Irregular Wavelet Frames and Gabor Frames

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Abstract

Given $g \in L^2(\mathbb{R}^n)$, we consider irregular wavelet systems of the form $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$, where $\lambda_j > 0$ and $b > 0$. Sufficient conditions for the wavelet system to constitute a frame for $L^2(\mathbb{R}^n)$ are given. For a class of functions $g \in L^2(\mathbb{R}^n)$ we prove that certain growth conditions on $\{\lambda_j\}$ will lead to frames, and that some other types of sequences exclude the frame property. We also give a sufficient condition for a Gabor system $\{e^{2\pi i b(j,x)} g(x-\lambda_k\}_{j\in\mathbb{Z}^n,k\in\mathbb{Z}}$ to be a frame.

1 Introduction

Recall that a family of elements $\{f_i\}$ in a Hilbert space H is a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_i |(f, f_i)|^2 \leq B\|f\|^2,$$

for every $f \in H$. The numbers A, B are called *frame bounds*. If $\{f_i\}$ is a frame, the *frame operator* defined on H is given by $Sf = \sum_i (f, f_i) f_i$.

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The frame operator is invertible, and every $f \in H$ can be represented $f = \sum_i (f, S^{-1}f_i)f_i$, a fact that makes frames very attractive in signal analysis.

Given a function $g \in L^2(\mathbb{R}^n)$, we consider the sequence of functions $\{g_{j,k}\}$ defined by

(i) $g_{j,k}(x) = \lambda_j^{\frac{n}{2}} g(\lambda_j x - kb), j \in \mathbb{Z},\ k \in \mathbb{Z}^n$, where $\{\lambda_j\}_{j\in\mathbb{Z}} \subseteq \mathbb{R}^+,\ \ b > 0$,

(ii) $g_{j,k}(x) = e^{2\pi i b(j,x)} g(x - \lambda_k), j \in \mathbb{Z}^n,\ k \in \mathbb{Z}$, where $\{\lambda_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}^n$, $b > 0$.

The purpose of this note is to give sufficient conditions for $\{g_{j,k}\}$ being a frame for $L^2(\mathbb{R}^n)$.

The set of functions $\{g_{j,k}\}$ defined by (i) will be called an *irregular wavelet system*; in the case (ii), we speak about an irregular Gabor system. The corresponding regular cases are well studied in the literature (see [1], [5], [8]) and appear as the special cases $\lambda_j = a^j$ for a certain $a > 1$ in (i) and $\lambda_k = ka$ for $a > 0$ $(k \in \mathbb{Z}^n)$ in (ii). The regular systems are clearly easier to deal with, but due to finite machine precision or measurement errors, one is sometimes forced to work with the irregular systems. One approach to the irregular case is to use perturbation methods, which however forces λ_j to be "close" to the regular case. Our approach is rather to give a direct argument, which is closer in spirit to the original proofs.

2 Wavelet frames

Before we consider the specific wavelet frames we make an observation for general frames. The elements in a frame $\{f_i\}$ can appear repeatedly. By deleting all repetitions of the elements in $\{f_i\}$ we obtain a family $\{g_k\}$ for which $g_k \neq g_l$ for $k \neq l$. Let n_k denote the number of times g_k appear in $\{f_i\}$. With this notation we have

Lemma 2.1 *Assume that $\{f_i\}$ is a frame with bounds A, B and that $||f_i||$ is bounded below by $C > 0$, for all i. Then*

$$N := \sup_k n_k \leq \frac{B}{C^2},$$

and $\{g_k\}$ *is a frame with bounds* $\frac{A}{N}, B$.

Proof. Given $k \in \mathbb{Z}$, the element g_k appears n_k times in $\{f_i\}$. Thus

$$n_k \cdot ||g_k||^4 \leq \sum_i |< g_k, f_i >|^2 \leq B \cdot ||g_k||^2,$$

and therefore $n_k \leq \frac{B}{||g_k||^2} \leq \frac{B}{C^2}$.

The family of elements consisting of all g_k, each of them repeated N times, clearly contains $\{f_i\}$. Therefore, for any f we have

$$N \sum_k |< f, g_k >|^2 \geq \sum_i |< f, f_i >|^2 \geq A||f||^2,$$

showing that $\{g_k\}$ has the lower bound $\frac{A}{N}$. ∎

Conversely, it is clear that if $\{g_k\}$ is a frame with bounds A and B and each element is repeated at most N times in $\{f_i\}$, then $\{f_i\}$ is a frame with bounds A and NB. Because of the Lemma above we will only consider sequences $\{\lambda_k\}_{k=1}^{\infty}$ without repetitions in (i) or (ii) of the introduction.

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform of f is the function $\hat{f}$ defined by

$$\hat{f}(x) = \int f(t) e^{-2\pi i (x,t)}\, dt, x \in \mathbb{R}^n.$$

We use frequently the Parseval formula, also called the Plancherel equality,

$$(f, g) = (\hat{f}, \hat{g}),\ \ f, g \in L^2(\mathbb{R}^n).$$

We first generalize a result from [1].

Theorem 2.2 *Let* $\{\lambda_j\}_{j \in \mathbb{Z}}$ *be a sequence of positive real numbers,* $b > 0$ *and* $g \in L^2(\mathbb{R}^n)$. *Suppose that*

$$A := \frac{1}{b^n} \text{ess}\inf_{x \in \mathbb{R}^n} \left(\sum_{j \in \mathbb{Z}} |\hat{g}(\frac{x}{\lambda_j})|^2 - \sum_{k \neq 0} \sum_{j \in \mathbb{Z}} |\hat{g}(\frac{x}{\lambda_j}) \hat{g}(\frac{x}{\lambda_j} + \frac{k}{b})| \right) > 0,$$

and

$$B := \frac{1}{b^n} \text{ess}\sup_{x \in \mathbb{R}^n} \sum_{k,j} |\hat{g}(\frac{x}{\lambda_j}) \hat{g}(\frac{x}{\lambda_j} + \frac{k}{b})| < \infty.$$

Then $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^n}$ *is a frame with frame bounds* A *and* B.

Proof. Let $f \in L^2$ and $b > 0$, then by the Parseval formula we get

$$(1) \qquad \sum_{j,k} |(f, g_{j,k})|^2 = \sum_{j,k} \lambda_j^n \left| \sum_{l \in \mathbb{Z}^n} \int_{I_b} \hat{f}(\lambda_j(y + \frac{l}{b}))\overline{\hat{g}}(y + \frac{l}{b}) e^{2\pi i b k y}\, dy \right|^2 ,$$

where $I_b = [0, \frac{1}{b}]^n = [0, \frac{1}{b}] \times [0, \frac{1}{b}] \times ... \times [0, \frac{1}{b}]$.

Assume that $\hat{f}$ has compact support. By the Parseval identity for multiple Fourier series (1) is equal to

$$(2) \qquad \sum_j (\frac{\lambda_j}{b})^n \int_{I_b} \left| \sum_{l \in \mathbb{Z}^n} \hat{f}(\lambda_j(y + \frac{l}{b}))\overline{\hat{g}}(y + \frac{l}{b}) \right|^2 dy.$$

To see that the integral in (2) is finite we have used the fact that $\hat{g} \in L^\infty(\mathbb{R}^n)$. We write the function in the above integral as $\sum_{l \in \mathbb{Z}^n} H_j(y + \frac{l}{b})\, G_j(y)$, where the function $H_j(y) = \hat{f}(\lambda_j y)\overline{\hat{g}}(y)$ and G_j is the $\frac{1}{b}$-periodic function $\sum_{m \in \mathbb{Z}^n} \overline{\hat{f}}(\lambda_j(y + \frac{m}{b}))\hat{g}(y + \frac{m}{b})$. Since $\hat{f}$ has compact support and $\hat{g}$ is bounded, $H_j G_j \in L^1(\mathbb{R}^n)$. Thus the expression in (2) is equal to

$$\sum_j (\frac{\lambda_j}{b})^n \int H_j(y) G_j(y)\, dy$$

$$(3) \qquad = \sum_j (\frac{\lambda_j}{b})^n \sum_{l \in \mathbb{Z}^n} \int \hat{f}(\lambda_j y)\overline{\hat{g}}(y)\overline{\hat{f}}(\lambda_j(y + \frac{l}{b}))\hat{g}(y + \frac{l}{b})\, dy.$$

Now, by a change of variables, the expression in (3) is

$$(4) \qquad \frac{1}{b^n} \sum_j \int |\hat{f}(x)\overline{\hat{g}}(\frac{x}{\lambda_j})|^2\, dx +$$

$$\frac{1}{b^n} \sum_j \sum_{l \neq 0} \int \hat{f}(x)\overline{\hat{g}}(\frac{x}{\lambda_j})\overline{\hat{f}}(x + \frac{\lambda_j l}{b})\hat{g}(\frac{x}{\lambda_j} + \frac{l}{b})\, dx$$

By the Cauchy-Schwarz inequality, the second term in (4) is bounded by

$$\frac{1}{b^n} \sum_{j,\, l \neq 0} (\int |\hat{f}(x)|^2 |\overline{\hat{g}}(\frac{x}{\lambda_j})\hat{g}(\frac{x}{\lambda_j} + \frac{l}{b})| dx \int |\overline{\hat{f}}(x + \frac{\lambda_j l}{b})|^2 |\overline{\hat{g}}(\frac{x}{\lambda_j})\hat{g}(\frac{x}{\lambda_j} + \frac{l}{b})| dx)^{\frac{1}{2}} =$$

$$\frac{1}{b^n}\sum_j\sum_{l\neq 0}(\int|\hat{f}(x)|^2|\overline{\hat{g}}(\frac{x}{\lambda_j})\hat{g}(\frac{x}{\lambda_j}+\frac{l}{b})|dx)^{\frac{1}{2}}(\int|\overline{\hat{f}}(x)|^2|\hat{g}(\frac{x}{\lambda_j}-\frac{l}{b})\overline{\hat{g}}(\frac{x}{\lambda_j})|dx)^{\frac{1}{2}}\leq$$

$$\frac{1}{b^n}\sum_j(\sum_{l\neq 0}\int|\hat{f}(x)|^2|\overline{\hat{g}}(\frac{x}{\lambda_j})\hat{g}(\frac{x}{\lambda_j}+\frac{l}{b})|dx)^{\frac{1}{2}}(\sum_{l\neq 0}\int|\overline{\hat{f}}(x+\frac{\lambda_j l}{b})|^2|\overline{\hat{g}}(\frac{x}{\lambda_j})\hat{g}(\frac{x}{\lambda_j}+\frac{l}{b})|dx)^{\frac{1}{2}}=$$

$$\frac{1}{b^n}\sum_j\int|\hat{f}(x)|^2\sum_{l\neq 0}|\overline{\hat{g}}(\frac{x}{\lambda_j})\hat{g}(\frac{x}{\lambda_j}+\frac{l}{b})|\,dx.$$

Thus $(Sf,f)\leq\frac{B}{b^n}\|f\|^2$, for all f in a dense class in L^2, so S can be continuously extended to $L^2(\mathbb{R}^n)$ with $\|S\|\leq\frac{B}{b^n}$. This implies that an upper frame bound is $\frac{B}{b^n}$. Also, by (4) and the inequality above we have

$$(Sf,f)\geq\frac{1}{b^n}\int|\hat{f}(x)|^2(\sum_j|\overline{\hat{g}}(\frac{x}{\lambda_j})|^2-\sum_j\sum_{l\neq 0}|\overline{\hat{g}}(\frac{x}{\lambda_j})\hat{g}(\frac{x}{\lambda_j}+\frac{l}{b})|)\,dx,$$

for a dense class in $L^2(\mathbb{R}^n)$. Thus, by the continuity of the operator S, we get $(Sf,f)\geq\frac{A}{b^n}$, for all $f\in L^2(\mathbb{R}^n)$. ∎

The result below is proved in one dimension and for the regular case $\lambda_j=a^j$ in [4], but the original proof works in our more general setting.

Lemma 2.3 *If $\{\lambda_j\}_{j\in\mathbb{Z}}\subseteq\mathbb{R}^+$ and $\{\lambda_j^{\frac{n}{2}}g(\lambda_j x-kb)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ is a frame with upper bound B, then*

$$\frac{1}{b^n}\sum_j|\hat{g}(\frac{x}{\lambda_j})|^2\leq B,\ \ a.e.x.$$

We now prove that this Lemma puts restrictions on the sequences $\{\lambda_j\}$ for which $\{\lambda_j^{\frac{n}{2}}g(\lambda_j x-kb)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ can be a frame. Following [3], we say that a a sequence $\{\lambda_j\}_{j\in\mathbb{Z}}$ of positive numbers is *logarithmically separated by $\lambda>1$* if

$$|\log\lambda_j-\log\lambda_k|\geq\log\lambda,\quad\forall k\neq j.$$

If $\{\lambda_j\}$ is ordered increasingly, this is equivalent to $\frac{\lambda_{j+1}}{\lambda_j}\geq\lambda,\ \forall j$.

Proposition 2.4 *Suppose that $\{\lambda_j^{\frac{n}{2}}g(\lambda_j x-kb)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ is a frame and that $\hat{g}$ is continuous in a point x_0 where $\hat{g}(x_0)\neq 0$. Then $\{\lambda_j\}_{j\in\mathbb{Z}}$ is a finite union of logarithmically separated sets.*

Proof. For notational convenience, we give the argument for $n = 1$. Let $s_j := \frac{1}{\lambda_j}$. According to Lemma 2.3,

$$\frac{1}{b^n}\sum_j |\hat{g}(xs_j)|^2 \leq B, \ a.e.x.$$

Assume that $x_0 > 0$. Let $c := |\hat{g}(x_0)|^2$ and choose $\delta > 0$ such that for all $x \in I_0 := [x_0, x_0 + \delta[$,

$$|\hat{g}(x)|^2 \geq \frac{c}{2}.$$

By taking $x = 1$, it is clear that the number N of elements from $\{\lambda_j\}$ in the interval I_0 satisfies $N\frac{c}{2} \leq Bb^n$, i.e., $N \leq \frac{2B}{c}b^n$. Now let $\sigma := \frac{x_0+\delta}{x_0}$ and define the intervals

$$I_n := [x_0\sigma^n, x_0\sigma^{n+1}[.$$

Clearly $\{I_n\}_{n=-\infty}^{\infty}$ is a disjoint covering of $I\!R^+$ ($I\!R^+ = \cup_{n=-\infty}^{\infty} I_n$), and for given $n \in Z\!\!Z$, the interval I_n contains at most N points from $\{S_j\}$. Now observe that each point in I_0 is logarithmically separated with each family we obtain by picking an arbitrary point from each of the intervals $I_{\pm 2}, I_{\pm 4}, \cdots$. Similarly, a point from I_1 is logarithmically separated with each family we obtain by picking an arbitrary point from each of the intervals $I_{-1}, I_{\pm 3}, I_{\pm 5}, \cdots$. Thus $\{s_j\}$ can be split into at most $2N$ logarithmically separated subsequences, from which the result follows. ∎

Let $B(0, r)$ denote the ball in $I\!R^n$ centered at the origin and with radius r.

Corollary 2.5 *Suppose that $g \in L^2(I\!R^n)$ and that supp$\hat{g} \subseteq B(0, \frac{1}{2b})$ for some $b > 0$. Then the following holds:*

(i) If

(5)
$$A = \frac{1}{b^n}\text{ess inf}_{x\in I\!R^n} \sum_{j\in Z\!\!Z} |\hat{g}(\frac{x}{\lambda_j})|^2 > 0, \ B = \frac{1}{b^n}\text{ess sup}_{x\in I\!R^n} \sum_{j\in Z\!\!Z} |\hat{g}(\frac{x}{\lambda_j})|^2 < \infty \ a.e.,$$

then $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j\in Z\!\!Z, k\in Z\!\!Z^n}$ is a frame with bounds A and B.

(ii) Conversely, if $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j\in Z\!\!Z, k\in Z\!\!Z^n}$ is a frame with frame bounds

A and B, then we have

$$A \leq \frac{1}{b^n} \sum_{j \in \mathbb{Z}} |\hat{g}(\frac{x}{\lambda_j})|^2 \leq B.$$

Therefore, the frame bounds A and B in (5) are the optimal frame bounds.

Proof. (i) is a consequence of Theorem 2.2. Now assume that the conditions in (ii) are satisfied. By (4) we have,

$$A\|f\|_2^2 \leq (Sf, f) = \int |\hat{f}(x)|^2 \frac{1}{b^n} \sum_j |\hat{g}(\frac{x}{\lambda_j})|^2 \leq B\|f\|^2,$$

for every $f \in L^2$. Now let E be any measurable set in $\mathbb{R}^n$ such that $0 < |E| < \infty$, where $|E|$ stands for the Lebesgue measure of the set E. Then $A \leq \frac{1}{|E|} \int_E \frac{1}{b^n} \sum_j |\hat{g}(\frac{x}{\lambda_j})|^2 dx \leq B$, thus we have

$$b^n A \leq \sum_{j \in \mathbb{Z}} |\hat{g}(\frac{x}{\lambda_j})|^2 \leq b^n B, a.e. x \in \mathbb{R}^n. \tag{6}$$ ∎

We say that a function f is a *radially increasing function* in $B(0, \frac{1}{2b})$ if

$$|x| \leq |y| < \frac{1}{2b} \Rightarrow |f(x)| \leq |f(y)|.$$

Since a radially increasing function in $B(0, \frac{1}{2b})$ is indeed a radial function in $B(0, \frac{1}{2b})$, we will often write $f(|x|)$ for $f(x)$, if $x \in B(0, \frac{1}{2b})$.

Given $0 < l \leq L$, $a > 1$, a real sequence $\{\lambda_j\}_{j \in \mathbb{Z}}$ is said to be of exponential type (a, l, L) if $\lambda_0 = 1$, $\lambda_j < \lambda_{j+1}$ and $la^j \leq \lambda_j \leq La^j$ for all $j \in \mathbb{Z}$. Note that if $L < la$ it is enough to select any $\lambda_j \in [la^j, La^j]$ in order to get a sequence of exponential type (a, l, L); in this case we also have $\frac{\lambda_{j+1}}{\lambda_j} \geq \frac{la^{j+1}}{La^j} = \frac{la}{L} > 1$, i.e., $\{\lambda_j\}$ is logarithmically separated. In case $L \geq la$ the intervals $[la^j, La^j]$ are overlapped; by selecting an *increasing* sequence $\{\lambda_j\}$ such that $\lambda_j \in [la^j, La^j]$, we again have a sequence of exponential type.

For further references we state the next remark.

Remark 2.6 *Let $\{\lambda_j\}_{j \in \mathbb{Z}}$ be a sequence of exponential type (a, l, L). It is easy to see that the set $\{j : 2b\lambda_k \leq \lambda_j \leq 2b\lambda_{k+1}\}$ contains at most $\lfloor \frac{2}{\log a} \log \frac{L}{l} + 1 \rfloor + 1$ elements; here $\lfloor x \rfloor$ denotes the integer part of x.*

In the next Theorem the assumptions on $\hat{g}$ are closely related with the so called *admissibility condition*, i.e.

$$0 < \int \frac{|\hat{g}(x)|^2}{|x|^n} dx < \infty.$$

In fact, let us assume that $\hat{g}$ is a radially increasing function in $B(0, \frac{1}{2b})$. Since $g \in L^2(I\!\!R^n)$ the above integral is finite if and only if $\sum_{-\infty}^{j} |\hat{g}(\frac{L}{l}a^l)|^2 < \infty$, for some $j \in Z\!\!\!Z$. Moreover if $\hat{g}(\frac{l}{L}a^{j-1}) \neq 0$, clearly the integral is positive.

Theorem 2.7 *Let $\{\lambda_j\}_{j\in Z\!\!\!Z}$ be a sequence of exponential type (a, l, L) and let j_0 be the greatest integer such that $\frac{L}{l}a^{j_0} < \frac{1}{2b}$. Let $g \in L^2(I\!\!R^n)$, suppose that $\hat{g}$ is bounded and supported on $B(0, \frac{1}{2b})$ and radially increasing on it. Assume $\hat{g}(\frac{l}{L}a^{j_0-1}) \neq 0$ and $\sum_{-\infty}^{j_0} |\hat{g}(\frac{L}{l}a^l)|^2 < \infty$, then $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j\in Z\!\!\!Z, k\in Z\!\!\!Z^n}$ is a frame with frame bounds*

$$\frac{1}{b^n}\sum_{j=-\infty}^{j_0-1} |\hat{g}(\frac{l}{L}a^j)|^2 \text{ and } \frac{1}{b^n}\sum_{j=-\infty}^{j_0} |\hat{g}(\frac{L}{l}a^j)|^2 + \frac{2}{b^n}(\lfloor\frac{2}{\log a}\log\frac{L}{l} + 1\rfloor + 1)\|\hat{g}\|_\infty^2.$$

Proof. Given $x \in I\!\!R^n$ there exists k such that $\lambda_k \leq |x| < \lambda_{k+1}$. Using that $\hat{g}$ has support in $B(0, \frac{1}{2b})$ and is radially increasing we have that

$$\sum_{j\in Z\!\!\!Z} |\hat{g}(\frac{x}{\lambda_j})|^2 = \sum_{j,\, 2b\lambda_k \leq \lambda_j} |\hat{g}(\frac{x}{\lambda_j})|^2$$

$$\leq \sum_{j,\, 2b\lambda_{k+1} < \lambda_j} |\hat{g}(\frac{\lambda_{k+1}}{\lambda_j})|^2 + \|\hat{g}\|_\infty^2 \natural\{j : 2b\lambda_k \leq \lambda_j \leq 2b\lambda_{k+1}\}$$

The second term is estimated by Remark 2.6 and the first one is bounded by

$$\sum_{j,\ 2b\lambda_{k+1} < \lambda_j} |\hat{g}(\frac{\lambda_{k+1}}{\lambda_j})|^2 \leq \sum_{j,\ 2b\lambda_{k+1} < \lambda_j,\ 2bLa^{k+1} < la^j} |\hat{g}(\frac{La^{k+1}}{la^j})|^2 +$$

$$\|\hat{g}\|_\infty^2 \natural \{j : 2b\frac{l}{L}a^{k+1} < a^j \leq 2b\frac{L}{l}a^{k+1}\}$$

$$\leq \sum_{j=-\infty}^{j_0} |\hat{g}(\frac{L}{l}a^j)|^2 + \|\hat{g}\|_\infty^2 \natural \{j : 2b\frac{l}{L}a^{k+1} < a^j \leq 2b\frac{L}{l}a^{k+1}\},$$

and we get the upper frame estimate by the hypothesis on $\hat{g}$ and the fact that $\natural\{j : 2b\frac{l}{L}a^{k+1} < a^j \leq 2b\frac{L}{l}a^{k+1}\} \leq \lfloor \frac{2}{\log a} \log \frac{L}{l} \rfloor + 1$.

To get the lower bound, let again $x \in \mathbb{R}^n$ and choose k such that $\lambda_k \leq |x| < \lambda_{k+1}$. Then

$$\sum_{j\in\mathbb{Z}} |\hat{g}(\frac{x}{\lambda_j})|^2 = \sum_{j,\, 2b\lambda_k<\lambda_j} |\hat{g}(\frac{x}{\lambda_j})|^2 \geq \sum_{j,\, 2b\lambda_{k+1}<\lambda_j} |\hat{g}(\frac{\lambda_k}{\lambda_j})|^2$$

$$\geq \sum_{j,\, 2bLa^{k+1}<la^j} |\hat{g}(\frac{\lambda_k}{\lambda_j})|^2 \geq \sum_{j,\, 2bLa^{k+1}<la^j} |\hat{g}(\frac{la^k}{La^j})|^2 = \sum_{j=-\infty}^{j_1} |\hat{g}(\frac{l}{L}a^j)|^2,$$

where j_1 is the largest integer j such that $\frac{L}{l}a^j a < \frac{1}{2b}$. ∎

Our next results show that certain types of sequences can not yield frames for the considered class of functions.

Corollary 2.8 *Let g be as in Theorem 2.7 and assume that $\hat{g}(x) \neq 0$ for $0 < |x| < \frac{1}{2b}$. Suppose that for $j > 0$, $\lambda_j = j^\alpha$ for some $\alpha > 0$ and that $\{\lambda_j\}_{j=-\infty}^{0}$ is an arbitrary decreasing sequence converging to zero as $j \to -\infty$. Then $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ is not a frame.*

Proof. To simplify the notation, assume that $2b \leq 1$. For $k^\alpha \leq |x| \leq k^{\alpha+1}$, we have

$$\sum_j |\hat{g}((\frac{x}{\lambda_j})^\alpha)|^2 \geq \sum_{j=k}^{\infty} |\hat{g}((\frac{k}{j})^\alpha)|^2 \geq \int_k^\infty |\hat{g}((\frac{k}{x})^\alpha)|^2\, dx = \frac{k}{\alpha}\int_0^1 |\hat{g}(t)|^2 \frac{dt}{t^{\frac{1+\alpha}{\alpha}}},$$

where the last expression goes to infinity as k tends to infinity. By Corollary 2.5 we obtain the desired conclusion. ∎

Corollary 2.9 *Set $\lambda_j = 2^{j^2}$ for $j \geq 0$, and $\lambda_j = 2^{-j^2}$, for $j \leq -1$ and let g be as in Theorem 2.7. Then $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ is not a frame.*

Proof. For $2^{-(k+1)^2} \leq |x| \leq 2^{-k^2}$,

$$\sum_{j=-\infty}^{\infty} |\hat{g}(\frac{1}{2^{k^2}\lambda_j})|^2 = \sum_{j=0}^{k-1} |\hat{g}(\frac{2^{j^2}}{2^{k^2}})|^2 + \sum_{j=1}^{\infty} |\hat{g}(\frac{1}{2^{j^2+k^2}})|^2 \leq$$

$$\sum_{j=0}^{k-1} |\hat{g}(\frac{2^{j^2}}{2^{k^2}})|^2 + \int_1^\infty |\hat{g}(\frac{1}{2^{x^2+k^2}})|^2\, dx \leq$$

$$\sum_{j=0}^{k-1} |\hat{g}(\frac{2^{j^2}}{2^{k^2}})|^2 + \int_0^{\frac{1}{2^{k^2+1}}} \frac{|\hat{g}(x)|^2}{x}\, dx \leq |k||\hat{g}(\frac{2}{2^{2|k|}})|^2 + \int_0^{\frac{1}{2^{k^2+1}}} \frac{|\hat{g}(x)|^2}{x}\, dx.$$

Since $\int_0^\infty \frac{|\hat{g}(x)|^2}{x}\, dx < \infty$ the last integral tends to 0 as $|k|$ tends to ∞. Also

$$\frac{1}{2}|\hat{g}(|x|)|log\frac{1}{|x|} \leq \int_{|x|}^{\sqrt{|x|}} \frac{|\hat{g}(t)|^2}{t}\, dt \to 0 \text{ for} |x| \to 0.$$

Since $\hat{g}$ is a radially increasing function we have that $|k||\hat{g}(\frac{2}{2^{2|k|}})|^2 \to 0$ as $|k| \to \infty$, if and only if $|\hat{g}(x)|^2 \log|x| \to 0$ as $x \to 0$. By Corollary 2.5 we conclude that the family $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ is not a frame. ∎

We now present a sufficient condition for $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ to be a frame. We need the following lemma.

Lemma 2.10 $\frac{1}{1+|a+b|^2} \leq 2(\frac{1+|a|^2}{1+|b|^2})^\varepsilon$ *for any* $0 < \varepsilon < 1$ *and* $a, b \in \mathbb{R}^n$.

Proof. It is clear that $1+|a+b|^2 \leq 2(1+|a|^2)(1+|b|^2)$, for $a, b \in \mathbb{R}^n$, which implies $\frac{1}{1+|a+b|^2} \leq 2\frac{1+|a|^2}{1+|b|^2}$. ∎

Theorem 2.11 *Let* $g \in L^2(\mathbb{R}^n)$ *and assume that* $|\hat{g}(x)| \leq c|x|^\alpha(1+|x|^2)^{\frac{-x}{2}}$, *with* $\alpha > 0$, $\gamma > \alpha + n$ *and* $\operatorname{ess\,inf}_{x\in\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |\hat{g}(\frac{x}{\lambda_j})|^2 > 0$. *If* $\{\lambda_j\}_{j\in\mathbb{Z}}$ *is a sequence of exponential type* (a, l, L), *then* $\{\lambda_j^{\frac{n}{2}} g(\lambda_j x - kb)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ *is a frame for all sufficiently small* $b > 0$.

Proof. We apply Theorem 2.2. C will denote a constant not necessarily the same on each occurrence.

$$\sum_{k\neq 0}\sum_{j\in\mathbb{Z}} |\hat{g}(\frac{x}{\lambda_j})\hat{g}(\frac{x}{\lambda_j}+\frac{k}{b})| \leq C\sum_{k\neq 0}\sum_{j\in\mathbb{Z}} \frac{|\frac{x}{\lambda_j}|^\alpha}{(1+|\frac{x}{\lambda_j}|^2)^{\frac{\gamma}{2}}} \frac{|\frac{x}{\lambda_j}+\frac{k}{b}|^\alpha}{(1+|\frac{x}{\lambda_j}+\frac{k}{b}|^2)^{\frac{\gamma}{2}}} \leq$$

$$C\sum_{k\neq 0}\sum_{j\in\mathbb{Z}}\frac{|\frac{x}{\lambda_j}|^\alpha}{(1+|\frac{x}{\lambda_j}|^2)^{\frac{\gamma}{2}}}\frac{1}{(1+|\frac{x}{\lambda_j}+\frac{k}{b}|^2)^{\frac{\gamma-\alpha}{2}}}\leq C\sum_{k\neq 0}\sum_{j\in\mathbb{Z}}\frac{|\frac{x}{\lambda_j}|^\alpha}{(1+|\frac{x}{\lambda_j}|^2)^{\frac{\gamma}{2}}}\frac{(1+|\frac{x}{\lambda_j}|^2)^{\frac{\varepsilon(\gamma-\alpha)}{2}}}{(1+|\frac{k}{b}|^2)^{\frac{\gamma-\alpha}{2}}},$$

where we have used Lemma 2.10 in the last inequality. Now, the above sums are estimated by

$$Cb^{\varepsilon(\gamma-\alpha)}\sum_j\frac{|\frac{x}{\lambda_j}|^\alpha}{(1+|\frac{x}{\lambda_j}|^2)^{\frac{\gamma-\varepsilon(\gamma-\alpha)}{2}}},$$

since $\sum_{k\in\mathbb{Z}^n,\,k\neq 0}\frac{1}{(1+|\frac{k}{b}|^2)^{\frac{\varepsilon(\gamma-\alpha)}{2}}}\leq Cb^{\varepsilon(\gamma-\alpha)}$, where the constant C depends only on the product $\varepsilon(\gamma-\alpha)$ and on n, here we have used that $\varepsilon(\gamma-\alpha)>n$.

Let i be such that $a^i\leq|x|\leq a^{i+1}$. Then

$$Cb^{\varepsilon(\gamma-\alpha)}\sum_j\frac{|\frac{x}{\lambda_j}|^\alpha}{(1+|\frac{x}{\lambda_j}|^2)^{\frac{\gamma-\varepsilon(\gamma-\alpha)}{2}}}\leq Cb^{\varepsilon(\gamma-\alpha)}\frac{L^{\gamma-\varepsilon(\gamma-\alpha)}}{l^\alpha}\sum_j\frac{a^{(i+1)\alpha}}{a^{j\alpha}(L^2+a^{(i-j)2})^{\frac{\gamma-\varepsilon(\gamma-\alpha)}{2}}}=$$

$$Cb^{\varepsilon(\gamma-\alpha)}\frac{L^{\gamma-\varepsilon(\gamma-\alpha)}a^\alpha}{l^\alpha}\sum_j a^{(i-j)\alpha}\frac{1}{(L^2+a^{(i-j)2})^{\frac{\gamma-\varepsilon(\gamma-\alpha)}{2}}}=$$

$$Cb^{\varepsilon(\gamma-\alpha)}\frac{L^{\gamma-\varepsilon(\gamma-\alpha)}a^\alpha}{l^\alpha}\sum_j a^{j\alpha}\frac{1}{(L^2+a^{j2})^{\frac{\gamma-\varepsilon(\gamma-\alpha)}{2}}}<Cb^{\varepsilon(\gamma-\alpha)}\frac{L^{\gamma-\varepsilon(\gamma-\alpha)}}{l^\alpha}.$$

Now, for $a^i\leq|x|\leq a^{i+1}$ we have

$$\sum_{j\in\mathbb{Z}}|\hat{g}(\frac{x}{\lambda_j})|^2\leq C\sum_j\frac{|\frac{x}{\lambda_j}|^{2\alpha}}{(1+|\frac{x}{\lambda_j}|^2)^\gamma}\leq C\frac{a^{2\alpha}L^{2\gamma}}{l^{2\alpha}}\sum_j\frac{a^{(i-j)2\alpha}}{(L^2+a^{(i-j)2})^\gamma}\leq D\frac{a^{2\alpha}L^{2\gamma}}{l^{2\alpha}}.$$

∎

Theorem 2.11 is related with Theorem 1 in [9]. In that paper the authors show that there exists $\delta>0$ such that for a sequence satisfying $(1-\delta)a^j\leq\lambda_j\leq(1+\delta)a^j$ there exists b_0 such that $\{\lambda_j^{\frac{1}{2}}g(\lambda_j x-bk)\}$ is a frame for $0<b<b_0$ and for a suitable function g. Theorem 2.11 above shows that a sequence of the form $la^j\leq\lambda_j\leq La^j$ *for any* l *and* L gives irregular frame sequences for any $0<b<b_0$, where b_0 is independent on l and L, provided $L\geq 1$.

3 Gabor Frames

We will use the following result of Casazza and Christensen [2].

Theorem 3.1 *Let $g \in L^2(\mathbb{R}^n)$ $b > 0$ and $\{\lambda_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}^n$. Suppose that*

$$A = \frac{1}{b^n}\text{essinf}_{y\in\mathbb{R}^n}(\sum_k |g(y-\lambda_k)|^2 - \sum_{m\neq 0} |\sum_{k\in\mathbb{Z}} g(y-\lambda_k)\overline{g}(y-\lambda_k+\frac{m}{b})|) > 0,$$

$$B = \frac{1}{b^n}\text{esssup}_{y\in\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} |\sum_{k\in\mathbb{Z}} g(y-\lambda_k)\overline{g}(y-\lambda_k+\frac{m}{b})| < \infty.$$

Then $\{e^{2\pi ib(j,x)}g(x-\lambda_k)\}_{j\in\mathbb{Z}^n,\, k\in\mathbb{Z}}$ is a frame with frame bounds A and B.

A result similarly to Corollary 2.5 holds for Gabor frames. We will now use Theorem 3.1 to obtain a sufficient condition for $\{e^{2\pi ib(j,x)}g(x-\lambda_k)\}_{j\in\mathbb{Z}^n,\, k\in\mathbb{Z}}$ to be a frame. We need a Lemma:

Lemma 3.2 *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be a sequence such that $\frac{|k|}{p+1} \leq |\lambda_k| \leq |k|$, for $k \in \mathbb{Z}$ and $p \geq 0$. Then*

$$\sup_{x\in\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \sum_{k\neq 0} \frac{1}{(1+|x-\lambda_m|)^\gamma} \frac{1}{(1+|x-\lambda_m-\frac{k}{b}|)^\gamma} \to 0, \text{ as } b \to 0,$$

for $\gamma > n$.

Proof. To simplify the notations we shall consider $n = 1$. Set

$$H_l(x) = \sum_{\{m:|x-\lambda_m|\geq l\}} \frac{1}{(1+|x-\lambda_m|)^\gamma}.$$

Then $\sup_{x\in\mathbb{R}} H_l(x) \to 0$ as $l \to \infty$. In fact, for $i \leq x \leq i+1$, we have $H_l(x) \leq \sum_{c_2|i-k|\geq l} \frac{1}{(1+c_1|i-k|)^\gamma}$.

Now, set $L_b(y) = \sum_{k\neq 0} \frac{1}{(1+|y-\frac{k}{b}|)^\gamma}$. First note that $L_b(y)$ is uniformly bounded for $y \in I\!R$ and for b bounded. To see that we can consider $\frac{l}{b} \leq y \leq \frac{l+1}{b}$ with $l \geq 0$, then

$$L_b(y) = \sum_{k=1}^{\infty} \frac{1}{(1+|y+\frac{k}{b}|)^\gamma} + \sum_{k=1}^{\infty} \frac{1}{(1+|y-\frac{k}{b}|)^\gamma} \leq$$

$$b^\gamma \sum_{k=1}^{\infty} \frac{1}{k^\gamma} + \sum_{k=1+l}^{\infty} \frac{1}{(1+\frac{k-l}{b})^\gamma} + \sum_{k=1}^{l-1} \frac{1}{(1+\frac{l-k}{b}|)^\gamma} + \frac{1}{(1+y-\frac{l}{b})^\gamma} \leq C_\gamma b^\gamma + 1.$$

Furthermore $\sup_{|y|\leq c} L_b(y) \to 0$ if $b \to 0$, since the series defining $L_b(y)$ can be dominated by a convergent numerical series in the region $|y| \leq c,\ 0 < b < 1$. Now we get

$$\sum_{m\in Z\!\!Z^n} \sum_{k\neq 0} \frac{1}{(1+|x-\lambda_m|)^\gamma} \frac{1}{(1+|x-\lambda_m-\frac{k}{b}|)^\gamma} \leq$$

$$(C_\gamma b^\gamma + 1) H_l(x) + \sum_{\{m:|x-\lambda_m|\leq l\}} L_b(x-\lambda_m) \leq (C_\gamma b^\gamma + 1) H_l(x) + c_l \sup_{|y|\leq l} L_b(y).$$

The proof is completed taking into account the properties of the functions H_l and L_b. ∎

Theorem 3.3 *Let $g \in L^2(I\!R^n)$ be such that $|g(x)| \leq \frac{C}{(1+|x|)^\gamma}$, with $\gamma > n$ and* $\mathrm{essinf}_{y\in I\!R^n} \sum_k |g(y-\lambda_k)|^2 > 0$. *Let $\{\lambda_m\}_{m\in Z\!\!Z}$ be a sequence in $I\!R^n$ such that $\frac{|m|}{p+1} \leq |\lambda_m| \leq |m|$, with $p \geq 0$. Then $\{e^{2\pi i b(j,x)} g(x-\lambda_m)\}_{j\in Z\!\!Z^n,\, m\in Z\!\!Z}$ is a frame for all b small enough.*

Proof. Follows from Theorem 3.1 and Lemma 3.2. ∎

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