Mathematical
Inequalities
& Applications
[5930] First Galley Proofs

ON THE MEASURE OF POLYNOMIALS ATTAINING MAXIMA ON A VERTEX

DAMIÁN PINASCO AND IGNACIO ZALDUENDO

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Abstract. We calculate the probability that a k-homogeneous polynomial in n variables attain a local maximum on a vertex in terms of the "sharpness" of the vertex, and then study the dependence of this measure on the growth of dimension and degree. We find that the behavior of vertices with orthogonal edges is markedly different to that of sharper vertices. If the degree k grows with the dimension n, the probability that a polynomial attain a local maximum tends to $1/2$, but for orthogonal edges the growth-rate of k must be larger than $n\ln n$, while for sharper vertices a growth-rate larger than $\ln n$ will suffice.

Introduction

Several results have been obtained regarding the size of the set of polynomials attaining relative maxima on given points [2], [7], [8]. The original motivation of these studies was the conjecture, by the second author, that the "probability" (to be defined below) of a k-homogeneous polynomial attaining its norm on a vertex of the unit ball of ℓ_∞^n would tend to one as the dimension n increases. Pérez-García and Vilanueva proved that an analogous result on the unit ball of ℓ_1^n is false for degree $k=2$, and in [8] the authors showed that for any degree $k>2$ the result is true on the unit ball of ℓ_1^n. The set of polynomials attaining a local maximum is given by inequalities between random variables, and the difference between degree 2 and higher degrees stems from the fact that the random variables involved are correlated for degree 2, but independent for all higher degrees.

The difficulty in proving the original conjecture – which still stands as such – is of a similar nature: the lack of independence of the random variables involved in the description of the set of polynomials of interest. Although the problem on the unit ball of ℓ_∞^n has remained intractable, the vertices of this unit ball have one simple property not shared by the vertices of the unit ball of ℓ_1^n: the angles between the edges meeting at the vertex are all right angles. In this paper we concentrate on vertices which have the property that the edges defining them form constant angles, but drop the requirement of orthogonality. Thus the vertices considered here may be more or less "sharp". We study the probability that a polynomial attain a local maximum on such a vertex. Although

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the random variables involved are not independent, we prove that the sharper the vertex, the higher the probability of attaining maxima. We quantify these results, study their dependence on dimension and degree, and find a clear-cut difference between the case of orthogonal edges and sharper vertices. Specifically, if the degree k grows with the dimension n, the probability that a polynomial attain a local maximum tends to $1/2$, but for orthogonal edges the growth-rate of k must be larger than $n\ln n$, while for sharper vertices a growth-rate larger than $\ln n$ suffices.

In the first section we define the basic Hilbertian structure on $\mathscr{P}^k(\mathbb{R}^n)$, and the Gaussian measure on the space of polynomials, and refer to the particularity of the case $k=2$. In the second section we calculate the measure of the set of k-homogeneous polynomials attaining a local maximum at an outward-pointing vertex (defined below), in terms of the "sharpness" of the vertex. We then study the dependence of this measure on the growth of dimension and degree.

1. Hilbertian structure and Gaussian measure on $\mathscr{P}^k(\mathbb{R}^n)$

For each k-homogeneous polynomial $P:\mathbb{R}^n \longrightarrow \mathbb{R}$ there is a unique symmetric k-linear function ϕ such that $P(x)=\phi(x,\dots,x)$. Thus we consider the space of k-homogeneous polynomials over $\mathbb{R}^n$ as the dual of the symmetric tensor product $\bigotimes_{k,s}\mathbb{R}^n$. Recall ([3], [6]) the Hilbert space structure on the full tensor product $\bigotimes_k \mathbb{R}^n$ given by the inner product

$$\langle v^1\otimes\cdots\otimes v^k, w^1\otimes\cdots\otimes w^k\rangle = \langle v^1,w^1\rangle\cdots\langle v^k,w^k\rangle.$$

Define also the symmetrization operator $S:\bigotimes_k\mathbb{R}^n \longrightarrow \bigotimes_k\mathbb{R}^n$ by setting its values on a basis as

$$S(e_{j_1}\otimes\cdots\otimes e_{j_k}) = \frac{1}{k!}\sum_{\sigma} e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(k)}}$$

where σ runs through all permutations of $\{1,\dots,k\}$. The image of S – the symmetric tensor product $\bigotimes_{k,s}\mathbb{R}^n$ – is a predual of the space of polynomials over $\mathbb{R}^n$, $\mathscr{P}^k(\mathbb{R}^n)$. We consider on $\bigotimes_{k,s}\mathbb{R}^n$ the Hilbert space structure induced by the ambient space $\bigotimes_k\mathbb{R}^n$, and on $\mathscr{P}^k(\mathbb{R}^n)$ the dual Hilbert space structure. The resulting norm on $\mathscr{P}^k(\mathbb{R}^n)$ is the Bombieri norm [1]

$$\|P\| = \left(\sum_{|\alpha|=k} a_\alpha^2\frac{\alpha!}{k!}\right)^{1/2},$$

if $P(x)=\sum_{|\alpha|=k}a_\alpha x^\alpha$ is the monomial-sum expression of P.

Every linear form $\varphi:\mathscr{P}^k(\mathbb{R}^n)\longrightarrow\mathbb{R}$ can be identified with an element of $\bigotimes_{k,s}\mathbb{R}^n$. For example, evaluation at x: $e_x(P)=P(x)$ is given by $x\otimes\cdots\otimes x$. We will mainly encounter the linear forms

$$\frac{\partial}{\partial v}(a):\mathscr{P}^k(\mathbb{R}^n)\longrightarrow\mathbb{R} \quad \text{given by } P\mapsto\frac{\partial P}{\partial v}(a),$$

which when identified with an element of $\bigotimes_{k,s}\mathbb{R}^n$ is

$$\frac{\partial}{\partial v}(a) = v\otimes a\otimes\cdots\otimes a + a\otimes v\otimes a\otimes\cdots\otimes a+\cdots+a\otimes\cdots\otimes a\otimes v.$$

We will need to calculate the inner products between such linear forms:

$$\begin{aligned}\left\langle \frac{\partial}{\partial v}(a), \frac{\partial}{\partial w}(b)\right\rangle &= \langle v\otimes a\otimes\cdots\otimes a+\cdots+a\otimes\cdots\otimes a\otimes v,\\ &\qquad w\otimes b\otimes\cdots\otimes b+\cdots+b\otimes\cdots\otimes b\otimes w\rangle\\ &= k\langle v,w\rangle\langle a,b\rangle^{k-1}+(k^2-k)\langle v,b\rangle\langle a,w\rangle\langle a,b\rangle^{k-2}.\end{aligned}$$

Note that if $\langle a,b\rangle = 0$, this is zero for $k>2$. However, when $k=2$ one has $2\langle a,w\rangle\langle v,b\rangle$ which can be non-zero. In fact, if $k=2$ and $n=2$,

$$\frac{\partial}{\partial(1,0)}(0,1) \quad\text{and}\quad \frac{\partial}{\partial(0,1)}(1,0)$$

are the same linear form.

We consider on $\mathscr{P}^k(\mathbb{R}^n)$ the standard Gaussian measure W corresponding to its Hilbert space structure, i.e., the measure

$$W(B) = \frac{1}{(2\pi)^{d/2}}\int_B e^{-\frac{\|P\|^2}{2}}\,dP, \qquad \text{for any Borel set } B\subset\mathscr{P}^k(\mathbb{R}^n),$$

where $d=\binom{n+k-1}{k}$, is the dimension of $\mathscr{P}^k(\mathbb{R}^n)$. We note that W is rotation-invariant, and also that if $T:\mathbb{R}^n\to\mathbb{R}^n$ is an orthogonal transformation, then

$$\widetilde{T}:\mathscr{P}^k(\mathbb{R}^n)\to\mathscr{P}^k(\mathbb{R}^n) \quad \text{such that } \widetilde{T}(P)=P\circ T$$

is a measure-preserving map.

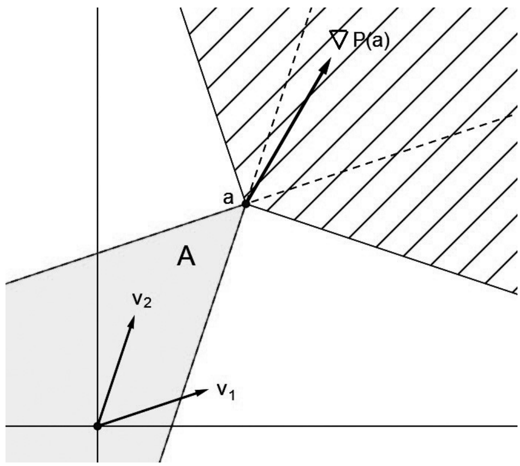
Recall also that if $\varphi:\mathscr{P}^k(\mathbb{R}^n)\longrightarrow\mathbb{R}$ is a linear form, φ is a normal random variable with mean zero and standard deviation $\|\varphi\|$.

2. Attaining maxima on a vertex

By "vertex" we mean the following generalization of a vertex of a hipercube. Consider n norm-one vectors $v_1,\dots,v_n$ in $\mathbb{R}^n$, such that the cosines of the angles between any pair remain constant:

$$\langle v_i,v_j\rangle = c \qquad \text{for } i\neq j.$$

We want to calculate the probability that a k-homogeneous polynomial in n variables attain a local maxima at the vertex a (relative to the set A – see figure).



We must therefore have $\langle \nabla P(a), v_i\rangle > 0$ for $i=1,\dots,n$, i.e.: $\frac{\partial P}{\partial v_i}(a) > 0$ for $i = 1,\dots,n$. Note that we have

$$\left\langle \frac{\partial}{\partial v_i}(a), \frac{\partial}{\partial v_j}(a)\right\rangle = k\langle v_i, v_j\rangle \|a\|^{2(k-1)} + (k^2-k)\langle a, v_j\rangle\langle v_i, a\rangle \|a\|^{2(k-2)}$$
$$= k\|a\|^{2(k-2)}\left[\langle v_i, v_j\rangle \|a\|^2 + (k-1)\langle a, v_j\rangle\langle v_i, a\rangle\right],$$

and thus

$$\left\|\frac{\partial}{\partial v_i}(a)\right\|^2 = k\|a\|^{2(k-2)}\left[\|a\|^2 + (k-1)\langle a, v_i\rangle^2\right].$$

Thus the measure of the set of polynomials attaining a local maximum at a is the Gaussian measure of an intersection of half-spaces directed by the n norm-one vectors $w_i = \frac{\frac{\partial}{\partial v_i}(a)}{\|\frac{\partial}{\partial v_i}(a)\|}$, for $i=1,\dots,n$, such that

$$\langle w_i, w_j\rangle = \frac{\langle v_i, v_j\rangle \|a\|^2 + (k-1)\langle a, v_j\rangle\langle v_i, a\rangle}{\sqrt{(\|a\|^2 + (k-1)\langle a, v_i\rangle^2)(\|a\|^2 + (k-1)\langle a, v_j\rangle^2)}}.$$

We will concentrate on "outward pointing" vertices, of the form $a = \alpha\sum_{i=1}^n v_i$. Note that in this case,

$$\|a\|^2 = \alpha^2 \sum_{i=1}^n \sum_{j=1}^n \langle v_i, v_j\rangle = \alpha^2 n(1+nc-c),$$

and

$$\langle a, v_j\rangle = \alpha \sum_{i=1}^n \langle v_i, v_j\rangle = \alpha(1+nc-c).$$

Thus

$$\begin{aligned}\langle w_i, w_j\rangle &= \frac{\alpha^2 nc(1+nc-c)+(k-1)\alpha^2(1+nc-c)^2}{\alpha^2 n(1+nc-c)+(k-1)\alpha^2(1+nc-c)^2}\\ &= \frac{nc+(k-1)(1+nc-c)}{n+(k-1)(1+nc-c)} \qquad \text{when } i \neq j.\end{aligned}$$

This expresses the cosine of the angle between w_i and w_j, which we will denote $\overline{c}$, in terms of c, k and n. Note that $\overline{c}$ is independent of α, therefore independent of the distance of our vertex a to zero.

Note also that since $\|a\|^2 = \alpha^2 n(1+nc-c) > 0$, the cosine c must be larger than $\frac{-1}{n-1}$. Hence, although for any given dimension n it can be negative, there is no negative value of c valid for all dimensions n. Since we will want to have n tending to infinity, we will only consider $c \geqslant 0$.

We now calculate the measure of the set A_a of k-homogeneous polynomials on $\mathbb{R}^n$ attaining a maximum at the vertex a.

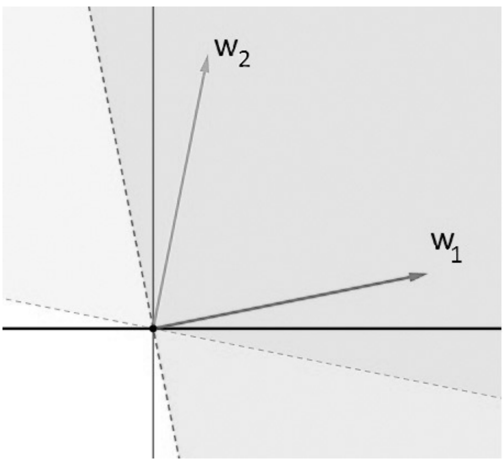
THEOREM 2.1. *The measure of A_a is*

$$W(A_a) = \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty e^{-\frac{t^2}{2\sigma^2}}\gamma_{n-1}(\Delta_t)\,dt,$$

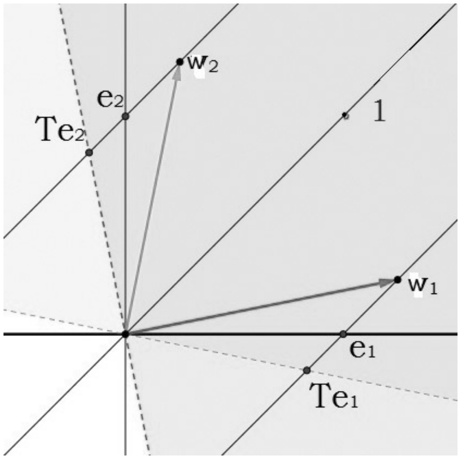
where $\gamma_{n-1}(\Delta_t)$ is the gaussian measure of the zero-centered $n-1$-dimensional simplex of size t, and

$$\sigma^2 = \frac{nk(nc+1-c)}{1-c}.$$

Proof. In order to calculate the measure $W(A_a)$, we will apply a linear transformation $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ to "open" the octant $\mathbb{R}^n_+ = \{x : x_i \geqslant 0, i = 1,\dots,n\}$.



To do this, we write $w_i = e_i + \mu\ \mathbf{1}$ for suitable μ and $Te_i = e_i - \lambda\ \mathbf{1}$ for suitable λ,



where by "suitable", we mean

μ such that $\cos(e_i + \mu\ \mathbf{1}, e_j + \mu\ \mathbf{1}) = \overline{c}$ for $i \neq j$, and
λ such that $e_i - \lambda\ \mathbf{1} \perp e_j + \mu\ \mathbf{1}$ for $i \neq j$.

Suitable μ:

$$\overline{c} = \frac{\langle e_i + \mu\ \mathbf{1}, e_j + \mu\ \mathbf{1}\rangle}{\|e_i + \mu\ \mathbf{1}\|\|e_j + \mu\ \mathbf{1}\|} = \frac{2\mu + n\mu^2}{1 + 2\mu + n\mu^2},$$

and thus

$$n(1-\overline{c})\mu^2 + 2(1-\overline{c})\mu - \overline{c} = 0,$$

from where

$$\mu = \frac{-1 + \sqrt{1 + \frac{n\overline{c}}{1-\overline{c}}}}{n}.$$

Suitable λ:

$$\langle e_i - \lambda\ \mathbf{1}, e_j + \mu\ \mathbf{1}\rangle = 0 \quad \text{for } i \neq j,$$

$$\begin{aligned} \lambda &= \frac{\mu}{1 + n\mu} \\ &= \frac{-1 + \sqrt{1 + \frac{n\overline{c}}{1-\overline{c}}}}{n\sqrt{1 + \frac{n\overline{c}}{1-\overline{c}}}}. \end{aligned}$$

Note that, in terms of the cosine c of the angles between our original vectors,

$$1+\frac{n\overline{c}}{1-\overline{c}}=\frac{k(nc+1-c)}{1-c},$$

and thus

$$\lambda=\frac{-1+\sqrt{\frac{k(nc+1-c)}{1-c}}}{n\sqrt{\frac{k(nc+1-c)}{1-c}}}$$

So now we set $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$, with $Te_i=e_i-\lambda\ \mathbf{1}$,

$$T=\begin{bmatrix} 1-\lambda & -\lambda & \cdots & -\lambda \\ -\lambda & 1-\lambda & \cdots & -\lambda \\ \vdots & \vdots & & \vdots \\ -\lambda & -\lambda & \cdots & 1-\lambda \end{bmatrix},$$

We have

$$\mathbb{R}^n_+ \xrightarrow[u\mapsto Tu=x]{T} T(\mathbb{R}^n_+) \xrightarrow{1\,d\gamma_n} \mathbb{R}$$

So

$$W(A_a)=\gamma_n(T(\mathbb{R}^n_+))=\int_{T(\mathbb{R}^n_+)} d\gamma_n(x)\,dx=\int_{\mathbb{R}^n_+} d\gamma_n(T(u))|\det T|\,du.$$

In order to calculate the determinant of T, we consider its eigenvalues:

a) $\mathbf{1}=(1,\ldots,1)$ is an eigenvector with eigenvalue $1-n\lambda$:

$$T(\ \mathbf{1})=T\left(\sum_{i=1}^{n} e_i\right)=\sum_{i=1}^{n}(e_i-\lambda\ \mathbf{1})=\ \mathbf{1}-n\lambda\ \mathbf{1}=(1-n\lambda)\ \mathbf{1}.$$

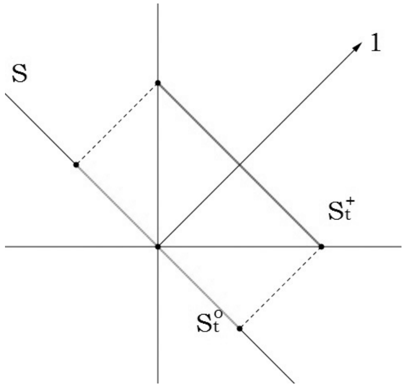
b) $S=\{x:x_1+\cdots+x_n=0\}$ is an eigenspace with eigenvalue 1:

$$Tx=T\left(\sum_{i=1}^{n} x_ie_i\right)=\sum_{i=1}^{n} x_i(e_i-\lambda\ \mathbf{1})=\sum_{i=1}^{n} x_ie_i-\lambda\sum_{i=1}^{n} x_i\ \mathbf{1}=x.$$

Thus $\det T=1-n\lambda=\frac{1}{\sqrt{\frac{k(nc+1-c)}{1-c}}}$. Recall the measure $W(A_a)$ of A_a,

$$W(A_a)=\int_{\mathbb{R}^n_+} |\det T|d\gamma_n(Tu)du.$$

We will consider $\mathbb{R}^n=[\ \mathbf{1}]\bigoplus S$:



where $S_t^+ = \{u \in \mathbb{R}_+^n : \sum_{i=1}^n u_i = t\}$ and $S_t^\circ = P_S(S_t^+)$.

Thus if $u \in S_t^+$, $u = \frac{t}{\sqrt{n}} \frac{\mathbf{1}}{\sqrt{n}} + x$, and we will write $du = \frac{dt}{\sqrt{n}} dx$. Also,

$$Tu = \frac{t}{\sqrt{n}} T\left(\frac{\mathbf{1}}{\sqrt{n}}\right) + Tx = \frac{t}{\sqrt{n}} \frac{1}{\sqrt{\frac{k(nc+1-c)}{1-c}}} \left(\frac{\mathbf{1}}{\sqrt{n}}\right) + x,$$

so $\|Tu\|^2 = \frac{t^2(1-c)}{nk(nc+1-c)} + \|x\|^2$. Thus

$$\begin{aligned}
W(A_a) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}_+^n} \frac{1}{\sqrt{\frac{k(nc+1-c)}{1-c}}} e^{-\frac{\|Tu\|^2}{2}} du \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty \int_{S_t^\circ} \frac{1}{\sqrt{\frac{k(nc+1-c)}{1-c}}} e^{-\frac{t^2(1-c)}{2nk(nc+1-c)}} e^{-\frac{\|x\|^2}{2}} dx \frac{dt}{\sqrt{n}} \\
&= \frac{\sqrt{1-c}}{\sqrt{2\pi nk(nc+1-c)}} \int_0^\infty e^{-\frac{t^2(1-c)}{2nk(nc+1-c)}} \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{S_t^\circ} e^{-\frac{\|x\|^2}{2}} dx dt \\
&= \frac{\sqrt{1-c}}{\sqrt{2\pi nk(nc+1-c)}} \int_0^\infty e^{-\frac{t^2(1-c)}{2nk(nc+1-c)}} \gamma_{n-1}(S_t^\circ) dt \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty e^{-\frac{t^2}{2\sigma^2}} \gamma_{n-1}(S_t^\circ)\, dt,
\end{aligned}$$

where

$$\sigma^2 = \frac{nk(nc+1-c)}{1-c}.$$

This completes the proof. $\square$

Note that σ^2 grows with c from nk to infinity in the interval $[0,1)$. Thus $W(A_a)$ grows with c and tends to $\frac{1}{2}$ as $c \to 1$, that is, as the vertex gets sharper.

We will calculate lower bounds for $W(A_a)$ for fixed c, under certain conditions on the relationship between degree k and dimension n. For this we need a lower bound for the gaussian measure of simplices.

LEMMA 2.2. *If S_t° denotes the zero-centered $n-1$-dimensional simplex of size t,*

$$\gamma_{n-1}(S_t^\circ) \geqslant \left(1 - e^{-\frac{t^2}{2n(n-1)}}\right)^{\frac{n}{2}}.$$

Proof. Note that S_t° is the orthogonal projection onto S of $S_t^+ = co\{te_i : i = 1,\ldots,n\}$, the convex hull of $te_1,\ldots,te_n$. Thus

$$S_t^\circ = co\left\{t\left(e_i - \frac{\mathbf{1}}{n}\right) : i = 1,\ldots,n\right\}.$$

This set is a simplex which contains an $n-1$-dimensional ball of radius $\frac{t}{\sqrt{n(n-1)}}$: indeed, the points on the boundary of S_t° closest to the origin are

$$w_j = \sum_{i\neq j} a_i t\left(e_i - \frac{\mathbf{1}}{n}\right)$$

with $\sum_{i\neq j} a_i = 1$ and minimizing the norm; but

$$\begin{aligned}
\left\|\sum_{i\neq j} a_i t(e_i - \frac{\mathbf{1}}{n})\right\|^2 &= \left\langle \sum_{i\neq j} a_i te_i - \frac{t}{n}\,\mathbf{1}, \sum_{k\neq j} a_k te_k - \frac{t}{n}\,\mathbf{1}\right\rangle \\
&= \sum_{i\neq j} a_i^2 t^2 - \frac{2t^2}{n}\sum_{i\neq j} a_i + \frac{t^2}{n} \\
&= \sum_{i\neq j} a_i^2 t^2 - \frac{t^2}{n},
\end{aligned}$$

which is minimal when all a_i are $\frac{1}{n-1}$. Thus the norm is

$$\sqrt{\frac{t^2}{n(n-1)}} = \frac{t}{\sqrt{n(n-1)}}, \quad \text{and } w_j = \sum_{i\neq j} \frac{t}{n-1}\left(e_i - \frac{\mathbf{1}}{n}\right).$$

S_t° may be described as the projection over S of the intersection of half-spaces given by the functionals $\phi_j(x) = \langle x, w_j\rangle$, for $j = 1,\ldots,n$:

$$S_t^\circ = \bigcap_{j=1}^n P_S\left\{x \in \mathbb{R}^n : \phi_j(x) \leqslant \frac{t}{\sqrt{n(n-1)}}\right\} \supset \bigcap_{j=1}^n P_S\left\{x \in \mathbb{R}^n : |\phi_j(x)| \leqslant \frac{t}{\sqrt{n(n-1)}}\right\}.$$

To calculate a lower bound for $\gamma_{n-1}(S_t^\circ)$ we use the Gaussian correlation inequality [9],

[5], [4]. Thus

$$\begin{aligned}
\gamma_{n-1}(S_t^\circ) &\geqslant \prod_{j=1}^{n} \gamma_{n-1}\left(P_S\left\{|\phi_j| \leqslant \frac{t}{\sqrt{n(n-1)}}\right\}\right)\\
&= \prod_{j=1}^{n} \gamma_n\left\{|\phi_j| \leqslant \frac{t}{\sqrt{n(n-1)}}\right\}\\
&= \left[\frac{1}{\sqrt{2\pi}}\int_{-\frac{t}{\sqrt{n(n-1)}}}^{\frac{t}{\sqrt{n(n-1)}}} e^{-\frac{s^2}{2}}\,ds\right]^n\\
&\geqslant \left(1-e^{-\frac{t^2}{2n(n-1)}}\right)^{\frac{n}{2}},
\end{aligned}$$

where in the last line we have used the well-known bound for a $N(0,1)$ normal random variable X

$$P\{|X| \leqslant \alpha\} \geqslant \left(1-e^{-\frac{\alpha^2}{2}}\right)^{\frac{1}{2}}. \quad \square$$

We now have the following.

THEOREM 2.3. *If $c>0$ and $(k_n)_{n\in\mathbb{N}}$ a sequence of natural numbers such that*

$$\lim_{n\to\infty} \frac{\ln n}{k_n} = 0,$$

then for every $\varepsilon>0$ there is an $n_\varepsilon \in \mathbb{N}$ such that for all $n \geqslant n_\varepsilon$ and $k \geqslant k_n$, the measure of polynomials attaining a local maximum at the vertex a is

$$W(A_a) > \frac{1}{2} - \varepsilon.$$

Proof. We have

$$W(A_a) = \frac{\sqrt{1-c}}{\sqrt{2\pi}\sqrt{nk(nc+1-c)}}\int_0^\infty e^{-\frac{t^2(1-c)}{2nk(nc+1-c)}}\gamma_{n-1}(S_t^\circ)\,dt.$$

Thus

$$W(A_a) \geqslant \frac{\sqrt{1-c}}{\sqrt{2\pi}\sqrt{nk(nc+1-c)}}\int_0^\infty e^{-\frac{t^2(1-c)}{2nk(nc+1-c)}}\left(1-e^{-\frac{t^2}{2n(n-1)}}\right)^{\frac{n}{2}} dt.$$

Now, setting $u = \frac{t\sqrt{1-c}}{\sqrt{nk(nc+1-c)}}$,

$$W(A_a) \geqslant \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{u^2}{2}}\left(1-e^{-\frac{k(nc+1-c)}{2(n-1)(1-c)}u^2}\right)^{\frac{n}{2}} du.$$

Now, clearly

$$\left(1-e^{-\frac{k(nc+1-c)}{2(n-1)(1-c)}u^2}\right)^{\frac{n}{2}}$$

is bounded by 1. We prove that it converges pointwise to 1, and apply Lebesgue's dominated convergence theorem in our lower bound for $W(A_a)$. To see the pointwise convergence to 1 consider $k \geqslant k_n$ and write

$$C_{n,k} = e^{-\frac{k(nc+1-c)}{2(n-1)(1-c)}u^2},$$

we have

$$(1-C_{n,k})^{\frac{n}{2}} = (1-C_{n,k})^{-\frac{1}{C_{n,k}}\frac{-nC_{n,k}}{2}} \qquad (*)$$

and note that $nC_{n,k} \to 0$: applying $\ln$,

$$\begin{aligned}\ln(nC_{n,k}) &= \ln n - \frac{k(nc+1-c)}{2(n-1)(1-c)}u^2\\ &= -k\left[\frac{(nc+1-c)}{2(n-1)(1-c)}u^2 - \frac{\ln n}{k}\right] \to -\infty, \text{ for } k\to\infty \text{ and } \frac{\ln n}{k} < \frac{\ln n}{k_n} \to 0.\end{aligned}$$

Thus in (*) $(1-C_{n,k})^{\frac{n}{2}} \to e^0 = 1$. We then have

$$W(A_a) \geqslant \frac{1}{\sqrt{2\pi}}\int_0^\infty (1-C_{n,k})^{\frac{n}{2}} e^{-\frac{u^2}{2}}\,du \to \frac{1}{2}.$$

This completes the proof. $\square$

THEOREM 2.4. *If $c=0$ and $(k_n)_{n\in\mathbb{N}}$ a sequence of natural numbers such that*

$$\lim_{n\to\infty}\frac{n\ln n}{k_n} = 0,$$

then for every $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that for all $n \geqslant n_\varepsilon$ and $k \geqslant k_n$, the measure of polynomials attaining a local maximum at the vertex a is

$$W(A_a) > \frac{1}{2} - \varepsilon.$$

Proof. The proof is as that of the previous theorem, but note that here we have

$$C_{n,k} = e^{-\frac{k}{2(n-1)}u^2},$$

thus, when applying $\ln$,

$$\begin{aligned}\ln(nC_{n,k}) &= \ln n - \frac{k}{2(n-1)}u^2\\ &= -\frac{k}{n}\left[\frac{nu^2}{2(n-1)} - \frac{n\ln n}{k}\right] \to -\infty. \quad \square\end{aligned}$$

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Damián Pinasco
Universidad Torcuato Di Tella
Av. Figueroa Alcorta 7350 (C1428BCW)
Buenos Aires, Argentina and CONICET
e-mail: dpinasco@utdt.edu

Ignacio Zalduendo
Universidad Torcuato Di Tella
Av. Figueroa Alcorta 7350 (C1428BCW)
Buenos Aires, Argentina and CONICET
e-mail: izalduendo@utdt.edu