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Hochschild cohomology of triangular string algebras and its ring structure

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ABSTRACT

We compute the Hochschild cohomology groups $\mathrm{HH}^*(A)$ in case A is a triangular string algebra, and show that its ring structure is trivial.

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1. Introduction

Let A be an associative, finite dimensional algebra over an algebraically closed field k. It is well known that there exists a finite quiver Q such that A is Morita equivalent to kQ/I, where kQ is the path algebra of Q and I is an admissible two-sided ideal of kQ.

A finite dimensional algebra is called *biserial* if the radical of every projective indecomposable module is the sum of two uniserial modules whose intersection is simple or zero, see [11]. These algebras have been studied by several authors and from different points of view since there are a lot of natural examples of algebras which turn out to be of this kind.

The representation theory of these algebras was first studied by Gel'fand and Ponomarev in [12]: they have provided the methods in order to classify all their indecomposable representations. This classification shows that special biserial algebras are always tame, see [21], and tameness of arbitrary biserial algebras was established in [10]. They are an important class of algebras whose representation theory has been very well described, see [2,6].

The subclass of *special biserial algebras* was first studied by Skowroński and Waschbüsch in [19] where they characterize the biserial algebras of finite representation type. A classification of the special biserial algebras which are minimal representation-infinite has been given by Ringel in [17].

An algebra is called a *string algebra* if it is Morita equivalent to a monomial special biserial algebra.

The purpose of this paper is to study the Hochschild cohomology groups of a string algebra A and describe its ring structure.

Since A is an algebra over a field k, the Hochschild cohomology groups $\mathrm{HH}^i(A, M)$ with coefficients in an A-bimodule M can be identified with the groups $\mathrm{Ext}^i_{A-A}(A, M)$. In particular, if M is the A-bimodule A, we simple write $\mathrm{HH}^i(A)$.

Even though the computation of the Hochschild cohomology groups $\mathrm{HH}^i(A)$ is rather complicated, some approaches have been successful when the algebra A is given by a quiver with relations. For instance, explicit formula for the dimensions of $\mathrm{HH}^i(A)$ in terms of those combinatorial data have been found in [4,7–9,15,16]. In particular, Hochschild cohomology of special biserial algebras has been considered in [5,20].

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In the particular case of monomial algebras, that is, algebras $A = KQ/I$ where I can be chosen as generated by paths, one has a detailed description of a minimal resolution of the A-bimodule A, see [3]. In general, the computation of the Hochschild cohomology groups using this resolution may lead to hard combinatoric computations. However, for string algebras the resolution, and the complex associated, are easier to handle.

The paper is organized as follows. In Section 2 we introduce all the necessary terminology. In Section 3 we recall the resolution given by Bardzell for monomial algebras in [3]. In Section 4 we present all the computations that lead us to Theorem 4.3 where we present the dimension of all the Hochschild cohomology groups of triangular string algebras. In Section 5 we describe the ring structure of the Hochschild cohomology of triangular string algebras.

2. Preliminaries

2.1. Quivers and relations

Let Q be a finite quiver with a set of vertices Q_0, a set of arrows Q_1 and $s, t : Q_1 \to Q_0$ be the maps associating to each arrow α its source $s(\alpha)$ and its target $t(\alpha)$. A path w of length l is a sequence of l arrows $\alpha_1 \dots \alpha_l$ such that $t(\alpha_i) = s(\alpha_{i+1})$. We denote by $|w|$ the length of the path w. We put $s(w) = s(\alpha_1)$ and $t(w) = t(\alpha_l)$. For any vertex x we consider e_x the trivial path of length zero and we put $s(e_x) = t(e_x) = x$. An oriented cycle is a non-trivial path w such that $s(w) = t(w)$. If Q has no oriented cycles, then A is said a *triangular algebra*.

We say that a path w divides a path u if $u = L(w)wR(w)$, where $L(w)$ and $R(w)$ are not simultaneously paths of length zero.

The path algebra kQ is the k-vector space with basis the set of paths in Q; the product on the basis elements is given by the concatenation of the sequences of arrows of the paths w and w' if they form a path (namely, if $t(w) = s(w')$) and zero otherwise. Vertices form a complete set of orthogonal idempotents. Let F be the two-sided ideal of kQ generated by the arrows of Q. A two-sided ideal I is said to be *admissible* if there exists an integer $m \geqslant 2$ such that $F^m \subseteq I \subseteq F^2$. The pair (Q, I) is called a *bound quiver*.

It is well known that if A is a basic, connected, finite dimensional algebra over an algebraically closed field k, then there exists a unique finite quiver Q and a surjective morphism of k-algebras $\nu : kQ \to A$, which is not unique in general, with $I_\nu = \operatorname{Ker} \nu$ admissible. The pair (Q, I_ν) is called a *presentation* of A. The elements in I are called *relations*, kQ/I is said a *monomial algebra* if the ideal I is generated by paths, and a relation is called *quadratic* if it is a path of length two.

2.2. String algebras

Recall from [19] that a bound quiver (Q, I) is *special biserial* if it satisfies the following conditions:

(S1) Each vertex in Q is the source of at most two arrows and the target of at most two arrows;
(S2) For an arrow α in Q there is at most one arrow β and at most one arrow γ such that $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

If the ideal I is generated by paths, the bound quiver (Q, I) is *string*.

An algebra is called *special biserial* (or *string*) if it is Morita equivalent to a path algebra kQ/I with (Q, I) a special biserial bound quiver (or a string bound quiver, respectively).

Since Hochschild cohomology is invariant under Morita equivalence, whenever we deal with a string algebra A we will assume that it is given by a string presentation $A = kQ/I$ with I satisfying the previous conditions. We also assume that the ideal I is generated by paths of minimal length, and we fix a minimal set $\mathcal{R}$ of paths, of minimal length, that generate the ideal I. Moreover, we fix a set $\mathcal{P}$ of paths in Q such that the set $\{\gamma + I, \gamma \in \mathcal{P}\}$ is a basis of $A = kQ/I$.

3. Bardzell's resolution

We recall that the Hochschild cohomology groups $\mathrm{HH}^i(A)$ of an algebra A are the groups $\mathrm{Ext}^i_{A-A}(A, A)$. Since string algebras are monomial algebras, their Hochschild cohomology groups can be computed using a convenient minimal projective resolution of A as A-bimodule given in [3].

In order to describe this minimal resolution, we need some definitions and notations.

Recall that we have fix a minimal set $\mathcal{R}$ of paths, of minimal length, that generate the ideal I. It is clear that no divisor of an element in $\mathcal{R}$ can belong to $\mathcal{R}$.

The n-concatenations are elements defined inductively as follows: given any directed path T in Q, consider the set of vertices that are starting and ending points of arrows belonging to T, and consider the natural order $<$ in this set. Let $\mathcal{R}(T)$ be the set of paths in $\mathcal{R}$ that are contained in the directed path T. Take $p_1 \in \mathcal{R}(T)$ and consider the set

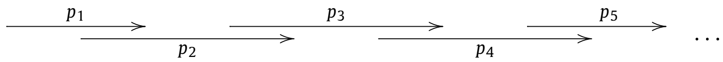
$$L_1 = \{p \in \mathcal{R}(T) \colon s(p_1) < s(p) < t(p_1)\}.$$

If $L_1 \neq \emptyset$, let p_2 be such that $s(p_2)$ is minimal with respect to all $p \in L_1$. Now assume that $p_1, p_2, \dots, p_j$ have been constructed. Let

$$L_{j+1} = \{p \in \mathcal{R}(T)\colon\ t(p_{j-1}) \leqslant s(p) < t(p_j)\}.$$

If $L_{j+1} \neq \emptyset$, let p_{j+1} be such that $s(p_{j+1})$ is minimal with respect to all $p \in L_{j+1}$. Thus $(p_1, \dots, p_{n-1})$ is an n-concatenation and we denote by $w(p_1, \dots, p_{n-1})$ the path from $s(p_1)$ to $t(p_{n-1})$ along the directed path T, and we call it the *support* of the concatenation.

These concatenations can be pictured as follows:



Let $AP_0 = Q_0$, $AP_1 = Q_1$ and AP_n the set of supports of n-concatenations.

The construction of the sets AP_n can also be done dually. Given any directed path T in Q take $q_1 \in \mathcal{R}(T)$ and consider the set

$$L_1^{op} = \{q \in \mathcal{R}(T)\colon\ s(q_1) < t(q) < t(q_1)\}.$$

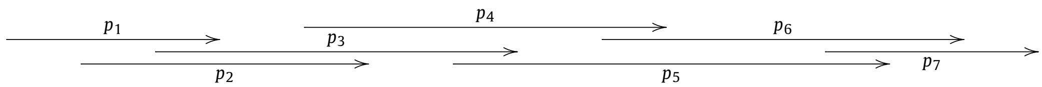
If $L_1^{op} \neq \emptyset$, let q_2 be such that $t(q_2)$ is maximal with respect to all $q \in L_1^{op}$. Now assume that $q_1, q_2, \dots, q_j$ have been constructed. Let

$$L_{j+1}^{op} = \{q \in \mathcal{R}(T)\colon\ s(q_j) < t(q) \leqslant s(q_{j-1})\}.$$

If $L_{j+1}^{op} \neq \emptyset$, let q_{j+1} be such that $t(q_{j+1})$ is maximal with respect to all $q \in L_{j+1}^{op}$. Thus $(q_{n-1}, \dots, q_1)$ is an n-op-concatenation, we denote by $w^{op}(q_{n-1}, \dots, q_1)$ the path from $s(q_{n-1})$ to $t(q_1)$ along the directed path T, we call it the support of the concatenation and we denote by AP_n^{op} the set of supports of n-op-concatenations constructed in this dual way. Moreover, we denote $w^{op}(q_{n-1}, \dots, q_1) = w^{op}(q^1, \dots, q^{n-1})$.

For any $w \in AP_n$ define $\mathrm{Sub}(w) = \{w' \in AP_{n-1}\colon\ w' \text{ divides } w\}$.

Example 1. Consider the following relations contained in a directed path T:



Then $w = w(p_1, p_2, p_4, p_5, p_7)$ is a 6-concatenation, $w = w^{op}(p_1, p_3, p_4, p_6, p_7)$ and

$$\mathrm{Sub}(w) = \{w(p_1, p_2, p_4, p_5), w(p_2, p_3, p_5, p_6), w(p_3, p_4, p_6, p_7)\}.$$

Lemma 3.1. *(See [3, Lemma 3.1].) If $n \geqslant 2$ then $AP_n = AP_n^{op}$.*

The previous lemma says that for any n-concatenation $(p_1, \dots, p_{n-1})$ there exists a unique n-op-concatenation $(q^1, \dots, q^{n-1})$ such that $w(p_1, \dots, p_{n-1}) = w^{op}(q^1, \dots, q^{n-1})$. We want to remark some facts in this construction that will be used later. First observe that $w(p_1) = w^{op}(p_1)$ and $w(p_1, p_2) = w^{op}(p_1, p_2)$. Assume that $n > 3$. It is clear that $q^{n-1} = p_{n-1}$ since they are relations in $\mathcal{R}$ contained in the same path and sharing target. When we look for q^{n-2} we can observe that the maximality of its target implies that $t(p_{n-2}) \leqslant t(q^{n-2})$. Since elements in $\mathcal{R}$ are paths of minimal length, $s(p_{n-2}) \leqslant s(q^{n-2})$. Now $t(q^{n-2}) < t(q^{n-1}) = t(p_{n-1})$ says that $q^{n-2} \neq p_{n-1}$ and the minimality of the starting point of p_{n-1} says that $s(q^{n-2}) < t(p_{n-3})$. Then

$$s(p_{n-2}) \leqslant s(q^{n-2}) < t(p_{n-3}) \quad \text{and} \quad t(p_{n-2}) \leqslant t(q^{n-2}) < t(p_{n-1}).$$

Since $s(q^{n-2}) < t(p_{n-3}) \leqslant s(p_{n-1}) = s(q^{n-1})$ we can continue this procedure in order to prove that, for $j = 2, 3, \dots, n-2$, the element q^{n-j} is such that

$$s(p_{n-j}) \leqslant s(q^{n-j}) < t(p_{n-j-1}), \qquad t(p_{n-j}) \leqslant t(q^{n-j}) < t(p_{n-j+1})$$

and

$$s(q^{n-j}) < t(p_{n-j-1}) \leqslant s(p_{n-j+1}) \leqslant s(q^{n-j+1}).$$

Finally the minimality of the source of p_2 and the inequality $t(p_1) \leqslant t(q^1) < t(p_2)$ shows that $q^1 = p_1$.

Lemma 3.2. *If $n, m \geqslant 0$, $n + m \geqslant 2$ then any $w(p_1, \dots, p_{n+m-1}) \in AP_{n+m}$ can be written in a unique way as*

$$w(p_1, \dots, p_{n+m-1}) = {}^{(n)}w\, u\, w^{(m)}$$

with ${}^{(n)}w = w(p_1, \dots, p_{n-1}) \in AP_n$, $w^{(m)} = w^{op}(q^{n+1}, \dots, q^{n+m-1}) \in AP_m^{op}$ and u a path in Q. Moreover, $p_n = a\,u\,b$ and $q^n = a'\,u\,b'$ with a, a', b, b' non-trivial paths, and hence $u \in \mathcal{P}$.

Proof. From Lemma 3.1 we know that $w(p_1,\dots,p_{n+m-1}) = w^{op}(q^1,\dots,q^{n+m-1})$. It is clear that $w(p_1,\dots,p_{n-1}) \in AP_n$ and $w^{op}(q^{n+1},\dots,q^{n+m-1}) \in AP_m^{op}$. In order to prove the existence of a path u we just have to observe that the construction explained after the previous lemma and the definition of concatenations imply that

$$t(p_{n-1}) \leqslant t\big(q^{n-1}\big) \leqslant s\big(q^{n+1}\big).$$

Finally, the relation of u with p_n and q^n follows from the inequalities

$$s(p_n) < t(p_{n-1}) \leqslant s\big(q^{n+1}\big) < t(p_n) \quad \text{and} \quad s\big(q^n\big) < t(p_{n-1}) \leqslant s\big(q^{n+1}\big) < t\big(q^n\big). \qquad \square$$

Now we want to study the sets $\mathrm{Sub}(w)$ in some particular cases. Observe that for any $w \in AP_n$, $\psi_1 = w^{op}(q^2,\dots,q^{n-1})$ and $\psi_2 = w(p_1,\dots,p_{n-2})$ belong to $\mathrm{Sub}(w)$ and $w = L(\psi_1)\psi_1 = \psi_2 R(\psi_2)$.

Lemma 3.3. *If $w = w(p_1,\dots,p_{n-1}) \in AP_n$ is such that p_i has length two for some i with $1 \leqslant i \leqslant n-1$, then $|\mathrm{Sub}(w)| = 2$.*

Proof. Assume that $p_i = \alpha\beta$. If $i = 1$, then any $(n-1)$-concatenation different from $(p_1,\dots,p_{n-2})$ and corresponding to an element in $\mathrm{Sub}(w)$ must correspond to a divisor of $w(p_2,\dots,p_{n-1})$, hence it is equal to $(p_2,\dots,p_{n-1})$. The proof for $i = n-1$ is similar. If $1 < i < n-1$ and $\hat{w} \in \mathrm{Sub}(w)$ then $\hat{w}$ also contains the quadratic relation p_i and by the previous lemma we have that

$$w = {}^{(i)}w\,w^{(n-i)}, \qquad \hat{w} = {}^{(j)}\hat{w}\,\hat{w}^{(n-1-j)}$$

with $t({}^{(i)}w) = t({}^{(j)}\hat{w})$ and $s(w^{(n-i)}) = s(\hat{w}^{(n-1-j)})$. Then $\hat{w}$ is ${}^{(i)}\hat{w}\hat{w}_1$ or $\hat{w}_2\hat{w}^{(n-i)}$, where $\hat{w}_1$ is the unique element in $\mathrm{Sub}(w^{(n-i)})$ sharing source with $w^{(n-i)}$ and $\hat{w}_2$ is the unique element in $\mathrm{Sub}({}^{(i)}w)$ sharing target with ${}^{(i)}w$. $\square$

Lemma 3.4. *If $w = w(p_1,\dots,p_{n-1}) = w^{op}(q^1,\dots,q^{n-1})$ and q^m has length two for some m such that $1 < m < n$ then $q^m = p_m$ and $q^{m-1} = p_{m-1}$.*

Proof. Let $q^m = \alpha\beta$. In the construction explained after Lemma 3.1 we have seen that

$$s\big(q^m\big) < t(p_{m-1}) \leqslant t\big(q^{m-1}\big).$$

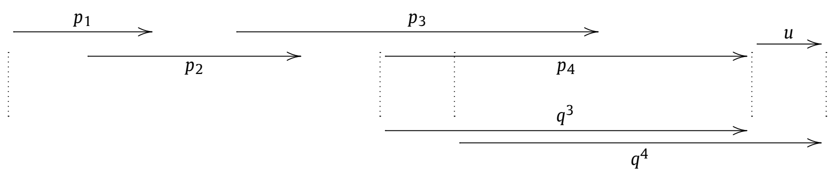
Now $s(q^m) = s(\alpha)$ and $t(q^{m-1}) = t(\alpha)$, so $t(p_{m-1}) = t(q^{m-1})$ and hence $p_{m-1} = q^{m-1}$. Analogously,

$$s\big(q^{m+1}\big) < t(p_m) \leqslant t\big(q^m\big),$$

$s(q^{m+1}) = s(\beta)$ and $t(q^m) = t(\beta)$, so $t(p_m) = t(q^m)$ and hence $p_m = q^m$. $\square$

In some results that will be shown in the following sections, we will need a description of right divisors of paths of the form wu, for w the support of a concatenation and $u \in \mathcal{P}$. Their existence depends on each particular case as we show in the following example.

Example 2. Let $w = w(p_1,p_2,p_3,p_4) \in AP_5$, $u \in \mathcal{P}$ with $t(w) = s(u)$. Observe that the existence of a divisor $\psi \in AP_n$, for $n = 4, 5$ such that $wu = L(\psi)\psi$ depends on the existence of appropriate relations. For instance, if wu is the following path



the existence of $\psi = \psi^{op}(q^1,q^2,q^3,q^4)$ depends on the existence of a relation whose ending point is between $s(q^3)$ and $s(q^4)$.

Part of the following lemma has also been proved in [3, Lemma 3.2].

Lemma 3.5. *If n is even let $w = w(p_1,\dots,p_{n-1}) \in AP_n$.*

(i) *If $v = v^{op}(q^2,\dots,q^{n-1}) \in AP_{n-1}$ is such that $wa = bv \notin AP_{n+1}$ with a, b paths in Q, $a \in \mathcal{P}$ then $t(p_1) \leqslant s(q_2)$, and*
(ii) *if $u = u^{op}(q^1,\dots,q^{n-1}) \in AP_n$ is such that $wa = bu \notin AP_{n+1}$ with a, b paths in $\mathcal{P}$ then there exists $z \in AP_{n+1}$ such that z divides the path T that contains w and u and $t(z) = t(u)$.*

Proof. (i) The assumption $a \in \mathcal{P}$ implies that $s(p_{n-1}) < s(q^{n-1}) < t(p_{n-1})$, and moreover $s(p_{n-1}) < s(q^{n-1}) < t(p_{n-2})$ since otherwise $wa \in AP_{n+1}$ because q^{n-1} would belong to the set considered in order to choose p_n. Now $q^{n-2} = p_{n-1}$ or $t(p_{n-1}) < t(q^{n-2})$, and hence

$$s(p_{n-1}) < s\big(q^{n-2}\big) < s\big(q^{n-1}\big).$$

Now q^{n-3} is such that $s(p_{n-3}) < s(q^{n-3}) < s(q^{n-1})$. The minimality of $s(p_{n-2})$ says that $s(p_{n-3}) < s(q^{n-3}) < t(p_{n-4})$. An inductive procedure shows that

$$s(p_{n-2j+1}) < s\big(q^{n-2j+1}\big) < t(p_{n-2j}) \quad \text{and} \quad q^{n-2j} = p_{n-2j+1} \quad \text{or} \quad t(p_{n-2j+1}) < t\big(q^{n-2j}\big).$$

Hence

$$s(p_{n-2j+1}) < s\big(q^{n-2j}\big) < s\big(q^{n-2j+1}\big)$$

for any j such that $1 \leqslant 2j-1, 2j \leqslant n-1$. In particular, since n is even we have that

$$q^2 = p_3 \quad \text{or} \quad s(p_3) < s\big(q^2\big) < s\big(q^3\big)$$

and hence $t(p_1) \leqslant s(q^2)$.

(ii) In order to prove the existence of z we have to show that there exists $q^0 \in \mathcal{R}(T)$ such that $z = z^{op}(q^0, q^1, \ldots, q^{n-1})$ belongs to AP_{n+1}, that is, we have to see that the set $\{q \in \mathcal{R}(T)\colon\ s(q^1) < t(q) \leqslant s(q^2)\}$ is not empty. Suppose it is empty. The assumption $b \in \mathcal{P}$ implies that $s(q^2) < t(p_1)$, a contradiction from (i). $\square$

Lemma 3.6. *(See [3, Lemma 3.3].) If $m \geqslant 1$ and $w \in AP_{2m+1}$ then $|\mathrm{Sub}(w)| = 2$.*

Now we are ready to describe the minimal resolution constructed by Bardzell in [3]:

$$\cdots \longrightarrow A \otimes kAP_n \otimes A \xrightarrow{d_n} A \otimes kAP_{n-1} \otimes A \longrightarrow \cdots \longrightarrow A \otimes kAP_0 \otimes A \xrightarrow{\mu} A \longrightarrow 0$$

where $kAP_0 = kQ_0$, $kAP_1 = kQ_1$ and kAP_n is the vector space generated by the set of supports of n-concatenations and all tensor products are taken over $E = kQ_0$, the subalgebra of A generated by the vertices.

In order to define the A–A-maps

$$d_n : A \otimes kAP_n \otimes A \to A \otimes kAP_{n-1} \otimes A$$

we need the following notations: if $m \geqslant 1$, for any $w \in AP_{2m+1}$ we have that $\mathrm{Sub}(w) = \{\psi_1, \psi_2\}$ where $w = L(\psi_1)\psi_1 = \psi_2 R(\psi_2)$; and for any $w \in AP_{2m}$ and $\psi \in \mathrm{Sub}(w)$ we denote $w = L(\psi)\psi R(\psi)$. Then

$$\mu(1 \otimes e_i \otimes 1) = e_i,$$
$$d_1(1 \otimes \alpha \otimes 1) = \alpha \otimes e_{t(\alpha)} \otimes 1 - 1 \otimes e_{s(\alpha)} \otimes \alpha,$$
$$d_{2m}(1 \otimes w \otimes 1) = \sum_{\psi \in \mathrm{Sub}(w)} L(\psi) \otimes \psi \otimes R(\psi),$$
$$d_{2m+1}(1 \otimes w \otimes 1) = L(\psi_1) \otimes \psi_1 \otimes 1 - 1 \otimes \psi_2 \otimes R(\psi_2).$$

The E–A bilinear map $c : A \otimes kAP_{n-1} \otimes A \to A \otimes kAP_n \otimes A$ defined by

$$c(a \otimes \psi \otimes 1) = \sum_{\substack{w \in AP_n \\ L(w)wR(w) = a\psi}} L(w) \otimes w \otimes R(w)$$

is a contracting homotopy, see [18, Theorem 1] for more details.

4. Hochschild cohomology

In this section we compute the dimension of all the Hochschild cohomology groups of triangular string algebras.

The Hochschild complex, obtained by applying $\mathrm{Hom}_{A-A}(-, A)$ to the Hochschild resolution we described in the previous section and using the isomorphisms

$$\mathrm{Hom}_{A-A}(A \otimes kAP_n \otimes, A) \simeq \mathrm{Hom}_{E-E}(kAP_n, A)$$

is

$$0 \longrightarrow \mathrm{Hom}_{E-E}(kAP_0, A) \xrightarrow{F_1} \mathrm{Hom}_{E-E}(kAP_1, A) \xrightarrow{F_2} \mathrm{Hom}_{E-E}(kAP_2, A) \cdots$$

where

$$F_1(f)(\alpha) = \alpha f(e_{t(\alpha)}) - f(e_{s(\alpha)})\alpha,$$
$$F_{2m}(f)(w) = \sum_{\psi \in \mathrm{Sub}(w)} L(\psi) f(\psi) R(\psi),$$
$$F_{2m+1}(f)(w) = L(\psi_1) f(\psi_1) - f(\psi_2) R(\psi_2).$$

In order to compute its cohomology, we need a manageable description of this complex: we will describe explicit basis of these k-vector spaces and study the behavior of the maps between them in order to get information about kernels and images.

Recall that we have fixed a set $\mathcal{P}$ of paths in Q such that the set $\{\gamma + I\colon \gamma \in \mathcal{P}\}$ is a basis of $A = kQ/I$. For any subset X of paths in Q, we denote $(X//\mathcal{P})$ the set of pairs $(\rho, \gamma) \in X \times \mathcal{P}$ such that ρ, γ are parallel paths in Q, that is

$$(X//\mathcal{P}) = \{(\rho, \gamma) \in X \times \mathcal{P}\colon\ s(\rho) = s(\gamma), t(\rho) = t(\gamma)\}.$$

Observe that the k-vector spaces $\mathrm{Hom}_{E-E}(kAP_n, A)$ and $k(AP_n//\mathcal{P})$ are isomorphic, and from now on we will identify elements $(\rho, \gamma) \in (AP_n//\mathcal{P})$ with basis elements $f_{(\rho,\gamma)}$ in $\mathrm{Hom}_{E-E}(kAP_n, A)$ defined by

$$f_{(\rho,\gamma)}(w) = \begin{cases} \gamma & \text{if } w = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Now we will introduce several subsets of $(AP_n//\mathcal{P})$ in order to get a nice description of the kernel and the image of F_n. For $n = 0$ we have that $(AP_0//\mathcal{P}) = (Q_0, Q_0)$. For $n = 1$, $(AP_1//\mathcal{P}) = (Q_1//\mathcal{P})$, and we consider the following partition

$$(Q_1//\mathcal{P}) = (1, 1)_1 \cup (0, 0)_1$$

where

$$(1, 1)_1 = \{(\alpha, \alpha)\colon\ \alpha \in Q_1\},$$
$$(0, 0)_1 = \{(\alpha, \gamma) \in (Q_1//\mathcal{P})\colon\ \alpha \neq \gamma\}.$$

For any $n \geqslant 2$ let

$$(0, 0)_n = \{(\rho, \gamma) \in (AP_n//\mathcal{P})\colon\ \rho = \alpha_1 \hat{\rho} \alpha_2 \text{ and } \gamma \notin \alpha_1 kQ \cup kQ\alpha_2\},$$
$$(1, 0)_n = \{(\rho, \gamma) \in (AP_n//\mathcal{P})\colon\ \rho = \alpha_1 \hat{\rho} \alpha_2 \text{ and } \gamma \in \alpha_1 kQ, \gamma \notin kQ\alpha_2\},$$
$$(0, 1)_n = \{(\rho, \gamma) \in (AP_n//\mathcal{P})\colon\ \rho = \alpha_1 \hat{\rho} \alpha_2 \text{ and } \gamma \notin \alpha_1 kQ, \gamma \in kQ\alpha_2\},$$
$$(1, 1)_n = \{(\rho, \gamma) \in (AP_n//\mathcal{P})\colon\ \rho = \alpha_1 \hat{\rho} \alpha_2 \text{ and } \gamma \in \alpha_1 kQ\alpha_2\}.$$

Remark 1.

(1) These subsets are a partition of $(AP_n//\mathcal{P})$.
(2) Any $(\rho, \gamma) \in (AP_n//\mathcal{P})$ verifies that ρ and γ have at most one common first arrow and at most one common last arrow: if $\rho = \alpha_1 \dots \alpha_s \beta \bar{\rho}$ and $\gamma = \alpha_1 \dots \alpha_s \delta \bar{\gamma}$, with β, δ different arrows, then $\alpha_s \beta \in I$. Since $\alpha_1 \dots \alpha_s$ is a factor of γ, and γ belongs to $\mathcal{P}$, then $\alpha_1 \dots \alpha_s \notin I$. But the n-concatenation associated to ρ must start with a relation in $\mathcal{R}(T)$, so $s = 1$, this concatenation starts with the relation $\alpha_1\beta$ and the second relation of this concatenation starts in $s(\beta)$.
(3) If $(\rho, \gamma) \in (1, 0)_n$, $\rho = \alpha_1 \bar{\rho}, \gamma = \alpha_1 \bar{\gamma}$ then $\bar{\rho} \in AP_{n-1}$ and $(\bar{\rho}, \bar{\gamma}) \in (0, 0)_{n-1}$. The same construction holds in $(0, 1)_n$. Finally, if $(\rho, \gamma) \in (1, 1)_n$, $\rho = \alpha_1 \hat{\rho} \alpha_2, \gamma = \alpha_1 \hat{\gamma} \alpha_2$ then $\hat{\rho} \in AP_{n-2}$ and $(\hat{\rho}, \hat{\gamma}) \in (0, 0)_{n-2}$.
(4) If $(\rho, \gamma) \in (AP_2//\mathcal{P}) = (\mathcal{R}//\mathcal{P})$, we have already seen that ρ and γ have at most one common first arrow and at most one common last arrow. Assume that $\rho = \alpha_1 \alpha_2 \bar{\rho}$, $\gamma = \alpha_1 \beta \bar{\gamma}$. Since A is a string algebra and $\gamma \notin I$ we have that $\alpha_1 \alpha_2 \in I$ and hence $\rho = \alpha_1 \alpha_2$. Since we are dealing with triangular algebras, we also have that ρ and γ cannot have simultaneously one common first arrow and one common last arrow. Then

$$(1, 1)_2 = \emptyset,$$
$$(1, 0)_2 = \{(\rho, \gamma) \in (\mathcal{R}, \mathcal{P})\colon\ \rho = \alpha_1 \alpha_2, \gamma \in \alpha_1 kQ, \gamma \notin kQ\alpha_2\},$$
$$(0, 1)_2 = \{(\rho, \gamma) \in (\mathcal{R}, \mathcal{P})\colon\ \rho = \alpha_1 \alpha_2, \gamma \notin \alpha_1 kQ, \gamma \in kQ\alpha_2\}.$$

We also have to distinguish elements inside each of the previous sets taking into account the following definitions:

$$^{+}(X//\mathcal{P}) = \{(\rho, \gamma) \in (X//\mathcal{P})\colon\ Q_1\gamma \not\subset I\},$$
$$^{-}(X//\mathcal{P}) = \{(\rho, \gamma) \in (X//\mathcal{P})\colon\ Q_1\gamma \subset I\}.$$

In an analogous way we define $(X//\mathcal{P})^+$, $(X//\mathcal{P})^-$, ${}^+(X//\mathcal{P})^+ = {}^+(X//\mathcal{P}) \cap (X//\mathcal{P})^+$ and so on. Finally we define

$$(1,0)_n^{--} = \{(\rho,\gamma) \in (1,0)_n^-:\ \rho = \alpha_1\hat{\rho}\alpha_2, \gamma = \alpha_1\hat{\gamma}, \hat{\gamma}Q_1 \subset I\},$$
$$(1,0)_n^{-+} = \{(\rho,\gamma) \in (1,0)_n^-:\ \rho = \alpha_1\hat{\rho}\alpha_2, \gamma = \alpha_1\hat{\gamma}, \hat{\gamma}Q_1 \not\subset I\},$$
$${}^{--}(0,1)_n = \{(\rho,\gamma) \in {}^-(0,1)_n:\ \rho = \alpha_1\hat{\rho}\alpha_2, \gamma = \hat{\gamma}\alpha_2, Q_1\hat{\gamma} \subset I\},$$
$${}^{+-}(0,1)_n = \{(\rho,\gamma) \in {}^-(0,1)_n:\ \rho = \alpha_1\hat{\rho}\alpha_2, \gamma = \hat{\gamma}\alpha_2, Q_1\hat{\gamma} \not\subset I\}.$$

Now we will describe the morphisms F_n restricted to the subsets we have just defined.

Lemma 4.1. *For any $n \geqslant 2$ we have*

(a) ${}^-(0,0)_{n-1}^- \cup (1,0)_{n-1}^- \cup {}^-(0,1)_{n-1} \cup (1,1)_{n-1} \subset \operatorname{Ker} F_n$;
(b) *the function F_n induces a bijection from ${}^-(0,0)_{n-1}^+$ to ${}^{--}(0,1)_n$;*
(c) *the function F_n induces a bijection from ${}^+(0,0)_{n-1}^-$ to $(1,0)_n^{--}$;*
(d) *there exist bijections $\phi_m : (1,0)_m^+ \to {}^+(0,1)_m$ and $\psi_m : (1,0)_m^{-+} \to {}^{+-}(0,1)_m$ such that*

$$\big(id + (-1)^{n-1}\phi_{n-1}\big)\big((1,0)_{n-1}^+\big) \subset \operatorname{Ker} F_n,$$

$$(-1)^n F_n\big((1,0)_{n-1}^+\big) = (1,1)_n$$

and

$$F_n\big({}^+(0,0)_{n-1}^+\big) = \big(id + (-1)^n\phi_n\big)\big((1,0)_n^+\big) \cup \big(id + (-1)^n\psi_n\big)\big((1,0)_n^{-+}\big).$$

Proof. (a) In order to check that (ρ,γ) belongs to $\operatorname{Ker} F_n$ we have to prove that for any $w \in AP_n$ such that ρ divides w, that is, $w = L(\rho)\rho R(\rho)$ and $|L(\rho)| + |R(\rho)| > 0$, then $L(\rho)\gamma R(\rho) \in I$.

If $(\rho,\gamma) \in {}^-(0,0)_{n-1}^-$ then $L(\rho)\gamma R(\rho) \in I$.

If $(\rho,\gamma) \in (1,0)_{n-1}^-$ then $\gamma R(\rho) \in I$ if $|R(\rho)| > 0$. On the other hand, if $w = L(\rho)\rho$ we can deduce that $L(\rho)\gamma \in I$ using Remark 1(2): if $L(\rho) \notin I$ then the first relation in the n-concatenation corresponding to w has α_1 as it last arrow and $\gamma = \alpha_1\hat{\gamma}$.

The proof for ${}^-(0,1)_{n-1}$ is analogous.

Finally, if $(\rho,\gamma) \in (1,1)_{n-1}$, the statement is clear for $n = 2, 3$. If $n > 3$ and $\rho = \alpha_1\hat{\rho}\alpha_2$, from Remark 1(2) we get that if $|L(\rho)| > 0$ then the first relation in the n-concatenation corresponding to w has α_1 as it last arrow, and if $|R(\rho)| > 0$ then the last relation has α_2 as it first arrow. The assertion is clear since $\gamma = \alpha_1\hat{\gamma}\alpha_2$ and hence $L(\rho)\gamma R(\rho) = L(\rho)\alpha_1\hat{\gamma}\alpha_2 R(\rho) \in I$.

(b) If $(\rho,\gamma) \in {}^-(0,0)_{n-1}^+$ there exists a unique arrow β such that $\gamma\beta \in \mathcal{P}$. It is clear that $\rho\beta \in AP_n$, $(\rho\beta,\gamma\beta) \in {}^{--}(0,1)_n$ and $F_n(f_{(\rho,\gamma)}) = (-1)^n f_{(\rho\beta,\gamma\beta)}$.

(c) Analogous to the previous one.

(d) If $(\alpha\hat{\rho},\alpha\hat{\gamma}) \in (1,0)_m^+$ then there exists a unique arrow β such $\alpha\hat{\gamma}\beta \in \mathcal{P}$. It is clear that $\hat{\rho} \in AP_{m-1}$, $(\hat{\rho},\hat{\gamma})^+(0,0)_{m-1}^+$, $(\hat{\rho}\beta,\hat{\gamma}\beta) \in {}^+(0,1)_m$ and $(\alpha\hat{\rho}\beta,\alpha\hat{\gamma}\beta) \in (1,1)_{m+1}$. The statement is clear if we define $\phi_m(\alpha\hat{\rho},\alpha\hat{\gamma}) = (\hat{\rho}\beta,\hat{\gamma}\beta)$ since

$$F_{m+1}(f_{(\hat{\rho}\beta,\hat{\gamma}\beta)}) = f_{(\alpha\hat{\rho}\beta,\alpha\hat{\gamma}\beta)} = (-1)^{m+1}F_{m+1}(f_{(\alpha\hat{\rho},\alpha\hat{\gamma})}).$$

In a similar way we can see that if $(\alpha\hat{\rho},\alpha\hat{\gamma}) \in (1,0)_m^{-+}$ then there exists a unique arrow β such $\hat{\gamma}\beta \in \mathcal{P}$. Now we have that $\hat{\rho} \in AP_{m-1}$, $(\hat{\rho},\hat{\gamma}) \in {}^+(0,0)_{m-1}^+$ and $(\hat{\rho}\beta,\hat{\gamma}\beta) \in {}^{+-}(0,1)_m$, so it is enough to define $\psi_m(\alpha\hat{\rho},\alpha\hat{\gamma}) = (\hat{\rho}\beta,\hat{\gamma}\beta)$.

Now if $(\hat{\rho},\hat{\gamma}) \in {}^+(0,0)_{m-1}^+$ there exist unique arrows α,β such that $\alpha\hat{\gamma} \in \mathcal{P}$ and $\hat{\gamma}\beta \in \mathcal{P}$. If $\alpha\hat{\gamma}\beta \in \mathcal{P}$ then $(\alpha\hat{\rho},\alpha\hat{\gamma}) \in (1,0)_m^+$ and $\phi_m(\alpha\hat{\rho},\alpha\hat{\gamma}) = (\hat{\rho}\beta,\hat{\gamma}\beta) \in {}^+(0,1)_m$. If $\alpha\hat{\gamma}\beta \in I$ then $(\alpha\hat{\rho},\alpha\hat{\gamma}) \in (1,0)_m^{-+}$ and $\psi_m(\alpha\hat{\rho},\alpha\hat{\gamma}) = (\hat{\rho}\beta,\hat{\gamma}\beta) \in {}^{+-}(0,1)_m$. In both cases

$$F_m(f_{(\hat{\rho},\hat{\gamma})}) = f_{(\alpha\hat{\rho},\alpha\hat{\gamma})} + (-1)^m f_{(\hat{\rho}\beta,\hat{\gamma}\beta)}. \qquad \square$$

Lemma 4.2. *For any $n \geqslant 2$ we have that*

$$\dim_k \operatorname{Ker} F_n = \big|{}^-(0,0)_{n-1}^-\big| + \big|(1,0)_{n-1}\big| + \big|{}^-(0,1)_{n-1}\big| + \big|(1,1)_{n-1}\big|$$

and

$$\dim_k \operatorname{Im} F_n = \big|{}^{--}(0,1)_n\big| + \big|(1,0)_n\big| + \big|(1,1)_n\big|.$$

Proof. From the previous lemma we have that

$$\dim_k \operatorname{Ker} F_n = \left|{}^-(0,0)_{n-1}^-\right| + \left|(1,0)_{n-1}^-\right| + \left|{}^-(0,1)_{n-1}\right| + \left|(1,1)_{n-1}\right| + \left|(1,0)_{n-1}^+\right|,$$

but

$$\left|(1,0)_{n-1}\right| = \left|(1,0)_{n-1}^-\right| + \left|(1,0)_{n-1}^+\right|.$$

Moreover

$$\dim_k \operatorname{Im} F_n = \left|{}^{--}(0,1)_n\right| + \left|(1,0)_n^{--}\right| + \left|(1,1)_n\right| + \left|(1,0)_n^+\right| + \left|(1,0)_n^{-+})\right|,$$

but

$$\left|(1,0)_n\right| = \left|(1,0)_n^{--}\right| + \left|(1,0)_n^{-+}\right| + \left|(1,0)_n^+\right|. \qquad \square$$

Theorem 4.3. *If A is a triangular string algebra, then*

$$\dim_k \mathrm{HH}^n(A) = \begin{cases} 1 & \text{if } n = 0, \\ |Q_1| + |{}^-(0,0)_1^-| - |Q_0| + 1 & \text{if } n = 1, \\ |{}^{+-}(0,1)_n| + |{}^-(0,0)_n^-| & \text{if } n \geqslant 2. \end{cases}$$

Proof. It is clear that $\mathrm{HH}^0(A) = \operatorname{Ker} F_1$ is the center of A, and has dimension 1 since A is triangular. This implies that

$$\dim_k \operatorname{Im} F_1 = \left|(Q_0//Q_0)\right| - \dim_k \operatorname{Ker} F_1 = |Q_0| - 1.$$

So

$$\dim_k \mathrm{HH}^1(A) = \dim_k \operatorname{Ker} F_2 - |Q_0| + 1 = \left|(1,1)_1\right| + \left|{}^-(0,0)_1^-\right| - |Q_0| + 1$$

since $(1,0)_1 = \emptyset = (0,1)_1$, and $|(1,1)_1| = |Q_1|$. Finally for $n \geqslant 2$

$$\begin{aligned} \dim_k \mathrm{HH}^n(A) &= \dim_k \operatorname{Ker} F_{n+1} - \dim_k \operatorname{Im} F_n \\ &= \left|(1,1)_n\right| + \left|(1,0)_n\right| + \left|{}^-(0,1)_n\right| + \left|{}^-(0,0)_n^-\right| - \left|(1,0)_n\right| - \left|(1,1)_n\right| - \left|{}^{--}(0,1)_n\right| \\ &= \left|{}^{+-}(0,1)_n\right| + \left|{}^-(0,0)_n^-\right|. \qquad \square \end{aligned}$$

The following corollary includes the subclass of gentle algebras, that is, string algebras $A = kQ/I$ such that I is generated by quadratic relations and for any arrow $\alpha \in Q$ there is at most one arrow β and at most one arrow γ such that $\alpha\beta \in I$ and $\gamma\alpha \in I$.

Corollary 4.4. *If A is a triangular quadratic algebra, then*

$$\dim_k \mathrm{HH}^n(A) = \begin{cases} 1 & \text{if } n = 0, \\ |Q_1| + |{}^-(0,0)_1^-| - |Q_0| + 1 & \text{if } n = 1, \\ |{}^-(0,0)_n^-| & \text{if } n \geqslant 2. \end{cases}$$

As a consequence, we recover [1, Theorem 5.1].

Corollary 4.5. *If A is a triangular string algebra, the following conditions are equivalent*:

(i) $\mathrm{HH}^1(A) = 0$;
(ii) *The quiver Q is a tree*;
(iii) $\mathrm{HH}^i(A) = 0$ *for $i > 0$*;
(iv) *A is simply connected.*

Proof. It is well known that for monomial algebras, A is simply connected if and only if Q is a tree. If $\mathrm{HH}^1(A) = 0$, observing that $|{}^-(0,0)_1^-| \geqslant 0$ we have that $|Q_1| - |Q_0| + 1 = 0$. Then the quiver Q is a tree. All the other implications are clear. $\square$

Example 3. For $n \geqslant 1$ let $A_n = kQ/I$ with

$$Q:\ 0 \underset{\beta_1}{\overset{\alpha_1}{\rightrightarrows}} 1 \underset{\beta_2}{\overset{\alpha_2}{\rightrightarrows}} 2 \qquad \cdots \qquad n-1 \underset{\beta_n}{\overset{\alpha_n}{\rightrightarrows}} n$$

and $I = \langle \alpha_i\alpha_{i+1}, \beta_i\beta_{i+1}\rangle_{\{i=1,\dots,n-1\}}$. Then

$$\dim_k \mathrm{HH}^i(A_1) = \begin{cases} 1 & \text{if } i=0,\\ 3 & \text{if } i=1,\\ 0 & \text{otherwise,}\end{cases} \qquad \dim_k \mathrm{HH}^i(A_{2m}) = \begin{cases} 1 & \text{if } i=0,\\ 2m & \text{if } i=1,\\ 0 & \text{otherwise}\end{cases}$$

and

$$\dim_k \mathrm{HH}^i(A_{2m+1}) = \begin{cases} 1 & \text{if } i=0,\\ 2m+1 & \text{if } i=1,\\ 2 & \text{if } i=2m+1,\\ 0 & \text{otherwise.}\end{cases}$$

5. Ring structure

In this section we prove that the product structure on the Hochschild cohomology of a triangular string algebra is trivial.

It is well known that the Hochschild cohomology groups $\mathrm{HH}^i(A)$ can be identified with the groups $\mathrm{Ext}^i_{A-A}(A,A)$, so the Yoneda product defines a product in the Hochschild cohomology $\sum_{i\geqslant 0}\mathrm{HH}^i(A)$ that coincides with the cup product as defined in [13,14].

Given $[f] \in \mathrm{HH}^m(A)$ and $[g] \in \mathrm{HH}^n(A)$, the cup product $[g \cup f] \in \mathrm{HH}^{n+m}(A)$ can be defined as follows: $g \cup f = gf_n$ where f_n is a morphism making the following diagram commutative

$$\begin{tikzcd}
A\otimes kAP_{m+n}\otimes A \arrow[r, "d_{m+n}"] \arrow[d, "f_n"] & A\otimes kAP_{m+n-1}\otimes A \arrow[r, "d_{m+n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "d_{m+1}"] & A\otimes kAP_m\otimes A \arrow[d, "f_0"] \arrow[dr, "f"] & & \\
A\otimes kAP_n\otimes A \arrow[r, "d_n"] \arrow[d, "g"] & A\otimes kAP_{n-1}\otimes A \arrow[r, "d_{n-1}"] & \cdots \arrow[r] & A\otimes kAP_0\otimes A \arrow[r, "\mu"] & A \arrow[r] & 0. \\
A & & & & &
\end{tikzcd}$$

In particular we are interested in maps $f \in \mathrm{Hom}_{A-A}(A\otimes kAP_m\otimes A, A)$ such that the associated morphism $\hat f \in \mathrm{Hom}_{E-E}(kAP_m, A)$ defined by $\hat f(w) = f(1\otimes w\otimes 1)$ is in the kernel of the morphism $F_{m+1}: \mathrm{Hom}_{E-E}(kAP_m, A) \to \mathrm{Hom}_{E-E}(kAP_{m+1}, A)$ appearing in the Hochschild complex.

For any $m > 0$ we will use Lemma 3.2 in order to the define maps f_n that complete the previous diagram in a commutative way. Recall that if $n > 0$ any $w = w(p_1,\dots,p_{m+n-1}) \in AP_{m+n}$ can be written in a unique way as

$$w = {}^{(n)}w\, u\, w^{(m)}$$

with ${}^{(n)}w = w(p_1,\dots,p_{n-1}) \in AP_n$ and $w^{(m)} = w^{op}(q^{n+1},\dots,q^{n+m-1}) \in AP_m$. Let

$$f_n : A\otimes kAP_{n+m}\otimes A \to A\otimes kAP_n\otimes A$$

be defined by

$$f_n(1\otimes w\otimes 1) = \begin{cases} 1\otimes 1\otimes \hat f(w) & \text{if } n=0,\\ \sum_{\psi\in AP_n, L(\psi)\psi R(\psi)={}^{(n)}wu} L(\psi)\otimes\psi\otimes R(\psi)\hat f(w^{(m)}) & \text{if } n>0.\end{cases}$$

Remark 2. From Lemma 3.2 and Lemma 3.6 we can deduce that if n is even then

$$f_n(1\otimes w\otimes 1) = 1\otimes {}^{(n)}w\otimes u\hat f\big(w^{(m)}\big)$$

because $w(p_1,\dots,p_n) = {}^{(n)}w\,u\,b$ and hence

$$\psi \in \mathrm{Sub}\big(w(p_1,\dots,p_n)\big) = \big\{\psi_1, \psi_2 = w(p_1,\dots,p_{n-1})\big\}.$$

But b is a non-trivial path, then $L(\psi)\psi R(\psi) = {}^{(n)}wu$ implies that $\psi = \psi_2 = {}^{(n)}w$.

Proposition 5.1. *Let $m>0$ and let $f\in \operatorname{Hom}_{A-A}(A\otimes kAP_m\otimes A, A)$ be such that $\hat{f}\in \operatorname{Ker} F_{m+1}$. Then $f=\mu f_0$ and $f_{n-1}d_{m+n}=d_nf_n$ for any $n\geqslant 1$.*

Proof. It is clear that $f=\mu f_0$ since for any $w\in AP_m$ we have that

$$\mu f_0(1\otimes w\otimes 1)=\mu\big(1\otimes 1\otimes \hat{f}(w)\big)=\hat{f}(w)=f(1\otimes w\otimes 1).$$

Let $n\geqslant 1$ and let $w\in AP_{n+m}$. By Lemma 4.1 we have that $\hat{f}$ is a linear combination of basis elements in ${}^{-}(0,0)^{-}_{m}$, $(1,0)^{-}_{m}$, ${}^{-}(0,1)_m$, $(1,1)_m$ and $(id+(-1)^m\phi_m)((1,0)^{+}_{m})$. The proof will be done in several steps considering $\hat{f}=f_{(\rho,\gamma)}$ with (ρ,γ) belonging to each one of the previous sets.

(i) Assume $(\rho,\gamma)\in {}^{-}(0,0)^{-}_{m}$. Using that $Q_1\hat{f}(w^{(m)})\subset I$ we have that

$$f_n(1\otimes w\otimes 1)=\sum_{\substack{\psi\in AP_n\\ L(\psi)\psi R(\psi)={}^{(n)}wu}} L(\psi)\otimes\psi\otimes R(\psi)\hat{f}\big(w^{(m)}\big)=L(\psi)\otimes\psi\otimes\hat{f}\big(w^{(m)}\big)$$

if there exists $\psi\in AP_n$ such that $L(\psi)\psi={}^{(n)}wu$ and zero otherwise. In the first case

$$\begin{aligned}d_nf_n(1\otimes w\otimes 1)&=\sum_{\substack{\phi\in AP_{n-1}\\ L(\phi)\phi R(\phi)=\psi}} L(\psi)L(\phi)\otimes\phi\otimes R(\phi)\hat{f}\big(w^{(m)}\big)\\&=L(\psi)L(\phi)\otimes\phi\otimes\hat{f}\big(w^{(m)}\big)\end{aligned}$$

for $\phi\in AP_{n-1}$ such that $L(\psi)L(\phi)\phi=L(\psi)\psi={}^{(n)}wu$. On the other hand, if $n+m$ is even,

$$f_{n-1}d_{n+m}(1\otimes w\otimes 1)=f_{n-1}\bigg(\sum_{\psi'\in \operatorname{Sub}(w)} L\big(\psi'\big)\otimes\psi'\otimes R\big(\psi'\big)\bigg)=L\big(\psi'\big)f_{n-1}\big(1\otimes\psi'\otimes 1\big)$$

with ψ' such that $L(\psi')\psi'=w$ since $\hat{f}(\psi'^{(m)})R(\psi')\subset\hat{f}(\psi'^{(m)})Q_1\subset I$ for any ψ' such that $|R(\psi')|>0$. In case $n+m$ is odd we get the same final result. Now $Q_1\hat{f}(\psi'^{(m)})\subset I$ and $\psi'={}^{(n-1)}\psi'uw^{(m)}$, then

$$L\big(\psi'\big)f_{n-1}\big(1\otimes\psi'\otimes 1\big)=L\big(\psi'\big)L\big(\phi'\big)\otimes\phi'\otimes\hat{f}\big(w^{(m)}\big)$$

if there exists $\phi'\in AP_{n-1}$ such that $L(\psi')L(\phi')\phi'=L(\psi')^{(n-1)}\psi'u={}^{(n)}wu$ and zero otherwise.
The desired equality holds because: if ψ and ϕ' do not exist, both terms vanish. If ψ exists, then it is clear that ϕ' also exists, in fact $\phi'=\phi$ and $L(\psi')L(\phi')\otimes\phi'=L(\psi)L(\phi)\otimes\phi$. Finally assume that there is no $\psi\in AP_n$ such that $L(\psi)\psi={}^{(n)}wu$ and that ϕ' exists with $L(\psi')L(\phi')\in\mathcal{P}$. If n is even, Lemma 3.5(i) applied on ${}^{(n)}w$ and ϕ' says that $t(p_1)\leqslant s(\phi')$ and hence $L(\psi')L(\phi')=0$. If n is odd, Lemma 3.5(ii) applied on ${}^{(n-1)}w$ and ϕ' implies the existence of ψ, a contradiction.

(ii) Assume $(\rho,\gamma)\in(1,0)^{-}_{m}$. Then $\rho=\alpha\hat{\rho}=\alpha\rho_1\cdots\rho_s$, $\gamma=\alpha\hat{\gamma}$ and $\alpha\rho_1\in\mathcal{R}$. Then $f_n(1\otimes w\otimes 1)=0$ if $w^{(m)}\neq\rho$. In this case $f_{n-1}d_{n+m}(1\otimes w\otimes 1)$ also vanishes: the assertion is clear if ρ does not divide w and, if it does, $w=L(\rho)\rho R(\rho)$ with $|R(\rho)|>0$ and $\hat{f}(\rho)R(\rho)$ vanishes. Assume now that $w^{(m)}=\rho$. This means that w contains the relation $q^{n+1}=\alpha\rho_1$, by Lemma 3.4 we have that $q^{n+1}=p_{n+1}$ and $q^n=p_n$, and by Lemma 3.3 we have that $|\operatorname{Sub}(w)|=2$. Then

$$\begin{aligned}f_{n-1}d_{n+m}(1\otimes w\otimes 1)&=L(\psi_1)f_{n-1}(1\otimes\psi_1\otimes 1)+(-1)^{n+m}f_{n-1}(1\otimes\psi_2\otimes 1)R(\psi_2)\\&=L(\psi_1)f_{n-1}(1\otimes\psi_1\otimes 1)\end{aligned}$$

since $\hat{f}(\psi_2^{(m)})=0$, and $\psi_1=w^{op}(q^2,\dots,q^{n+m-1})={}^{(n-1)}\psi_1vw^{(m)}={}^{(n-1)}\psi_1v\rho$. If n is odd, by Remark 2 we have that

$$f_{n-1}d_{n+m}(1\otimes w\otimes 1)=L(\psi_1)\otimes{}^{(n-1)}\psi_1\otimes v\hat{f}\big(w^{(m)}\big)=L(\psi_1)\otimes{}^{(n-1)}\psi_1\otimes v\gamma,$$

and if n is even

$$f_{n-1}d_{n+m}(1\otimes w\otimes 1)=\sum_{\substack{\phi\in AP_{n-1}\\ L(\phi)\phi R(\phi)={}^{(n-1)}\psi_1v}} L(\psi_1)L(\phi)\otimes\phi\otimes R(\phi)\gamma.$$

On the other hand, $w={}^{(n)}wuw^{(m)}={}^{(n)}wu\rho$ and

$$f_n(1\otimes w\otimes 1)=\sum_{\substack{\psi\in AP_n\\ L(\psi)\psi R(\psi)={}^{(n)}wu}} L(\psi)\otimes\psi\otimes R(\psi)\hat f\big(w^{(m)}\big)$$
$$=\sum_{\substack{\psi\in AP_n\\ L(\psi)\psi R(\psi)={}^{(n)}wu}} L(\psi)\otimes\psi\otimes R(\psi)\alpha\hat\gamma.$$

By definition we have that $s(p_{n+1})<t(p_n)<t(p_{n+1})$, and $p_{n+1}=\alpha\rho_1$ so $t(p_n)=t(\alpha)$. Then ${}^{(n)}w\,u\,\alpha={}^{(n+1)}w$ and

$$\big\{\psi\in AP_n,\ L(\psi)\psi R(\psi)={}^{(n)}wu\big\}=\mathrm{Sub}\big({}^{(n+1)}w\big)\setminus\{\hat\psi\}$$

where $\hat\psi=w^{op}(q^2,\dots,q^n)={}^{(n-1)}\psi_1 v\alpha$ and ${}^{(n+1)}w=L(\psi_1)\hat\psi$. So

$$f_n(1\otimes w\otimes 1)=d_{n+1}\big(1\otimes{}^{(n+1)}w\otimes 1\big)\hat\gamma-(-1)^{n+1}L(\psi_1)\otimes\hat\psi\otimes\hat\gamma,$$

and hence

$$d_nf_n(1\otimes w\otimes 1)=(-1)^nL(\psi_1)d_n(1\otimes\hat\psi\otimes 1)\hat\gamma.$$

If n is odd

$$d_n(1\otimes\hat\psi\otimes 1)=L(\hat\psi_1)\otimes\hat\psi_1\otimes 1-1\otimes\hat\psi_2\otimes R(\hat\psi_2)$$

with $\hat\psi_2={}^{(n-1)}\psi_1$, $R(\hat\psi_2)=v\alpha$ and $L(\psi_1)L(\hat\psi_1)=0$ because

$$s\big(L(\psi_1)\big)=s(p_1)<t(p_1)=t\big(q^1\big)\leqslant s\big(q^3\big)=s(\hat\psi_1)=t\big(L(\hat\psi_1)\big)$$

implies that p_1 divides $L(\psi_1)L(\hat\psi_1)$. So

$$d_nf_n(1\otimes w\otimes 1)=L(\psi_1)\otimes\hat\psi_2\otimes R(\hat\psi_2)\hat\gamma=L(\psi_1)\otimes{}^{(n-1)}\psi_1\otimes v\gamma.$$

Finally, if n is even

$$d_nf_n(1\otimes w\otimes 1)=\sum_{\substack{\phi\in AP_{n-1}\\ L(\phi)\phi R(\phi)=\hat\psi}} L(\psi_1)L(\phi)\otimes\phi\otimes R(\phi)\hat\gamma$$

and the desired equality holds since

$$\big\{\phi\in AP_{n-1}\colon L(\phi)\phi R(\phi)={}^{(n-1)}\psi_1 v\big\}=\big\{\phi\in AP_{n-1}\colon L(\phi)\phi R(\phi)=\hat\psi\big\}\setminus\{\hat\psi_1\}$$

and, as we have already seen, $L(\psi_1)L(\hat\psi_1)=0$.

(iii) If $(\rho,\gamma)\in(1,1)_m$ then $\rho=\alpha\hat\rho\beta=\alpha\rho_1\dots\rho_s\beta$, $\gamma=\alpha\hat\gamma\beta$ and $\alpha\rho_1,\rho_s\beta\in\mathcal{R}$. Then $f_n(1\otimes w\otimes 1)=0$ if $w^{(m)}\neq\rho$. In this case $f_{n-1}d_{n+m}(1\otimes w\otimes 1)$ also vanishes: the assertion is clear if ρ does not divide w and, if it does, Lemma 3.3 says that $|\mathrm{Sub}(w)|=2$ and then

$$f_{n-1}d_{n+m}(1\otimes w\otimes 1)=L(\psi_1)f_{n-1}(1\otimes\psi_1\otimes 1)+(-1)^{n+m}f_{n-1}(1\otimes\psi_2\otimes 1)R(\psi_2).$$

The first summand vanishes since $\psi_1^{(m)}=w^{(m)}\neq\rho$. For the second one, observe that it vanishes if $\psi_2^{(m)}\neq\rho$. If $\psi_2^{(m)}=\rho$ then $\psi_2=w(p_1,\dots,p_{n+m-2})$ with $p_{n+m-2}=\rho_s\beta$ and $p_{n+m-1}=\beta R(\psi_2)$. Then $\hat f(\psi_2^{(m)})R(\psi_2)=\alpha\hat\gamma\beta R(\psi_2)=0$.
Assume now that $w^{(m)}=\rho$. This means that w contains the relation $q^{n+1}=\alpha\rho_1$ and the proof follows exactly as in (ii).

(iv), (v) Similar to the previous ones. □

In order to describe the product $[g\cup f]$ we need to choose convenient representatives of the classes $[f]$ and $[g]$, see [5]. Given $f\in\mathrm{Hom}_{A-A}(A\otimes kAP_m\otimes A,A)$ we define $f^{\leqslant}$ and $f^{\geqslant}$ as follows: we start by considering basis elements $f_{(\rho,\gamma)}$

$$f^{\leqslant}_{(\rho,\gamma)}=\begin{cases} f_{(\rho,\gamma)} & \text{if } (\rho,\gamma)\in(0,0)_m,\\ (-1)^{m-1}f_{\phi_m(\rho,\gamma)} & \text{if } (\rho,\gamma)\in(1,0)_m^{+},\\ (-1)^{m-1}f_{\psi_m(\rho,\gamma)} & \text{if } (\rho,\gamma)\in(1,0)_m^{-+},\\ 0 & \text{if } (\rho,\gamma)\in(1,0)_m^{--},\\ f_{(\rho,\gamma)} & \text{if } (\rho,\gamma)\in(0,1)_m,\\ 0 & \text{if } (\rho,\gamma)\in(1,1)_m\end{cases}$$

and

$$f^{\geqslant}_{(\rho,\gamma)} = \begin{cases} f_{(\rho,\gamma)} & \text{if } (\rho,\gamma) \in (0,0)_m, \\ (-1)^{m-1} f_{\phi_m^{-1}(\rho,\gamma)} & \text{if } (\rho,\gamma) \in {}^{+}(0,1)_m, \\ (-1)^{m-1} f_{\psi_m^{-1}(\rho,\gamma)} & \text{if } (\rho,\gamma) \in {}^{+-}(0,1)_m, \\ 0 & \text{if } (\rho,\gamma) \in {}^{--}(0,1)_m, \\ f_{(\rho,\gamma)} & \text{if } (\rho,\gamma) \in (1,0)_m, \\ 0 & \text{if } (\rho,\gamma) \in (1,1)_m \end{cases}$$

and then we extend by linearity. By Lemma 4.1 we have that $f - f^{\leqslant}, f - f^{\geqslant} \in \operatorname{Im} F_m$ for any $m > 0$, and hence $[f] = [f^{\leqslant}] = [f^{\geqslant}]$. Moreover, observe that $f^{\leqslant}$ is a linear combination of basis elements in $(0,0)_m \cup (0,1)_m$ and $f^{\geqslant}$ is a linear combination of basis elements in $(0,0)_m \cup (1,0)_m$.

Theorem 5.2. *If A is a triangular string algebra and $n, m > 0$ then $\mathrm{HH}^n(A) \cup \mathrm{HH}^m(A) = 0$.*

Proof. Let $[f] \in \mathrm{HH}^m(A)$ and $[g] \in \mathrm{HH}^n(A)$. We will show that $[g^{\leqslant} \cup f^{\geqslant}] = 0$. Let $w \in AP_{n+m}$, $w = {}^{(n)}w\ u\ w^{(m)}$ then

$$g^{\leqslant} \cup f^{\geqslant}(1 \otimes w \otimes 1) = \sum_{\substack{\psi \in AP_n \\ L(\psi)\psi R(\psi) = {}^{(n)}wu}} L(\psi)\hat{g}^{\leqslant}(\psi)R(\psi)\hat{f}^{\geqslant}\big(w^{(m)}\big).$$

Since $f \in \operatorname{Ker} F_m$ and $g \in \operatorname{Ker} F_n$, we know that f and g are linear combination of basis elements as described in Lemma 4.1. Moreover, $f^{\geqslant}$ is a linear combination of basis elements in ${}^{-}(0,0)^{-} \cup (1,0)$ and $g^{\leqslant}$ is a linear combination of basis elements in ${}^{-}(0,0)^{-} \cup (0,1)$.

The vanishing of the previous computation is clear if $f^{\geqslant}$ or $g^{\leqslant}$ are basis elements associated to pairs in ${}^{-}(0,0)^{-}$ because for $n, m > 0$ we have that $|\hat{g}^{\leqslant}(\psi)| > 0$ and $|\hat{f}^{\geqslant}(w^{(m)})| > 0$. Finally, if $f^{\geqslant}$ is a basis element associated to a pair $(\alpha\rho, \alpha\gamma) \in (1,0)$ and $g^{\leqslant}$ is a basis element associated to a pair $(\rho'\beta, \gamma'\beta) \in (0,1)$ then we only have to consider the summand with $\psi = \rho'\beta$ and $w^{(m)} = \alpha\rho$. In this case w verifies the following conditions: $p_{n+1} = q^{n+1} = \alpha\rho_1$, ${}^{(n)}wu$, and hence also ${}^{(n+1)}w$, contains the quadratic relation $\rho_s'\beta$, by Lemma 3.3 the element ${}^{(n+1)}w$ has exactly two divisors, one of them sharing the ending point with ${}^{(n+1)}w$, so $\psi = {}^{(n)}w$ and $p_n = \beta u \alpha$. Now the summand we are considering is

$$\begin{aligned} L(\psi)\hat{g}^{\leqslant}(\psi)R(\psi)\hat{f}^{\geqslant}\big(w^{(m)}\big) &= \hat{g}^{\leqslant}\big({}^{(n)}w\big)\, u\, \hat{f}^{\geqslant}\big(w^{(m)}\big) \\ &= \gamma'\beta\, u\, \alpha\gamma \\ &= \gamma' p_n \gamma = 0. \qquad \Box \end{aligned}$$

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