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On the A_∞ conditions for general bases

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Abstract We discuss several characterizations of the A_∞ class of weights in the setting of general bases. Although they are equivalent for the usual Muckenhoupt weights, we show that they can give rise to different classes of weights for other bases. We also obtain new characterizations for the usual A_∞ weights.

Keywords Weights · Muckenhoupt bases · A_p Classes

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1 Introduction

Muckenhoupt defined in [25] the A_∞ class of weights in $\mathbb{R}^n$ as the collection of nonnegative locally integrable functions w satisfying the following condition: given $\epsilon > 0$ there exists

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$\delta > 0$ such that if Q is a cube, $E \subset Q$ and $|E| < \delta |Q|$, then $w(E) < \epsilon w(Q)$, where $|E|$ denotes the Lebesgue measure of the set E and $w(E)$ the integral of w on E. Then he proved that w is in A_∞ if and only if $w \in A_p$ for some $p > 1$.

Coifman and Fefferman [7] defined A_∞ as follows: there are constants $C, \delta > 0$ such that for any cube Q and any measurable set $E \subset Q$ it holds that

$$\frac{w(E)}{w(Q)} \leq C \left(\frac{|E|}{|Q|} \right)^\delta . \tag{1.1}$$

They also proved that this condition is equivalent to being in A_p for some $p > 1$. Condition (1.1) appeared previously in [6] without proving the equivalence (see also [18]).

Other characterizations of A_∞ were given by Fujii [12], an apparently less-known paper. Later, Hruščev [20] and independently García-Cuerva and Rubio de Francia in the book [13] introduced another characterization, obtained as the limit when p goes to infinity of the A_p condition, namely,

$$\frac{1}{|Q|} \int_Q w \leq C \exp \left(\frac{1}{|Q|} \int_Q \log w \right). \tag{1.2}$$

All the mentioned characterizations including those in Fujii's paper were also collected in the survey [11].

Reviewing several presentations of the theory of A_p weights we can see that the authors do not agree in using a particular condition as the definition of the A_∞ class. The condition of Muckenhoupt's paper mentioned in the first paragraph is chosen in [30] and the variant in which the condition is required only for some $\epsilon, \delta < 1$ is preferred in [11] and [33]. Condition (1.1) is the definition given in [9, 13, 14]; the definition in [17] is (1.2); and the authors of [8] use the union of the A_p classes to define A_∞. A different condition using medians [(labeled as (P5) in Definition 2.5] is chosen in [31].

The authors of this paper studied in [10] the weights associated to the maximal operator defined on $(0, +\infty)$ by the basis of intervals of the form $(0, b)$, with $b > 0$, which are also the weights associated to the Calderón operator. We proved that the corresponding classes of A_p weights do not behave as the usual ones: their union is strictly contained in the class of weights given by an exponential condition of type (1.2), or they need not satisfy a reverse Hölder inequality, for instance. This last property has been observed also for weights associated to other bases.

Our aim in this paper is to compare the different characterizations of A_∞ in a general context. For this, we use a basis in an arbitrary measure space and write in Sect. 2 a list of conditions that when particularized to the bases of cubes (or balls) in $\mathbb{R}^n$ are known to be equivalent. We first prove in Sect. 3 that some of them are equivalent without further assumptions on the basis, and then in Sect. 4 some one-way implications. Section 5 is devoted to the counterexamples. In a few cases, we have not been able to decide whether the conditions are somehow related. For the basis of intervals associated to the Calderón weights studied in [10], and also for the basis of Carleson cubes in a half-space, we prove in Sect. 6 the precise inclusions for all the classes of weights defined by the conditions of Sect. 2. In Sect. 7 we consider bases of cubes and rectangles, and introduce other characterizations of A_∞. Finally, we define a *BMO* space associated to the basis and show that the classical result relating functions in *BMO* to logarithms of A_∞ weights does not hold in general.

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2 Characterizations of A_∞

Given a measure space (X, μ) with a σ-finite measure, a basis $\mathcal{B}$ is a collection of μ-measurable subsets B of X such that $0 < \mu(B) < \infty$. The maximal function associated to $\mathcal{B}$ is

$$M_{\mathcal{B}} f(x) = \sup_{B \in \mathcal{B}: x \in B} \frac{1}{\mu(B)} \int_B |f| \, d\mu, \tag{2.1}$$

where f is a (real or complex) μ-measurable function and the supremum can be infinity. To avoid technicalities we assume that the union of the elements of $\mathcal{B}$ is X up to a set of μ-measure zero. Thus (2.1) is defined for almost every $x \in X$. If the basis is formed by open sets or if it is countable, for instance, $M_{\mathcal{B}} f$ is measurable, which we assume when needed.

A weight w for a given basis $\mathcal{B}$ is a μ-measurable nonnegative function in X such that the integral of w over the sets of $\mathcal{B}$ is finite. For a weight w and a μ-measurable set E we write

$$w(E) = \int_E w \, d\mu.$$

If $w \equiv 1$, we write $\mu(E)$ instead. The average of w on B with respect to μ is denoted by w_B, that is,

$$w_B = \frac{w(B)}{\mu(B)}. \tag{2.2}$$

A function f is in $L^p(w)$ if

$$\int_X |f|^p w \, d\mu < +\infty.$$

Definition 2.1 Let $\mathcal{B}$ be a basis and $1 < p < \infty$. A weight w is in $A_{p,\mathcal{B}}$ if it satisfies

$$\sup_{B \in \mathcal{B}} \left(\frac{1}{\mu(B)} \int_B w \, d\mu \right) \left(\frac{1}{\mu(B)} \int_B w^{1-p'} \, d\mu \right)^{p-1} < \infty. \tag{2.3}$$

We say that the basis $\mathcal{B}$ is a Muckenhoupt basis if the maximal operator $M_{\mathcal{B}}$ is bounded on $L^p(w)$ for each p, $1 < p < \infty$, and for every $w \in A_{p,\mathcal{B}}$.

Definition 2.2 Let $\mathcal{B}$ be a basis and $1 < q < \infty$. A weight w is in $RH_{q,\mathcal{B}}$ if for some $C > 0$ and every $B \in \mathcal{B}$ it satisfies

$$\left(\frac{1}{\mu(B)} \int_B w^q \, d\mu \right)^{1/q} \leq \frac{C}{\mu(B)} \int_B w \, d\mu. \tag{2.4}$$

Remark 2.3 Let the measure ν be given by $d\nu = w d\mu$. The $A_{p,\mathcal{B}}$ condition (2.3) can be written as

$$\left(\frac{1}{\nu(B)} \int_B (w^{-1})^{p'} d\nu \right)^{1/p'} \leq \frac{C}{\nu(B)} \int_B w^{-1} \, d\nu, \tag{2.5}$$

which is a reverse Hölder inequality of w^{-1} with respect to ν with exponent p'.

Definition 2.4 Let w be a weight and B a set in the basis. The median of w in B is a number $m(w; B)$ such that

$$\mu(\{x \in B : w(x) < m(w; B)\}) \leq \frac{1}{2} \mu(B)$$

and

$$\mu(\{x \in B : w(x) > m(w; B)\}) \leq \frac{1}{2} \mu(B).$$

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If there is more than one value satisfying the conditions, the median can be chosen as the largest possible value, for instance.

The choice of the largest value does not coincide with the definition of [31]. Nevertheless, any number satisfying the stated conditions can be used as the median in the results appearing in this paper.

Note that

$$m(w; B) \leq 2w_B, \tag{2.6}$$

due to the inequality

$$\mu(\{x \in B : w(x) \geq m(w; B)\}) \geq \frac{1}{2}\mu(B).$$

Definition 2.5 Let $\mathcal{B}$ be a basis and w a weight such that $0 < w(B) < \infty$ for every $B \in \mathcal{B}$. We define the following properties that w may satisfy or not.

(P1) $w \in \bigcup_{1<p<\infty} A_{p,\mathcal{B}}$.

(P1') There exist $\delta, C > 0$ such that for every $B \in \mathcal{B}$ and every μ-measurable set E contained in B it holds that

$$\frac{\mu(E)}{\mu(B)} \leq C\left(\frac{w(E)}{w(B)}\right)^{\delta}.$$

(P2) There exists $C > 0$ such that for every $B \in \mathcal{B}$ it holds that

$$\frac{1}{\mu(B)}\int_B w\, d\mu \leq C \exp\left(\frac{1}{\mu(B)}\int_B \log w\, d\mu\right).$$

(P2') There exists $C > 0$ such that for every $B \in \mathcal{B}$ and for every $s \in (0, 1)$ it holds that

$$\frac{1}{\mu(B)}\int_B w\, d\mu \leq C\left(\frac{1}{\mu(B)}\int_B w^s\, d\mu\right)^{1/s}.$$

(P3) $w \in \bigcup_{1<q<\infty} RH_{q,\mathcal{B}}$.

(P3') There exist $\delta, C > 0$ such that for every $B \in \mathcal{B}$ and every μ-measurable set E contained in B it holds that

$$\frac{w(E)}{w(B)} \leq C\left(\frac{\mu(E)}{\mu(B)}\right)^{\delta}.$$

(P4) There exist $\alpha, \beta \in (0, 1)$ such that for every $B \in \mathcal{B}$ and every μ-measurable set E contained in B for which $\mu(E) < \alpha\mu(B)$ it holds that $w(E) \leq \beta w(B)$.

(P4') There exist $\alpha, \beta \in (0, 1)$ such that for every $B \in \mathcal{B}$ it holds that

$$\mu(\{x \in B : w(x) \leq \alpha w_B\}) \leq \beta\mu(B).$$

(P5) There exists $C > 0$ such that for every $B \in \mathcal{B}$ it holds that

$$w_B \leq C m(w; B).$$

(This inequality is the reverse of (2.6).)

(P6) There exists $C > 0$ such that for every $B \in \mathcal{B}$ it holds that

$$\int_B w \log^+ \frac{w}{w_B} d\mu \leq C w(B).$$

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(P7) There exists $C > 0$ such that for every $B \in \mathcal{B}$ it holds that

$$\int_B M_{\mathcal{B}}(w\chi_B)d\mu \leq Cw(B).$$

(P8) There exist $C, \beta > 0$ such that for every $B \in \mathcal{B}$ and for every $\lambda > w_B$ it holds that

$$w(\{x \in B : w(x) > \lambda\}) \leq C\lambda\mu(\{x \in B : w(x) > \beta\lambda\}).$$

When $\mathcal{B}$ is the basis of cubes (or Euclidean balls) in $\mathbb{R}^n$ equipped with the Lebesgue measure, all these properties are equivalent. The reader will easily find in the literature the equivalence of most of them.

Condition (P1') corresponds to the characterization of the restricted weak-type for the Hardy–Littlewood maximal operator by Kerman and Torchinsky [23]. Condition (P2') for fixed s is the same as saying that w^s is in $RH_{1/s,\mathcal{B}}$. The fact that the usual A_∞ weights hold this property was observed in [32, Lemma 6.1]. We need the uniformity of the constant C for $s \in (0, 1)$ to obtain the equivalence of (P2) and (P2'), but in the case of the usual A_∞ weights just one value of s in $(0, 1)$ is enough for the characterization. Condition (P5) using medians was introduced by Strömberg and Torchinsky [31], as the definition of the A_∞ class. Conditions (P6) and (P7) appear in the characterizations of the A_∞ class by Fujii [12]. Condition (P7) was also used by Wilson [34] and subsequent works, and has received special attention in recent times because its constant (the smallest value of C satisfying the inequality) has been used to write sharp bounds for the norms of some operators as in [21,22,28], for instance. Condition (P6) is presented in [4,5] as a limit case of the reverse Hölder inequalities and the value of the best constant in such inequality is compared with the constant in (P2) for cubes and for dyadic cubes in $\mathbb{R}^n$. Condition (P8) appears in [7] as a step in the proof of the reverse Hölder inequality, and in [26,27] it is presented as one of the characterizations of A_∞.

Fix w and consider the measure ν given by $d\nu = wd\mu$. Let us say that a weight v satisfies $(\text{Pj})_\nu$ ($j = 1, 2, \ldots, 6, 8$) if it satisfies (Pj) with v instead of w and ν instead of μ. As indicated in Remark 2.3, (P1) for w is equivalent to $(\text{P3})_\nu$ for w^{-1}. It is easy to check that (P4) for w and $(\text{P4})_\nu$ for w^{-1} are equivalent (apply (P4) to $B\backslash E$ to reverse the inequalities). In the terminology of [7] this means that the measures μ and ν are equivalent to each other. Based on this symmetry, in the case of the usual A_∞ weights one can write characterizations of the type "w^{-1} satisfies $(\text{Pj})_\nu$" that are not listed above (see, for instance, inequality (7.1) in Sect. 7). Other conditions involving medians and also the measure ν are in [31]. Finally, let us also mention the Orlicz type conditions in [16] and the recent characterization of [19] involving a restricted weak-type inequality for the maximal operator.

3 Equivalences

In this section we prove some equivalences among the previous conditions.

Theorem 3.1 (a) *Conditions* (P2) *and* (P2') *are equivalent.*
(b) *Conditions* (P3) *and* (P3') *are equivalent.*
(c) *Conditions* (P1) *and* (P1') *are equivalent.*
(d) *Conditions* (P4) *and* (P4') *are equivalent.*

Proof (a) The equivalence of (P2) and (P2') follows from the fact that the function

$$\varphi(s) = \left(\frac{1}{\mu(B)}\int_B w^s\, d\mu\right)^{1/s}$$

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is increasing and

$$\lim_{s\to 0+}\varphi(s)=\exp\left(\frac{1}{\mu(B)}\int_B \log w\,d\mu\right).$$

(b) To see that (P3) implies (P3') we have

$$w(E)=\int_E w\,d\mu\le\left(\int_E w^q\,d\mu\right)^{1/q}\mu(E)^{1/q'}\le Cw(B)\left(\frac{\mu(E)}{\mu(B)}\right)^{1/q'},$$

which is (P3') with $\delta=1/q'$.
To prove the converse, assume $\delta<1$. Set $E_\lambda=\{x\in B: w(x)>\lambda\}$. From (P3') we obtain

$$\lambda\mu(E_\lambda)\le w(E_\lambda)\le Cw(B)\left(\frac{\mu(E_\lambda)}{\mu(B)}\right)^{\delta}.$$

Take $1<q<1/(1-\delta)$. Then

$$\begin{aligned}\int_B w^q\,d\mu&=q\int_0^\infty \lambda^{q-1}\mu(E_\lambda)d\lambda\\&\le q\int_0^A\lambda^{q-1}d\lambda\;\mu(B)+q\int_A^\infty\lambda^{q-1-1/(1-\delta)}d\lambda\left(\frac{Cw(B)}{\mu(B)^\delta}\right)^{1/(1-\delta)}\\&=A^q\mu(B)+C(q,\delta)A^{q-1/(1-\delta)}\left(\frac{Cw(B)}{\mu(B)^\delta}\right)^{1/(1-\delta)}.\end{aligned}$$

Choose $A=w_B$ to obtain (P3).

(c) Let ν the measure given by $d\nu=wd\mu$. If we write (P3') for the weight w^{-1} with respect to the measure ν, we get (P1'). On the other hand, Remark 2.3 shows that the condition $w\in A_{p,\mathcal{B}}$ is a reverse Hölder inequality for w^{-1} with respect to ν. Then (c) follows from (b).

(d) Assume (P4) for some α and β. Fix $\beta_1\in(1-\alpha,1)$ and $\alpha_1\in(0,1-\beta)$. We claim that (P4') holds with α_1 and β_1, that is, for every $B\in\mathcal{B}$ we have

$$\mu(\{x\in B: w(x)\le\alpha_1 w_B\})\le\beta_1\mu(B).$$

If not, we would have

$$\mu(\{x\in B: w(x)\le\alpha_1 w_B\})>\beta_1\mu(B).$$

Set $E=\{x\in B: w(x)>\alpha_1 w_B\}$. Then

$$\mu(B)=\mu(E)+\mu(\{x\in B: w(x)\le\alpha_1 w_B\})>\mu(E)+\beta_1\mu(B).$$

Therefore, $\mu(E)<(1-\beta_1)\mu(B)<\alpha\mu(B)$, and using (P4) we deduce $w(E)\le\beta w(B)$. It follows that

$$w(\{x\in B: w(x)\le\alpha_1 w_B\})\ge(1-\beta)w(B).$$

On the other hand,

$$w(\{x\in B: w(x)\le\alpha_1 w_B\})\le\alpha_1 w_B\mu(B)=\alpha_1 w(B).$$

This is in contradiction with our choice of α_1.

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Assume now (P4') and write α_1 and β_1 the values of α and β appearing in the condition. Let $E \subset B$ such that $w(E) > \beta w(B)$ where β will be chosen later. Set $S = B \backslash E$. Then $w(S) < (1-\beta)w(B)$. Write $S = S_1 \cup S_2$ where

$$S_1 = \{x \in S : w(x) > \alpha_1 w_B\}, \quad S_2 = S \backslash S_1.$$

Then $\mu(S_2) \le \beta_1 \mu(B)$ by the assumption and

$$\mu(S_1) \le \frac{1}{\alpha_1 w_B} w(S) \le \frac{1-\beta}{\alpha_1} \mu(B).$$

If we choose $\alpha, \beta \in (0,1)$ such that $\beta_1 + \frac{1-\beta}{\alpha_1} < 1-\alpha$, we have $\mu(S) < (1-\alpha)\mu(B)$, that is, $\mu(E) > \alpha\mu(B)$. □

4 Implications and restricted implications

In the first theorem of this section we establish several implications without additional assumptions. The second theorem presents several restricted implications, that is, implications for which we require boundedness assumptions on the maximal operator $M_{\mathcal{B}}$.

Theorem 4.1 *The following chains of implications hold:*

(a) (P1) ⇒ (P2) ⇒ (P5) ⇒ (P4).
(b) (P8) ⇒ (P3) ⇒ (P6) ⇒ (P4).

Proof (a) Assume that w is in $A_{p_0,\mathcal{B}}$ for some p_0. Then it is in $A_{p,\mathcal{B}}$ for $p > p_0$ and satisfies (2.3) with uniform constant in the right-hand side. Taking the limit in (2.3) when p goes to infinity, we get (P2).

Assume now (P2'). Set $E^+ = \{x \in B : w(x) > m(w; B)\}$. With use of Hölder's inequality, (P2') and the definition of the median we have for $s < 1$:

$$\int_{E^+} w^s \, d\mu \le (w(B))^s \mu(E^+)^{1-s} \le C^s 2^{s-1} w^s(B),$$

where C is the constant of (P2'). Choose s such that $C^s 2^{s-1} < 3/4$. Then

$$m(w; B)^s \mu(B) \ge \int_{B \backslash E^+} w^s \, d\mu \ge \frac{1}{4} w^s(B) \ge \frac{1}{4C} \mu(B)^{1-s} (w(B))^s,$$

from which it follows that $w_B \le (4C)^{1/s} m(w; B)$. This is (P5).

Assume that (P5) holds. Take $\alpha = 1/4$ and let E be a subset of B such that $\mu(E) < \mu(B)/4$. From the definition of the median we deduce

$$\mu(\{x \in B \backslash E : w(x) \ge m(w; B)\}) \ge \frac{1}{4}\mu(B).$$

Therefore

$$w(B \backslash E) \ge \frac{1}{4} m(w; B)\mu(B) \ge \frac{1}{4C} w(B).$$

Thus (P4) holds for $\beta < 1 - 1/(4C)$.

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(b) Assume that (P8) holds. Then

$$\begin{aligned}\frac{1}{\mu(B)}\int_B w^{1+\delta}\,d\mu &= \frac{\delta}{\mu(B)}\int_0^\infty \lambda^{\delta-1} w(\{x\in B : w(x)>\lambda\})\,d\lambda\\ &= \frac{\delta}{\mu(B)}\int_0^{w_B} \lambda^{\delta-1} w(\{x\in B : w(x)>\lambda\})\,d\lambda\\ &\quad + \frac{\delta}{\mu(B)}\int_{w_B}^\infty \lambda^{\delta-1} w(\{x\in B : w(x)>\lambda\})\,d\lambda\\ &\le (w_B)^{1+\delta} + \frac{C\delta}{\mu(B)}\int_{w_B}^\infty \lambda^{\delta}\mu(\{x\in B : w(x)>\beta\lambda\})\,d\lambda\\ &\le (w_B)^{1+\delta} + \frac{C\delta}{\beta^{1+\delta}}\frac{1}{\mu(B)}\int_B w^{1+\delta}\,d\mu.\end{aligned}$$

Therefore w is in $RH_{1+\delta,\mathcal{B}}$ if δ is such that $C\delta\beta^{-(1+\delta)} < 1$.

Assume now (P3), that is, w is in $RH_{q,\mathcal{B}}$ for some $q > 1$. Assume $w_B = 1$, or equivalently $w(B) = \mu(B)$. Set $E_k = \{x \in B : 2^{k-1} < w(x) \le 2^k\}$. From (2.4), we deduce $2^{(k-1)q}\mu(E_k) \le C^q\mu(B)$. Then

$$\int_B w\log^+\frac{w}{w_B}\,d\mu \le C\sum_{k=1}^{\infty} k2^k\mu(E_k) \le C\mu(B),$$

and we obtain (P6).

Assume finally (P6) and let $E \subset B$ such that $\mu(E) < \alpha\mu(B)$. The inequality $ab \le a\log a - a + e^b$ holds for $a > 1$ and $b > 0$. Assume $w_B = 1$. We have

$$\begin{aligned}w(E) &= \int_{E\cap\{w\le 1\}} w\,d\mu + \int_{E\cap\{w>1\}} w\,d\mu\\ &\le \mu(E) + \frac{1}{b+1}\int_E (w\log^+ w + e^b)\,d\mu\\ &\le \left(1+\frac{e^b}{b+1}\right)\mu(E) + \frac{C}{b+1}\mu(B),\end{aligned}$$

where C is the constant of (P6). With $b = 2C-1$, choose α such that $\alpha\left(1+\frac{e^b}{b+1}\right) < 1/4$. As $w(B) = \mu(B)$, (P4) holds for such α and $\beta = 3/4$.

Note that a direct proof of the implication (P3) $\Rightarrow$ (P4) is easy using (P3'). Indeed, it is enough to take α such that $\beta = C\alpha^\delta < 1$ □

Theorem 4.2 (a) *Assume that $M_{\mathcal{B}}$ is a Muckenhoupt basis. Then* (P1) *implies* (P7).
(b) *Assume that $M_{\mathcal{B}}$ is bounded on $L^p(\mu)$ for every $p > 1$. Then* (P3) *implies* (P7).
(c) *Assume that $M_{\mathcal{B}}$ is of weak-type* (1, 1). *Then* (P6) *implies* (P7).
(d) *Assume that $M_{\mathcal{B}}$ is bounded on $L^p(\mu)$ for some $p > 1$. Assume also that for each B and $x \in B$, $M_{\mathcal{B}}(w\chi_B)(x)$ coincides with the supremum when the averages are taken only on elements of $\mathcal{B}$ contained in B. Then* (P2) *implies* (P7).

Proof (a) If $w \in A_{p,\mathcal{B}}$, then $w^{1-p'} \in A_{p',\mathcal{B}}$. Since $M_{\mathcal{B}}$ is bounded on $L^{p'}(w^{1-p'})$ we have

$$\int_B M_{\mathcal{B}}(w\chi_B)\,d\mu \le \left(\int_B M_{\mathcal{B}}(w\chi_B)^{p'} w^{1-p'}\,d\mu\right)^{1/p'} w(B)^{1/p} \le Cw(B).$$

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(b) Let w be in $RH_{q,\mathcal{B}}$ for some $q > 1$. Using Hölder's inequality and the boundedness of $M_{\mathcal{B}}$ on L^q we have

$$\int_B M_{\mathcal{B}}(w\chi_B)\,d\mu \le \left(\int_B M_{\mathcal{B}}(w\chi_B)^q\,d\mu\right)^{1/q} \mu(B)^{1/q'}$$
$$\le C\left(\int_B w^q\right)^{1/q} \mu(B)^{1/q'} \le Cw(B).$$

(c) If $M_{\mathcal{B}}$ is weak (1, 1), we have

$$\mu(\{x : M_{\mathcal{B}}f(x) > \lambda\}) \le \frac{C}{\lambda}\int_{\{f(x)>\lambda/2\}} f\,d\mu.$$

Then

$$\int_B M_{\mathcal{B}}(w\chi_B)\,d\mu = \int_0^\infty \mu(\{x \in B : M_{\mathcal{B}}(w\chi_B)(x) > \lambda\})d\lambda$$
$$\le \int_0^{2w_B} \mu(B)d\lambda + \int_{2w_B}^\infty \frac{C}{\lambda}\int_{\{x\in B:w(x)>\lambda/2\}} w\,d\mu\,d\lambda$$
$$\le C\int_B w(1+\log^+\frac{w}{w_B})\,d\mu.$$

(d) From the assumption and (P2') we have

$$M_{\mathcal{B}}(w\chi_B)(x) \le CM_{\mathcal{B}}(w^s\chi_B)(x)^{1/s}$$

for $s < 1$. Take $s = 1/p$ and use the L^p boundedness of $M_{\mathcal{B}}$.

□

The condition required in (d) holds for the bases of cubes or rectangles in $\mathbb{R}^n$, or for the basis $\mathcal{B} = \{(0, b) : b > 0\}$ in $(0, +\infty)$, for instance.

5 Counterexamples

We build the counterexamples for $\mathcal{B} = \{(0, b) : b > 0\}$ considered as a basis in $(0, \infty)$ and take μ to be the Lebesgue measure.

Counterexample 1 Let w be the characteristic function of $(0, 1)\cup(3, \infty)$. This weight satisfies (P8) with $C = 3$ and $\beta = 1$. Just take into account that

$$w(\{x < b : w(x) > \lambda\}) = |\{x < b : w(x) > \lambda\}|$$

and $w_B \ge 1/3$ for every $B \in \mathcal{B}$. As a consequence, w also satisfies (P3), (P4), (P6) and (P7). On the other hand, w does not satisfy (P5) because $m(w; B) = 0$ for $B = (0, b)$ and $b \in (2, 4)$; hence, neither satisfies (P1) nor (P2).

If w is the characteristic function of $(0, 2) \cup (3, \infty)$, then w satisfies (P5), because $m(w; B) = 1$ for all B, but it does not satisfy (P2).

Counterexample 2 The weight w appearing in Proposition 4.4 of [10] satisfies (P2) but not (P1). It is defined as $w(x) = (x - 2^k)^k$ for $x \in (2^k, 2^k + 1)$ $(k = 1, 2, \ldots)$, and $w(x) = 1$ otherwise.

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Counterexample 3 Let $I_i = (2^i + 2^{-i}, 2^i + 1)$, let $\Omega = \cup_{i=1}^{\infty} I_i$ and let Ω^c be its complement. The weight

$$w(x) = \chi_{\Omega^c}(x) + \sum_{i=1}^{\infty} \frac{1}{(x - 2^i)^2} \chi_{I_i}(x)$$

satisfies (P1) (actually, it is in $A_{p,\mathcal{B}}$ for all p). In [10] (see Remark 4.5) it is proved that it does not satisfy a reverse Hölder inequality. But even (P6) fails, because

$$\int_0^{2^k} w \le C_1 2^k \quad \text{and} \quad \int_0^{2^k} w \log^+ w \ge C_2 k 2^k.$$

Counterexample 4 The weight

$$w(x) = \chi_{(0,1)\cup(7/2,+\infty)} + \frac{1}{(x-3)\log^2(x-3)} \chi_{(3,7/2)}$$

satisfies (P7), but neither (P6) nor (P5) hold for w. Consequently, none of (P3), (P8), (P2) and (P1) can be fulfilled.

To see that (P7) holds it is enough to realize that

$$\frac{b}{2} \le \int_0^b w \le 2b$$

for all $b > 0$, and that this implies $M_{\mathcal{B}}(w\chi_{(0,b)})(x) \le 2$.

It is clear that w does not satisfy (P6) because $w \log^+ w$ is not integrable in $(0, b)$ for $b > 3$. On the other hand, w does not satisfy (P5) as in the first case of Counterexample 1.

Counterexample 5 Replacing $\log^2(x - 2)$ with $\log^3(x - 2)$ in the previous example gives a weight satisfying (P6) but not (P3).

Counterexample 6 The weight

$$w(x) = \chi_{(0,1)\cup(2,+\infty)} + \frac{1}{(x-1)^{1/2}} \chi_{(1,2)}$$

satisfies (P3) but not (P8). It is clear that the reverse Hölder inequality holds for $q < 2$. Let $B = (0, 1 + \epsilon)$ and note that $w_B < 2$. We can assume that $\beta < 1/2$. Take $\lambda = 1/\beta$. For $\epsilon < \beta^2$ we have

$$\{x \in B : w(x) > \lambda\} = \{x \in B : w(x) > \beta\lambda\} = (1, 1 + \epsilon).$$

Assuming that (P8) holds, we would get $2\epsilon^{1/2} \le C(2 + 1/\beta)\epsilon$, which is impossible for small ϵ.

Counterexample 7 The weight $w(x) = 2xe^{x^2}$ satisfies (P7) but not (P4). Since $w(0, x)/x$ is increasing, we deduce that

$$M_{\mathcal{B}}(w\chi_{(0,b)})(x) = \frac{w(0, b)}{b}$$

for every $x \in (0, b)$, so that (P7) holds.

For fixed $\alpha \in (0, 1)$ we have

$$\lim_{b\to\infty} \frac{w(b - \alpha b, b)}{w(0, b)} = \lim_{b\to\infty} \frac{e^{b^2} - e^{b^2(1-\alpha)^2}}{e^{b^2} - 1} = 1,$$

and (P4) fails.

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Remark 5.1 The reader can check that all the previous counterexamples can be easily adapted to the basis of the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$ formed by the Carleson cubes, that is,

$$\mathcal{B} = \{Q \times (0, l) : Q \text{ cube in } \mathbb{R}^n \text{of sidelength } l\}.$$

It is enough to consider weights depending only on the last variable.

The $A_{p,\mathcal{B}}$ weights corresponding to this basis when $n = 1$ are those characterizing the weighted L^p boundedness of the Bergman projection in the upper half-plane and are known as Békollé-Bonami weights. See [2,3] for the analogous characterization in the unit ball, or [24].

6 Eight different $A_{\infty,\mathcal{B}}$ classes

We have neither a proof nor a counterexample to decide whether (P4) implies (P7) in general. For the basis for which the counterexamples of the previous section have been built, and also for the basis of Carleson cubes, we can indeed prove the implication.

Theorem 6.1 *Let $\mathcal{B}$ be either $\mathcal{B} = \{(0, b) : b > 0\}$ as a basis in $(0, \infty)$ or the basis of Carleson cubes in the upper half-space. In both cases take μ to be the Lebesgue measure. Then* (P4) *implies* (P7).

Proof (i) We consider first the basis of intervals starting at 0. Fix $b > 0$. Using (P4) we know that

$$w(0, \alpha^n b) \le \beta w(0, \alpha^{n-1} b) \le \cdots \le \beta^n w(0, b).$$

For $x \in (\alpha^n b, \alpha^{n-1} b)$, we have

$$\frac{w(0, x)}{x} \le \frac{w(0, \alpha^{n-1} b)}{\alpha^n b} \le \frac{\beta^{n-1} w(0, b)}{\alpha^n b}.$$

If $\beta \le \alpha$, we deduce that $M_{\mathcal{B}}(w\chi_{(0,b)})(x) \le w(0, b)/\alpha b$, and (P7) holds. If $\beta > \alpha$ and $x \in (\alpha^n b, \alpha^{n-1} b)$, we have

$$M_{\mathcal{B}}(w\chi_{(0,b)})(x) \le \frac{\beta^{n-1} w(0, b)}{\alpha^n b}.$$

Therefore

$$\int_0^b M_{\mathcal{B}}(w\chi_{(0,b)})(x)\, dx \le \sum_{n=1}^{\infty} (\alpha^{n-1} b - \alpha^n b) \frac{\beta^{n-1} w(0, b)}{\alpha^n b}$$
$$= \frac{1 - \alpha}{\alpha(1 - \beta)} w(0, b).$$

(ii) We deal now with the basis of Carleson cubes. For simplicity, we write the proof in the case of the half-plane, but it can be extended to any number of dimensions. We can assume that $\alpha = 1/N$ for some integer N.

We work with the cube $Q = [0, 1] \times [0, 1]$. Any other Carleson cube is obtained from Q by translation and dilation. To compute $M_{\mathcal{B}}(w\chi_Q)(x)$ for $x \in Q$ we only need to consider Carleson cubes containing x and contained in Q.

First we define a discretized maximal operator as follows. For $j = 0, 1, 2, \ldots$, and $k = 1, 2, \ldots, N^j$, let

$$Q_{k,j} = [(k - 1)N^{-j}, kN^{-j}] \times [0, N^{-j}].$$

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Denote $\tilde{Q}_{k,j} = Q_{k-1,j} \cup Q_{k,j} \cup Q_{k+1,j}$ (where $Q_{0,j}$ and $Q_{N^j+1,j}$ are assumed to be empty). Define the maximal operator

$$\mathcal{M}f(x) = \max_{k,j:x\in Q_{k,j}} N^{2j} \int_{\tilde{Q}_{k,j}} |f|. \tag{6.1}$$

(each x belongs to only a finite number of $Q_{k,j}$'s). Given a Carleson cube $Q' \subset Q$ containing x, there exists a cube $Q_{k,j}$ with side at most N times the side of Q' such that $Q' \subset \tilde{Q}_{k,j}$. Thus

$$M_{\mathcal{B}}(w\chi_Q)(x) \le N^2 \mathcal{M}(w\chi_Q)(x),$$

and it is enough to prove (P7) with $\mathcal{M}(w\chi_Q)$ instead of $M_{\mathcal{B}}(w\chi_Q)$.

Let $R_{k,j} = [(k-1)N^{-j}, kN^{-j}] \times [N^{-j-1}, N^{-j}]$, the top of the cube $Q_{k,j}$. For fixed j there are exactly N rectangles of this type placed under a unique rectangle of the form $R_{k',j-1}$. Note that $\mathcal{M}(w\chi_Q)$ is constant on $R_{k,j}$. Either this constant coincides with the one obtained for the rectangle $R_{k',j-1}$ of the previous level or it is

$$\mathcal{M}(w\chi_Q)(x) = N^{2j} \int_{\tilde{Q}_{k,j}} w.$$

Let $\overline{R}_j = \bigcup_{k=1}^{N^j} R_{k,j}$. We claim that for some constant $C(N)$ depending only on N we have

$$\int_{\overline{R}_j} \mathcal{M}(w\chi_Q) \le C(N)\beta^j w(Q). \tag{6.2}$$

The proof is by induction. It clearly holds for $j = 0$. Assume that it holds for $j - 1$. We have

$$\int_{\overline{R}_j} \mathcal{M}(w\chi_Q) \le \frac{1}{N} \int_{\overline{R}_{j-1}} \mathcal{M}(w\chi_Q) + \sum_{k=1}^{N^j} |R_{k,j}| \frac{w(\tilde{Q}_{k,j})}{|Q_{k,j}|},$$

where the first term is the value obtained if the maximal function repeats the values of the previous level (the factor $1/N$ comes from the ratio of the areas of the rectangles in two consecutive levels). We use the induction hypothesis to bound the first term by $N^{-1}C(N)\beta^{j-1}w(Q)$. On the other hand, we use (P4) several times to get

$$w(\overline{R}_j) \le \beta^j w(Q).$$

Then we have

$$\int_{\overline{R}_j} \mathcal{M}(w\chi_Q) \le \left(\frac{C(N)}{\beta N} + 3\right) \beta^j w(Q),$$

from which (6.2) holds if

$$C(N) \ge \frac{3\beta N}{\beta N - 1}.$$

Summing the estimate (6.2) in j we get (P7). □

A consequence of the results in the previous section and the counterexamples in this section is the following corollary.

Corollary 6.2 *Let $\mathcal{B}$ be one of the bases of the preceding theorem. Define for $j = 1, 2, \ldots, 8$ the classes*

$$A^j_{\infty,\mathcal{B}} = \{w : w \text{ is a weight and satisfies (P}j\text{)}\}.$$

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No two of them coincide. More precisely, only the inclusions

$$A^1_{\infty,\mathcal{B}} \subset A^2_{\infty,\mathcal{B}} \subset A^5_{\infty,\mathcal{B}} \subset A^4_{\infty,\mathcal{B}} \subset A^7_{\infty,\mathcal{B}}$$

and

$$A^8_{\infty,\mathcal{B}} \subset A^3_{\infty,\mathcal{B}} \subset A^6_{\infty,\mathcal{B}} \subset A^4_{\infty,\mathcal{B}} \subset A^7_{\infty,\mathcal{B}}$$

hold, and all of them are strict.

7 Bases of cubes and rectangles: a new characterization

Properties (P1)–(P8) are equivalent for the basis of all the cubes of $\mathbb{R}^n$ with the Lebesgue measure, so that for this basis the $A^j_{\infty,\mathcal{B}}$ classes defined in Corollary 6.2 are all the same.

Theorem 4.2 can be applied to the bases of cubes to obtain (P7) from several other conditions. On the other hand, the Hardy–Littlewood maximal operator satisfies a reverse weak-type (1, 1) inequality:

$$\frac{1}{\lambda}\int_{\{x:f(x)>\lambda\}} f(x)dx \leq C|\{x : Mf(x) > \lambda\}|.$$

This inequality is due to Stein [29] and can be found also in the books of the references at the end of the paper. For each cube Q, apply the inequality to $f = w\chi_Q$ and integrate in $\lambda \in (w_Q, \infty)$ to deduce that (P7) implies (P6).

To obtain the equivalence of all the properties it remains to prove that (P4) implies (P1) and (P8). In [7] (P8) is obtained from (P4') using the Calderón–Zygmund decomposition of w adapted to the cube Q (see also [27] or [26]). On the other hand, since (P1) is equivalent to $(\text{P3})_\nu$ for w^{-1} (we use here the notation introduced at the last paragraph of Sect. 2), applying the first implication of part (b) of Theorem 4.1, (P1) follows from $(\text{P8})_\nu$ for w^{-1}: there exist $C, \beta > 0$ such that for every $B \in \mathcal{B}$ and for every $\lambda > w_B^{-1}$ it holds that

$$\mu(\{x \in B : w(x)^{-1} > \lambda\}) \leq C\lambda w(\{x \in B : w(x)^{-1} > \beta\lambda\}). \tag{7.1}$$

As for the proof of (P8) from (P4) this can be done using the Calderón–Zygmund decomposition of w^{-1} adapted to the cube Q with respect to the measure ν, which is a doubling measure (this is again a consequence of (P4)).

We introduce in this section another property from which we obtain new characterizations. It is somehow related to the Orlicz type characterization of [16].

Definition 7.1 Let $\Phi : (1, +\infty) \to [0, +\infty)$ be a nondecreasing function such that $\lim_{t\to+\infty} \Phi(t) = +\infty$.

(P9): There exists $C > 0$ such that for every $B \in \mathcal{B}$ it holds that

$$\int_{\{x\in B:w(x)>w_B\}} w(x)\Phi\left(\frac{w(x)}{w_B}\right) d\mu(x) \leq C\int_B w d\mu.$$

Actually, this is not a unique condition because it differs for each Φ under the stated assumptions. In particular, $\Phi(t) = \log t$ is the same as (P6), and $\Phi(t) = t^{q-1}$ with $q > 1$ is the same as saying that $w \in RH_{q,\mathcal{B}}$.

Theorem 7.2 (a) (P9) *implies* (P4).

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(b) *Assume that* Φ *grows slowly in the sense that it satisfies*

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t^{q-1}} = 0 \quad \textit{for all } q > 1. \tag{7.2}$$

Then (P3) *implies* (P9).

Proof (a) For each $s > \Phi(1)$ define $\Phi^{-1}(s) = \sup\{t : \Phi(t) \le s\}$.
Given $E \subset B$, let $E_1 = \{x \in E : w(x) \le \Phi^{-1}(2C)w_B\}$, where C is the constant in (P9), and $E_2 = E\setminus E_1$. We have the estimates

$$\int_{E_1} w d\mu \le \Phi^{-1}(2C)w_B\mu(E)$$

and

$$\int_{E_2} w d\mu \le \frac{1}{2C}\int_{\{x\in B:w(x)>w_B\}} w(x)\Phi\left(\frac{w(x)}{w_B}\right) d\mu(x) \le \frac{1}{2}w(B).$$

Adding both estimates we get

$$w(E) \le \left(\Phi^{-1}(2C)\frac{\mu(E)}{\mu(B)} + \frac{1}{2}\right) w(B).$$

Choosing α such that $\alpha\Phi^{-1}(2C) < 1/4$, (P4) holds with $\beta = 3/4$.

(b) If $w \in RH_{q,\mathcal{B}}$, use that $\Phi(t) \le Ct^{q-1}$ for $t > t_0$ to majorize the integrand of the left-hand side of (P9) by $w^q(w_B)^{1-q}$.

□

Corollary 7.3 *If* Φ *satisfies* (7.2)*, then* (P9) *characterizes the usual* A_∞ *class of weights for cubes.*

We can take $\Phi(t) = \log\log t$ or other functions growing slowly to infinity and obtain new characterizations of A_∞ (related characterizations are in [16]).

Remark 7.4 Gehring's lemma is an improvement of the exponent of a reverse Hölder inequality [15]. It can be stated as follows: if w is in $RH_{q,\mathcal{B}}$, where $\mathcal{B}$ is the basis of cubes with the Lebesgue measure, then it is in $RH_{q+\delta,\mathcal{B}}$ for some $\delta > 0$. Note that Gehring's lemma together with (P2') (for just one value of s) gives the reverse Hölder inequality for the weight $w \in A_\infty$.

Alberto de la Torre obtained a simplified proof of the reverse Hölder inequality for A_p weights using (P2'). Actually, it is equivalent to proving Gehring's lemma. De la Torre did not publish his proof, but it can be found in [28], for instance. Although only one value of $s \in (0, 1)$ is enough to obtain a reverse Hölder inequality for w, the advantage of having (P2') for every $s \in (0, 1)$ with uniform constant is that the exponent of the reverse Hölder inequality can be given in terms of the constant as in [28].

The equivalence of (P9) and (P3) for the basis of cubes can be phrased as an improvement of the local integrability as in Gehring's lemma. Indeed, if w satisfies (P9) for some Φ fulfilling (7.2), then it satisfies a reverse Hölder inequality and, in particular, is locally in L^r for some $r > 1$.

On the other hand, the counterexamples of Sect. 5 can be adapted to see that both converses of Theorem 7.2 fail for the basis of intervals starting at 0.

We consider now the basis obtained transforming the basis of cubes with a linear transformation. Let T be an invertible linear transformation in $\mathbf{R}^n$ ($T \in GL(n, \mathbf{R})$). Consider the basis

$$\mathcal{B}_T = \{T(Q) : Q \text{ cube in } \mathbf{R}^n\}.$$

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Define $f_T(x) = f(Tx)$. Given $B \in \mathcal{B}_T$, let $B = T(Q)$. Then

$$\frac{1}{|B|}\int_B f(x)dx = \frac{1}{|\det T||Q|}\int_{T(Q)} f(x)dx = \frac{1}{|Q|}\int_Q f_T(x)dx.$$

Therefore, $M_{\mathcal{B}_T} f(x) = M f_T(T^{-1}x)$, where M is the usual Hardy–Littlewood maximal operator on cubes.

An immediate consequence of the equivalence of the characterizations of A_∞ for the basis of cubes is that the corresponding characterizations are also equivalent for $\mathcal{B}_T$.

Corollary 7.5 *Let $\mathcal{T}$ be a subfamily of $GL(n, \mathbf{R})$ and $\mathcal{B} = \bigcup_{T\in\mathcal{T}} \mathcal{B}_T$. All the characterizations except* (P7) *are equivalent for $\mathcal{B}$. They are also equivalent to the following:*

(P10)*: there exists $C > 0$ such that*

$$\sup_{T\in\mathcal{T}} \sup_{B\in\mathcal{B}_T} \int_B M_{\mathcal{B}_T}(w\chi_B)d\mu \le Cw(B).$$

Moreover, (P7) *clearly implies* (P10)*, hence it implies all the other characterizations.*

7.1 Bases of rectangles

Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in (0, +\infty)^n$. Define $T_\Lambda(x) = (\lambda_1 x_1, \ldots, \lambda_n x_n)$. Then $\mathcal{B}_{T_\Lambda}$ is the (one-parameter) family of rectangles in $\mathbf{R}^n$ with sides parallel to the coordinate axes and dimensions $\lambda_1 l \times \cdots \times \lambda_n l$ with $l \in (0, +\infty)$. The basis $\mathcal{B} = \bigcup_{\Lambda\in(0,+\infty)^n} \mathcal{B}_{T_\Lambda}$ is then the family of all rectangles in $\mathbf{R}^n$ with sides parallel to the coordinate axes. Then Corollary 7.5 applies.

In this case, the maximal function associated to the basis of rectangles is the strong maximal function. Since we know that it is bounded on L^p for $p > 1$, we can use part (b) of Theorem 4.2 to deduce that (P7) is also an equivalent characterization.

7.2 Bases of all rectangles in all directions

By using all the transformations in $GL(n, \mathbf{R})$ we obtain the basis formed by all rectangles in $\mathbf{R}^n$ with arbitrary orientation. We can apply Corollary 7.5 again. Since the associated maximal function, that is, the maximal function over all rectangles, is unbounded on L^p for finite p, we cannot apply Theorem 4.2. In this case, we have not been able to prove or disprove the equivalence of (P7) with the other characterizations. If we adopt the notation of Corollary 6.2, all the classes except $A^7_{\infty,\mathcal{B}}$ coincide. This one is contained in the others, but we do not know if the inclusion is strict.

7.3 Bases of rectangles in a set of directions

If we fix a subset of directions in the unit sphere and consider only rectangles whose largest side is in one of the directions of the fixed subset we obtain the equivalence of all the characterizations whenever the corresponding maximal operator is bounded on some L^p for finite p. The equivalence of (P7) holds in such a case, because part (d) of Theorem 4.2 applies. A characterization of the L^p boundedness of such maximal functions in the plane is in [1].

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8 *BMO* for general bases

Given a basis $\mathcal{B}$ we can define an associated *BMO* space in a naive way saying that $BMO_{\mathcal{B}}$ is the class of measurable functions f, integrable on every $B \in \mathcal{B}$, such that

$$\sup_{B\in\mathcal{B}} \frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu < +\infty.$$

In the case of the usual Muckenhoupt weights for the Hardy–Littlewood maximal operator there is a close relationship between *BMO* and the A_p weights: (i) if $w \in A_\infty$, then $\log w \in BMO$; (ii) if $f \in BMO$, for some $\alpha \in \mathbb{R}$, then $e^{\alpha f}$ is in A_p.

In the general setting we consider the weights satisfying (P2) and show that the first part remains true, but not the second one. For all the other conditions except (P1), which implies (P2), the weights can vanish in a set of positive Lebesgue measure (for the basis for which the counterexamples have been built) and it does not make sense to consider their logarithm.

Theorem 8.1 *Let $BMO_{\mathcal{B}}$ as before.*

(a) If w satisfies (P2)*, then* $\log w \in BMO_{\mathcal{B}}$.
(b) There exists a basis $\mathcal{B}$ and a function $f \in BMO_{\mathcal{B}}$ such that $e^{\alpha f}$ does not satisfy (P2) *for any $\alpha \in \mathbb{R}$.*

Proof (a) Let $(\log w)_B = m$. Using (P2) we have

$$\frac{1}{\mu(B)} \int_B w \le Ce^m.$$

Let $B_+ = \{x \in B : e^{-m} w(x) > 1\}$ and $B_- = \{x \in B : e^{-m} w(x) < 1\}$. Then,

$$\begin{aligned}
&\frac{1}{\mu(B)} \int_B |\log w - m|\, d\mu \\
&\quad = \frac{1}{\mu(B)} \int_{B_+} (\log w - m)\, d\mu + \frac{1}{\mu(B)} \int_{B_-} (m - \log w)\, d\mu \\
&\quad = \frac{2}{\mu(B)} \int_{B_+} (\log w - m)\, d\mu \le \frac{2}{\mu(B)} \int_{B_+} e^{-m} w\, d\mu \le 2C.
\end{aligned}$$

(b) With the basis $\mathcal{B} = \{(0, b) : b > 0\}$ in $(0, \infty)$ and the Lebesgue measure, let us define

$$f(x) = \begin{cases} 1, & \text{if } x \in (0, 1) \cup (3, \infty), \\ -\dfrac{1}{(x-1)^{1/2}}, & \text{if } x \in (1, 2), \\ \dfrac{1}{(x-2)^{1/2}}, & \text{if } x \in (2, 3). \end{cases}$$

To check that $f \in BMO_{\mathcal{B}}$ it is enough to observe that

$$\frac{1}{b} \int_0^b |f| \le 3.$$

But for any $\alpha \in \mathbb{R}$ the function $e^{\alpha f}$ is not integrable on the interval $(0, 3)$, so that it cannot fulfil (P2).

□

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