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Abduction: A categorical characterization

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ABSTRACT

Scientific knowledge is gained by the informed (on the basis of *theoretic* ideas and criteria) examination of data. This can be easily seen in the context of quantitative data, handled with statistical methods. Here we are interested in other forms of data analysis, although with the same goal of extracting meaningful information. The idea is that data should guide the construction of suitable models, which later may lead to the development of new theories. This kind of inference is called *abduction* and constitutes a central procedure called *Peircean qualitative induction*. In this paper we will present a category-theoretic representation of abduction based on the notion of *adjunction*, which highlights the fundamental fact that an abduction is the most efficient way of capturing the information obtained from a large body of evidence.

1. Introduction

A possible classification of scientific activities focuses on the kinds of inference mechanisms applied to gain further knowledge. Pure theoretical branches usually use mathematics and therefore *deductive* inference[1] while more empirically oriented ones use statistical (i.e. *inductive*) inference disguised in various forms. The question to be raised by a logician in the Peircean tradition is how to accommodate the third type of inference, *qualitative induction* or *abduction*.[2] To answer this question, let us note first that the key procedure in this kind of induction (which should be clearly distinguished from statistical induction) is the inference from *evidence* to *explanation*. That is, abduction does not predict which evidence should be observed given a theory—a deductive inference—nor does it build a general description (*prototype*) of the evidence—a statistical inference. Rather, in scientific matters, abduction is the reasoning process that helps

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[1] On the other hand, the burden of mathematical activity lies in conjecturing possible results or arguments to prove statements. This is also an abductive reasoning, of a kind that will not be discussed here.

[2] For a full characterization of this type of induction see [30].

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to find theoretical constructs providing plausible explanations for how the data drawn from the real world were in fact generated.

In the rest of this paper we will discuss the role of abductive reasoning and extend it to a more general line of inquiry: to find qualitative descriptions of the information found in crude data. Section 2 is devoted to presenting a discussion on the meaning of abduction. Section 3 presents a category-theoretical environment for abduction, seen as a functor among "data" and "theoretical" categories. Sections 4 and 5 characterize in two different, albeit related, ways the abduction functor, one in terms of an *a priori* selection of potential "theoretical" outcomes and the other on constraints on the admissible selections *given* the "data" inputs. Finally, Section 6 discusses the conclusions of this work and presents possible lines for further inquiry.

2. Abductive reasoning and data analysis

Peirce emphasized the importance of Kepler's example for understanding how abduction works. The German astronomer, working with the huge database of planetary observations collected by Tycho Brahe, used his knowledge of geometry to conjecture that the planets follow elliptic paths around the Sun [26]. Without this insightful result, Newtonian physics would not have been possible. Similarly, a great deal of scientific theorizing arises from the insights provided by the examination of data. Any scientist, faced with some data, always tries to detect a pattern. In our terms, she tries to perform an informal abductive inference.

One of the relevant contexts in which abduction could be applied is analogous to Kepler's example. Although it sounds rather obvious, let us emphasize that there is a gap between the formulation of a question to be answered through measurement and the *actual* measurement providing the right answer. This difference arises from the fact that problems are usually stated in qualitative terms while data can be quantitative. In consequence, rough data (which certainly includes the quantitative counterparts of qualitative concepts) must be organized according to the qualitative structure to be tested. That is, a correspondence between theory and data must be sought. So, for example, in many socially oriented disciplines there exists a crucial distinction between ordinal and cardinal magnitudes in the characterization of *preferences*. But once measurements are involved it is clear that the theoretical relational structure must be assumed to be homomorphic to a numerical structure [19].

This implies that if there exists a database of numerical observations about the behavior of a phenomenon or a system, we might want to infer the properties of the qualitative relational structure to which the given numerical structure is homeomorphic. Of course, this is impaired by many factors:

- The representation of the qualitative structure may not have a unambiguous syntactic characterization [6].
- Heterogeneity in the representation hampers the unification of data sources.[3]
- Even if the observations fall in a numerical scale, the real world is too noisy to ensure a neat description of phenomena under consideration.
- There are complexity issues that make it highly convenient to just look for approximations, instead of a characterization that may make sense of every detail.

These factors, which usually preclude a clear cut characterization of the observations, leave ample room for arbitrary differences. In this sense, the intuition and experience of the analyst determine the limits of arbitrariness. Yet the reasoning process that justifies the decisions actually made is not often made clear. More generally, since empirical scientists spend a great deal of their time looking for relations hidden in the data, the process they apply to uncover those relations cries out for clarification.

[3] On the heterogeneity of data see [38]. See also the discussions on the convenience of having *iconic* representations in [8].

As an example, consider the question "Did the Argentine economy grow in the last year?" To provide an answer, first, one has to define clearly what it means for an economy to grow and which variables can be used to measure the phenomenon of growth. Economic theory states that economic growth means growth of the national income. But in order to answer the question an economist has to define what real world data will represent national income; i.e. she has to embed the available data into the framework given by the theory. In this case the national product is an available variable which is easy to measure and is considered (theoretically) equivalent to the national income. Therefore it is easy to check out whether the economy grew or not. But in the case where the question is something like "Did poverty increase in the last twenty years?" the procedure is far less straightforward. How do we define poverty and moreover, how do we make the concept operational? This is where intuition is called in. Although theoretical concepts may be lacking, a set of alternative models of the notion of poverty and its evolution in time should be provided in order to check out which one better fits the real world data. Only when this question is settled it is possible to consider the development of a theory formalizing the properties verified in the chosen model.

The inferences allowing for the detection of patterns in data cannot be reduced just to statistical inductions. They are more a result of a detective-like approach to scarce and unorganized information, where the goal is to get clues out of unorganized data bases of observations and to disclose hidden explanations that would make them meaningful. In other words: it is a matter of making guesses, which later can be put in a deductive framework and tested by statistical procedures. So far, it seems that it is just an "artistic" feat, which can only be performed by experts.[4] This means that some degree of expertise in the area may be useful to perceive patterns in a seemingly unorganized set of numerical values or to choose the way to state a question. Of course, this does not preclude the possibility of looking at the problem at hand in new ways. Heuristic tools like plotting the data, extracting statistics, running simulations or looking at *barcodes* [15] may help to state hypotheses about the features of the process that in fact generated the data [25].

In any case, abduction is fundamental in the process of model building. That is, given a theory to be tested or an informal question to be answered, a *model* has to be built, representing either the intended interpretation of the theory to be tested or, more interestingly, *the intended interpretation of a theory yet to be formulated*. In the process of choosing a model from among a potentially very large number of alternatives is where the ability of the scientist is shown. Although statistics provides tools to calibrate models, these methods are based on pure quantitative considerations without regard to the context of application.

The increasing interest of logicians in abductive reasoning has been catapulted by the requirements of Artificial Intelligence [13]. One of the main goals in AI is to design a full architecture able to perform something like the three kinds of Peircean inference. One of the hardest tasks is, of course, to build an abductive engine. This requires a formalization of the procedure of abductive inference. Although previous attempts to reproduce historical examples have been partially successful (for example several versions of a system called BACON [34]), they have been the result of pure ad-hoc procedures. One of the problems we find in those approaches is that they are based on a fundamental confusion between statistical induction and abduction. More specifically, it is rather evident that any attempt to build theories up from data can only yield very simple theoretical structures, more like prototypes than conceptual frameworks. These structures are sometimes called *phenomenological theories*, and are distinguished from more elaborated structures, which involve non-observable entities, called *representational theories* [2].

What clearly separates abduction from statistical induction is that it requires a previous meta-theoretical commitment. Kepler, for example, was committed to the idea of simplicity and elegance of Nature. More precisely, his idea was that there had to exist an optimal geometrical configuration explaining the data. Then, for years, he tried to fit the data to different geometrical structures until he found that conic curves provided the best match. Then, using the basic observation that the positions of the celestial bodies cycled,

[4] This might be a reason for why formal logicians, until recently, did not intensively study abduction in contrast to the other forms of inference.

he concluded that only a closed conic could explain the data. Therefore, since ellipses fitted better than circles, he concluded his search, postulating his well-known laws of planetary motion.

This historical example was carefully studied by the American polymath Charles Peirce, who drew from it prescriptions of how to perform an abduction. First of all, data had to be structured by means of Peirce's own classifications of signs [27]. From his point of view, every set of data constitutes a sign, which therefore can be classified according to Peirce's exhaustive taxonomy. The advantage of this approach is that there exists only a finite set of possibilities to match with the real world information. Once one of the possibilities is chosen, it is assumed to provide a clear statement of the kind of structure hidden in the data, although not necessarily as complex as a functional form.

The problem with Peirce's approach is that on one hand his classification may seem rather arbitrary and on the other the involved logical system can appear cumbersome and contrived. Despite this, if we take it only as a heuristic guide, it seems promising. The available pieces of information cannot be all put on the same level. In fact, to classify a set of data in terms of the meaningfulness of the information conveyed is very useful in order to construct a testable hypothesis. While this is already a hard task, the remaining chore is still harder: to work on the classified data base, trying to fit it to one of a bundle of possible functional forms.

The mainstream in contemporary approaches to the formalization of abduction processes is summarized by what Gabbay and Woods [14] call the *AKM schema* for abduction, advanced in, among others, Aliseda [3,4], Kakas et al. [18], Kuipers [20] and Magnani [24]. Roughly, given a knowledge base K and a conjecture H about any event E that cannot be deduced from the former, the abduction yields a minimal consistent revision of the knowledge base joint with the conjecture, $K(H)$ such that E can be deduced from this new knowledge base. The *GW model*, advanced by Gabbay and Woods, aims instead to address the failure of K to deduce E with a refurbishing of the deductive apparatus, modifying the consequence relation. In any case, the literature on abduction has boomed in the last years, both along these two great lines of work as well as in other approaches, like the role of extra-theoretical information in abduction discussed by Magnani [25].[5]

In this paper we pursue an alternative formal framework, less dependent on consequence relations, in which to represent the form in which qualitative information is drawn from quantitative data. The key idea is to introduce two categories, one for data and the other for conceptual structures. Abduction is captured by a functor among both categories. Of particular importance in the characterization of this functor are the *criteria* that can be imposed on the abduction. The resulting structure is a *topos* in which the sub-object classifier may yield different truth values corresponding to the degree in which the abduction captures the real world data.[6]

3. A framework for abduction

We will try to make this discussion a bit more formal and develop an approach to qualitative model building. In the first place, we should note that we intend **model** to be understood as in first-order logic[7]:

Definition 1. We define a **structure** Δ as $\Delta = \langle \mathcal{D}, \Pi \rangle$, where $\mathcal{D}$ is the domain while Π is a set of relations among elements of $\mathcal{D}$. Given a first-order language $\mathcal{L}$, Δ becomes an **interpretation** of any consistent set of well formed formulas of $\mathcal{L}$, $\mathcal{T}(\mathcal{L})$, if there exists a correspondence from $\mathcal{T}(\mathcal{L})$ to Δ that obtains from

[5] A less than exhaustive list includes the following: Meheus and Batens [28], D'Agostino et al. [9] and Beirlaen and Aliseda [7] on logical systems for abduction; Schurtz [31] on the different patterns that abduction may exhibit and Woods [39] on the cognitive resources at play in abduction.

[6] This is in line with previous work, like Lamma et al. [21] on the use of multi-valued logic in the process of learning concepts up from data and Li and Pereira [23] on the use of category theory in the description of diagnoses in system behavior.

[7] For a precise characterization of these notions see [32].

assigning constants in $\mathcal{L}$ to elements in $\mathcal{D}$ and function symbols and predicate symbols of the language to Π. A **model** of $\mathcal{T}(\mathcal{L})$ is an interpretation where every interpreted formula is verified by Δ.

A structure can be thought of as a database plus the relations and functions that are, implicitly or explicitly, true in it. In the case of the empirical sciences, it is assumed that the chosen structure Δ is a *representation* of the real world features of interest. In many cases it adopts a quantitative form. That is, it includes any kind of numerical relations among objects in the domain. Of course, other kinds of mathematical relations are admissible. Therefore, Δ may include quantitative as well qualitative features. An interpretation is a structure associated to a certain set of well-formed formulas (when deductively closed this set is called a *theory*). If, when replacing the constants by elements in the interpretation and the predicate symbols by relations in the structure, all the formulas are made true in the interpretation, the structure is called a model. To say that abduction helps in model building means that it is a process that embeds the real-world information in a certain structure that is assumed to be the model of a theory or at least of a coherent part of one.

In many areas of inquiry it is usual to find that there is not a clear distinction between what is meant by "theory" and by "model". One reason is that for most applications, it is excessive to demand a theory, which has to be deductively closed, which means that all its consequences should be immediately available. In usual practice, statements are far from being deduced in a single stroke. On the other hand—and this explains clearly the confusion between theory and model—most scientific theories have an intended meaning more or less clear in its statements. This does not preclude the formulation of general and abstract theories, but their confrontation with data are always mediated by an intended model [35].

One concern that may arise from our approach is related to the limitations given by working within the framework of a first order language. With respect to this observation, we first notice that first-order is sufficient to capture set structures *à la* Zermelo–Fraenkel [11]. Furthermore, in a Peircean vein, the existential graphs (EG), which can represent statements in a graphic way, can be translated into a first-order language, at least for α or β graphs [36].[8] Analogously, other graphical forms of representation like [17] can be translated into a first-order language interpretable in terms of elements and relations among them.

Of course more complex statements (quantifying over predicates) are no longer representable in a first-order language. In this work, however, we care for the translation of the data base of observations in a formal structure satisfying the following (first-order) conditions [22]:

- Each element of interest in the data has a symbolic representation.
- For each (simple) relationship in the data, there must be a connection among the elements in the representation.
- There exist one-to-one correspondences between relationships and connections, and between elements in the data and in the representation.

This representation of the real world information which we denote with Λ, facilitates the abduction by means of its comparison with alternative structures. Notice that a numerical data base may in this respect be taken at face value, that is, the variables and their values already immediately constitute a symbolic representation.

To analyze how to get this abduction we start by considering a first-order language $\mathcal{L}$ and defining two categories, $\mathbf{S}$ and $\mathbf{DR}$ of structures and data representations, respectively:

[8] On the other hand, γ graphs can be translated into a modal framework, which in turn has been shown to be reducible to a first-order one [29].

Definition 2. Given a first-order language $\mathcal{L}$,

- The category **S** has as objects structures Δ, each one a model for some consistent class of statements of $\mathcal{L}$, and morphisms among them, $f : \Delta \to \Delta'$ where:
  - $\Delta = \langle \mathcal{D}, \Pi \rangle$, $\Delta' = \langle \mathcal{D}', \Pi' \rangle$ and f is a one-to-one function from $\mathcal{D}$ to $\mathcal{D}'$
  - for any relation $\pi(d_1, \ldots, d_n) \in \Pi$ of arity n there exists a $\bar{\pi}(f(d_1), \ldots, f(d_n)) \in \Pi'$ of the same arity.
- The category **DR** has objects of the form $\Lambda = \langle O, L, R \rangle$, where O is a class of *rough data*, L is a set of constants from $\mathcal{L}$, one for each element of interest in O, and R is a class of relations over L drawn from the relations among elements of interest in O and morphisms g among them where[9]
  - $g : \Lambda \to \Lambda'$ is a one-to-one function from $L \in \Lambda$ to $L' \in \Lambda'$.
  - For any n-ary relation $r(a_1, \ldots, a_n) \in R$ there exists a $\bar{r}(f(a_1), \ldots, f(a_n)) \in R'$.

We can see that both **S** and **DR** are basically the same categories of structures over $\mathcal{L}$, but the latter's objects include "rough data". That is, observations, pieces of evidence, experimental results, etc., not yet expressed in $\mathcal{L}$. This is in order to distinguish structures according to their origin, theoretical (the objects of **S**) and empirical (those in **DR**). For each object Δ in **S** an associated object Λ can be defined, constructing a set O of potential data pieces that could be translated as Δ. Thus $\Lambda = \langle O, \Delta \rangle$. With this proviso we have that:

Proposition 1. **S** *and* **DR** *are well defined categories.*

Proof. Trivial. In both cases the morphisms are one-to-one functions. Identity is of course one-to-one. And one-to-one functions can be composed and this composition is associative. $\square$

It is natural to define an ordered set $\langle \mathrm{Obj}(\mathbf{S}), \preceq_{\mathbf{S}} \rangle$, where $\mathrm{Obj}(\mathbf{S})$ is the class of objects from **S** such that given $\Delta, \Delta' \in \mathrm{Obj}(\mathbf{S})$, we say that $\Delta \preceq_{\mathbf{S}} \Delta'$ iff there exists a morphism $f : \Delta \to \Delta'$ in **S**. Analogously, we define $\preceq_{\mathbf{DR}}$ over $\mathrm{Obj}(\mathbf{DR})$. We have:

Proposition 2. $\langle \mathit{Obj}(\mathbf{S}), \preceq_{\mathbf{S}} \rangle$ *and* $\langle \mathit{Obj}(\mathbf{DR}), \preceq_{\mathbf{DR}} \rangle$ *are partially ordered sets.*

Proof. Both $\preceq_{\mathbf{S}}$ and $\preceq_{\mathbf{DR}}$ satisfy *reflexivity* and *transitivity*, since they are defined in terms of morphisms in a category and thus there exist identity morphisms for each object and morphisms can be composed. *Antisymmetry* obtains by noticing that since morphisms are one-to-one functions, two objects of these categories that have morphisms back and forth between them, must be identical. $\square$

Furthermore:

Lemma 1. *The categories* **S** *and* **DR** *have products and coproducts as well as initial objects.*

Proof. Consider **S** (the case of **DR** is completely analogous). An object $\Delta^{\emptyset}$, with $\Delta^{\emptyset} = \langle \mathcal{D}^{\emptyset}, \Pi^{\emptyset} \rangle$ and $\mathcal{D}^{\emptyset} = \Pi^{\emptyset} = \emptyset$. Thus, for any other Δ there exists an injection $f : \Delta^{\emptyset} \to \Delta$, which can be identified with the set-inclusion among components. That is, $\Delta^{\emptyset}$ is an initial object in the category. On the other hand, since $\langle \mathrm{Obj}(\mathbf{S}), \preceq_{\mathbf{S}} \rangle$ is a poset, the min and max over objects are defined as the *product* and *coproduct*, respectively. For instance, in the case of two objects Δ_1 and Δ_2, $\Delta_1 \times \Delta_2 = \min(\Delta_1, \Delta_2)$ is such that there exists morphisms $p_i : \Delta_1 \times \Delta_2 \to \Delta_i$ $(i = 1, 2)$, i.e. $\min(\Delta_1, \Delta_2) \preceq_{\mathbf{S}} \Delta_i$ and for every other object Δ and morphisms $f_i : \Delta \to \Delta_i$ (or $\Delta \preceq_{\mathbf{S}} \Delta_i$) there exists a unique morphism $f : \Delta \to \Delta_1 \times \Delta_2$ or $\Delta \preceq_{\mathbf{S}} \min(\Delta_1, \Delta_2)$. The characterization of $\Delta_1 + \Delta_2$ as $\max(\Delta_1, \Delta_2)$ is obtained in the same form. $\square$

[9] Notice that O is immaterial. g captures the structural relations between Λ and Λ', seen as *reduced models* [33].

4. The abduction functor: *a priori* criteria

We can now see how abduction works. An important aspect of this procedure is the way in which additional criteria guide the process. For this, let us assume that these criteria are given *a priori*, i.e. they restrict the possible outcomes without considering the concrete features of the data to which the procedure could be applied. That is:

Definition 3. Given Λ, an object in $\mathbf{DR}$ and a class of structures $\{\Delta\}^{\mathcal{C}} \subseteq \mathrm{Obj}(\mathbf{S})$, selected for verifying a set of criteria $\mathcal{C}$, an abduction chooses one of them, say Δ^*.

In words, given a class of criteria, there might exist several possible structures that may explain the data in Λ. To *abduce* Λ, is to choose one of them.

The abduction procedure can be captured by a functor $\mathcal{A}$ from $\mathbf{DR}$ to $\mathbf{S}$, which given an object Λ from $\mathbf{DR}$ yields a structure $\mathcal{A}(\Lambda)$ in $\mathbf{S}$. A *trivial abduction* would be one yielding $\langle \mathcal{D}, \Pi \rangle = \langle L, R \rangle$, for $\Lambda = \langle O, L, R \rangle$. Much more interesting cases arise when the abduction has to respect some given theoretical criteria.

Before going into the formalism, let us emphasize two elements in the characterization of $\mathcal{A}$. One is the class of *criteria* and what they might be and the other is how a single structure may be selected. With respect to the criteria, notice that in the case of Kepler's abduction he had at least one criterion in mind: trajectories of celestial bodies should be described by simple geometrical expressions. Under this criterion, Kepler had to choose one among a few structures comparing the movements implied by them with the behavior of a given set of real-world elements (the known planets of the solar system). Each of those structures was a simple geometric representation of the solar system. He eventually chose the one that fitted the data best.

In general, the criteria represent all the elements that a scientist may want to find incorporated into the chosen structure. Given the criteria in $\mathcal{C}$, the structures that satisfy them form a class $\{\Delta\}^{\mathcal{C}}$, which in turn leads to a subcategory of $\mathbf{S}$, denoted $\mathbf{S}_{\mathcal{C}}$, where the objects are the structures in $\{\Delta\}^{\mathcal{C}}$ and the morphisms are those of $\mathbf{S}$ restricted to these structures.

Of course, this set of possible structures may be empty, if $\mathcal{C}$ cannot be satisfied by any structure. If this is not the case, we can define an abduction according to the criteria $\mathcal{C}$ as a functor $\mathcal{A}_{\mathcal{C}} : \mathbf{DR} \to \mathbf{S}_{\mathcal{C}}$. Notice that in this setting it may no longer be possible to define $\mathcal{A}_{\mathcal{C}}$ as a trivial abduction. This is because $\mathcal{A}_{\mathcal{C}}(\Lambda) = \langle \mathcal{D}, \Pi \rangle$ must be an object of $\mathbf{S}_{\mathcal{C}}$ while $\langle L, R \rangle$ may not, for $\Lambda = \langle O, L, R \rangle$.

This last consideration indicates that there are many ways in which $\mathcal{A}_{\mathcal{C}}$ could be defined. To avoid a proliferation of alternative definitions we care only for the class of possible *abduction* functors modulo a *natural transformation*, and from them select the most appropriate one. More precisely:

Definition 4. Given two functors $\mathcal{A}_{\mathcal{C}} : \mathbf{DR} \to \mathbf{S}_{\mathcal{C}}$ and $\mathcal{B}_{\mathcal{C}} : \mathbf{DR} \to \mathbf{S}_{\mathcal{C}}$, a natural transformation η between them is a map $\eta : \mathcal{A}_{\mathcal{C}} \to \mathcal{B}_{\mathcal{C}}$ such that for any pair Λ, Λ' of objects in $\mathbf{DR}$, the diagram in Fig. 1 is commutative.

This allows for the selection of a *representative* of each class of abduction functors that are natural transformations one of the other. As said, there are many candidates to represent abduction under a class

$$\begin{tikzcd} \mathcal{A}_{\mathcal{C}}(\Lambda) \arrow[r, "\eta_{\Lambda}"] \arrow[d, "\mathcal{A}_{\mathcal{C}}(\Lambda \preceq_{\mathbf{DR}} \Lambda')"'] & \mathcal{B}_{\mathcal{C}}(\Lambda) \arrow[d, "\mathcal{B}_{\mathcal{C}}(\Lambda \preceq_{\mathbf{DR}} \Lambda')"] \\ \mathcal{A}_{\mathcal{C}}(\Lambda') \arrow[r, "\eta_{\Lambda'}"'] & \mathcal{B}_{\mathcal{C}}(\Lambda') \end{tikzcd}$$

Fig. 1. Natural transformation between $\mathcal{A}_{\mathcal{C}}$ and $\mathcal{B}_{\mathcal{C}}$.

$$\begin{tikzcd} \Lambda \arrow[r, "\nu_\Lambda"] \arrow[dr, "g"'] & \mathcal{F} \circ \mathcal{A}_\mathcal{C}(\Lambda) \arrow[d, "\mathcal{F}(f)"] & & \mathcal{A}_\mathcal{C}(\Lambda) \arrow[d, "!f"] \\ & \mathcal{F}(\Delta_\mathcal{C}) & & \Delta_\mathcal{C} \end{tikzcd}$$

Fig. 2. Left adjoint of $\mathcal{F}$.

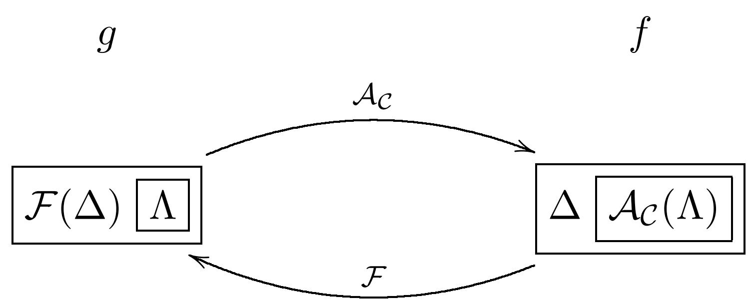

Fig. 3. $\mathcal{A}_\mathcal{C}$ and $\mathcal{F}$ minimize hypotheses and maximize consequences.

of criteria. But we can restrict them further taking into account a functor $\mathcal{F} : \mathbf{S}_\mathcal{C} \to \mathbf{DR}$ representing the way in which from theoretical structures we can derive *expected* observations and therefore data structures:

Definition 5. Consider $\mathcal{F} : \mathbf{S}_\mathcal{C} \to \mathbf{DR}$ and $\mathcal{A}_\mathcal{C} : \mathbf{DR} \to \mathbf{S}_\mathcal{C}$ such that $\nu : Id_{\mathbf{DR}} \to \mathcal{F} \circ \mathcal{A}_\mathcal{C}$ is a natural transformation such that for every Λ object of $\mathbf{DR}$ and $\Delta_\mathcal{C}$ and object of $\mathbf{S}_\mathcal{C}$ for every $g : \Lambda \to \mathcal{F}(\Delta_\mathcal{C})$ there exists a unique $f : \mathcal{A}_\mathcal{C}(\Lambda) \to \Delta_\mathcal{C}$ such that g is obtained as $\mathcal{F}(f) \circ \nu_\Lambda$ as indicated in the commutative diagram of Fig. 2.

In categorical terms, $\mathcal{A}_\mathcal{C}$ is the *left adjoint* of $\mathcal{F}$. As shown in Fig. 3, we can interpret this adjointness as a kind of "Ockham's Razor": abduction minimizes hypotheses while $\mathcal{F}$ maximizes the number of possible consequences. It is worthwhile to notice that the very formalism of Category Theory as applied in this context gives us the efficiency property of abduction seen as the counterpart of the "instantiation" of theoretical structures. Adjunction provides the formal machinery for that in the most natural way.

Since $\mathbf{S}_\mathcal{C}$ and $\mathbf{DR}$ generate two posets, $\langle \mathrm{Obj}(\mathbf{S}_\mathcal{C}), \preceq_{\mathbf{S}_\mathcal{C}} \rangle$ and $\langle \mathrm{Obj}(\mathbf{DR}), \preceq_{\mathbf{DR}} \rangle$ adjointness is captured by means of a *Galois connection*. That is:

$$\Delta \ \preceq_{\mathbf{S}_\mathcal{C}} \ \mathcal{A}_\mathcal{C} \circ \mathcal{F}[\Delta]$$

and

$$\Lambda \ \preceq_{\mathbf{DR}} \ \mathcal{F} \circ \mathcal{A}_\mathcal{C}[\Lambda],$$

where both $\mathcal{F}$ and $\mathcal{A}_\mathcal{C}$ are monotonic [10].

Such an $\mathcal{F}$ may be conceived as a functor that up from a structure, understood as a model of a set of theoretical statements, obtains the largest number of possible observations that would satisfy those statements. This idea is captured by the following characterization:

Definition 6. Given a structure $\bar{\Delta}$, an object of $\mathbf{S}_\mathcal{C}$ we define $\mathcal{F}(\bar{\Delta})$ as $\Lambda = \langle O, \bar{\Delta}^+ \rangle$ where $\bar{\Delta}^+$ is a maximal element in $\{\Delta : \bar{\Delta} \preceq_{\mathbf{S}_\mathcal{C}} \Delta \text{ and } \exists \langle O, \Delta \rangle \in \mathrm{Obj}(\mathbf{DR})\}$.

Notice that $\mathcal{F}$ is well-defined since $\{\Delta : \bar{\Delta} \preceq_{\mathbf{S}_\mathcal{C}} \Delta \text{ and } \exists \langle O, \Delta \rangle \in \mathrm{Obj}(\mathbf{DR})\} \neq \emptyset$. The worst case is when this set is a singleton, i.e. $\mathcal{F}(\bar{\Delta}) = \bar{\Delta}$. On the other hand, we have that:

Theorem 1. *The left adjoint of $\mathcal{F}$ is $\mathcal{A}_\mathcal{C}(\Lambda) = \min\{\Delta : \Lambda \preceq_{\mathbf{DR}} \mathcal{F}(\Delta)\}$ for $\Lambda = \langle O, \bar{\Delta} \rangle$.*

$$\begin{tikzcd}
\mathcal{A}(\Lambda) \arrow[r, "\eta_\Lambda"] \arrow[d, "\mathcal{A}(\Lambda \preceq_{\mathbf{DR}} \Lambda')"'] & \mathcal{A}'(\Lambda) \arrow[d, "\mathcal{A}'(\Lambda \preceq_{\mathbf{DR}} \Lambda')"] \\
\mathcal{A}(\Lambda') \arrow[r, "\eta_{\Lambda'}"'] & \mathcal{A}'(\Lambda')
\end{tikzcd}$$

Fig. 4. Natural transformation between $\mathcal{A}$ and $\mathcal{A}'$.

Proof. Just consider an object Δ. $\mathcal{F}(\Delta)$ gives a Λ with component Δ^+ such that $\Delta \preceq_{\mathbf{S}_\mathcal{C}} \Delta^+$. Suppose by way of contradiction that $\mathcal{A}_\mathcal{C}(\Lambda)$ returns a $\Delta' \prec_{\mathbf{S}_\mathcal{C}} \Delta$. But then, by the monotonicity of $\mathcal{F}$, $\mathcal{F}(\Delta') \prec_{\mathbf{DR}} \mathcal{F}(\Delta)$. But by the definition of $\mathcal{F}$, and the transitivity of $\preceq_{\mathbf{S}_\mathcal{C}}$, we would have $\mathcal{F}(\Delta') = \mathcal{F}(\Delta)$. Contradiction. □

To provide an operational characterization of $\mathcal{A}_\mathcal{C}$ consider $\Lambda = \langle O, \Delta^\Lambda \rangle$ and $\mathbf{F}(\Lambda) = \{\lambda \in \mathrm{WFF}(\mathcal{L}) : \Delta^\Lambda \models \lambda\}$, where $\mathrm{WFF}(\mathcal{L})$ is the class of well formed formulas of language $\mathcal{L}$ (i.e. without free variables) and $\models$ is the classical relation of semantical consequence. That is, $\mathbf{F}(\Lambda)$ consists of those well formed formulas satisfied by the data in Λ.

Consider now

$$\mathbf{S}_\mathcal{C}^\Lambda = \{\Delta \in \mathrm{Obj}(\mathbf{S}_\mathcal{C}) : \Delta \models \lambda \text{ for every } \lambda \in \mathbf{F}(\Lambda)\}$$

which are the objects in the category $\mathbf{S}_\mathcal{C}$ that support the data in Λ. Then we have:

Proposition 3. $\mathcal{A}_\mathcal{C}(\Lambda) = \min(\mathbf{S}_\mathcal{C}^\Lambda)$.

Proof. Consider $\Lambda = \langle O, \Delta \rangle$. Each $\Delta' \in \mathbf{S}_\mathcal{C}^\Lambda$ is such that $\mathcal{F}(\Delta') = \langle O^*, \Delta^* \rangle$ with $\{\lambda \in \mathrm{WFF}(\mathcal{L}) : \Delta \models \lambda\} \subseteq \{\lambda \in \mathrm{WFF}(\mathcal{L}) : \Delta^* \models \lambda\}$ and thus $\Lambda \preceq_{\mathbf{DR}} \mathcal{F}(\Delta')$. By Theorem 1, $\mathcal{A}_\mathcal{C}(\Lambda)$ is the product of all these Δ', i.e. $\min(\mathbf{S}_\mathcal{C}^\Lambda)$. □

5. The abduction functor: restrictions over the process

As we have seen, abduction can be well defined if criteria are applied *a priori* in an analysis of the data, in order to select which structures might be appropriate candidates. But criteria can also be defined in terms of the data, that is, in order to perform data-guided abductions. In that case, we are interested not in some constraints on the category $\mathbf{S}$ but on functors between $\mathbf{DR}$ and possible subsets of $\mathbf{S}$.

More precisely, we will call an *abduction* any functor that, given an object Λ in $\mathbf{DR}$ yields a set of acceptable structures in $\mathbf{S}$, hopefully a singleton. Furthermore, we will try to capture the idea that the less information given by an object in $\mathbf{DR}$ the larger must be the number of potential structures supporting it. This means that we are interested in objects (abduction procedures) of the category *Functors*$(\mathbf{DR}^{\mathrm{op}}, 2^{\mathrm{Obj}(\mathbf{S})})$, where[10]:

- The objects are *contravariant* functors from $\mathbf{DR}$ to the category of sets in $\mathrm{Obj}(\mathbf{S})$.[11] Any such object $\mathcal{A}$ is a contravariant functor because, if $\Lambda \preceq_{\mathbf{D}R} \Lambda'$ in $\mathbf{DR}$ then $\mathcal{A}(\Lambda') \subseteq \mathcal{A}(\Lambda)$ in $\mathrm{Obj}(\mathbf{S})$.
- Given two objects in the category, $\mathcal{A}$ and $\mathcal{A}'$, a morphism between them is a natural transformation $\eta : \mathcal{A} \to \mathcal{A}'$ such that the diagram of Fig. 4 commutes.

Each object in *Functors*$(\mathbf{DR}^{\mathrm{op}}, 2^{\mathrm{Obj}(\mathbf{S})})$ is a *presheaf* on $\mathbf{DR}$ valued in $\mathbf{S}$. The category of such presheaves is a *topos* [12]. In particular we have that:

[10] See definitions and related results in [16], [5] and [1].

[11] The latter category is trivially well defined: its objects are sets $S \subseteq \mathrm{Obj}(\mathbf{S})$ and a morphism $f : S \to S'$ between two objects S and S' in the category exists if and only if $S \subseteq S'$.

Theorem 2. *Functors*$(\mathbf{DR}^{\mathrm{op}}, 2^{\mathrm{Obj}(\mathbf{S})})$ *has a sub-object classifier* $\Omega : \mathbf{DR} \to 2^{\mathrm{Obj}(\mathbf{S})}$ *such that:*

(i) *If* Λ *is an object in* $\mathbf{DR}$, $\Omega(\Lambda) = \{L^{\Lambda'}\}_{\Lambda' \preceq_{\mathbf{D}R} \Lambda}$, *where* $L^{\Lambda'} = \{\bar{\Delta} : \bar{\Lambda} = \langle \bar{O}, \bar{\Delta}\rangle \in \mathrm{Obj}(\mathbf{S})$ *and* $\bar{\Lambda} \preceq_{\mathbf{DR}} \Lambda'\}$.
(ii) *Given* $\Lambda' \preceq_{\mathbf{DR}} \Lambda$, $\Omega_{\Lambda',\Lambda} : \Omega(\Lambda) \to \Omega(\Lambda')$ *is such that for any* $S \in \Omega(\Lambda)$, $\Omega_{\Lambda',\Lambda}(S) = S \cap L^{\Lambda'}$.

Proof. Trivial. In a category of presheaves on a category $\mathbf{C}$, the sub-object classifier Ω yields on any object A of $\mathbf{C}$ the set of all *sieves* on A. In the case that $\mathbf{C}$ is a poset, a sieve corresponds to the lower sets below A. To obtain (i), given any Λ, we take all the downward sets with respect to $\preceq_{\mathbf{DR}}$ below Λ and take from them the components corresponding to structures in $\mathbf{S}$. Similarly, given $p \preceq q$ in $\mathbf{C}$ with order $\preceq$, $\Omega_{p,q} : \Omega_q \to \Omega_p$ is such that for $S \in \Omega_q$, $\Omega_{p,q}(S)$ is the intersection of S with the downward set $\{r \in \mathbf{C} : r \preceq p\}$. Replacing $p \preceq q$ by $\Lambda' \preceq_{\mathbf{DR}} \Lambda$ and $\{r \in \mathbf{C} : r \preceq p\}$ by $L^{\Lambda'}$ we obtain (ii). $\square$

The elements in $\Omega(\Lambda)$ for any object Λ in $\mathbf{DR}$ can be seen as "truth values" or validity degrees in a question for which structures in $\mathcal{S}$ correspond better to the data in Λ. The higher in the hierarchy of such values the more appropriate would be the answer. This would be then L^{Λ}, which yields all the structures in $\mathbf{S}$ that are included in the corresponding structures in Λ and in all objects below Λ in $\mathbf{DR}$.

Abduction, thus, is captured by taking the highest ranked sub-object given Λ. That is, $\mathcal{A}^*(\Lambda) = L^{\Lambda}$, where $\mathcal{A}^*$ is abduction seen as an object in *Functors*$(\mathbf{DR}^{\mathrm{op}}, 2^{\mathrm{Obj}}(\mathbf{S}))$, instead of a functor between $\mathbf{DR}$ and $\mathbf{S}$ as in the previous section.

If we consider an alternative to $\mathbf{S}^{\Lambda}_{\mathcal{C}}$,[12]

$$\mathbf{S}^{\Lambda}_{*} = \bigl\{\Delta \in \mathrm{Obj}(\mathbf{S}) : \Delta \models \lambda \text{ for every } \lambda \in \mathbf{F}\bigl(\Lambda'\bigr) \text{ with } \Lambda' \preceq_{\mathbf{D}R} \Lambda\bigr\}$$

we obtain the following variant of Proposition 3:

Proposition 4. $\mathcal{A}^*(\Lambda) = \mathbf{S}^{\Lambda}_{*}$.

Proof. Each $\Delta \in L^{\Lambda}$ by definition corresponds to a $\Lambda' \preceq_{\mathbf{D}R} \Lambda$ and thus $\Delta \models \lambda$ for every $\lambda \in \mathbf{F}(\Lambda')$. Then $L^{\Lambda} \subseteq \mathbf{S}^{\Lambda}_{*}$. On the other hand, for every $\Delta \in \mathbf{S}^{\Lambda}_{*}$, since it supports a class of sentences $\mathbf{F}(\Lambda')$ for $\Lambda' \preceq_{\mathbf{D}R} \Lambda$, there exists $\Lambda'' = \langle O'', \Delta\rangle$ in $\mathbf{DR}$ with $\Lambda'' \preceq_{\mathbf{D}R} \Lambda'$ and $\Lambda' \preceq_{\mathbf{D}R} \Lambda''$ and so, by the antisymmetry of $\preceq_{\mathbf{D}R}$, $\Lambda'' = \Lambda' \in L^{\Lambda}$. $\square$

But so far, it remains that many structures may be chosen. To reduce the number of selected structures, leading to a restricted version of $\mathcal{A}^*$, denoted $\mathcal{A}^*_{\mathcal{C}}$, a class of criteria $\mathcal{C}$ can be imposed on $\mathcal{A}^*$. Among them consider the following:

Definition 7.

- $\mathbf{c}^{mi}$ (**Maximal Information**): given two structures Δ_1, Δ_2 in $\mathbf{S}$, such that $\mathbf{F}(\Delta_i) = \{\lambda \in \mathrm{WFF}(\mathcal{L}) : \Delta_i \models \lambda\}$,[13] if we have that $\mathbf{F}(\Delta_1) \subseteq \mathbf{F}(\Delta_2)$ while $\mathbf{F}(\Delta_2) \not\subseteq \mathbf{F}(\Delta_1)$, discard Δ_1.
- $\mathbf{c}^{comp}$ (**Completeness w.r.t.** Λ): given two structures Δ_1, Δ_2 in $\mathbf{S}$ such that $\mathbf{F}(\Lambda) \subseteq \mathbf{F}(\Delta_2)$ but $\mathbf{F}(\Lambda) \not\subseteq \mathbf{F}(\Delta_1)$, discard Δ_1.
- $\mathbf{c}^{conc}$ (**Concordance w.r.t.** Λ): a given structure Δ in $\mathbf{S}$ is kept if for every $\lambda \in \mathbf{F}(\Lambda)$ either λ or $\neg\lambda$ belongs to $\mathbf{F}(\Delta)$.

Then we have the following result:

[12] Notice the different quantifier. In $\mathbf{S}^{\Lambda}_{\mathcal{C}}$ it is universal over $\mathbf{F}(\Lambda)$ while in $\mathbf{S}^{\Lambda}_{*}$ it is universal over $\mathbf{F}(\Lambda')$ for a Λ' in the downward set from Λ.
[13] Analogously to the definition of $\mathbf{F}(\Lambda)$.

Proposition 5. *If* $\mathcal{C} = \{\mathbf{c}^{mi}, \mathbf{c}^{com}\}$, $\mathcal{A}^*_{\mathcal{C}}(\Lambda) = \max(\mathbf{S}^{\Lambda}_*)$.

Proof. Trivial: According to $\mathbf{c}^{mi}$ and given the ordering $\preceq_{\mathcal{S}}$ over $\mathbf{S}^{\Lambda}_*$ only the largest Δ is kept. Furthermore, according to $\mathbf{c}^{com}$ no larger Δ is included. This leaves only $\max(\mathbf{S}^{\Lambda}_*)$. □

Similarly:

Proposition 6. *If* $\mathcal{C} = \{\mathbf{c}^{mi}, \mathbf{c}^{conc}\}$, *then* $\mathcal{A}^*_{\mathcal{C}}(\Lambda) = \max(\mathbf{S}^{\Lambda}_*)$.

These results show that a unique structure can be selected if the restrictions on structures that can be abduced obey certain methodological criteria like Maximal Information, Completeness or Concordance. This is not without a cost: if the only wffs in the chosen structure are the ones drawn from the database, it is not possible to provide more than a description (data fitting) of the available information. This means in turn that if only methodological criteria are to be used, the result of the inference is the generation of a prototype, i.e. only a statistical inference is performed.[14]

Finally, if both types of criteria are included, say $\mathcal{C}^{ap}$ and $\mathcal{C}^{m}$ (for *a priori* and *methodological*, respectively) we obtain a more general kind of abduction, denoted $\mathcal{A}^*_{\mathcal{C}^{ap}\cup\mathcal{C}^{m}}$ such that:

Theorem 3. *If* $\mathcal{C}^m = \{\mathbf{c}^{mi}, \mathbf{c}^{com}\}$ *or* $\mathcal{C}^m = \{\mathbf{c}^{mi}, \mathbf{c}^{conc}\}$, *we have that* $\mathcal{A}^*_{\mathcal{C}^{ap}\cup\mathcal{C}^{m}}(\Lambda) = \min(\mathbf{S}^{\Lambda}_{\mathcal{C}}) = \max(\mathbf{S}^{\Lambda}_*)$.

Proof. Trivial since by any of the two versions of $\mathcal{C}^m$ a single structure, $\max(\mathbf{S}^{\Lambda}_*)$ is obtained (in $\mathbf{S}_{\mathcal{C}^{ap}}$) and this object equals the result under $\mathcal{C}^m$, i.e. $\max(\mathbf{S}^{\Lambda}_*)$. □

A sensible question is whether a general abduction functor like $\mathcal{A}^*_{\mathcal{C}^{ap}\cup\mathcal{C}^{m}}$ behaves adequately in the presence of both negative and positive pieces of information in Λ. Any such abduction faces two different risks. One, already mentioned above, is the possibility of overfitting the resulting structure. That is, to select a Δ that accommodates "too much" to the data, which may come from noisy or unreliable sources. But the presence of substantial *a priori* criteria may reduce the chance of this happening, since in most cases an excessive precision in the incorporation of empirical information hampers the quality sought out by concrete criteria.

Similarly, the other possible risk is that the predictive power of Δ may be severely hampered by the presence of inconsistencies among the positive and negative instances in the data. But the fact that Δ corresponds to the highest ranked sub-object in $\Omega(\Lambda)$ ensures that all the knowledge that can be obtained up from Λ is the closest approximation to the data evidence. This may leave room for the presence of ambiguous pieces of information in Δ, which as said above, is not necessarily a deductively closed set of well-formed formulas.

6. Conclusions

In any case, a relational structure Δ^* chosen according to both *a priori* and methodological criteria, is conceived as a model of a theoretical body $\mathcal{T}$. To derive it, one might choose one from a collection of closed sets of wffs of $\mathcal{L}$, each one having Δ^* as a model. One candidate is just $\mathbf{F}(\Delta^*)$. Other possible theories may involve information not present in the data. In the case that $\mathcal{T}$ is $\mathbf{F}(\Delta^*)$, the theory is called *phenomenological*. Otherwise, the theory is said to be *representational* and involves non-observable properties and entities. In the case of Kepler's abduction, the corresponding phenomenological theory is provided by

[14] In statistical analyses these criteria are usually violated since sometimes inferences are drawn from partial samples from a bigger database (violation of $\mathbf{c}^{comp}$), some observations are discarded as outliers (violation of $\mathbf{c}^{mi}$), or some information in the database is not used (violation of $\mathbf{c}^{conc}$).

the three laws abduced by Kepler from Tycho Brahe's databases. On the other hand, the finally accepted corresponding representational theory is *Newtonian mechanics*, which yields a rationale for the planetary motions.

Nonetheless, in many empirical fields, the scientist who performs the abduction does not feel compelled to find a $\mathcal{T}$ corresponding to Δ^* since that's a task she leaves to theoreticians. It is interesting, however, to see that while the criteria used to select Δ^* may be made public, those for choosing a $\mathcal{T}$ do not respond to more than a few prescriptions of scientific "hygiene".[15] Beyond that, theoreticians are free to select any possible set of formulas that may correspond to the accepted model.

At any rate it seems that abductive reasoning must abandon the realm of implicit activity to become an open activity, one that may be discussed with the same seriousness as the values of statistical estimates.

In this respect, the scientific community might rightly ask that abductive inferences be publicly reported, providing:

- The set of methodological criteria to be considered, precisely stated.
- The alternative hypotheses that are postulated (obeying *a priori* criteria). Each one should be represented by a system of relations, which constitutes a necessary condition for the respective hypothesis.
- The tests showing which of the hypotheses is accepted. The acceptance criteria should be already stated in the set of methodological criteria.

Therefore, any discussion of the inference can be based either on the criteria used or on the set of postulated hypotheses. In the first case, the criteria may be wrong, biased, insufficient, etc. In the second case, any new hypotheses added to the list should conform to the originally stated criteria or provide justification for why they deviate. Both types of discussion may enliven the scientific evaluation of the available information.[16]

Above and beyond the interest of such critical and methodological consequences, however, the strongest potential of a category-theoretical approach to the problem of abduction based on adjoint functors between "theoretical" and "empirical" categories resides perhaps in what the present paper leaves almost entirely tacit, namely the naturally controlled variability of possible mappings, due to the structure-preserving character of categorical functors, between organized complexity that may already be explicit in the empirical data on the one hand and systems of theoretical representation (such as distinct languages and notations) on the other.

In effect, the characterization given above treats the collection of data O as essentially a structureless aggregate of elements, that is, a mere set. However, the model of abduction via adjoints easily accommodates collections of data endowed with much richer structure, particularly if such structure may be represented in category-theoretical terms (which is often the case). Because of the generality of the role of adjoint functors in our account of abduction—the structure of which may be characterized independently of how the rough data is actually given on the one hand and what particular theoretical language is used on the other, that is, how each of these is organized internally—a broad spectrum emerges of various possible combinations of forms of data-presentation (in the empirical category) and data-representation (in the theoretical category). What matters for the account is the field of possible correlations between the mode of presentation of the data and the mode of representing it in some theoretical framework, not the specific details of one or the other term of any particular correlation.

Thus, corresponding to the possibility of tracking more richly structured collections of data in the set O of the "empirical" category is the possibility of constructing highly variable frameworks for the formal-language component of the "theoretical" category. In such constructions, it will generally be necessary at a minimum

[15] Ockham's razor and consistency with the accepted theories, among them.

[16] For a description of how this may work for Economics, but can be extended easily to other disciplines see [37].

to accommodate the basic operations of first-order logic, such as universal and existential quantification and the representability of n-ary relations, but such operations are always available in any category that is also a topos, which means that a large space of different kinds of theoretical representation (one that includes both classical and non-classical logics) becomes available for possible application and experimentation in conjunction with various organized forms of raw data. Because the use and results of distinct correlations between forms of data and structures of representation would effectively characterize an open-ended variety of different modalities of abductive inference, the door is opened for a systematic, scientific investigation into the possibilities and limits of general abductive processes taken in their aggregate as a unique, albeit highly complex theoretical object.

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