

# A geometric description of the spatial coherence and Babinet's like principle for the fringe visibility

## Una descripción geométrica de la coherencia espacial y un principio similar al de Babinet para explicar la visibilidad de franjas

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**ABSTRACT:** The concept of spatial coherence is usually hard to comprehend the first time that it is studied. We propose here a fully intuitive geometric description that does not contain mathematical difficulties, and permits one to understand how a Young's fringe stationary system is obtained with a source not spatially coherent. It is based in a very simple experiment that permits the detection of spatial coherence in a scene. Experimental results are shown. Using this approach, a Babinet-like principle for the visibility of the Young's fringes is proposed and experimentally demonstrated.

**Key words:** optical coherence, spatial coherence, Van Cittert-Zernike Theorem, Babinet's principle

**RESUMEN:** El concepto de coherencia espacial es, usualmente, difícil de comprender la primera vez que es abordado. Nosotros proponemos en este trabajo una descripción geométrica totalmente intuitiva, que no contiene dificultades matemáticas y permite entender cómo un sistema estacionario de franjas de Young es obtenido utilizando una fuente de luz espacialmente incoherente. Este método está basado en un experimento muy simple que permite la detección de la coherencia espacial en una escena, de la que se muestran los resultados experimentales obtenidos. Mediante el uso de esta aproximación, un principio similar al de Babinet es propuesto y experimentalmente demostrado, a fin de explicar la visibilidad de las franjas de Young.

**Palabras clave:** óptica coherente, coherencia espacial, Teorema de van Cittert-Zernike, principio de Babinet.

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## 1. Introduction

Interference, diffraction and polarization phenomena are extensively developed in most texts for undergraduate students. Nevertheless, light source coherence is only mentioned in brief [1-3] or poorly explained [4], in spite of the fact that this concept is fundamental to understanding the above mentioned phenomena. Hecht [5] makes a more complete analysis of coherence, but its mathematical complexity may be too high for clear comprehension by a significant proportion of the students and even some professors. This mathematical development also favours against the following of the intuitive meaning of the subject.

There seems to exist a gap between these two extremes. We hope that the following approaches can be of help to fill the gap before introducing the more difficult complete description [6-7].

The concept of spatial coherence is usually difficult to understand. An introductory approach to the subject was developed by M. L. Calvo [8]. A rigorous study of spatial coherence has been developed by Van Cittert and Zernike (VCZ Theorem) [6]. It states that the normalized degree of coherence is the Fourier transform of the intensity distribution for uncorrelated emitters.

It is generally admitted that for stationary interference fringes to be observed, light must come from a coherent source, for example, a laser. Nevertheless, even from the early works by Verdet [9] it is possible to observe interference fringes in an everyday scene illuminated by a natural light source. How can this phenomenon be interpreted?

In this work, we propose a very simple description of spatial coherence for undergraduate students, using a geometric approach that is simple from a mathematical point of view and shows how Young's fringes with observable visibility can be obtained with a source that is not spatially coherent, as many everyday scenes contain independent light sources.

A simple experiment, sometimes employed for illustration of Young's fringes and can be conducted in the presence of any source, shows that irradiance discontinuities in the object give rise to perceivable fringes. Fainter fringes are also present in sources with no discontinuities, but are difficult for the naked eye to perceive.

Using this approach, in this paper we propose a Babinet-like principle fulfillment for the visibility of Young's fringes. It is a non-evident result, but is both predicted by this approach and observed in the experiment. We show some experimental results.

## 2. Theory

When the interference phenomenon is studied, it usually starts with the calculation of the irradiance in an observation plane due to the superposition of two waves.

In the classical elementary calculation of the field, due to plane waves coming from two sources, they are assumed to be coherent and monochromatic, that is, to have the same frequency and to keep a constant phase difference along the time.

These two assumptions, when taken rigorously, have the same meaning. If the waves are strictly monochromatic their phase difference is effectively constant. Thus, we are going to assume that the light source is quasi-monochromatic.

Following the description given in Ref. [5], the electric field of the two interfering waves can be described using the following equation:

$$\vec{E}_i(\vec{r}, t) = \vec{E}_{0i} \cos(\vec{k}_i \cdot \vec{r} - \omega t + \varepsilon_i); \quad i = 1, 2; \quad (1)$$

where  $\varepsilon_i$  is the initial phase of the wave.

In any point where both waves overlap, the total field is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 \quad (2)$$

In the optical range, the waves frequencies are too high to be resolved by conventional detectors (eye, cameras, photometers). The magnitude that can be actually measured is the mean value (along time) of the Poynting vector, named *irradiance*, which is proportional to the squared value of the electrical field peak value.

The irradiance  $I$  due to the superposition of two *monochromatic* waves is then (angular brackets indicate time mean value):

$$I = \varepsilon \cdot v \langle E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2 \rangle = I_1 + I_2 + I_{12}; \quad (3)$$

$$I_{12} = 2\varepsilon \cdot v \langle \vec{E}_1 \cdot \vec{E}_2 \rangle; \quad (4)$$

$$\begin{aligned} I_{12} &= 2\varepsilon \cdot v \vec{E}_{01} \cdot \vec{E}_{02} \langle \cos(\vec{k}_1 \cdot \vec{r} - \omega t + \varepsilon_1) \cos(\vec{k}_2 \cdot \vec{r} - \omega t + \varepsilon_2) \rangle \\ &= 2\varepsilon \cdot v \vec{E}_{01} \cdot \vec{E}_{02} \langle \cos(\phi_1 - \omega t) \cos(\phi_2 - \omega t) \rangle; \end{aligned} \quad (5)$$

where  $\varepsilon$  is the dielectric constant and  $v$  is the speed of light in the medium. By using the identities

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B; \quad (6)$$

$$\langle \cos^2(\omega t) \rangle = \langle \sin^2(\omega t) \rangle = \frac{1}{2}; \quad (7)$$

$$\langle \cos(\omega t) \rangle = \langle \sin(\omega t) \rangle = 0; \quad (8)$$

$$I_{12} = \varepsilon \cdot v \vec{E}_{01} \cdot \vec{E}_{02} \cos \delta_{12}; \quad (9)$$

with

$$\delta_{12} = (\vec{k}_1 \cdot \vec{r}_1 - \vec{k}_2 \cdot \vec{r}_2 + \varepsilon_1 - \varepsilon_2) \quad (10)$$

If the field vectors are parallel:

$$I_{12} = 2\sqrt{I_1 I_2} \cos \delta_{12} \quad (11)$$

The total irradiance is found to be [5]:

$$I = 2I_0 (1 + \cos \delta_{12}) = 4I_0 \cos^2 \left( \frac{\delta_{12}}{2} \right); \quad (12)$$

where  $I_0$  is the irradiance of the two waves and  $\delta_{12}$  is the phase difference between them. We assumed that the electrical fields of both waves are parallel.

The phase difference  $\delta_{12}$  in the case of incoherent quasi-monochromatic sources is not constant but instead changes rapidly, and ordinary detectors cannot detect the interference phenomenon. The irradiance resulting from two such sources is thus the direct sum of the irradiances of each of the sources.

Nevertheless, when we observe a light source through a pupil composed by two thin slits, we find a fringe system. Why does it happen? How can we justify it?

How is it possible that two (or more) incoherent elements of a source could give rise to stable non-zero visibility fringes in time, in spite of the fact that their relative phases are fluctuating randomly?

Each idealised point element of the source produces high contrast fringes, but different elements are not supposed to give rise to stable interference patterns.

Accordingly, the only possibility for high visibility to subsist when both point source elements are present should be when the source points are separated by such a distance that makes the individual fringe systems to coincide.

Emphasis is exerted on the fact that the existence of measurable fringe visibility is due to  $2\pi$  congruent superposition of multiple fringe systems originating in different source elements that are incoherent between them. That is, both fringe systems are shifted from one respect to the other so that their maxima and minima coincide.

This idea, originally used to calculate visibility in times before the VCZ theorem was stated, can be exemplified by using two very small and close narrow slits very near to the eyes and observing outdoor scenarios through them. Photography of a natural scene is shown in Figure 1. Even if the available light is not strictly monochromatic, fringes can be observed principally in luminance discontinuities, such as edges, wires or poles; images of the sun in dew drops or cylindrical surfaces also show fringes with visibility high enough to be discerned. It is easy, then, to figure out that low or zero visibility in extended sources is due to the superposition of shifted fringe systems.

We suggest here a description using elementary trigonometric identities to explain how a Young's fringe system can be obtained from a source constituted by incoherent point sources. The visibility in the fringes with a compound source is found with the coincidence of several shifted fringe systems, each coming from every single source point. These are added on an intensity basis.

This approach leads, in a natural way, to the same visibility result as the Van Cittert-Zernike Theorem for any arbitrary source distribution.

We use a simple experiment to illustrate this proposal. It consists of the observation of a scene through a card with two very small and close slits.

### 3. Experiments

#### 3.a. Simple experiment with a natural scene

If we observe a point like a light source with an optical system limited by two parallel slit apertures, a Young's fringe pattern can be observed in the image.

If the source is composed of several incoherent emitters, the observable irradiance is too fast to be detected in the optical range; the average irradiance is zero and no fringes are observed.



Fig. 1. Photography of a natural scene showing Young's fringes (slits horizontal)

Nevertheless, if this optical system is pointed at any natural scene, the scene can appear to be covered with fringes. As an example, in Figure 1 we show a natural scene (a backyard) though an optical system

(the camera) limited by two thin parallel slits. The corresponding Young's fringes can be observed in the irradiance discontinuities.

How we can solve this contradiction? Why do fringes appear?

To search for the answer, we are going to consider very simple sources and to ask what happens with the fringes produced by every point of the sources.

### 3.b. Two mutually non coherent point light sources

In Figure 2,  $S_1$  and  $S_2$  in plane  $\pi$  represent two quasi-monochromatic point sources separated by a distance  $X_0$ . They have the same mean wavelength  $\lambda$  and the same irradiance  $I_0$ . Narrow slits  $P_1$  and  $P_2$  are separated by a distance  $d$ . The lens  $L$ , with focal distance  $f$  conjugates the planes  $\pi$  and  $\pi'$ . The distance  $z$  is much bigger than  $d$  distances.

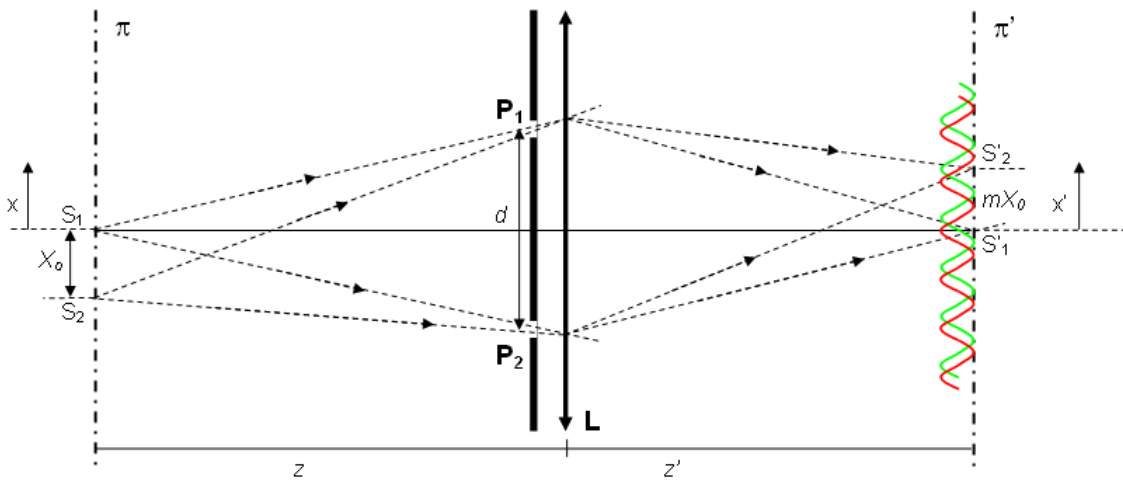


Fig. 2. Young's fringe pattern obtained from two mutually non-coherent point sources.

Each quasi-monochromatic source corresponds in plane  $\pi$  to the light distribution found in a Young's fringes experiment in plane  $\pi'$ .

The irradiance distributions  $I_1$  and  $I_2$  in plane  $\pi'$  due to the sources  $S_1$  and  $S_2$  respectively can be described as follows:

$$I_1(x') = 4I_0 \cos^2 \left[ \frac{\pi d}{\lambda z'} x' \right]; \tag{13}$$

$$I_2(x') = 4I_0 \cos^2 \left[ \frac{\pi d}{\lambda z'} (x' - mX_0) \right]; \tag{14}$$

where  $m$  is the lateral magnification due to the lens. The visibility of the fringe system produced by  $I_1$  and  $I_2$ , when both overlap, is:

$$V = \frac{I_{MAX} - I_{MIN}}{I_{MAX} + I_{MIN}} = \left| \cos \left( \frac{\pi m X_0 d}{\lambda z'} \right) \right|; \tag{15}$$

(see Appendix I for a step-by-step calculation involving elementary trigonometric identities).

The visibility depends on the separation  $d$  between  $P_1$  and  $P_2$  (points that are used in the correlation in the Van Cittert-Zernike Theorem) as well as the relationship between  $z'$  and the mean wavelength  $\lambda$ .

Imposing a  $2\pi$  (or integer multiples of  $2\pi$ ) shift between both fringe systems will superpose them, and the visibility of the composed system will be at maximum. This is due to coincidence of the fringes rather than to interference between the light comings from the different sources.

### 3.c. Continuous source distributions

Following the same line of reasoning as before, if there are  $N$  discrete point sources with irradiance  $I_i$ , located at points  $x_i$ , the intensity distribution in plane  $\pi'$  results as follows:

$$I(x') = \sum_{i=1}^N I_i \cos^2 [f_x (x' - mx_i)]; \quad (16)$$

where

$$f_x = \frac{\pi d}{\lambda z'} \quad (17)$$

For the case of a continuous quasi-monochromatic incoherent intensity distribution source, the visibility of the Young's fringe systems becomes:

$$I(x') = \int I(x) \cos^2 (f_x (x' - mx)) dx = B + \frac{1}{2} A \cos(2f_x x' - \delta); \quad (18)$$

with

$$A \cos \delta = \int I(x) \cos(2f_x mx) dx; \quad (19)$$

and

$$A \sin \delta = \int I(x) \sin(2f_x mx) dx; \quad (20)$$

$$B = \frac{1}{2} \int I(x) dx \quad (21)$$

So that (for detailed description, see Appendix II)

$$A = \left\{ \left| \int I(x) \cos(2f_x mx) dx \right|^2 + \left| \int I(x) \sin(2f_x mx) dx \right|^2 \right\}^{1/2}; \quad (22)$$

where  $I(x)$  is the density of irradiance per unit length, and the integrals extend to include the entire source. Subsequently, the maximum and the minimum irradiances will be, respectively:

$$I_{Max} = B + \frac{1}{2} A; \quad I_{Min} = B - \frac{1}{2} A; \quad (23)$$

and the visibility is:

$$V = \frac{I_{Max} - I_{Min}}{I_{Max} + I_{Min}} = \frac{A}{2B} \quad (24)$$

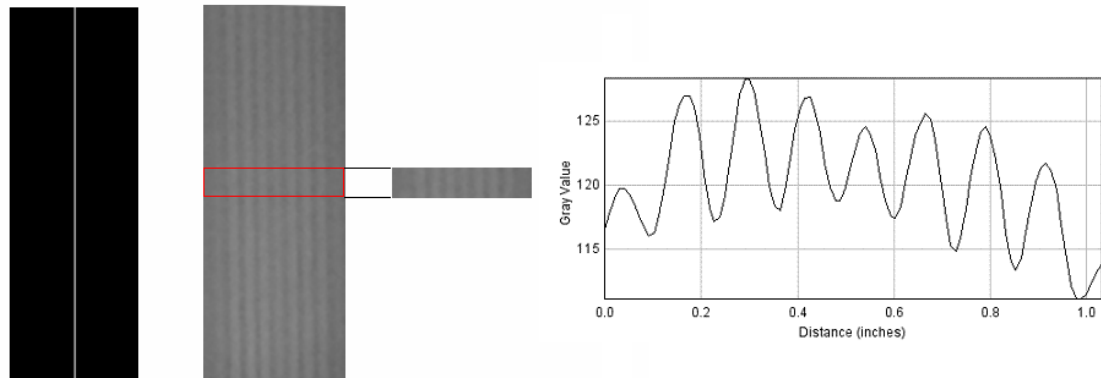
$$V = \frac{\left\{ \left| \int I(x) \cos(2f_x mx) dx \right|^2 + \left| \int I(x) \sin(2f_x mx) dx \right|^2 \right\}^{1/2}}{\int I(x) dx} \quad (25)$$

The visibility is given by the modulus of the normalized Fourier transform of the intensity-density distribution of the source. This is the result of the Van Cittert-Zernike theorem. Visibility of the fringes depends on the distance  $d$  (coordinates difference) between points  $P_1$  and  $P_2$ , and not on their actual position in front of the lens.

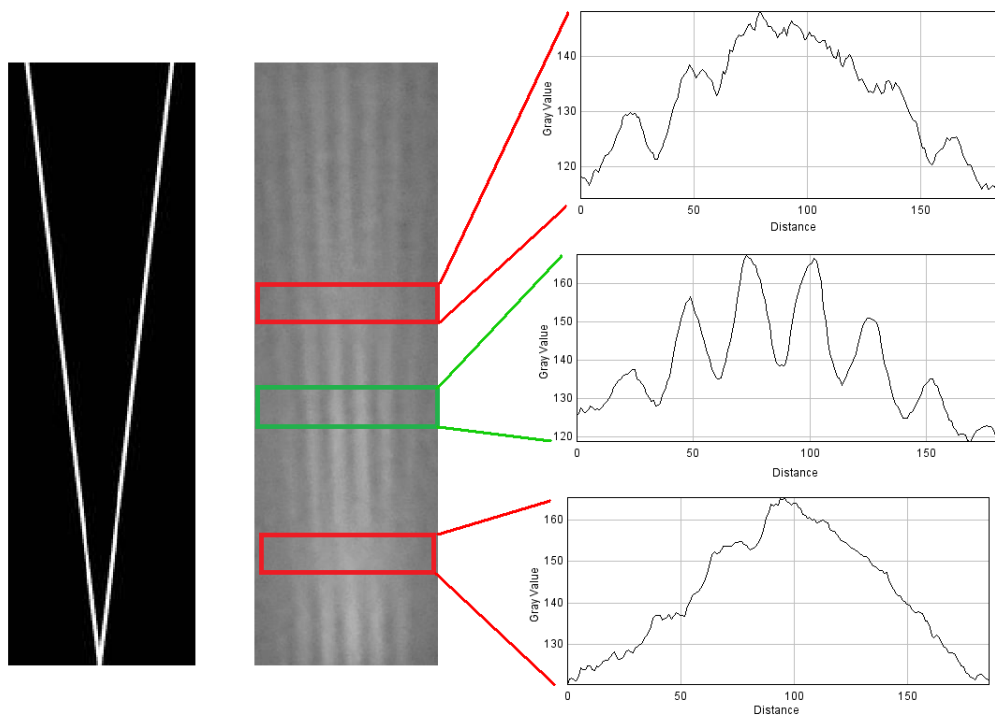
Notice that only elementary calculus and trigonometric identities are used in this description, and at every step, a clear understanding of their meanings easily maintained.

From the Eq. (25), when the source consists in two point incoherent sources, the results given by Eq. (15) are re-obtained.

An experimental verification can be seen in Figure 3a), where a vertical straight bright line is used as an object (width = 0.13 mm) with vertical Young's slits (width = 0.3 mm and separation  $d = 1.16$  mm) also in front of the camera. The fringes appear with high visibility.



a)



b)

Fig. 3. a) Fringes produced by an object consisting of a single vertical slit (width = 0.13 mm) and its corresponding profile; b) Fringes produced by a V-shaped double slit and used as incoherent objects and its corresponding profile. In both cases, we use two parallel slits (width = 0.3 mm and separation  $d = 1.16$  mm), in front of the object, as in Young's experiment (The object is not shown in its actual scale). Notice the visibility variation and its contrast reversals in the vertical direction, in agreement with our proposal. The visibility in the green region is comparatively higher than in the red region.

Another experimental verification can be seen in Figure 3b). With vertical Young's slits in front of the camera, a V-shaped object (a transparency mask with two bright convergent slits) was used so that the horizontal separation of the points of the V is a linear function of its vertical coordinate. In this way, we can simultaneously observe different values of the two points' separation.

When observed through the double parallel vertical slits (Young's experiment), each pair of points in the horizontal direction of the V produces fringe systems that add their effects or cancel them, according to their distance. This results in a periodic (chromatic, as it depends also on wavelength) variation of visibility. Notice that in the image of the V, there are regions with high visibility fringes alternated in the vertical direction with low visibility ones, in agreement with our proposal. Also, we show irradiance profiles in two different regions. The visibility in the green region is comparatively higher than in the red region. Here,  $X_0$  is a linear function of the vertical coordinate, and the visibility changes accordingly with Eq. (15).

#### 4. Babinet-like principle of the visibility of fringes.

So far, we have given an intuitive description of the origin of the visibility in fringes produced by partially coherent sources. Here, we are going to show that the same idea can give insight to a not obvious phenomenon.

Babinet's principle [5] states that the diffraction pattern from two complementary masks (that is, one of them is transparent in the region where the other is opaque and conversely) is identical except for the mean value of the forward beam intensity. The last one is concentrated in the center of the diffraction pattern and is called *zero order*. Does *visibility* fulfill a Babinet-like principle?

In this case, we propose that the visibility of the Young's fringes when the object is a certain mask is identical to that of the other complementary mask. The analogous to the zero order of the diffraction pattern corresponding to the classical Babinet's principle is not relevant here because it corresponds to  $d = 0$  in Eq. (17).

Let us suppose a trivial example where the object is a uniform intensity distribution. In accordance with our intuitive model, each point acts as a point source and produces a Young's fringe system. All these systems add their intensity incoherently, the fringes cancel each other out, and the obtained visibility is zero.

When Mask M covers part of the uniform intensity distribution, then the contributions of the hidden points are not present in the image. Those fringes that were cancelled by the hidden points are now restored. The visibility is then increased but the observed fringes are contrast-reversed to the hidden fringes.

If we now replace M with its complementary mask, the hidden points are now exposed conversely. It is expected that the same fringes observed in the former description will now be replaced by the complementary ones.

On the other hand, if the visibility itself, as shown in Eq. (25), is the modulus of the normalized Fourier transform of the source distribution, it could be expected that complementary sources would give rise to similar fringes, but contrast-reversed.

Figure 4a) shows the result of the experiment when the source is a uniform distribution covered with a thin transparent curved slit. Young's fringes can be seen against the uniform background.

In Figure 4b), the slit in the object is replaced by its complementary slit (a dark curved line on a bright background), and the resulting fringes can be seen.

In the literature [10], the mathematical treatment of a classical Babinet's principle for the Fraunhofer diffraction follows reasoning similar to that of the following.

If P is the transmittance in the intensity function of a certain binary mask, then

$$\bar{P} = 1 - P ; \tag{26}$$

describes the transmittance of its complementary screen. The visibility of the Young's fringes obtained with the latter, as a function of the distance  $d$  between slits, consists of a delta distribution in the origin of frequencies, corresponding to  $d = 0$ , minus the Fourier transform of P. The change of sign indicates contrast reversal. Thus, a Babinet-like principle holds *for visibility* but with a minor change in its interpretation.



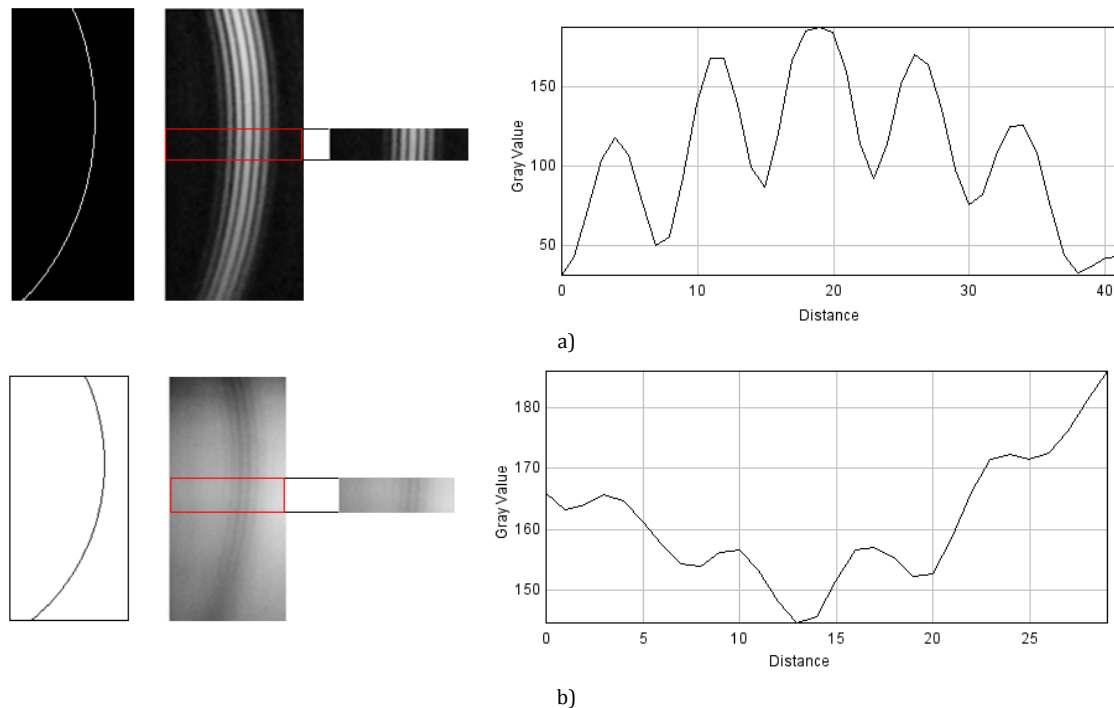


Fig. 4. Fringes obtained with a) an object in the shape of a curved slit with the corresponding profiles at the center, and b) its complementary slit showing a Babinet-like principle for visibility. Notice the fringes on the bright uniform background where the visibility is zero.

It can be seen that the same result is obtained with our intuitive approach as with the rigorous mathematical description.

Figure 4) shows the results obtained using a curved slit and its complementary slit as the object. It can be seen that the complementary slit gives rise to Young's fringes, which are contrast-reversed with respect to those produced by the former lit on a uniform background with null visibility.

## 5. Conclusions

In this paper, we present an introduction to the concept of spatial coherence using a geometric approach as an intuitive alternative to the rigorous study of the spatial coherence developed by Van Cittert and Zernike (VCZ Theorem). We consider that this approach is a useful tool for undergraduate students of Science and Engineering.

When a single point-like source is observed through two narrow slits, we obtain a Young's fringe pattern, where the contributions of each aperture are added on a field basis.

For the spatial extended light sources, high visibility Young's fringes can still be observed if every point of the light source gives rise to a fringe system that coincides with the produced by the others points of the source. In this case, the addition of these elementary contributions is in intensity basis.

For the fringe systems to coincide and obtain a good visibility result, source points should not exist too close to each other, which will spoil their visibility. This is particularly perceptible when the source exhibits spatial discontinuities. It is the presence of source discontinuities that gives rise to higher visibility.

This is also true when the source is a uniform field and there are isolated discontinuities (e.g., a diluted dark object on a bright uniform field). Thus, visibility behaves as though fulfilling a Babinet-like complementary property, although it gives rise to fringes that are contrast-reversed between those given by two complementary masks.

A holodiagram description of some of these phenomena can be found in [11], where this approach is extended to somewhat more complex source distributions. When the source distribution is not entirely contained in a plane perpendicular to the optic axis, the calculation turns out to be slightly more involved but can still be described using this geometric approach [11].

Small departures in experimental visibility observation can be expected if the slits are thin enough.

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**Appendix I**

Calculation of the visibility in the fringes obtained with two infinitely narrow object slits.

Incoherent addition:

$$I_1(x') = 4I_0 \cos^2 \left[ \frac{\pi d}{\lambda z'} \cdot x' \right] \quad (I)$$

$$I_2(x') = 4I_0 \cos^2 \left[ \frac{\pi d}{\lambda z'} \cdot (x' - mX_0) \right] \quad (II)$$

Adding (I) and (II):

$$I = I_1(x') + I_2(x') \quad (III)$$

$$I = 4I_0 \left\{ \cos^2 \left[ \frac{\pi d}{\lambda z'} \cdot x' \right] + \cos^2 \left[ \frac{\pi d}{\lambda z'} \cdot (x' - mX_0) \right] \right\} \quad (IV)$$

By using the identity

$$\cos^2 \alpha = \frac{1}{2} + \frac{1}{2} \cos 2\alpha \quad (V)$$

Then

$$\begin{aligned} I &= 4I_0 \left\{ \frac{1}{2} + \frac{1}{2} \cos \left( \frac{2\pi d}{\lambda z'} \cdot x' \right) + \frac{1}{2} + \frac{1}{2} \cos \left( \frac{2\pi d}{\lambda z'} \cdot (x' - mX_0) \right) \right\} \\ &= 4I_0 \left\{ 1 + \frac{1}{2} \left[ \cos \left( \frac{2\pi d}{\lambda z'} \cdot x' \right) + \cos \left( \frac{2\pi d}{\lambda z'} \cdot (x' - mX_0) \right) \right] \right\} \end{aligned} \quad (VI)$$

And now using:

$$\cos A + \cos B = 2 \cos \left[ \frac{1}{2}(A + B) \right] \cdot \cos \left[ \frac{1}{2}(A - B) \right] \quad (VII)$$

with

$$A = \frac{2\pi d}{\lambda z'} \cdot x' \quad (VIII)$$

$$B = \frac{2\pi d}{\lambda z'} \cdot (x' - mX_0) \quad (IX)$$

It results in

$$\begin{aligned} \cos A + \cos B &= 2 \cos \left[ \frac{1}{2} \left( \frac{2\pi d}{\lambda z'} \cdot (x' + x' - mX_0) \right) \right] \cos \left[ \frac{1}{2} \left( \frac{2\pi d}{\lambda z'} \cdot mX_0 \right) \right] \\ I &= 4I_0 \left\{ 1 + \frac{1}{2} \left( 2 \cos \left[ \frac{1}{2} \frac{2\pi d}{\lambda z'} \cdot (2x' - mX_0) \right] \cos \left[ \frac{1}{2} \frac{2\pi d}{\lambda z'} \cdot mX_0 \right] \right) \right\} = \\ &= 4I_0 \left\{ 1 + \cos \left[ \frac{\pi d}{\lambda z'} \cdot (2x' - mX_0) \right] \cos \left[ \frac{\pi d}{\lambda z'} \cdot mX_0 \right] \right\} \end{aligned} \quad (X)$$

$$I_{MAX} = 4I_0 \left\{ 1 + MAX \left\{ \cos \left[ \frac{\pi d}{\lambda z'} \cdot (2x' - mX_0) \right] \cos \left[ \frac{\pi d}{\lambda z'} \cdot mX_0 \right] \right\} \right\} \quad (XI)$$

$$\cos \left[ \frac{\pi d}{\lambda z'} \cdot mX_0 \right] = \pm \left| \cos \frac{\pi d}{\lambda z'} \cdot mX_0 \right| \quad (XII)$$

$$MAX = \left| \cos \frac{\pi d}{\lambda z'} \cdot mX_0 \right| \cdot MAX \left[ \pm \cos \frac{\pi d}{\lambda z'} \cdot (2x' - mX_0) \right] \quad (XIII)$$

The maximum value is obtained when the factor inside the keys is +1 and its value is

$$MAX = \left| \cos \frac{\pi d}{\lambda z'} \cdot mX_0 \right| \quad (XIV)$$

The minimum value is obtained when the factor inside the key is -1 and its value is

$$MIN = - \left| \cos \frac{\pi d}{\lambda z'} \cdot mX_0 \right| \quad (XV)$$

Then, by definition the visibility is

$$V = \frac{I_{MAX} - I_{MIN}}{I_{MAX} + I_{MIN}} \quad (XVI)$$

$$V = \frac{4I_0 [1 + MAX] - 4I_0 [1 + MIN]}{4I_0 [1 + MAX] + 4I_0 [1 + MIN]} = \frac{MAX - MIN}{2 + MAX + MIN}$$

$$V = \frac{\left| \cos \frac{\pi d}{\lambda z'} \cdot mX_0 \right| - \left( - \left| \cos \frac{\pi d}{\lambda z'} \cdot mX_0 \right| \right)}{2} \quad (XVII)$$

$$\therefore V = \left| \cos \frac{\pi d}{\lambda z'} \cdot mX_0 \right| \quad (XVIII)$$

## Appendix II

Calculation of Irradiance.

$$\begin{aligned} I(x') &= \int I(x) \cos^2(f_x(x' - mx)) dx \\ &= B + \frac{1}{2} A \cos(2f_x x' - \delta) \end{aligned} \quad (i)$$

By using the identities:

$$\cos^2 \alpha = \frac{1}{2} + \frac{1}{2} \cos 2\alpha \quad (ii)$$

So

$$\cos[2f_x(x' - mx)] = \cos 2f_x x' \cdot \cos 2f_x mx + \sin 2f_x x' \cdot \sin 2f_x mx \quad (iii)$$

Then

$$I(x') = \int I(x) \cos^2(f_x(x' - mx)) dx \quad (iv)$$

$$I(x') = \frac{1}{2} \int I(x) dx + \frac{1}{2} \int I(x) \cos(2f_x(x' - mx)) dx \quad I(x') = \frac{1}{2} \int I(x) dx + \frac{1}{2} \int I(x) \cos(2f_x(x' - mx)) dx$$

$$I(x') = B + C \cos 2f_x x' + D \sin 2f_x x'$$

with

$$C = \int I(x) \cos(2f_x mx) dx \quad (vii)$$

and

$$D = \int I(x) \sin 2f_x mx dx \quad (viii)$$

If we now define

$$C = A \cos \delta \quad (ix)$$

and

$$D = A \sin \delta \quad (x)$$

then

$$A \cos \delta = \int I(x) \cos(2f_x mx) dx \quad (xi)$$

and

$$A \sin \delta = \int I(x) \sin(2f_x mx) dx \quad (xii)$$

Where

$$\delta = \operatorname{tg}^{-1} \left[ \frac{\int I(x) \sin 2f_x mx dx}{\int I(x) \cos(2f_x mx) dx} \right] \quad (xiii)$$

and

$$A = \left\{ \left[ \int I(x) \cos(2f_x mx) dx \right]^2 + \left[ \int I(x) \sin(2f_x mx) dx \right]^2 \right\}^{1/2} \quad (xiv)$$