On the Interaction Between Focus and Distributional Properties in Multidimensional Poverty Measurement

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Abstract

In the multidimensional poverty measurement literature, most measures satisfy the deprivation focus property, which means that they disregard any improvement in non-deprived achievements. Such measures cannot satisfy strong distributional properties as traditionally defined, because the distributional transformations among the poor are allowed to take place among their non-deprived achievements. We formally address this incompatibility and propose a set of alternative definitions of distributional properties that restrict distributional transformations to take place only among deprived achievements. This alternative definition allows discerning within the set of measures that satisfy the deprivation focus property, those that are strongly sensitive to distributional transformations from those that are not. With this new lens, we review some of the most prominent multidimensional poverty measures proposed in the literature and illustrate how measures within the same class as well as measures across different classes can be discerned from each other based on the alternative definitions.

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1 Introduction

Since the early 2000s, there has been a surge of multidimensional poverty measures building upon the rich literature on unidimensional axiomatic poverty measures. The various sets of properties that these measures satisfy implicitly entail different concepts of poverty and are rooted in notions of fairness and justice. Awareness about the properties that each poverty measure should satisfy is not trivial: in the end, the choice of a poverty measure may affect policies in general and, in particular, the allocation of resources for alleviating poverty.

In the unidimensional context, the properties of poverty measurement can be classified into *invariance properties* and *dominance properties* (Foster 2006). Invariance properties are those that require the poverty level to remain unchanged under certain data transformations; whereas *dominance properties* are those that require the poverty level to change in a particular direction under certain data transformations. Among the invariance properties a fundamental one is *poverty focus*, which requires that a poverty measure should not reflect any change in the level of poverty owing to any improvement among the non-poor population. Similarly, among the dominance properties, a fundamental one is *transfer* (Sen 1976), which is concerned with the dispersion of the underlying distributional achievements among the poor. Although these properties are compatible in the unidimensional context, their interplay between focus properties and distributional properties becomes complicated when poverty evaluation is based on two or more (multiple) dimensions.

In the multidimensional context, there is an additional focus property called *deprivation focus*, besides the poverty focus property. Measures that satisfy the deprivation focus property are required not to reflect any change in the level of poverty when there are improvements in non-deprived dimensions.[1] On the contrary, measures that do not satisfy the deprivation focus property *allow* compensations to take place between non-deprived and deprived dimensions of each poor person. Thus, the key line of distinction between these two cases is whether improvement in a non-deprived dimension of a poor person can somehow compensate for her deprived dimension or not. Although the poverty measurement literature unanimously agrees on the poverty focus property, the deprivation focus property has been a subject of debate. There are arguments in support of both views in the literature. For example, Tsui (2002, p. 74), favouring the deprivation focus property, argues that incorporating this property 'in a sense, emphasizes the essentiality of each attribute', and that its reasonability depends on 'how imperative the considered attributes are to lead a meaningful life'. Bresson (2009, pp. 2–3), on the other hand, argues that different attributes may have relationships in terms of well-being such that larger deprivation short-fall in one dimension may entail greater needs in any other dimension(s) of poverty, and questions the justification to restrict the compensation to take place only within bounded domains of dimensions (i.e., below deprivation thresholds) rather than their entire domains. Despite the existing disagreement in the literature, however, most of the proposed poverty indices so far satisfy the deprivation focus property, as the argument for 'intrinsic importance' of each dimension is both theoretically and empirically appealing.[2]

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[1] Bourguignon and Chakravarty (2003) refer to the deprivation focus property and the poverty focus property as 'strong focus' and 'weak focus', respectively.

[2] Measures that satisfy the deprivation focus property include the ones proposed by Chakravarty et al. (1998), Tsui (2002), Bourguignon and Chakravarty (2003), Bossert et al. (2013), Alkire and Foster (2011), Aaberge and Peluso (2012), Chakravarty and D'Ambrosio (2006), Nicholas and Ray (2012), and two of the three measures proposed by Maasoumi and Lugo (2008). Whereas, measures that do not satisfy the depri-

Like the focus properties, in the multidimensional context, there are two types of distribution properties: *transfer* and *rearrangement*. Transfer, as in the unidimensional context, is concerned with the dispersion of dimensional distributions among the poor; whereas, *rearrangement* is concerned with the association of achievements among the poor. The dominance properties, such as transfer and rearrangement, typically have a weaker version and a stronger version. The weaker version requires that poverty measures 'should not change in a particular direction' (either 'should not increase' or 'should not decrease') due to certain data transformations and thus allows the possibility for poverty measures to remain insensitive to the corresponding transformation. The primary objective of the weaker versions is to ensure that a measure should not move to an undesired direction due to a particular data transformation. The stronger version of each dominance property requires that poverty measures *should* change in a particular direction (either 'should increase' or 'should decrease') in response to these data transformations.

It has been noted in the multidimensional poverty measurement literature that poverty measures that satisfy the deprivation focus property cannot satisfy the strong versions of distributional properties (Tsui 2002; Datt 2013; Alkire et al. 2015). However, this issue has not been fully addressed. Our paper is similar in motivation to that of Donaldson and Weymark (1986) in the case of unidimensional poverty measurement, which analyses the interrelationships (compatibilities and incompatibilities) between two sets of dominance properties (monotonicity and transfer) and two alternative definitions of the poor. In this paper, we formally identify the incompatibility between the deprivation focus property with the strong version of distributional properties as usually defined in the literature. We then propose an alternative way to define strong versions of distributional properties that are compatible with the deprivation focus property. Finally, we revisit some of the most prominent existing measures that satisfy deprivation focus property and distinguish them by their sensitiveness to different types of distributional transformations.

It must be noted that in this paper the discussion is relevant to the context of multidimensional poverty measures (a) that are absolute and (b) where the considered dimensions are cardinal. In the presence of at least one ordinal variable, the range of applicable multidimensional poverty measures is substantially reduced and the distributional properties as defined need reconsideration.[3]

The paper is organised as follows. Section 2 presents the framework and the distributional properties as they are traditionally presented. Section 3 formally outlines the incompatibility results between the deprivation focus property and the strong versions of traditional distributional properties. Section 4 introduces the alternative versions of distributional properties and discusses how they are compatible with different focus properties. Section 5 summarises how different poverty measures that have been proposed in the literature satisfy the alternative versions of distributional properties. Section 6 provides concluding remarks.

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2 Framework: Multidimensional Focus and Distributional Properties

Suppose a hypothetical society consists of a fixed set of n persons and poverty is assessed by a fixed set of d dimensions. The achievement of any person i in any dimension j is denoted by $x_{ij} \in \mathbb{R}_+$ for all $i = 1, \ldots, n$ and all $j = 1, \ldots, d$. All $n \times d$ achievements are summarized by the *achievement matrix* $X \in \mathbb{R}_+^{n \times d}$, where $x_{i\cdot}$ denotes its ith row and $x_{\cdot j}$ denotes its jth column. We denote the set of people in X by $N(X)$ and the set of all possible $n \times d$-dimensional achievement matrices by $\mathcal{X}$.

Let us provide an example to facilitate our understanding of a typical achievement matrix. Suppose, a society is composed of five people with achievements in three (cardinal) dimensions. Their achievements in all three dimensions are summarized by the following achievement matrix:

$$X = \begin{bmatrix} 8 & 20 & 15 \\ 4 & 19 & 13 \\ 3 & 16 & 12 \\ 3 & 16 & 14 \\ 1 & 18 & 6 \end{bmatrix}.$$

Each row within X summarises the achievements of each person in three dimensions. For example, the achievements of the first person are presented by the vector $x_{1\cdot} = (8, 20, 15)$; whereas, the achievements of the fourth person are presented by the vector $x_{4\cdot} = (3, 16, 14)$. Likewise, each column within X summarises the achievements of all five people in each dimension. For instance, the achievements of all people in the second dimension are presented by the vector $x_{\cdot 2} = (20, 19, 16, 16, 18)$. We will be revisiting the example-matrix X to illustrate various concepts throughout the paper.

There are three stages in poverty measurement. In the first stage, a space for measuring poverty is defined, i.e., set $\mathcal{X}$, which may be functionings, capabilities, resources or other. The other two steps follow Sen (1976).

In the second stage, an identification strategy ρ is applied to discern the poor from the non-poor population. Different types of identification strategies have been proposed and been used both in the academic literature as well as in practice. One may use a (meaningful) aggregation function to add up the achievements in different dimensions to obtain an overall welfare aggregate for each person and then use a poverty threshold to discern the poor population from their non-poor counterpart, as was proposed by Maasoumi and Lugo (2008), for example, referring it to the 'aggregate poverty line approach'. Decancq et al. (2014) pursue the same approach while assessing multidimensional poverty incorporating individual preferences. This type of identification procedure is also commonly used in the statistical literature for assessing multidimensional poverty, where first an aggregate achievement value for each person is obtained by aggregating her achievements using multivariate statistical methods and then a certain percentile of the aggregate achievement values is used as poverty threshold.[4]

A second route may be to define a deprivation threshold for each dimension to identify the deprived dimensions and then identify the poor population based on these deprived dimensions, such as by counting the number of deprivations as in the 'counting

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[4] See Alkire et al. (2015), chapter 3 and Asselin (2009).

approach' Atkinson (2003).[5] Within this second route, the counting approach has been observed to have wider empirical applications, primarily following to the axiomatisation of the approach by Alkire and Foster (2011). Within the counting approach, there are two extreme-possible criteria: *union* and *intersection*. A *union criterion* identifies a person as poor if the person is deprived in at least one of multiple dimensions under consideration; whereas, an *intersection criterion* identifies a person as poor only whenever the person is simultaneously deprived in all dimensions under consideration. There is a full range of alternative options between these two extreme criteria.

Further alternatives for identification of the poor may be through geography or ethnicity, which are frequently observed as part of various targeting programmes. Let us denote the set of all identification strategies by $\boldsymbol{\rho}$. Based on the identification strategy $\rho \in \boldsymbol{\rho}$, we denote the set of poor population in X by $Z(X;\rho) \subseteq N(X)$.

In the third stage, a poverty index $P : \mathcal{X} \times \boldsymbol{\rho} \to \mathbb{R}$ is constructed to assess the level of poverty within the society. Assuming that an identification strategy has already been selected, for notational simplicity in the paper, we denote the set of poor corresponding to X by $Z(X)$ and the level of poverty in X by $P(X)$.

2.1 Multidimensional Focus Properties

The multidimensional poverty measurement literature unanimously agrees that any poverty measure should satisfy the *poverty focus property*, which requires that a poverty measure should not register any change in the level of poverty owing to any improvement among the non-poor population. The poverty focus property can be stated as follows.

Poverty Focus For any $X, Y \in \mathcal{X}$, if $Y \neq X$ and Y is obtained from X such that $y_i. \geq x_i.$ for some $i \notin Z(X)$ but $y_i. = x_i.$ for all $i \in Z(X)$, then $P(Y) = P(X)$.

As discussed above, some identification strategies are based on the identification of deprivations, in which case we are required to first identify whether each person is deprived or not in each dimension. Suppose, the set of *deprivation cut-offs* is denoted by $z \in \mathbb{R}_{++}^d$, such that person i is deprived in dimension j if $x_{ij} < z_j$. In other words, dimension j is a *deprived dimension* for person i if $x_{ij} < z_j$. From there one can define the *deprivation-censored achievement matrix* as $\tilde{X}(z)$, whose ijth element $\tilde{x}_{ij}(z)$ can be obtained such that $\tilde{x}_{ij}(z) = x_{ij}$ if $x_{ij} < z_j$ and $\tilde{x}_{ij}(z) = z_j$ otherwise.[6] In this way, the achievements where a person has met the relevant deprivation threshold are censored. Let us revisit our example-matrix X to illustrate how a deprivation-censored achievement matrix is obtained. Suppose, the deprivation cut-offs for the three dimensions are summarised by the vector $z = (\,5\ 19\ 12\,)$.

$$X = \begin{bmatrix} 8 & 20 & 15 \\ 4 & 19 & 13 \\ 3 & 16 & 12 \\ 3 & 16 & 14 \\ 1 & 18 & 6 \end{bmatrix}; \tilde{X}(z) = \begin{bmatrix} 5 & 19 & 12 \\ 4 & 19 & 12 \\ 3 & 16 & 12 \\ 3 & 16 & 12 \\ 1 & 18 & 6 \end{bmatrix}.$$

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[5] These include measures proposed by the already cited papers in footnote 2.

[6] This definition is analogous to the weak identification of the poor by Donaldson and Weymark (1986).

We have highlighted the deprived achievements within X, which are lower that the corresponding deprivation cut-offs in z, in bold. The relevant deprivation-censored achievement matrix $\tilde{X}(z)$ is obtained from X by replacing the non-deprived achievements by the corresponding deprivation cut-offs. The censored achievements in $\tilde{X}(z)$ are in italics.

Note that $\tilde{X}(z) = X$ whenever $x_{ij} < z_j$ for all i and for all j. In this case, every dimension of every person is a deprived dimension. We denote the set of all deprivation cut-offs by **z**. Then, we define the second focus property in the multidimensional context—the *deprivation focus property*, which unlike the poverty focus property, requires that a poverty measure should not register any change in the level of poverty owing to any improvement in any dimension in which a person is not deprived. The deprivation focus property is defined as follows.

Deprivation Focus For any $X, Y \in \mathcal{X}$ and for any $z \in \mathbf{z}$, if $Y \geq X$ and $Y \neq X$ but $\tilde{Y}(z) = \tilde{X}(z)$, then $P(Y) = P(X)$.

How are these two focus properties different from each other? The poverty focus property forbids *inter-personal* compensations to take place between the poor population and the non-poor population in the sense that when a non-poor person's achievement(s) improves, then it should not reduce the society's poverty level. The deprivation focus property, in turn, forbids *intra-personal* compensations between any deprived achievement and any non-deprived achievement. Contrary to the poverty focus property, there is no unanimous agreement regarding the essentiality of the deprivation focus property. For example, while contending against the deprivation focus property, Bresson (2009) argues that if two persons experience the same disability (i.e., same health deprivation) but no income deprivation, then the person with more income may take better care of her health deprivation than the person with much lower income. Similarly, while pursuing a well-being based approach to capture individual preferences, Decancq et al. (2014) argue on the importance of reflecting all achievements rather than just the censored ones.

However, in the emblematic example of the '*old* beggar' by Bourguignon and Chakravarty (2003), the deprivation focus property ensures that enjoying longevity does not compensate for having low income. The deprivation focus property is also consistent with considering the alleviation of every dimensional deprivation as *essential* supported by the capability approach (Sen 1999, 2009) as well as by a human rights point of view.

Let us use our example-matrix X to facilitate the understanding of these properties. Recall, within X, that an achievement in bold denotes a deprived achievement. Clearly, regardless of the identification strategy, the first person should not be included in the set of the poor, since the person is not deprived in any dimension. Thus, the poverty focus property requires that –holding everything else fixed– an increment in achievement of this person in any of the three dimensions should not affect the society's overall poverty level. Likewise, if an identification criterion additionally considers the second, third and fourth persons to be non-poor, then the requirement for the poverty focus property applies to these persons as well. In turn, the deprivation focus property requires that a society's overall poverty level remains unaltered, whenever there is an increment in any non-deprived achievement or, in this particular illustration, an increment in any of the non-bold achievements in X.

Multidimensional poverty measures may satisfy one of the two focus properties without satisfying the other (Alkire and Foster 2011, p. 481). A poverty measure satisfies the poverty focus property but not the deprivation focus property whenever a person's aggregate achievement is obtained by summing up her dimensional achievements and then she is identified as poor if the aggregate achievement is lower than an aggregate poverty cutoff. This is equivalent to a standard unidimensional poverty measurement framework. In contrast, a

poverty measure satisfies the deprivation focus property but not the poverty focus property if someone is required to experience deprivation in all considered dimensions simultaneously in order to be identified as poor, and yet the poverty measure is based on all deprivations of all poor and all non-poor persons (Alkire and Foster 2011, p. 481). Is there any situation when one focus property implies the other? It turns out that within the counting approach framework, when the poor people are identified using a *union criterion of identification*, the deprivation focus property implies the poverty focus property; whereas, when the poor people are identified using an *intersection criterion of identification*, the poverty focus axiom implies the deprivation focus property (Alkire and Foster 2011). In terms of our example-matrix X and the example-deprivation cut-off vector z, a union criterion identifies only the first person as non-poor. The deprivation focus property already requires the society's poverty level to remain unchanged whenever there is an increase in any of the achievements of the first person and thus automatically complies with the poverty focus property. Conversely, an intersection criterion identifies only the fifth person as poor. Focusing exclusively on the fifth person, who is deprived in all dimensions, automatically ignores changes in other achievement values, including the non-deprived ones.

We formally denote the class of poverty measures that satisfy the poverty focus property by $\mathcal{P}^1$ and the class of poverty measures that satisfy the deprivation focus property by $\mathcal{P}^2$.

2.2 Multidimensional Distributional Properties

Unlike in the unidimensional measurement context, there are two types of distributional properties in the context of multidimensional poverty measurement: *transfer* and *rearrangement*. The literature has typically defined the transfer and rearrangement properties using the non-censored matrix of achievements based on what we refer here as *transfer among the poor* and *rearrangement among the poor*.

The transfer transformation makes use of the concepts of the *bistochastic matrix* and the *permutation matrix*. A bistochastic matrix (B) is a non-negative square matrix with elements in each row and each column summing to one; i.e., if the ijth element of B is denoted by b_{ij}, then $\sum_i b_{ij} = 1$ for all j and $\sum_j b_{ij} = 1$ for all i. A permutation matrix is also a square matrix with one element in each row and each column equal to one and the rest are zeros.

Transfer among the poor (TP) For any $X, Y \in \mathbb{R}_+^{n \times d}$, Y is obtained from X such that (i) $Y = BX$ where B is an $n \times n$-dimensional bistochastic matrix and $b_{ii} = 1$ for all $i \notin Z(X)$ and (ii) Y is not a permutation of X.

In words, a transfer among the poor means that the achievements of the poor, either deprived or non-deprived achievements, are redistributed among them such that the resulting distribution is less concentrated: "…this transformation is equivalent to replacing the original bundles of attributes of any pair of individuals by a convex combination of them" (Bourguignon and Chakravarty 2003, pp. 30–31).

Continuing with our example-matrix X, the example-deprivation cut-off vector z, and additionally assuming that the poor persons are those that experience two or more deprivations, matrix Y may be stated to have been obtained from X by a transfer among the poor.[7] One may observe that the average achievement within every dimension in Y is the same as

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[7] The bistochastic matrix used to obtain matrix Y is: $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \end{bmatrix}$.

the average achievement within the corresponding dimension in X, but the achievement values (in italics) between the third and fifth persons in Y are now equally shared.

$$Y = \begin{bmatrix} 8 & 20 & 15 \\ \mathbf{4} & 19 & 13 \\ 2 & \mathit{17} & 9 \\ \mathbf{3} & \mathbf{16} & 14 \\ 2 & \mathit{17} & 9 \end{bmatrix}$$

Let us now look at the rearrangement transformation, which makes use of the minimum and maximum vector transformation. For vectors $a, b, c \in \mathbb{R}^d$, $c = a \vee b$ implies $c_j = \max\{a_j, b_j\}$ for all j; whereas $c = a \wedge b$ implies $c_j = \min\{a_j, b_j\}$ for all j.

Rearrangement among the poor (RP) For any $X, Y \in \mathbb{R}_+^{n \times d}$, Y is obtained from X such that (i) $y_{l.} = x_{l.} \vee x_{k.}$ and $y_{k.} = x_{l.} \wedge x_{k.}$ for some $l, k \in Z(X)$, (ii) $y_{i.} = x_{i.}$ for all $i \neq l, k$, (iii) $Z(Y) = Z(X)$, and (iv) Y is not a permutation of X.

In words, a rearrangement among the poor implies that achievements are switched between two poor people in such a way that one of them ends up having an equal or higher amount in all achievements, and strictly higher amount in at least one achievement. That is, in the pre-rearrangement achievement matrix X, person l must have strictly lower achievement in at least one dimension as well as strictly higher achievement in at least one dimension than person k; this is ensured by condition (iv) in the definition of RP.

Formally, there is no vector dominance between $x_{l.}$ and $x_{k.}$.[8] In the post-rearrangement situation, person l must have no lower achievement than person k in any dimension and strictly higher achievement in some dimensions. Formally, there is vector dominance between $x_{l.}$ and $x_{k.}$ in this case. Intuitively, the association among deprivations increases by a rearrangement among the poor.

Let us revisit our example-matrix X and the example-deprivation cut-off vector z, and compare the achievements of the fourth person with the fifth person. Note that the fourth person has higher achievements in the first and third dimensions, but the fifth person has higher achievement in the second dimension. Suppose now that matrix L is obtained from X by a rearrangement among the poor, whereby the fourth and fifth persons have swapped their achievements (in italics) in the second dimension. Clearly, the fourth person now has higher achievement than the fifth person in every dimension.

$$L = \begin{bmatrix} 8 & 20 & 15 \\ \mathbf{4} & 19 & 13 \\ \mathbf{3} & \mathbf{16} & 12 \\ \mathbf{3} & \mathit{18} & 14 \\ \mathbf{1} & \mathit{16} & \mathbf{6} \end{bmatrix}$$

Based on transformations TP and RP, the transfer and rearrangement properties have typically been defined in their weak forms.

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[8] Following Alkire and Foster (2011), Alkire et al. (2015) define the property using an *association decreasing rearrangement* among the poor with an attempt to keep the set of poor unchanged. We however restate the property in terms of association increasing rearrangement, but requiring the set of the poor to remain unchanged (condition (iii)) which guarantees consistency between the definition of the two related transformations and excludes the controversial case in which the number of the poor increases due to an association increasing rearrangement. See Donaldson and Weymark (1986) for a relevant discussion in the unidimensional context.

Weak Transfer (WT) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by a TP, then $P(Y) \leq P(X)$.

Weak Rearrangement (WR) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by an RP, then $P(Y) \geq P(X)$.

Converse Weak Rearrangement (CWR) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by an RP, then $P(Y) \leq P(X)$.

The first rearrangement property is applicable to situations when dimensions are considered substitutes and thus higher association among deprivations is considered detrimental to poverty; whereas the second rearrangement property is applicable when dimensions are complements and thus higher association among deprivations is considered to be ameliorating poverty (see Bourguignon and Chakravarty 2003).

These properties are of fundamental importance as they prevent a poverty measure to move in an *undesired* direction. However, they do not guarantee that the measure will be strictly sensitive to the defined transformations. Thus, it is natural to state an extension of these properties to their strong forms.

Strong Transfer (ST) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by a TP, then $P(Y) < P(X)$.

Strong Rearrangement (WR) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by an RP, then $P(Y) > P(X)$.

Converse Strong Rearrangement (CWR) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by an RP, then $P(Y) < P(X)$.

3 Incompatibilities

While natural, the strong forms of the distributional properties defined in the previous section are not typically stated in the literature of multidimensional poverty measurement and that is not a coincidence. The reason is that multidimensional poverty measures, which satisfy not only poverty focus but also deprivation focus, cannot satisfy these strong versions of distributional properties. The intuition behind such incompatibility is that transformations TP and RP, on which these properties are based, are so generally defined that they may affect non-deprived achievements without affecting any deprived achievement. In this case, a poverty measure that satisfies the deprivation focus property registers no variation, which we state in Proposition 1.

Proposition 1 *There does not exist any poverty measure $P \in \mathcal{P}^2$ that satisfies strict transfer, strict rearrangement, and converse strict rearrangement for all $X \in \mathcal{X}$, for all $z \in \mathbf{z}$ and for all $\rho \in \boldsymbol{\rho}$.*

Proof By definition, any $P \in \mathcal{P}^2$ satisfies the deprivation focus property. We need to show that there is no $P \in \mathcal{P}^2$ that satisfies the three properties—strict transfer, strict rearrangement, and converse strict rearrangement for all $X \in \mathcal{X}$ and for all $z \in \mathbf{z}$.

First, we show that no $P \in \mathcal{P}^2$ satisfies strict transfer for all $X \in \mathcal{X}$, for all $z \in \mathbf{z}$ and for all $\rho \in \boldsymbol{\rho}$. Our proof of this part will be complete if we show that no $P \in \mathcal{P}^2$ satisfies strict transfer for some $X \in \mathcal{X}$, for some $z \in \mathbf{z}$ and for some $\rho \in \boldsymbol{\rho}$. Suppose there exists some X^1 and X^2 in $\mathcal{X}$ and some $z' \in \mathbf{z}$ such that $Z(X^1) \neq \emptyset$ and $x_{ij}^1 < z_j'$ for some (i, j). In words, at least one person in X^1 is poor and there is at least one person who has at least one deprived dimension in X^1. Suppose further that X^2 has been obtained from X^1 by a *transfer among the poor*,

such that $\tilde{X}^1(z') = \tilde{X}^2(z')$. In words, X^2 has been obtained from X^1 by reducing inequality among the poor but there is no change in the deprived achievements. Given that $\tilde{X}^1 = \tilde{X}^2$, the deprivation focus property requires however that $P(X^2) = P(X^1)$ for all $P \in \mathcal{P}^2$. The strict transfer property however requires that $P(X^2) < P(X^1)$, which is a contradiction. Hence, no poverty measure $P \in \mathcal{P}^2$ can satisfy strict transfer for all $X \in \mathcal{X}$ and for all $z \in \mathbf{z}$.

Next, we can similarly show that no poverty measure $P \in \mathcal{P}^2$ satisfies strict rearrangement and converse strict rearrangement for all $X \in \mathcal{X}$, for all $z \in \mathbf{z}$ and for all $\rho \in \boldsymbol{\rho}$. In this case, suppose there exists some X^3 and X^4 in $\mathcal{X}$, some $z'' \in \mathbf{z}$ and some $\rho \in \boldsymbol{\rho}$ such that $Z(X^3) \neq \emptyset$ and $x_{ij}^3 < z_j''$ for some (i, j). Suppose further that X^4 has been obtained from X^3 by a *rearrangement among the poor*, but $\tilde{X}^3(z'') = \tilde{X}^4(z'')$. In words, the rearrangement does not affect the deprived dimensions. Thus, the deprivation focus property requires that $P(X^3) = P(X^4)$ for all $P \in \mathcal{P}^2$. However, the strict rearrangement property requires that $P(X^4) < P(X^3)$ and the converse strict rearrangement property requires that $P(X^4) > P(X^3)$ which are again contradictions. Hence, no poverty measure $P \in \mathcal{P}^2$ can satisfy strict rearrangement and converse strict rearrangement for all $X \in \mathcal{X}$ and for all $z \in \mathbf{z}$. This completes our proof. $\qquad\square$

Proposition 1 is a generalisation of Proposition 2 in Tsui (2002, p. 77) in two ways. First, Tsui's proposition was restricted to the class of measures satisfying symmetry, replication invariance, monotonicity, deprivation focus, continuity and subgroup consistency. Here, the incompatibility is stated for a broader class of measures that only satisfy the deprivation focus property.[9] Second, our proposition not only shows the incompatibility between the deprivation focus property and the strong transfer property, but also between the deprivation focus property and the strong rearrangement properties.

Proposition 1 should be interpreted carefully. It may be possible that a particular poverty measure satisfying the deprivation focus property also satisfies the strong distributional properties for a particular *poverty identification criterion*. For example, a counting based measure with an intersection criterion to identification of the poor requires the poor people to be deprived in all dimensions, and thus there is no opportunity for the strong distributional properties not to be satisfied by the measure in this situation. However, the point of Proposition 1 is that although such compatibility holds for a particular poverty identification criterion, it would be violated when some other poverty identification criterion is used with the same poverty measure.

Let us provide two illustrations. Suppose, matrix W is obtained from our example-matrix X by a transfer among the poor (considering poor people are those that are deprived in two or more dimensions).[10] Clearly, with this transformation, the achievements of the third and fourth persons are now equally shared. Yet, the only achievements whose values effectively change belong to the third dimension, a dimension in which none of the two persons was initially deprived. A poverty measure satisfying the deprivation focus property would certainly not register a change under this transformation.

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[9] Tsui (2002) does not distinguish between the poverty focus property and the deprivation focus property as he adopts a union approach to identification.

[10] The bistochastic matrix used to obtain matrix W is: $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

$$W = \begin{bmatrix} 8 & 20 & 15 \\ \mathbf{4} & 19 & 13 \\ \mathit{3} & \mathit{16} & \mathit{13} \\ \mathit{3} & \mathit{16} & \mathit{13} \\ \mathbf{1} & \mathbf{18} & \mathbf{6} \end{bmatrix}$$

Analogously, matrix *H* below is obtained from our example-matrix *X* by a rearrangement among the poor, where, we assume that the poor people are those that are deprived in at least one dimension. With this rearrangement, the second and fourth persons have swapped their achievements (in italics) in the third dimension. Yet this is a non-deprived achievement for both. Again, a poverty measure satisfying deprivation focus will obviously not register a change under this transformation.

$$H = \begin{bmatrix} 8 & 20 & 15 \\ \mathbf{4} & 19 & \mathit{14} \\ \mathbf{3} & \mathbf{16} & 12 \\ \mathbf{3} & \mathbf{16} & \mathit{13} \\ \mathbf{1} & \mathbf{16} & \mathbf{6} \end{bmatrix}$$

The incompatibility between the deprivation focus axiom and the strong versions of transfer and rearrangement properties leaves essentially *three alternatives*.

One alternative is to stick to measures that satisfy both focus properties and adopt weak forms of distributional properties. By satisfying the deprivation focus property, this alternative does not allow compensations between non-deprived achievements and deprived achievements when identifying the poor, considering each dimension as fundamental. Naturally, these poverty measures reflect no change under redistribution of achievements among the poor when such redistributions involve only non-deprived achievements: it is understood that they are irrelevant for the status of the poor. It is worth noting however that these measures may register a change if redistributions involve at least one deprived achievement.

A *second alternative* is to use multidimensional poverty measures that satisfy only the poverty focus property and thus satisfy the distributional properties in their strong forms. This alternative acknowledges potential interactions between achievements across dimensions at all levels and therefore allows compensations between non-deprived achievements and deprived achievements while identifying the poor as well as while evaluating distributional transformations. In this case, poverty registers a change when there are redistributions, even if these involve only non-deprived achievements, as they *are* relevant for the status of the poor.

A *third alternative* is to stick to the first normative position of no compensation between non-deprived and deprived achievements but define more restricted forms of the strong distributional properties—essentially over deprived achievements only. These re-defined strong forms of distributional properties are compatible with both focus properties. In this case, compensations are not allowed between deprived achievements and non-deprived achievements across dimensions, and the poverty level changes strictly whenever redistributions among the poor occur over their deprived achievements.

The first alternative has been in fact the route most commonly undertaken in the literature. The second alternative has also been pursued, for example, by Maasoumi and Lugo (2008), Bresson (2009) and Decancq et al. (2014). The third alternative was suggested

by Tsui ([2002](), fn. 19). We explore this alternative in the next section and offer further considerations.

4 Redefined Distributional Properties and Compatibilities

We now define a variant of the TP and RP transformations: a ***deprivation*** *transfer among the poor* and a ***deprivation*** *rearrangement among the poor*.

Deprivation transfer among the poor (DTP) For any $X, Y \in \mathcal{X}$ and $z \in \mathbf{z}$, Y is obtained from X such that (i) $Y = BX$ where B is an $n \times n$-dimensional bistochastic matrix and $b_{ii} = 1$ for all $i \notin Z(X)$ and (ii) $\tilde{Y}(z)$ is not a permutation of $\tilde{X}(z)$.

As in case of TP, DTP also smooths achievements among the poor using a bistochastic matrix. Yet, in TP the smoothing can include deprived as well as non-deprived achievements of the poor, even when the smoothing is affecting only the non-deprived achievements, as it is in the case of the example-matrix W. On the contrary, with DTP, condition (ii) guarantees that the smoothing affects at least one deprived achievement among the poor, as it is in the case of the transformation from example-matrix X to example-matrix Y.

Deprivation rearrangement among the poor (DRP) For any $X, Y \in \mathcal{X}$ and $z \in \mathbf{z}$, Y is obtained from X such that (i) $\tilde{Y}(z)$ is not a permutation of $\tilde{X}(z)$, (ii) $\tilde{y}_{l.}(z) = \tilde{x}_{l.}(z) \vee \tilde{x}_{k.}(z)$ and $\tilde{y}_{k.}(z) = \tilde{x}_{l.}(z) \wedge \tilde{x}_{k.}(z)$ for some $l, k \in Z(X)$, (iii) $\tilde{y}_{i.}(z) = \tilde{x}_{i.}(z)$ for all $i \neq l, k$, and (iv) $Z(Y) = Z(X)$.

Similarly, as in case of RP, DRP increases the association among dimensions through a rearrangement between some poor l and some poor k. However, as with transfer, the difference is that while in RP such rearrangement may include either deprived or non-deprived achievements of the poor, even occurring over non-deprived achievements only (as it is in the case of example- matrix H), with DRP, condition (iv) guarantees that the rearrangement occurs between deprived achievements among the poor, as it is in the case of the transformation from example-matrix X to example-matrix L.[11]

Based on the more restricted type of transformations, DTP and DRP, we now re-define the distributional properties[12]:

Weak Deprivation Transfer (WDT) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by a DTP, then $P(Y) \leq P(X)$.

Strong Deprivation Transfer (SDT) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by a DTP, then $P(Y) < P(X)$.

Weak Deprivation Rearrangement (WDR) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by a DRP, then $P(Y) \geq P(X)$.

Strong Deprivation Rearrangement (SDR) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by a DRP, then $P(Y) > P(X)$.

Converse Strong Deprivation Rearrangement (CSDR) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by a DRP, then $P(Y) < P(X)$.

Converse Weak Deprivation Rearrangement (CWDR) For any $X, Y \in \mathcal{X}$, if Y is obtained from X by a DRP, then $P(Y) \leq P(X)$.

With the transfer and rearrangement properties defined over the censored domain of dimensions, measures satisfying the deprivation focus property can satisfy the strong

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[11] It must be noted that Bourguignon and Chakravarty ([2003](), p.47) define an analogous transformation to DRP called correlation increasing switch for any two deprived achievements.

[12] For the rearrangement properties, we use the same acronyms introduced in Alkire et al. ([2015]()).

forms of distributional properties. In other words, the level of poverty is required to strictly decrease only when redistributions among the poor occur over deprived achievements (the level of poverty may be required to increase due to DRP if dimensions are considered complements); whereas it should remain unchanged when redistributions among the poor occur over non-deprived achievements only.[13]

Clearly, in an inter-temporal policy perspective, the eradication and alleviation of poverty requires transfers from the non-poor to the poor. Poverty eradication is impossible to achieve by redistribution of the achievements merely among the poor, especially if the distributions are performed over deprived achievements. The redefined distributional properties from this point of view may appear irrelevant to poverty alleviation policies. However, these redefined distributional properties are relevant if one thinks in terms of comparisons across societies, such as across groups, regions or countries. It is sensible to demand that a society with a more equal distribution of achievements among the poor—whether in terms of a lower dispersion within dimensions, or a lower association across achievements if dimensions are considered substitutes, or a higher association across achievements if dimensions are considered complements—is deemed less poor than another society with similar level of average deprivations but more unequal distribution. Revisiting the example-matrices, it appears that the example-matrix Y is preferable than the example-matrix X; whereas, the example-matrix L appears to be less preferable than the example-matrix X, if one considers the dimensions to be substitutes, and to be preferable if the dimensions are considered to be complements.

Note that the properties defined on the restricted domain of dimensions are subsets of properties defined on the entire domain. Thus, all measures in $\mathcal{P}^1$ that satisfy properties defined over the entire domain satisfy the relevant properties defined over the restricted domain. The converse clearly does not hold, as it is obvious from our discussions in the Sect. 6. We formally present these results in Proposition 2 and Corollary 1. Proposition 2 presents the relationship between the weak versions of distributional properties.

Proposition 2 *For all $X \in \mathcal{X}$, (i) if any measure $P \in \mathcal{P}^1$ satisfies weak transfer, then it satisfies weak deprivation transfer, (ii) if any measure $P \in \mathcal{P}^1$ satisfies weak rearrangement, then it satisfies weak deprivation rearrangement, and (iii) if any measure $P \in \mathcal{P}^1$ satisfies converse weak rearrangement, then it satisfies converse weak deprivation rearrangement.*

Proof Suppose there exists some $X, Y \in \mathcal{X}$ and any $z \in \mathbf{z}$ such that Y is obtained from X by a deprivation transfer among the poor. It means that $\tilde{Y}(z)$ is not a permutation of $\tilde{X}(z)$. Now we know that if Y is a permutation of X, then it implies that $\tilde{Y}(z)$ is a permutation of $\tilde{X}(z)$ for all $z \in \mathbf{z}$. Then if $\tilde{Y}(z)$ is not a permutation of $\tilde{X}(z)$ for any $z \in \mathbf{z}$, then it implies that Y is not a permutation of X. Therefore, using the definitions, we find that if Y is obtained from X by a deprivation transfer among the poor, then it must have been the case that Y is obtained from X by a transfer among the poor. Therefore, if a poverty measure $P \in \mathcal{P}^1$ satisfies weak transfer, then it must satisfy weak deprivation transfer. This proves part (i).

The next two parts of the proof use the same logic that if $\tilde{Y}(z)$ is not a permutation of $\tilde{X}(z)$ for any $z \in \mathbf{z}$, then Y is not a permutation of X. Therefore, using the definitions we find

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Table 1 Compatibility between focus and distributional properties for multidimensional poverty indices

Distributional property	Poverty focus	Deprivation focus
Weak transfer (WT)	Yes	Yes
Strong transfer (ST)	Yes	No
Weak rearrangement (WR)	Yes	Yes
Strong rearrangement (SR)	Yes	No
Converse weak rearrangement (CWR)	Yes	Yes
Converse strong rearrangement (CSR)	Yes	No
Weak deprivation transfer (WDT)	Yes	Yes
Strong deprivation transfer (SDT)	Yes	Yes
Weak deprivation rearrangement (WDR)	Yes	Yes
Strong deprivation rearrangement (SDR)	Yes	Yes
Converse strong deprivation rearrangement (CSDR)	Yes	Yes
Converse weak deprivation rearrangement (CWDR)	Yes	Yes

that if Y is obtained from X by a deprivation rearrangement among the poor, then it must have been the case that Y is obtained from X by a rearrangement among the poor. Hence, if a poverty measure $P \in \mathcal{P}^1$ satisfies weak rearrangement, then it must satisfy weak deprivation rearrangement, which proves part (ii). Finally, if a poverty measure $P \in \mathcal{P}^1$ satisfies converse weak rearrangement, then it must satisfy converse weak deprivation rearrangement, which proves part (iii). □

Corollary 1 presents the relationship between the strong versions of distributional properties.

Corollary 1 *For all $X \in \mathbb{R}_+^{n \times d}$, (i) if any measure $P \in \mathcal{P}^1$ satisfies strong transfer, then it satisfies strong deprivation transfer, (ii) if any measure $P \in \mathcal{P}^1$ satisfies strong rearrangement, then it satisfies strong deprivation rearrangement, and (iii) if any measure $P \in \mathcal{P}^1$ satisfies converse strict rearrangement, then it satisfies converse strong deprivation rearrangement.*

Proof The proof follows the same structure as Proposition 2. □

Table 1 presents which of the two focus properties are compatible with which distributional properties. In the first column of the table, we present different strong and weak versions of distributional properties. In the second column, we present whether the distributional properties are compatible with the poverty focus; whereas in the third column we present whether the distributional properties are compatible with the deprivation focus property. It is to be noted that by 'Yes' we do not mean that all indices that satisfy a focus property must satisfy a particular distributional property, but rather that some indices can satisfy both properties (i.e., compatible); whereas by 'No' we mean that it is not possible for any measure to satisfy both properties simultaneously (incompatible).

Table 2 Multidimensional poverty measures that satisfy the focus property and distributional properties over restricted domain

Literature	Measure	Parameter restriction	Properties								
			Entire domain			Restricted domain					
			WT	WR	CWR	WDT	SDT	WDR	SDR	CWDR	CSDR
Chakravarty et al. (1998)[a]	$P_{CMR1}(X;z) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^d w_j \left[1 - \left(\frac{\bar{x}_{ij}}{z_j} \right) \right]^\alpha$	$0 < \alpha < 1;\ w_j > 0\ \forall j$	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	No
	$P_{CMR2}(X;z) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^d w_j \left(\frac{z_j - \bar{x}_{ij}}{z_j} \right)^\alpha$	$\alpha > 1;\ w_j > 0\ \forall j$	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	No
	$P_{CMR3}(X;z) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^d w_j \left(\frac{z_j - \bar{x}_{ij}}{z_j} \right)$	$\alpha = 1;\ w_j > 0\ \forall j$	Yes	Yes	Yes	Yes	No	Yes	No	Yes	No
Tsui (2002)	$P_{T1}(X;z) = \frac{1}{n} \sum_{i=1}^n \left[\prod_{j=1}^d \left(\frac{\bar{x}_{ij}}{z_j} \right)^{-\alpha_j} - 1 \right]$	$\alpha_j > 0\ \forall j$	Yes	Yes	–	Yes	Yes	Yes	Yes	–	–
	$P_{T2}(X;z) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^d w_j \ln \left(\frac{z_j}{\bar{x}_{ij}} \right)$	$w_j > 0\ \forall j$	Yes	Yes	–	Yes	Yes	Yes	No	–	–
Bourguignon and Chakravarty (2003)	$P_{BC}(X;z) = \frac{1}{n} \sum_{i=1}^n \left[\sum_{j=1}^d w_j \left(\frac{z_j - \bar{x}_{ij}}{z_j} \right)^\beta \right]^{\alpha/\beta};$ $\beta > 1$ and $\alpha \geq 0$	$\alpha > \beta;\ w_j > 0\ \forall j$	Yes	Yes	–	Yes	Yes	Yes	Yes	–	–
		$\alpha < \beta;\ w_j > 0\ \forall j$	Yes	–	Yes	Yes	Yes	–	–	Yes	Yes
Alkire and Foster (2011)[b]	$P_{AF}(X;z,k) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^d w_j \left(\frac{z_j - \bar{x}_{ij}}{z_j} \right)^\alpha$	$\alpha = 1;\ w_j > 0\ \forall j$	Yes	Yes	Yes	Yes	No	Yes	No	Yes	No
		$\alpha = 2;\ w_j > 0\ \forall j$	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	No

[a] If a union approach to identification is used, $P_{CMR3}(X;z) = P_{AF}(X;z)$ for $\alpha = 1$ and for $\alpha = \beta = 1$, $P_{CMR3}(X;z) = P_{BC}(X;z)$

[b] Note that this family of measures, unlike other measure reported in the table, departs from considering only the union criterion of identification and allows any intermediate identification cutoffs $k > 0$. Unlike in the union criterion, poor people are identified as those whose sum of weighted deprivations is equal or higher than the poverty cutoff k

5 Classification of Poverty Measures

What differences do the redefined properties make in practice? In this penultimate section, we consider four well-known classes of multidimensional poverty indices proposed in the literature that are useful for cardinal variables and satisfy both the deprivation focus property and the poverty focus property. In cases where a class allows a range of parameter values, we specify the subranges of such values for which the measures in the class satisfy specific weak and/or strong distributional properties. Table 2 thus intends to provide a synthetic menu of available options, exhibiting how measures in each class are compatible with various distributional properties.

It must be noted that all measures considered in Table 2 satisfy the *weak* versions of transfer and rearrangement properties (WT and WR and/or CWR), which prevents these measures from changing counterintuitively under certain distributional transformations. Given that these measures also satisfy the poverty focus property, by Proposition 2, they also satisfy the weak versions of deprivation transfer and *deprivation* rearrangement properties. The main contribution of the table is that it allows distinguishing the measures that satisfy the strong versions of the deprivation transfer and deprivation rearrangement properties from those that do not—both within the same class of measures as well as across different classes of measures. The first two columns of the table report the research studies and the classes of measures proposed in these studies. The third column presents different parametric restrictions on these measures. The final nine columns report various distributional properties that these measures satisfy. The first three of these nine columns report whether the measures satisfy the weak versions of transfer and rearrangement properties; whereas the rest of the six columns report whether these measures satisfy the weak as well as strong versions of *deprivation* transfer and *deprivation* rearrangement properties.

Let us first consider the measures proposed by Chakravarty et al. (1998), which we classify into three categories: P_{CMR1}, P_{CMR2} and P_{CMR3}. Measures in all three categories satisfy the weak transfer property and both weak rearrangement properties, in the sense that the poverty measures do not change due to a rearrangement among the poor (RP). However, the P_{CMR3} measure is simply the average of normalized deprivation gaps and it is well-known in the poverty measurement literature that the average of normalized gaps is not strictly sensitive to distributional changes. Measures in classes P_{CMR1} and P_{CMR2}, on the other hand, are strictly sensitive to transfer. This subtle difference is not captured by the weak transfer (WT) property, but the strong deprivation transfer (SDT) property captures this subtle difference. In fact, both P_{CMR1} and P_{CMR2} would register a decrease under transformations from the example-matrix X to the example-matrix Y, as they satisfy SDT; whereas, P_{CMR3} would not register any change. Owing to the additive functional form, none of the measures proposed by Chakravarty et al. (1998) satisfy the strong versions of deprivation rearrangement properties.

Next, we present two classes of poverty measures proposed by Tsui (2002), which we refer to as P_{T1} and P_{T2}. By Proposition 5 of Tsui (2002), measures in both classes satisfy WT and WR, but not CWR. Although Tsui presented the measures for $\alpha_j \geq 0$ and $w_j \geq 0$ for all j, respectively, we purposefully consider the measures with restrictions $\alpha_j > 0$ and $w_j > 0$ for all j, respectively, so as to draw a meaningful distinction between two classes. Under the strict restrictions on the parameters, measures in both classes satisfy SDT, but only measures in P_{T1} satisfy SDR. This means that all these measures would register a decrease while transforming the example-matrix X into the example-matrix Y, but only the

measures in P_{T1} would register an increase when transforming the example-matrix X into the example-matrix L.[14]

Let us now review the pioneering class of measures proposed by Bourguignon and Chakravarty ([2003]) denoted by P_{BC}. Although all measures satisfy WT, measures in P_{BC} satisfy WR for $\alpha > \beta$; whereas measures in P_{BC} satisfy CWR for $\alpha < \beta$. All measures with the given parametric restrictions, however, satisfy SDT, exhibiting a decrease when going from the example-matrix X to the example-matrix Y. Moreover, for $\alpha > \beta$, the measures satisfy SDR, registering an increase while going from the example matrix X to the example-matrix L; whereas for $\alpha < \beta$, the measures satisfy CSDR, registering a decrease under the same transformation.

Our framework is also useful for distinguishing the various measures in the well-known class of multidimensional poverty measures, proposed by Alkire and Foster ([2011]). We consider two particular subclasses: P_{AF} for $\alpha = 1$ and P_{AF} for $\alpha = 2$. Measures in both classes satisfy WT as well as both WR and CWR in the same spirit as Chakravarty et al. ([1998]). However, measures in P_{AF} for $\alpha = 2$ are obtained by averaging the squared normalized gaps; whereas measures in P_{AF} for $\alpha = 1$ are obtained by averaging the normalized gaps. This crucial distinction is not unfortunately captured by WT. However, SDT can spot this distinction and, clearly, measures in P_{AF} for $\alpha = 2$ satisfy SDT but measures in P_{AF} for $\alpha = 1$ do not. Again, while P_{AF} for $\alpha = 2$ will register a decrease when transforming the example-matrix X into the example-matrix Y, P_{AF} for $\alpha = 1$ will not.

Similar to the distinction between measures in same classes of measures, our framework is also useful in distinguishing measures across classes. Note that except for P_{CMR3} and P_{AF} for $\alpha = 1$, all measures in Table 2 satisfy the strong deprivation transfer property (SDT). In words, whenever there is a smoothing of deprived achievements among the poor, as exemplified in the transformation from the example-matrix X to the example-matrix Y, measures satisfying SDT exhibit a reduction in poverty. However, not all well-known classes of measures reported in the table satisfy the strong deprivation rearrangement properties. Only the P_{BC} and P_{T1} classes of measures satisfy the strong deprivation rearrangement properties, registering a change under transformations, when going from the example-matrix X into the example-matrix L.

It is worth discussing however why some of the well-known multidimensional measures do not satisfy the strong deprivation rearrangement properties. Note that a policy relevant property, which requires the overall poverty to be decomposed into a weighted sum of deprivations in different dimensions—differently known as *factor decomposability* due to Chakravarty et al. ([1998]), *dimensional break-down* due to Alkire and Foster ([2011]), and *additive decomposability in attributes* due to Bossert et al. ([2013]), conflicts with the strong deprivation rearrangement properties. We thus observe here another form of incompatibility between a property that requires poverty measures to be additively decomposable across dimensions and the strong deprivation rearrangement properties.[15] The property of decomposability across dimensions may be privileged in certain cases for policy purposes, at the cost of sensitivity to strong rearrangement sensitivity. However, if interactions among dimensions are considered crucial while evaluating poverty involving multiple dimensions, strict sensitivity to rearrangements among the poor may be privileged at the cost of the decomposability property across dimensions.

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[14] While Tsui acknowledges that measures in P_{T2} do not satisfy SDR, he does not address it formally.

[15] For a discussion on such incompatibilities in case of ordinal variables, see Alkire and Foster ([2016]).

6 Concluding Remarks

In this paper, we address an important incompatibility between the focus properties and the traditional definitions of distributional properties in the multidimensional poverty measurement framework and propose a refinement of the distributional properties. Specifically, when poverty measures are required to disregard improvements in the non-deprived achievements within dimensions (deprivation focus), the measures may remain unchanged due to distributional transformations among the poor in their non-deprived achievements based on the traditional definitions. Owing to this incompatibility, measures are traditionally required to be weakly sensitive to distributional transformations rather than being strongly sensitive. The main limitation of allowing measures to be weakly sensitive to distributional transformations is that it lets measures to be invariant to these crucial transformations.

We formally address this shortcoming and redefine the distributional properties on a restricted domain of achievements, so that the distributional transformations are restricted to occur *among the poor, in their deprived achievements only*. This redefinition allows us to discern the measures that are only weakly sensitive to distributional transformations from the measures that are strongly sensitive to such transformations. If such strong sensitivity is deemed important, then our redefinition should be crucial in distinguishing measures. We, in fact, analyse some of the well-known multidimensional measures in the penultimate section of the paper and illustrate how measures within the same class as well as measures across different classes can be differentiated from each other.

We acknowledge that the selection of a poverty measure in normative framework for a specific use requires certain choices to be made. This paper, thus, alongside several others in the literature, exposes the fact that the compliance with one property entails ceding on some other properties. For example, if potential interactions among dimensions at all levels of achievements are considered relevant, then preference may be placed over selecting poverty measures that do not satisfy the deprivation focus property and the strong distributional properties as traditionally defined may be sufficient. If, however, potential interactions among dimensions are not considered to be relevant at all levels of achievement, then one must adjust the distributional properties in order to distinguish measures that are sensitive to distributional transformations among the poor in their deprived achievements from the ones that are not. Similarly, for some purposes, it may be considered key for the measures to be broken-down into dimensional deprivations, at the expense of being insensitive to rearrangement transformations; whereas, in some other purposes, capturing strong interactions across dimensions may be considered more significant and thus the possibility of decomposing a measure into dimensional deprivations may be sacrificed. Our redefinition of the distributional properties offered in this paper, as well as with the surrounding discussion, should facilitate the choice among the menu of available measures, according to the requirements of the measurement exercise at hand.

References

Aaberge, R., & Peluso, E. (2012). *A counting approach for measuring multidimensional deprivation.* IZA discussion paper no. 6589. Retrieved from https://ssrn.com/abstract=2085176. Accessed July 2018.

Alkire, S., & Foster, J. E. (2011). Counting and multidimensional poverty measurement. *Journal of Public Economics, 95*(7), 487–487.

Alkire, S., & Foster, J. E. (2016). Dimensional and distributional contributions to multidimensional poverty. OPHI Working Paper 100, University of Oxford.

Alkire, S., Foster, J. E., Seth, S., Santos, M. E., Roche, J. M., & Ballon, P. (2015). *Multidimensional poverty measurement and analysis*. Oxford: Oxford University Press.

Arcagni, A., Belgiojoso, E. B., Fattore, M., & Rimoldi, S. M. (2019). Multidimensional analysis of deprivation and fragility patterns of migrants in Lombardy, using partially ordered sets and self-organizing maps. *Social Indicators Research, 141,* 551–579.

Asselin, L. M. (2009). *Analysis of multidimensional poverty: Theory and case studies*. New York: Springer.

Atkinson, A. B. (2003). Multidimensional deprivation: Contrasting social welfare and counting approaches. *Journal of Economic Inequality, 1*(1), 51–65.

Bossert, W., Chakravarty, S. R., & D'Ambrosio, C. (2013). Multidimensional poverty and material deprivation with discrete data. *Review of Income and Wealth, 59*(1), 29–43.

Bourguignon, F. J., & Chakravarty, S. R. (2003). The measurement of multidimensional poverty. *Journal of Economic Inequality, 1*(1), 25–49.

Bresson, F. (2009). *Multidimensional poverty measurement with the weak focus axiom*. Nice: Mimeo, Université de Nice Sophia Antipolis.

Chakravarty, S. R., & D'Ambrosio, C. (2006). The measurement of social exclusion. *Review of Income and Wealth, 52*(3), 377–398.

Chakravarty, S. R., Mukherjee, D., & Ranade, R. (1998). On the family of subgroup and factor decomposable measures of multidimensional poverty. *Research on Economic Inequality, 8,* 175–194.

Datt, G. (2013). *Making every dimension count: Multidimensional poverty without the "dual cut"*. Discussion paper 32/13, Monash University, Department of Economics.

Decancq, K., Fleurbaey, M., & Maniquet, F. (2014). *Multidimensional poverty measurement with individual preferences.* Research paper no. 058, Princeton University, William S. Dietrich II Economic Theory Center. Retrieved from https://doi.org/10.2139/ssrn.2388959.

Donaldson, D., & Weymark, J. A. (1986). Properties of fixed-population poverty indices. *International Economic Review, 27*(3), 667–688.

Fattore, M. (2016). Partially ordered sets and the measurement of multidimensional ordinal deprivation. *Social Indicators Research, 128,* 835–858.

Foster, J. E. (2006). Poverty indices. In A. de Janvry & R. Kanbur (Eds.), *Poverty, inequality and development: Essays in honor to Erik Thorbecke*. New York: Springer.

Maasoumi, E., & Lugo, M. A. (2008). The information basis of multivariate poverty assessments. *Quantitative approaches to multidimensional poverty measurement* (pp. 1–29). Basingstoke: Palgrave Macmillan.

Nicholas, A., & Ray, R. (2012). Duration and persistence in multidimensional deprivation: Methodology and Australian application. *Economic Record, 88*, 106–126.

Sen, A. K. (1976). Poverty: An ordinal approach to measurement. *Econometrica, 44*(2), 219–231.

Sen, A. K. (1999). *Development as freedom*. Oxford: Oxford University Press.

Sen, A. K. (2009). *The idea of justice*. Cambridge, Massachusetts: Harvard University Press.

Tsui, K.-Y. (2002). Multidimensional poverty indices. *Social Choice and Welfare, 19*(1), 69–93.