbb{V} \subseteq \mathbb{W} \subseteq \mathbb{G} \cap \widetilde{\mathbb{V}}$.

Hence, for any subvariety $\mathbb{V}$ of $\mathbb{R} \mathbb{R}$,

$$
[\mathbb{V}, \mathbb{G} \cap \widetilde{\mathbb{V}}]=\left\{\mathbb{W} \in \boldsymbol{L}^{v}(\mathbb{R} \mathbb{L}): \mathbb{V} \subseteq \mathbb{W} \subseteq \mathbb{G} \cap \widetilde{\mathbb{V}}\right\}
$$

is the family of all Glivenko varieties $\mathbb{W}$ such that $\mathbb{I W}=\mathbb{V}$.

## 7. The Kolmogorov and Glivenko properties: logical interpretations

In this section, we will explore the logical implications of the algebraic results presented so far. First, we will see that the connection between a variety $\mathbb{V}$ and the associated class $R(\mathbb{V})$ can be expressed in terms of the respective equational consequence relations; this will be called Kolmogorov translation property (see [8, Section 8.6]). We will also study the corresponding notion for axiomatic extensions of $\mathbf{F} \mathbf{L}_{\text {ew }}$, the Full Lambek Calculus with exchange and weakening, whose equivalent algebraic semantics is the variety $\mathbb{R L}$.

We shall use the following abbreviations:

- if $\Sigma$ is a set of $\{\wedge, \vee, *, \rightarrow, 0,1\}$-terms, then $\widetilde{\Sigma}=\{\tilde{t}: t \in \Sigma\}$,
- if $E$ is a set of $\{\wedge, \vee, *, \rightarrow, 0,1\}$-equations, then $\widetilde{E}=\{\widetilde{t} \approx \widetilde{s}: t \approx s \in E\}$.

Given varieties of residuated lattices $\mathbb{V}$ and $\mathbb{W}$, we say that the Kolmogorov translation property holds for $\mathbb{V}$ relative to $\mathbb{W}$ if for every set of equations $(\{\wedge, \vee, *, \rightarrow, 0,1\}$-equations) $E \cup\{t \approx s\}$, we have:

$$
\begin{equation*}
E \models_{\mathbb{W}} t \approx s \text { if and only if } \widetilde{E} \models_{\mathbb{V}} \widetilde{t} \approx \widetilde{s}, \tag{7.1}
\end{equation*}
$$

where $\models_{\mathbb{V}}$ and $\models_{\mathbb{W}}$ are the equational consequence relations determined by $\mathbb{V}$ and $\mathbb{W}$, respectively.

Theorem 7.1. Let $\mathbb{V}$ and $\mathbb{W}$ be varieties of residuated lattices. Then the Kolmogorov translation property holds for $\mathbb{V}$ relative to $\mathbb{W}$ if and only if $\mathbb{W}=$ $S R(\mathbb{V})$.

Proof. Observe that if the Kolmogorov translation property holds for $\mathbb{V}$ relative to $\mathbb{W}$, then, for any terms $t$ and $s$,

$$
\mathbb{W} \models t \approx s \text { iff } \models_{\mathbb{W}} t \approx s \text { iff } \models_{\mathbb{V}} \widetilde{t} \approx \widetilde{s} \text { iff } \mathbb{V} \models \widetilde{t} \approx \widetilde{s}
$$

Then $\mathbb{W}$ is the variety given by the set of equations $\{t \approx s: \mathbb{V} \models \widetilde{t} \approx \widetilde{s}\}$, and so by Corollary 4.2 and Lemma 3.7, $\mathbb{W}=S R(\mathbb{V})$.

To see that the Kolmogorov translation property holds for $\mathbb{V}$ relative to $S R(\mathbb{V})$, it is enough to check the property in (7.1) for a finite set $E$ of equations, i.e., $E=\left\{t_{1} \approx s_{1}, \ldots, t_{k} \approx s_{k}\right\}$, because, since $\mathbb{V}$ and $S R(\mathbb{V})$ are varieties,
the consequence relations $\models_{\mathbb{V}}$ and $\models_{S R(\mathbb{V})}$ are finitary (e.g., see [7, Chapter Q] and the references given here).

Suppose that $E=_{S R(\mathbb{V})} t \approx s$, and assume that all variables appearing in $E \cup\{t \approx s\}$ belong to $\left\{x_{1}, \ldots, x_{n}\right\}$. Let $\boldsymbol{A} \in \mathbb{V}$ and $a_{1}, \ldots, a_{n} \in A$ be such that $\widetilde{t}_{i}^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right)=\widetilde{s}_{i}^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right)$ for $1 \leqslant i \leqslant k$; then by Lemma 4.1, we have that for $1 \leqslant i \leqslant k, t_{i}^{\boldsymbol{\operatorname { R e g } ( \boldsymbol { A } )}}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right)=s_{i}^{\boldsymbol{\operatorname { R e g } ( \boldsymbol { A } )}}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right)$. Hence, by the assumption and Lemma 4.1 again, we obtain

$$
\begin{aligned}
\widetilde{t}^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right) & =t^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right) \\
& =s^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right)=\widetilde{s}^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right) .
\end{aligned}
$$

The arbitrariness in the choice of $\boldsymbol{A}$ and $a_{1}, \ldots, a_{n}$ shows that $\widetilde{E} \models_{\mathrm{V}} \widetilde{t} \approx \widetilde{s}$.
Conversely, assume that $\widetilde{E} \models_{\mathbb{V}} \widetilde{t} \approx \widetilde{s}$. Let $\boldsymbol{A} \in S R(\mathbb{V})$ and $a_{1}, \ldots, a_{n} \in A$ such that $t_{i}^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right)=s_{i}^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right)$ for $1 \leqslant i \leqslant k$. Consider $\boldsymbol{B} \in \mathbb{V}$ such that $\boldsymbol{A} \subseteq \boldsymbol{\operatorname { R e g }}(\boldsymbol{B})$. Then $a_{1}, \ldots, a_{n} \in \operatorname{Reg}(\boldsymbol{B})$, and for $1 \leqslant i \leqslant k$, we have

$$
\begin{aligned}
{\widetilde{t_{i}}}^{\boldsymbol{B}}\left(a_{1}, \ldots, a_{n}\right) & =t_{i}^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{B})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right)=t_{i}^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{B})}\left(a_{1}, \ldots, a_{n}\right) \\
& =s_{i}^{\boldsymbol{\operatorname { R e g } ( \boldsymbol { B } )}}\left(a_{1}, \ldots, a_{n}\right)=s_{i}^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{B})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right) \\
& ={\widetilde{s_{i}}}^{\boldsymbol{B}}\left(a_{1}, \ldots, a_{n}\right),
\end{aligned}
$$

Thus, by the assumption, $\widetilde{t}^{\boldsymbol{B}}\left(a_{1}, \ldots, a_{n}\right)=\widetilde{s}^{\boldsymbol{B}}\left(a_{1}, \ldots, a_{n}\right)$, and since we have $a_{1}, \ldots, a_{n} \in A \subseteq \operatorname{Reg}(\boldsymbol{B})$, we also have

$$
\begin{aligned}
t^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right) & =t^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{B})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right) \\
& =s^{\boldsymbol{\operatorname { R e g }}(\boldsymbol{B})}\left(\neg \neg a_{1}, \ldots, \neg \neg a_{n}\right)=s^{\boldsymbol{A}}\left(a_{1}, \ldots, a_{n}\right) .
\end{aligned}
$$

Therefore, $E \models_{S R(\mathbb{V})} t \approx s$.
By Corollary 3.9, in general, $S R(\mathbb{V})$ is not contained in $\mathbb{V}$, and thus it cannot coincide with $\mathbb{I V}$, hence there is a mistake in [8, line 11, page 373], which renders Theorems 8.43 and 8.44 not true.

We say simply that the Kolmogorov translation property holds in $\mathbb{V}$ when the Kolmogorov translation property holds for $\mathbb{V}$ relative to $\mathbb{I V}$; by Theorem 7.1 and Lemma 3.7, this is equivalent to $R(\mathbb{V}) \subseteq \mathbb{V}$, that is, $\mathbb{V}$ is regular. Then, taking into account Remark 2.3 and using Theorem 4.5, Corollary 4.6, and Theorem 7.1, we obtain the following result.

Theorem 7.2. For each variety $\mathbb{V}$ of residuated lattices, the following are equivalent:
(1) the Kolmogorov translation property holds in $\mathbb{V}$,
(2) $\mathbb{V}$ is regular,
(3) $\boldsymbol{S}_{\mathbb{V}}^{r}(\neg \neg \bar{X})$ is a homomorphic image of $\boldsymbol{F}_{\mathbb{V}}(\bar{X})$,
(4) for any term $t, \mathbb{V} \models t \approx 1$ implies $\mathbb{V} \models \widetilde{t} \approx 1$.

We recall that $\mathbb{V}$ has the Glivenko property provided that for any set of equations $E \cup\{s \approx t\}$, we have $E \models_{\mathbb{V}} s \approx t$ if and only if $E \models_{\mathbb{V}} \neg \neg s \approx \neg \neg t$, or equivalently, setting $\neg \neg E=\{\neg \neg t \approx \neg \neg s: t \approx s \in E\}$, we have $E \models_{\mathbb{I V}} s \approx t$ if
and only if $\neg \neg E \models_{\mathbb{V}} \neg \neg s \approx \neg \neg t$. Then from the results given in [5] (see also [8]), we deduce the following:

Theorem 7.3. A variety $\mathbb{V}$ of residuated lattices has the Glivenko property if and only if it is a Glivenko variety.

Then since Glivenko varieties are regular, we deduce that the Glivenko property implies the Kolmogorov property:

Corollary 7.4. If $\mathbb{V}$ has the Glivenko property, then the Kolmogorov translation property holds in $\mathbb{V}$.

As is done in [5] for the Glivenko property, the Kolmogorov translation property has a logical version. The reason for this is the fact that residuated lattices in our sense, i.e., bounded commutative integral residuated lattices, are the algebraic counterpart of $\mathbf{F} \mathbf{L}_{\text {ew }}$, the Full Lambek Calculus with exchange and weakening. In fact, the variety $\mathbb{R L}$ is the equivalent algebraic semantics of $\mathbf{F L} \mathbf{L}_{\text {ew }}$ in the sense of Blok and Pigozzi (see [1]). More precisely, for every set of formulas (terms) $\Sigma \cup\{\varphi, \psi\}$, the following hold.
(al1): $\Sigma \vdash_{\mathbf{F L}_{\mathrm{ew}}} \varphi$ if and only if $\{\gamma \approx 1: \gamma \in \Sigma\} \models_{\mathbb{R L L}} \varphi \approx 1$.
(al2): $\varphi \approx \psi=\models_{\mathbb{R L}}(\varphi \rightarrow \psi) *(\psi \rightarrow \varphi) \approx 1$.
Or equivalently, for every set of equations $E$ and for any terms $\varphi, \psi$ :
(al3): $E \models_{\mathbb{R L}} \varphi \approx \psi$ if and only if

$$
\{(\gamma \rightarrow \xi) *(\xi \rightarrow \gamma): \gamma \approx \xi \in E\} \vdash_{\mathbf{F L}_{\mathrm{ew}}}(\varphi \rightarrow \psi) *(\psi \rightarrow \varphi)
$$

(al4): $\varphi \vdash_{\mathbf{F L}_{\text {ew }}}(\varphi \rightarrow 1) *(1 \rightarrow \varphi)$.
It follows, see [1], that any axiomatic extension $\mathbf{L}$ of $\mathbf{F L}_{\mathrm{ew}}$ is also algebraizable, and its equivalent algebraic semantics is the following subvariety of $\mathbb{R} \mathbb{L}$ :
$\mathbb{V}_{\mathbf{L}}=\left\{\boldsymbol{A} \in \mathbb{R} \mathbb{L}: \boldsymbol{A} \models \varphi \approx 1\right.$, for every formula $\varphi$ such that $\left.\vdash_{\mathbf{L}} \varphi\right\}$.
In other words, (al1), (al2), (al3), and (al4) hold if $\mathbf{F L}_{\text {ew }}$ and $\mathbb{R} \mathbb{L}$ are replaced by $\mathbf{L}$ and $\mathbb{V}_{\mathbf{L}}$, respectively. This correspondence is one to one and onto; indeed, any subvariety $\mathbb{V}$ of $\mathbb{R L}$ is the equivalent algebraic semantics of the axiomatic extension $\mathbf{L}_{\mathbb{V}}$ given by $\vdash_{\mathbf{L}_{\mathbb{V}}} \varphi$ if and only if $\mathbb{V} \models \varphi \approx 1$. Moreover, $\mathbf{L}_{\mathbb{V}_{\mathbf{L}}}=\mathbf{L}$ and $\mathbb{V}_{\mathbf{L}_{\mathbb{V}}}=\mathbb{V}$.

From these correspondences we can translate the results proven in the previous sections to the axiomatic extensions of $\mathbf{F L} \mathbf{L}_{\text {ew }}$. For any axiomatic extension $\mathbf{L}$ of $\mathbf{F} \mathbf{L}_{\text {ew }}$, denote by $\mathbf{I n v L}$ the axiomatic extension of $\mathbf{L}$ resulting after adding the axioms $\neg \neg \varphi \rightarrow \varphi$ for every formula $\varphi$. Thus, $\mathbb{V}_{\text {InvL }}=\mathbb{I}\left(\mathbb{V}_{\mathbf{L}}\right)$.

Following in part the nomenclature used in [8], given two axiomatic extensions of $\mathbf{F} \mathbf{L}_{\text {ew }}, \mathbf{L}$ and $\mathbf{K}$, we say that the Kolmogorov translation property holds for $\mathbf{L}$ relative to $\mathbf{K}$ if, for every set of formulas (terms) $\Sigma \cup\{\varphi\}$,

$$
\Sigma \vdash_{\mathbf{K}} \varphi \text { if and only if } \widetilde{\Sigma} \vdash_{\mathbf{L}} \widetilde{\varphi}
$$

Now, from (al1)-(al4) and Theorem 7.1, we deduce the following result.

Theorem 7.5. Let $\mathbf{L}$ and $\mathbf{K}$ be axiomatic extensions of $\mathbf{F L}_{\text {ew }}$. Then the following conditions are equivalent:
(1) the Kolmogorov translation property holds for $\mathbf{L}$ relative to $\mathbf{K}$,
(2) the Kolmogorov translation property holds for $\mathbb{V}_{\mathbf{L}}$ relative to $\mathbb{V}_{\mathbf{K}}$,
(3) $\mathbf{K}=\mathbf{L}_{S R\left(\mathbb{V}_{\mathbf{L}}\right)}$, i.e. $\mathbb{V}_{\mathbf{K}}=S R\left(\mathbb{V}_{\mathbf{L}}\right)$.

Proof. (2) iff (3): This follows since $\mathbf{K}=\mathbf{L}_{S R\left(\mathbb{V}_{\mathbf{L}}\right)}$ if and only if $\mathbb{V}_{\mathbf{K}}=S R\left(\mathbb{V}_{\mathbf{L}}\right)$. (2) implies (1): We have

$$
\begin{aligned}
& \Sigma \vdash_{\mathbf{K}} \varphi \text { iff }\{\psi \approx 1: \psi \in \Sigma\} \not \models_{\mathbb{V}_{\mathbf{K}}} \varphi \approx 1 \text { iff }\{\widetilde{\psi} \approx 1: \psi \in \Sigma\} \models_{\mathbb{V}_{\mathbf{L}}} \widetilde{\varphi} \approx 1 \\
& \text { iff }\{\gamma \approx 1: \gamma \in \widetilde{\Sigma}\} \models_{\mathbb{V}_{\mathbf{L}}} \widetilde{\varphi} \approx 1 \text { iff } \widetilde{\Sigma} \vdash_{\mathbf{L}} \widetilde{\varphi} .
\end{aligned}
$$

(1) implies (3): Using Theorem 7.1, for any formula $\varphi$, we obtain

$$
\begin{gathered}
\mathbb{V}_{\mathbf{K}} \models \varphi \approx 1 \text { iff } \models_{\mathbb{V}_{\mathbf{K}}} \varphi \approx 1 \text { iff } \vdash_{\mathbf{K}} \varphi \text { iff } \vdash_{\mathbf{L}} \widetilde{\varphi} \text { iff } \models_{\mathbb{V}_{\mathbf{L}}} \widetilde{\varphi} \approx 1 \\
\text { iff } \models_{S R\left(\mathbb{V}_{\mathbf{L}}\right)} \varphi \approx 1 \text { iff } S R\left(\mathbb{V}_{\mathbf{L}}\right) \models \varphi \approx 1 .
\end{gathered}
$$

Remark 7.6. The logic $\mathbf{L}_{S R\left(\mathbb{V}_{\mathbf{L}}\right)}$ can be characterized directly in terms of $\mathbf{L}$. In fact, $\mathbf{L}_{S R\left(\mathbb{V}_{\mathbf{L}}\right)}$ is the axiomatic extension of $\mathbf{F} \mathbf{L}_{\text {ew }}$ by the axioms $\left\{\varphi: \vdash_{\mathbf{L}} \widetilde{\varphi}\right\}$.

If $\mathbf{L}$ is an axiomatic extension of $\mathbf{F} \mathbf{L}_{\text {ew }}$, we say that the Kolmogorov translation property holds for $\mathbf{L}$ provided that for every set of formulas $\Sigma \cup\{\varphi\}$,

$$
\Sigma \vdash_{\operatorname{InvL}} \varphi \text { if and only if } \widetilde{\Sigma} \vdash_{\mathbf{L}} \widetilde{\varphi}
$$

Thus, from (al1)-(al4) and Theorem 7.2, we deduce the following:
Theorem 7.7. Let $\mathbf{L}$ be an axiomatic extension of $\mathbf{F} \mathbf{L}_{\mathrm{ew}}$, then the following are equivalent:
(1) the Kolmogorov translation property holds for $\mathbf{L}$,
(2) the Kolmogorov translation property holds for $\mathbb{V}_{\mathbf{L}}$,
(3) for any formula $\varphi, \vdash_{\mathbf{L}} \widetilde{\varphi}$ implies $\vdash_{\mathbf{I n v L}} \varphi$.

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