MIXED BOHR RADIUS IN SEVERAL VARIABLES

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ABSTRACT. Let $K(B_{\ell_p^n}, B_{\ell_q^n})$ be the n-dimensional (p,q)-Bohr radius for holomorphic functions on $\mathbb{C}^n$. That is, $K(B_{\ell_p^n}, B_{\ell_q^n})$ denotes the greatest constant $r \geq 0$ such that for every entire function $f(z) = \sum_\alpha c_\alpha z^\alpha$ in n-complex variables, we have the following (mixed) Bohr-type inequality

$$\sup_{z \in r \cdot B_{\ell_q^n}} \sum_\alpha |c_\alpha z^\alpha| \leq \sup_{z \in B_{\ell_p^n}} |f(z)|,$$

where $B_{\ell_r^n}$ denotes the closed unit ball of the n-dimensional sequence space ℓ_r^n.

For every $1 \leq p, q \leq \infty$, we exhibit the exact asymptotic growth of the (p,q)-Bohr radius as n (the number of variables) goes to infinity.

1. INTRODUCTION

At the early twentieth century, during the course of his investigations on the famous Riemann ζ function, Harald Bohr [Boh13, Boh14] devoted great efforts in the study and development of a general theory of Dirichlet series. A Dirichlet series is just an expression of the form

$$D(s) = \sum_{n \geq 1} \frac{a_n}{n^s},$$

where $a_n \in \mathbb{C}$ and $s = \sigma + it$ is a complex variable. The regions of convergence, absolute convergence and uniform convergence of these series define half-planes of the form $[Re(s) > \sigma_0]$ in the complex field. Bohr was mainly interested in controlling the region of convergence of a series. To achieve this, he related different types of convergence and focused on finding the width of the greatest strip for which a Dirichlet series can converge uniformly but not absolutely. This question is popular and known nowadays as the *Bohr's absolute convergence problem.*

Although the solution of this problem problem appeared two decades after it was proposed (given by Bohnenblust and Hille [BH31] who showed that the maximum width of this strip is $\frac{1}{2}$), Bohr made major contributions in the area (arguably, even more important than the solution of the problem

2010 *Mathematics Subject Classification.* 32A05,32A22 (primary),46B15,46B20,46G25,46E50 (secondary).

Key words and phrases. Bohr radius, power series, homogeneous polynomials, unconditional bases, domains of convergence for monomial expansions.

This work was partially supported by projects CONICET PIP 1122013010032, ANPCyT PICT 2015-2224, ANPCyT PICT 2015-2299, UBACyT 20020130300057BA, UBACyT20020130300052BA,UBACyT 20020130100474B . The second author was supported by a CONICET doctoral fellowship.

itself) in order to tackle it. He discovered a deep connection among Dirichlet series and power series in infinitely many variables. Given a Dirichlet series $D(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$, he considered for each $n \in \mathbb{N}$ the prime decomposition $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ (where $(p_k)_{k\in\mathbb{N}}$ denote the sequence formed by the ordered primes) and defined $z = (p_1^{-s}, \dots, p_r^{-s})$. Thus,

$$D(s) = \sum_{n\geq 1} a_n (p_1^{-s})^{\alpha_1} \dots (p_r^{-s})^{\alpha_r} = \sum a_n z_1^{\alpha_1} \dots z_r^{\alpha_r}.$$

This correspondence, known as the *Bohr transform* is not just formal: it gives an isometry between suitable spaces of Dirichlet series and power series [HLS97]. The Bohr transform allows to transform/translate problems about Dirichlet series in terms of power series and tackle them with complex analysis techniques. This cycle of ideas brought Bohr to ask whether is possible to compare the absolute value of a power series in one complex variable with the sum of the absolute value of its coefficients. He manged to prove the following result nowadays referred as *Bohr's inequality*:

The radius $r = \frac{1}{3}$ is the largest value for which the following inequality holds:

$$\sum_{n\geq 0} |a_n| r^n \leq \sup_{z\in\mathbb{D}} |\sum_{n\geq 0} a_n z^n|, \tag{1}$$

for every entire function $f(z) = \sum_{n\geq 0} a_n z^n$ on the unit disk $\mathbb{D}$ such that $\sup_{z\in\mathbb{D}} |f(z)| < \infty$.

As a matter of fact, Bohr's paper [Boh14], compiled by G. H. Hardy from correspondence, indicates that Bohr initially obtained the radius $\frac{1}{6}$, but this was quickly improved to the sharp result by M. Riesz, I. Schur, and N. Wiener, independently. Bohr's article presents both his own proof and the one of his colleagues.

This interesting inequality was overlooked during many years until the end of the twentieth century. In particular, Dineen and Timoney [DT89], Dixon [Dix95], Boas and Khavinson [KB97], Aizenberg [Aiz00] and Boas [Boa00] retook this work and use it in different contexts and/or generalize it. Several of these authors analyzed if a similar phenomenon occurs for power series in many variables. For each Reinhardt domain $\mathscr{R}$, they introduced the notion of the *Bohr radius* $K(\mathscr{R})$ as the biggest $r \geq 0$ such that for every analytic function $f(z) = \sum_\alpha a_\alpha z^\alpha$ bounded on $\mathscr{R}$, it holds:

$$\sup_{z\in r\cdot\mathscr{R}} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z\in\mathscr{R}} |f(z)|. \tag{2}$$

Note that with this notation, Bohr's inequality can be formulated simply as $K(\mathbb{D}) = \frac{1}{3}$. Surprisingly, the exact value of the Bohr radius is unknown for any other domain. The central results of [KB97, Boa00] contained a (partial) successful estimate for the Bohr radius for the complex unit balls of ℓ_p^n, $1 \le p \le \infty$.

The gap between the upper and lower estimates in this papers leaded many efforts to compute the exact asymptotic order of $K(B_{\ell_p^n})$, for $1 \le p \le \infty$.

To obtain the upper bounds Boas [Boa00] generalized in a very ingenious way a theorem of Kahane-Salem-Zygmund on random trigonometric polynomials [Kah93, Theorem 4 in Chapter 6], which gives (by the use of a probabilistic argument) the existence of homogeneous polynomials with "large coefficients" and uniform norm "relatively small". This technique (and some refinements of it, for example [DGM04, Bay12]) do the work when dealing with upper bounds.

The lower bound is a horse of a different color. In [DGM03] Defant, García and Maestre related the Bohr radius with some non-elementary concepts of the local theory of Banach spaces: unconditionality in spaces of homogeneous polynomials via some Banach-Mazur distance estimates. Although at that moment this did not give optimal asymptotic bounds, it started a way with which $K(B_{\ell_p^n})$ would be obtained.

They were Defant, Frerick, Ortega-Cerdá, Ounaïes and Seip [DFOC$^+$11] who made an incredible contribution in the problem and managed to exhibit the exact asymptotic value of $K(B_{\ell_\infty^n})$. The authors involved again into the game the classical Bohnenblust-Hille inequality, which was used to compute Bohr's convergence width eighty years before. This inequality asserts that the $\ell_{\frac{2m}{m+1}}$-norm of the coefficients of a given m-homogeneous polynomial in n-complex variables is bounded by a constant *independent of n* times its supreme norm on the polydisk. Precisely, given $m \in \mathbb{N}$. there is a constant $C_m > 0$ such that for every m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ in $\mathbb{C}^n$,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \le C_m \sup_{z \in \mathbb{D}^n} |P(z)|. \tag{3}$$

The groundbreaking progress consisted in showing that C_m is in fact hypercontractive; that is, C_m can be taken less than or equal to C^m for some absolute constant $C > 0$. With this at hand they proved that $K(B_{\ell_\infty^n})$ behaves asymptotically as $\sqrt{\frac{\log(n)}{n}}$ (other cornerstone of the paper is that they have also described the Sidon constant for the set of frequencies $\{\log(n) : n \text{ a positive integer } \le N\}$). This paper, arguably in some sense, marked the path of the whole area over the last years. In fact much more can be said about $K(B_{\ell_\infty^n})$: Bayart, Pellegrino and Seoane [BPSS14] managed to push these techniques further in an amazingly ingenious way to obtain that $\lim_{n\to\infty} \frac{K(B_{\ell_\infty^n})}{\sqrt{\frac{\log(n)}{n}}} = 1$.

Since $K(B_{\ell_\infty^n})$ bounds from below the radius $K(\mathscr{R})$ for any other Reinhardt domain $\mathscr{R}$, the range where $p \geq 2$ easily follows. The solution of the case $p < 2$, required quite different methods. A celebrated theorem proved independently by Pisier [Pis86] and Schüt [Sch78] allows to study unconditional bases in spaces of multilinear forms in terms of some invariants such as the local unconditional structure or the Gordon-Lewis property. These results have their counterpart in the context of spaces of polynomials as shown in [DDGM01], replacing the full tensor product by the symmetric one.

Defant and Frerick [DF11] (continuing their previous work given in [DF06]) established some sort of extension of Pisier-Schüt result to the symmetric tensor product with accurate bounds and gave a new estimate on the Gordon-Lewis constant of the symmetric tensor product. As a byproduct, they found the exact asymptotic growth for the Bohr radius on the unit ball of the spaces ℓ_p^n.

The aforementioned results give the following relation for the Bohr radius.

Theorem 1.1. [DFOC⁺11, DF11] *For* $1 \leq p \leq \infty$*, we have*

$$K(B_{\ell_p^n}) \sim \left(\frac{\log(n)}{n}\right)^{1-\frac{1}{\min\{p,2\}}}. \tag{4}$$

The proof of the exact asymptotic behavior of $K(B_{\ell_p^n})$ given in [DF11] for $p < 2$ as mentioned before use "sophisticated machinery" from the Banach space theory. Inspired by recent results from the general theory of Dirichlet series, in [BDS16] Bayart, Defant and Schlüters managed to give upper estimates for the unconditional basis constants of spaces of polynomials on ℓ_p spanned by finite sets of monomials, which avoid the use of this "machinery". This perspective gives a new and, in a sense, clear proof of Theorem 1.1 for the case $p < 2$.

The study of the Bohr radius together with the techniques developed for this purpose have been enriching many mathematical areas such us number theory [CDG⁺15, DFOC⁺11], complex analysis [BDF⁺17, BDS16, DMP09], operator algebras [PPS02, Dix95], random polynomials [Boa00] (together with several works influenced by this article such as [DGM04, Bay12, GMSP15]), the study of functions on the boolean cube [DMP17a, DMP17b] (which are fundamental in theoretical computer science, graph theory, social choice, etc.) and even in quantum information [DMP17b, Mon12].

Our aim is to continue the study of the Bohr phenomenon for mixed Reinhardt domains. Let $\mathscr{R}$ and $\mathscr{S}$ be two Reinhardt domain in $\mathbb{C}^n$. The mixed Bohr radius $K(\mathscr{R},\mathscr{S})$ is defined as the biggest number $r \geq 0$ such that for every analytic function $f(z) = \sum_\alpha a_\alpha z^\alpha$ bounded on $\mathscr{R}$, it holds:

$$\sup_{z\in r\cdot\mathscr{S}} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z\in\mathscr{R}} |f(z)|. \tag{5}$$

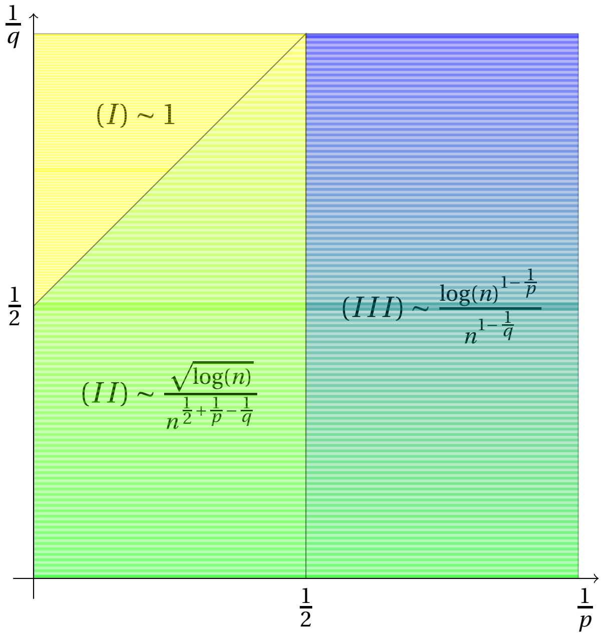

FIGURE 1. Graphical overview of the mixed Bohr radius described in Theorem 1.2.

We will focus in the case where $\mathscr{R}$ and $\mathscr{S}$ are the closed unit balls of ℓ_p and ℓ_q for $1 \leq p, q \leq \infty$. Note that $K(B_{\ell_p^n})$ in the previous notation is just $K(B_{\ell_p^n}, B_{\ell_p^n})$. Our contribution is the following theorem which provides the correct asymptotic estimates for the full range of p's and q's.

Theorem 1.2. *Let $1 \leq p, q \leq \infty$, with $q \neq 1$. The asymptotic growth of the (p,q)-Bohr radius is given by*

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \sim \begin{cases} 1 & \textit{if (I): } 2 \leq p \leq \infty \wedge \frac{1}{2}+\frac{1}{p} \leq \frac{1}{q}, \\ \frac{\sqrt{\log(n)}}{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}} & \textit{if (II): } 2 \leq p \leq \infty \wedge \frac{1}{2}+\frac{1}{p} \geq \frac{1}{q}, \\ \frac{\log(n)^{1-\frac{1}{p}}}{n^{1-\frac{1}{q}}} & \textit{if (III): } 1 \leq p, q \leq 2. \end{cases}$$

For $q = 1$ and every $1 \leq p \leq \infty$, $K(B_{\ell_p^n}, B_{\ell_q^n}) \sim 1$.

As for $K(B_{\ell_p^n})$, the upper bound are obtained using random polynomials with adequate coefficients and relatively small norm [Boa00, DGM04, Bay12]. To obtain the lower bounds the proof is divided in several cases. For $p < 2$ we have combined an appropriate way to divide and distinguish certain subsets of monomials together with the upper estimates for the unconditional basis constants of spaces of polynomials on ℓ_p spanned by finite sets of monomials given in [BDS16]. The interplay between monomial convergence and mixed unconditionality for spaces of homogeneous polynomials presented in [DMP09, Theorem 5.1.] (which, of course, gives information on the Bohr radius) is crucial for the case $p > 2$. We have strongly used some recent inclusion for the set of monomial convergence $\mathrm{dom} H_\infty(B_{\ell_p})$ $p \geq 2$ given in [DMP09, BDF$^+$17]. Therefore, it is worth noting that the techniques and results developed in the last years were fundamental for our proof.

The article is organized as follows. In Section 2 we present some basic background and results that we will use to prove Theorem 1.2. We also give in this section some of the notation and concepts that appeared in this introduction. Moreover, we include a heuristic argument as to why one should find, when studying the asymptotic behavior of the mixed Bohr radius, three differentiated regions in Theorem 1.2 (see Figure 1). In Sections 3 and 4 we show the upper and lower estimates for the theorem respectively. In Sections 3 and 4 we show the upper and lower estimates for the theorem respectively.

2. Preliminaries

We write by $\mathbb{D}$ the closed unit disk in the complex plane $\mathbb{C}$. As usual we denote ℓ_p^n for the Banach space of all n-tuples $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ endowed with the norm $\|(z_1, \ldots, z_n)\|_p = \left(\sum_{i=1}^n |z_i|^p\right)^{1/p}$ if $1 \leq p < \infty$, and $\|(z_1, \ldots, z_n)\|_\infty = \max_{i=1,\ldots,n} |z_i|$ for $p = \infty$. The unit ball of ℓ_p^n is denoted by $B_{\ell_p^n}$. For $1 \leq p \leq \infty$ we write p' for its conjugate exponent (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$).

For every $x, y \in \mathbb{C}^{\mathbb{N}}$ we denote $|x| = (|x_1|, \ldots, |x_n|, \ldots)$, and $|x| \leq |y|$ will mean that $|x_i| \leq |y_i|$ for every $i \in \mathbb{N}$. Recall that a Banach sequence space is a Banach space $(X, \|\cdot\|_X)$ with $\ell_1 \subset X \subset \ell_\infty$; and such that whenever $y \in X$, $x \in \mathbb{C}^{\mathbb{N}}$ and $|x| \leq |y|$ it follows $x \in X$ and $\|x\|_X \leq \|y\|_X$. A non-empty open set $\mathscr{R} \subset X$ is called a Reinhardt domain whenever given $x \in \mathbb{C}^{\mathbb{N}}$ and $y \in \mathscr{R}$ such that $|x| \leq |y|$ then it holds $x \in \mathscr{R}$.

Given a Banach sequence space X and fixed $n \in \mathbb{N}$ its n-th projection X_n is defined as the quotient space induced by the mapping

$$\begin{aligned} \pi_n : X &\to \mathbb{C}^n \\ x &\mapsto (x_1, \ldots, x_n). \end{aligned}$$

An m-homogeneous polynomial in n variables is a function $P : \mathbb{C}^n \to \mathbb{C}$ of the form

$$P(z_1, \ldots, z_n) = \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha,$$

where $\Lambda(m,n) := \{\alpha \in \mathbb{N}_0^n : |\alpha| := \alpha_1 + \cdots + \alpha_n = m\}$, $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $a_\alpha \in \mathbb{C}$. We will use the notation $a_\alpha =: a_\alpha(P)$.

Another way of writing a polynomial P is as follows:

$$P(z_1,\dots,z_n)=\sum_{\mathbf{j}\in\mathscr{J}(m,n)} c_{\mathbf{j}} z_{\mathbf{j}},$$

where $\mathscr{J}(m,n):=\{\mathbf{j}=(j_1,\dots,j_k):1\le j_1\le\ldots\le j_k\le n\}$, $z_{\mathbf{j}}:=z_{j_1}\cdots z_{j_k}$ and $c_{\mathbf{j}}\in\mathbb{C}$. Note that $c_{\mathbf{j}}=a_\alpha$ with $\mathbf{j}=(1,\overset{\alpha_1}{\dots},1,\dots,n,\overset{\alpha_n}{\dots},n)$. For some fixed $\mathbf{j}\in\mathscr{J}(m,n)$ and some $\mathbf{i}=(i_1,\dots,i_m)\in\mathbb{N}^m$ we say $\mathbf{i}\in[\mathbf{j}]$ if the exists some permutation $\sigma\in S_m$ such that $(i_{\sigma(1)},\dots,i_{\sigma(m)})=\mathbf{j}$ and $|\mathbf{j}|$ will denote the number of elements in $[\mathbf{j}]$. Observe that $|\mathbf{j}|=\frac{m!}{\alpha!}$ if $\mathbf{j}=(1,\overset{\alpha_1}{\dots},1,\dots,n,\overset{\alpha_n}{\dots},n)$.

The elements $(z^\alpha)_{\alpha\in\Lambda(m,n)}$ (equivalently, $(z_{\mathbf{j}})_{\mathbf{j}\in\mathscr{J}(m,n)}$) are commonly refereed as *the monomials*.

Given a subset $\mathscr{J}\subset\mathscr{J}(m,n)$, we call

$$\mathscr{J}^*=\{\mathbf{j}\in\mathscr{J}(m-1,n):\text{ there is } k\ge 1, (\mathbf{j},k)\in\mathscr{J}\}.$$

For $1\le p\le\infty$ we denote by $\mathscr{P}(^m\ell_p^n)$ the Banach space of all m-homogeneous polynomials in n complex variables equipped with the uniform (or sup) norm

$$\|P\|_{\mathscr{P}(^m\ell_p^n)}:=\sup_{z\in B_{\ell_p^n}}\left|P(z)\right|.$$

Given two Banach sequence spaces X and Y, for $n,m\in\mathbb{N}$ let $\chi_M(\mathscr{P}(^mX_n),\mathscr{P}(^mY_n))$ be the best constant $\lambda>0$ such that

$$\sup_{z\in B_{Y_n}}\left|\sum_{\alpha\in\Lambda(m,n)}\theta_\alpha a_\alpha z^\alpha\right|\le\lambda\sup_{z\in B_{X_n}}\left|\sum_{\alpha\in\Lambda(m,n)}a_\alpha z^\alpha\right|,$$

for every $(a_\alpha)_{\alpha\in\Lambda(m,n)}\subset\mathbb{C}$ and every choice of complex numbers $(\theta_\alpha)_{\alpha\in\Lambda(m,n)}$ of modulus one.

When $X=\ell_p$ and $Y=\ell_q$ we will denote $\chi_M(\mathscr{P}(^mX_n),\mathscr{P}(^mY_n))$, the (p,q)-mixed unconditionally constant for the monomial basis of $\mathscr{P}(^m\mathbb{C}^n)$, as $\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))$. It should be mentioned that, for any fixed $m\in\mathbb{N}$, the asymptotic growth of $\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))$ as $n\to\infty$ was studied in [GMM16].

Every entire function $f:\mathbb{C}^n\to\mathbb{C}$ can be written as

$$f=\sum_{m\ge0}\sum_{\alpha\in\Lambda(m,n)}a_\alpha(f)z^\alpha.$$

Recall that $K(B_{\ell_p^n},B_{\ell_q^n})$ stands for the n-dimensional (p,q)-Bohr radius. That is, $K(B_{\ell_p^n},B_{\ell_q^n})$ denotes the greatest constant $r>0$ such that for every entire function $f=\sum_\alpha a_\alpha z^\alpha$ in n-complex variables, we have the following (mixed) Bohr-type inequality

$$\sup_{z\in r\cdot B_{\ell_q^n}}\sum_\alpha|a_\alpha z^\alpha|\le\sup_{z\in B_{\ell_p^n}}|f(z)|.$$

In the same way, the m-homogeneous mixed Bohr radius, $K_m(B_{\ell_p^n}, B_{\ell_q^n})$, is defined as the greatest $r>0$ such that for every $P \in \mathscr{P}(^m\mathbb{C}^n)$ it follows

$$\sup_{z\in B_{\ell_q^n}} \sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| r^m = \sup_{z\in r\cdot B_{\ell_q^n}} \sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| \leq \|P\|_{\mathscr{P}(^m\ell_p^n)}$$

It is plain that $K(B_{\ell_p^n}, B_{\ell_q^n}) \leq K_m(B_{\ell_p^n}, B_{\ell_q^n})$.

Remark 2.1.

$$K_m(B_{\ell_p^n}, B_{\ell_q^n}) = \frac{1}{\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n))^{1/m}}.$$

Proof. Given $P \in \mathscr{P}(^m\mathbb{C}^n)$ and for any $(\theta_\alpha)_{\alpha\in\Lambda(m,n)}$ we have

$$\begin{aligned}\Big\|\sum_{\alpha\in\Lambda(m,n)} \theta_\alpha a_\alpha(P) z^\alpha\Big\|_{\mathscr{P}(^m\ell_q^n)} &\leq \Big\|\sum_{\alpha\in\Lambda(m,n)} |a_\alpha(P) z^\alpha|\Big\|_{\mathscr{P}(^m\ell_q^n)} \\ &= \Big\|\sum_{\alpha\in\Lambda(m,n)} |a_\alpha(P) z^\alpha| (K_m(B_{\ell_p^n}, B_{\ell_q^n}))^m\Big\|_{\mathscr{P}(^m\ell_q^n)} \frac{1}{(K_m(B_{\ell_p^n}, B_{\ell_q^n}))^m} \\ &\leq \frac{1}{(K_m(B_{\ell_p^n}, B_{\ell_q^n}))^m} \|P\|_{\mathscr{P}(^m\ell_p^n)},\end{aligned}$$

which leads to the inequality $\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n))^{1/m} \leq \frac{1}{K_m(B_{\ell_p^n}, B_{\ell_q^n})}$. On the other hand, for $P \in \mathscr{P}(^m\mathbb{C}^n)$ take $\theta_\alpha = \frac{\overline{a_\alpha(P)}}{|a_\alpha(P)|}$. Then we have

$$\begin{aligned}\Big\|\sum_{\alpha\in\Lambda(m,n)} |a_\alpha(P) z^\alpha|\Big\|_{\mathscr{P}(^m\ell_q^n)} &= \Big\|\sum_{\alpha\in\Lambda(m,n)} \theta_\alpha a_\alpha(P) z^\alpha\Big\|_{\mathscr{P}(^m\ell_q^n)} \\ &\leq \chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n)) \|P\|_{\mathscr{P}(^m\ell_p^n)},\end{aligned}$$

or equivalently,

$$\sup_{z\in B_{\ell_q^n}} \sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| \left(\frac{1}{\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n))^{1/m}}\right)^m \leq \|P\|_{\mathscr{P}(^m\mathbb{C}^n)},$$

which means $\frac{1}{\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n))^{1/m}} \leq K_m(B_{\ell_p^n}, B_{\ell_q^n})$. □

It will be useful to remember a classic result due to F. Wiener (see [KB97]) which asserts that for every holomorphic function f written as the sum of m-homogeneous polynomials as $f = \sum_{m\geq 1} P_m + a_0$ and such that $\sup_{z\in B_{\ell_p^n}} |f(z)| \leq 1$ it holds

$$\|P_m\|_{\mathscr{P}(^m\ell_p^n)} \leq 1 - |a_0|^2, \tag{6}$$

for every $m \in \mathbb{N}$.

In general this inequality is presented for the uniform norm on the polydisk $\|\cdot\|_{\mathscr{P}(^m\ell_\infty^n)}$ (i.e., $p=\infty$), but this version easily follows by a standard modification of the original argument (given $z \in B_{\ell_p^n}$ consider the auxiliary function $g: \mathbb{C}^n \to \mathbb{C}$ given by $g(w) := f(w\cdot z)$).

The next lemma is an adaption of the case $p=q$, see [DGM03, Theorem 2.2.] and constitutes the basic link between Bohr radius and unconditional basis constants of spaces of polynomials on the mixed context (p not necessarily equal to q).

Lemma 2.2. *For every $n\in\mathbb{N}$ and $1\le p,q\le\infty$ it holds*

$$\frac{1}{3}\frac{1}{\sup_{m\ge1}\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))^{1/m}}\le K(B_{\ell_p^n},B_{\ell_q^n})\le\min\left\{\frac{1}{3},\frac{1}{\sup_{m\ge1}\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))^{1/m}}\right\}.$$

Proof. Form Remark 2.1 we have $K_{p,q(n)}\le\inf_{m\ge1}\frac{1}{\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))^{1/m}}=\frac{1}{\sup_{m\ge1}\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))^{1/m}}$ and due to Bohr's inequality we know $K(\mathbb{D})=\frac{1}{3}$ as it is clear that $K(B_{\ell_p^n},B_{\ell_q^n})\le K(\mathbb{D})$ for every $n\in\mathbb{N}$ the right hand side inequality holds. For the left hand side inequality let us take some holomorphic function f, without loss of generality let us assume $\sup_{z\in B_{\ell_p^n}}|f(z)|\le1$, and consider its decomposition as a sum of m-homogeneous polynomials $f=\sum\limits_{m\ge0}P_m$. For every $m\in\mathbb{N}_0$ it holds $P_m(z)=\sum\limits_{\alpha\in\Lambda(m,n)}a_\alpha(f)z^\alpha$, thus taking $\rho=\sup_{m\ge1}\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))^{1/m}$ and using Remark 2.1 again it follows

$$\Big\|\sum_{\alpha\in\Lambda(m,n)}|a_\alpha(f)|\left(\frac{z}{\rho}\right)^\alpha\Big\|_{\mathscr{P}(^m\ell_q^n)}\le\Big\|\sum_{\alpha\in\Lambda(m,n)}a_\alpha(f)z^\alpha\Big\|_{\mathscr{P}(^m\ell_p^n)}.$$

Applying the above mentioned Wiener's result for some $w\in B_{\ell_q^n}$ we have that

$$\begin{aligned}\sum_{m\ge0}\sum_{\alpha\in\Lambda(m,n)}|a_\alpha(f)|\left(\frac{w}{3\rho}\right)^\alpha&\le|a_0(f)|+\sum_{m\ge1}\frac{1}{3^m}\Big\|\sum_{\alpha\in\Lambda(m,n)}a_\alpha(f)z^\alpha\Big\|_{\mathscr{P}(^m\ell_p^n)}\\&\le|a_0(f)|+\sum_{m\ge1}\frac{1}{3^m}(1-|a_0(f)|^2)\\&\le|a_0(f)|+\frac{1-|a_0(f)|^2}{2}\le1,\end{aligned}$$

where last inequality holds as $|a_0(f)|\le\sup_{z\in B_{\ell_p^n}}|f(z)|\le1$. The last chain of inequalities and the maximality of mixed Bohr radius lead us to $\frac{1}{3\rho}\le K(B_{\ell_p^n},B_{\ell_q^n})$ as we wanted to prove. □

The previous lemma shows that understanding $K(B_{\ell_p^n},B_{\ell_q^n})$ translates into seeing how the constant $\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))^{1/m}$ behaves. It should be mentioned that, for any fixed $m\in\mathbb{N}$, the asymptotic growth of $\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))$ as $n\to\infty$ was studied in [GMM16]. These results unfortunately are not useful because, as can be seen in Lemma 2.2, one needs to comprehend how $\chi_M(\mathscr{P}(^m\ell_p^n),\mathscr{P}(^m\ell_q^n))^{1/m}$ grows by moving both the number of variables, n, and the degree of homogeneity, m. But beyond this, they give a guideline of what to expect (at least what the different regions in Figure 1 should look like). Indeed, in [GMM16] we have proved the following

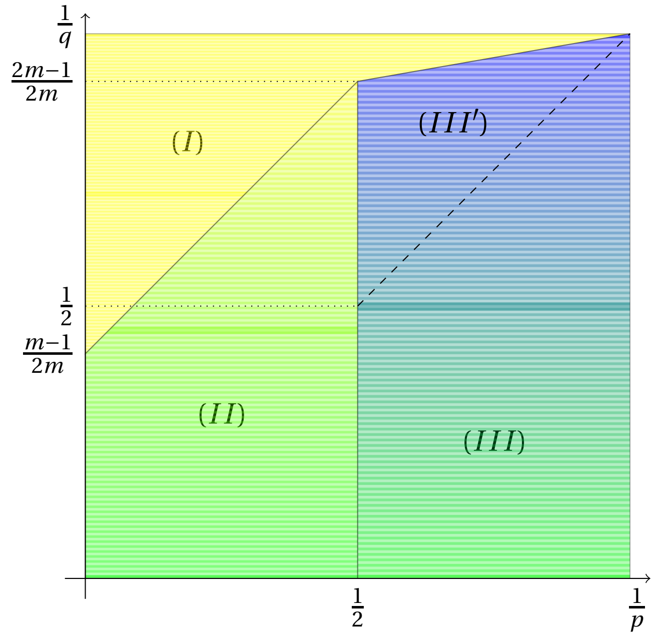

FIGURE 2. Graphical overview of the mixed unconditional constant described in Theorem 2.3.

Theorem 2.3.

$$\begin{cases} \chi_{p,q}(\mathscr{P}(^m\mathbb{C}^n)) \sim 1 & \textit{for } (I): [\frac{1}{p} + \frac{m-1}{2m} \leq \frac{1}{q} \wedge \frac{1}{p} \leq \frac{1}{2}] \textit{ or } [\frac{m-1}{m} + \frac{1}{mp} < \frac{1}{q} \wedge \frac{1}{2} \leq \frac{1}{p}], \\ \chi_{p,q}(\mathscr{P}(^m\mathbb{C}^n)) \sim n^{m(\frac{1}{2}+\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} & \textit{for } (II)\ [\frac{1}{p} + \frac{m-1}{2m} \geq \frac{1}{q} \wedge \frac{1}{p} \leq \frac{1}{2}], \\ \chi_{p,q}(\mathscr{P}(^m\mathbb{C}^n)) \sim n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}} & \textit{for } (III): [\frac{1}{p} \geq \frac{1}{q} \wedge \frac{1}{2} \leq \frac{1}{p}], \\ \chi_{p,q}(\mathscr{P}(^m\mathbb{C}^n)) \sim_\varepsilon n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}} & \textit{for } (III'): [1 - \frac{1}{m} + \frac{1}{mp} \geq \frac{1}{q} \geq \frac{1}{p} \wedge \frac{1}{2} < \frac{1}{p} < 1]. \end{cases}$$

where $\chi_{p,q}(\mathscr{P}(^m\mathbb{C}^n)) \sim_\varepsilon n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}}$ *means that*

$$n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}} \ll \chi_{p,q}(\mathscr{P}(^m\mathbb{C}^n)) \ll n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}+\varepsilon},$$

for every $\varepsilon > 0$.

The heuristic to interpret the different regions in Theorem 1.2 is the following: If one thinks that the correct order of the region (III') coincides with that of (III) in Theorem 2.3 (in fact, this is what we believe) and assumes that the homogeneity degree is very very large ($m \to \infty$) then the graph in Figure 2 transforms into the one presented in Figure 1. All this, together with the upper bounds that one gets after using classical random polynomials (see Section 3, somehow the easy part) helped us to define where to aim to prove lower bounds. We highlight that, the logarithmic factors that appear in Theorem 1.2, are missing in Theorem 2.3. This is, somehow, not coincidental and their presence is due to the interplay between the number of variables and the degree of homogeneity in Lemma 2.2.

We continue with some definitions that will be useful later. Given X a Banach sequence space we denote $H_\infty(B_X)$ to the space of holomorphic functions over B_X endowed with the norm given by $\|f\|_{H_\infty(B_X)} := \sup_{z\in B_X} |f(z)|$. The domain of monomial convergence of $H_\infty(B_X)$ is defined as

$$\operatorname{dom}(H_\infty(B_X)) := \Big\{z \in \ell_\infty : \sum_{m\geq 0} \sum_{\alpha\in\Lambda(m,n)} |a_\alpha(f) z^\alpha| < \infty \text{ for every } f \in H_\infty(B_X)\Big\}.$$

The next theorem appears in [DMP09, Theorem 5.1] and relates monomial convergence with the study of the mixed unconditional constant of the monomial basis.

Theorem 2.4. *For a couple of Banach sequence spaces X, Y the following are equivalent*

(1) $rB_Y \subset domH_\infty(B_X)$ *for some* $r > 0$.
(2) *There exists a constant* $C > 0$ *independent of* m *such that*

$$\sup_{n\geq 1} \chi_M(\mathscr{P}(^m X_n), \mathscr{P}(^m Y_n)) \leq C^m.$$

If $(a_n)_n$ and $(b_n)_n$ are two sequences of real numbers we will write $a_n \ll b_n$ if there exists a constant $C > 0$ (independent of n) such that $a_n \leq C b_n$ for every n. We will write $a_n \sim b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$. We will use repeatedly the Stirling formula which asserts

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{7}$$

We use the letters C, C_1, C_2, etc. to denote absolute positive constants (which from one inequality to the other may vary and sometimes denoted in the same way).

3. Upper bounds

Upper bounds constitute the easy part: we will use the classical probabilistic approach. Bayart in [Bay12, Corollary 3.2] (see also [Boa00, DGM03, DGM04]) exhibited polynomials with small sup-norm on the unit ball of ℓ_p^n. He showed that for each $1 \leq p \leq \infty$ there exists an m-homogeneous polynomial in n complex variables, $P(z) := \sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha$, with $\varepsilon_\alpha = \pm 1$ for every α, such that $\|P\|_{\mathscr{P}(^m\ell_p^n)} \leq D_p(m,n)$ where

$$D_p(m,n) := C_p \times \begin{cases} (\log(m)m!)^{1-\frac{1}{p}}\, n^{1-\frac{1}{p}} & \text{if } 1 \leq p \leq 2, \\ (\log(m)m!)^{\frac{1}{2}}\, n^{m(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}} & \text{if } 2 \leq p \leq \infty, \end{cases} \tag{8}$$

and C_p depends exclusively on p.

We will also need the following remark which is an easy calculus exercise.

Remark 3.1. For every positive numbers $a, b > 0$ and $n \in \mathbb{N}$, the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by $f(x) = x^a n^{\frac{b}{x}}$ attains its minimum at $x = \log(n)\frac{b}{a}$.

Proof of the upper bounds of Theorem 1.2. Upper bounds for the case $\frac{1}{p} + \frac{1}{2} \le \frac{1}{q}$ in Theorem 1.2 are trivial.

Suppose $\frac{1}{p} + \frac{1}{2} \ge \frac{1}{q}$ and let $(\varepsilon_\alpha)_{\alpha \in \Lambda(m,n)} \subset \{-1, 1\}$ signs such that

$$\Big\| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big\|_{B_{\ell_p^n}} \le D_p(m,n),$$

as (8). Taking $z_0 = (\frac{1}{n^{1/q}}, \dots, \frac{1}{n^{1/q}}) \in B_{\ell_q}$ we can conclude that

$$\begin{aligned} n^{m(1-\frac{1}{q})} &= \sum_{\alpha \in \Lambda(m,n)} \frac{m!}{\alpha!} \left(\frac{1}{n^{\frac{1}{q}}} \right)^m \\ &\le \Big\| \sum_{\alpha \in \Lambda(m,n)} |\varepsilon_\alpha| \frac{m!}{\alpha!} z^\alpha \Big\|_{B_{\ell_q^n}} \\ &\le \chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n)) \Big\| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big\|_{B_{\ell_p^n}} \\ &\le \chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n)) \cdot D_p(m,n). \end{aligned}$$

For $1 \le p \le 2$ we have by Stirling formula (7),

$$\begin{aligned} \frac{1}{\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n))^{1/m}} &\le \left(C_p n^{\frac{1}{p'}} (\log(m) m!)^{\frac{1}{p'}} \right)^{1/m} \frac{1}{n^{\frac{1}{q'}}} \\ &\le C \frac{1}{n^{\frac{1}{q'}}} m^{\frac{1}{p'}} n^{\frac{1}{p'm}}, \end{aligned}$$

where $C > 0$ depends only on p. Thanks to Lemma 2.1, Remark 3.1 and the previous inequality

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \le C \frac{1}{n^{\frac{1}{q'}}} \inf_{m \ge 1} m^{\frac{1}{p'}} n^{\frac{1}{2m}} \le C(p,q) \frac{\log(n)^{\frac{1}{p'}}}{n^{\frac{1}{q'}}}.$$

On the other hand, for $p \ge 2$ and $\frac{1}{q} \le \frac{1}{p} + \frac{1}{2}$ it follows

$$\begin{aligned} \frac{1}{\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n)))^{1/m}} &\le \left(C_p n^{\frac{1}{2}} (\log(m) m!)^{\frac{1}{2}} \right)^{1/m} \frac{1}{n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}} \\ &\le C \frac{1}{n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}} m^{\frac{1}{2}} n^{\frac{1}{2m}}. \end{aligned}$$

Thus minimizing $m^{\frac{1}{2}} n^{\frac{1}{2m}}$ as in the previous case we get,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \le C \frac{\sqrt{\log(n)}}{n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}},$$

as we wanted to prove. □

4. Lower bounds

For the proof of the lower bounds we need to consider four different cases. We begin with the case $q=1$ and the case $p \leq q$, which are the easy ones. Then we study the case $1<q\leq p\leq 2$ where we use tools from unconditionality and finally the case $p\geq 2$ where the key tool is monomial convergence.

4.1. **The case $q=1$.** By [Aiz00],

$$K(B_{\ell_1^n}) \sim 1.$$

Thus, for any $f(z)=\sum_\alpha a_\alpha z^\alpha$, it follows that

$$\sup_{z\in K(B_{\ell_1^n})\cdot B_{\ell_1^n}} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z\in B_{\ell_1^n}} |f(z)| \leq \sup_{z\in B_{\ell_p^n}} |f(z)|,$$

which implies that $K(B_{\ell_p^n}, B_{\ell_1^n}) \geq K(B_{\ell_1^n}) \sim 1$.

4.2. **The case $p\leq q$.** For this case we will strongly use Theorem 1.1. The case $p\leq q$ is an easy corollary of this result.

Proof for the lower bound of Theorem 1.2: the case $p\leq q$. Taking $m\in\mathbb{N}$, for any $P(z)=\sum\limits_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha$, it follows that

$$\begin{aligned}\Big\| \sum_{\alpha\in\Lambda(m,n)} |a_\alpha| z^\alpha \Big\|_{\mathscr{P}(^m\ell_q^n)} &\leq n^{m(\frac{1}{p}-\frac{1}{q})} \Big\| \sum_{\alpha\in\Lambda(m,n)} |a_\alpha| z^\alpha \Big\|_{\mathscr{P}(^m\ell_p^n)} \\ &\leq n^{m(\frac{1}{p}-\frac{1}{q})} K(B_{\ell_p^n})^{-m} \Big\| \sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha \Big\|_{\mathscr{P}(^m\ell_p^n)},\end{aligned}$$

which implies that $K_m(B_{\ell_p^n}, B_{\ell_q^n}) \geq K(B_{\ell_p^n}) n^{\frac{1}{q}-\frac{1}{p}}$ for every $m\in\mathbb{N}$. Using Lemma 2.2 and Theorem 1.1 we have, for $p\leq 2$,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \geq \frac{1}{3} n^{\frac{1}{q}-\frac{1}{p}} K(B_{\ell_p^n}) \sim n^{\frac{1}{q}-\frac{1}{p}} \left(\frac{\log(n)}{n}\right)^{1-\frac{1}{p}} = \frac{\log(n)^{1-1/p}}{n^{1-1/q}},$$

and, for $p\geq 2$,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \geq \frac{1}{3} n^{\frac{1}{q}-\frac{1}{p}} K(B_{\ell_p^n}) \sim n^{\frac{1}{q}-\frac{1}{p}} \left(\frac{\log(n)}{n}\right)^{1-\frac{1}{2}} = \frac{\sqrt{\log(n)}}{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}},$$

which concludes the proof. □

4.3. **The case** $1 < q \le p \le 2$**.**

Lemma 4.1. [BDS16, Lemma 3.5.] *Let* $1 \le p \le \infty$ *and* P *be an* m*-homogeneous polynomial in* $\mathscr{P}(^m\ell_p^n)$*. Then for any* $\mathbf{j} \in \mathscr{J}(m-1,n)$

$$\left(\sum_{k=j_{m-1}}^{n} |c_{(\mathbf{j},k)}(P)|^{p'}\right)^{1/p'} \le me^{1+\frac{m-1}{r}} |\mathbf{j}|^{1/p} \|P\|_{\mathscr{P}(^m\ell_p^n)}.$$

The next lemma is an adaptation of the case $r \le 2$ in [BDS16, Theorem 3.2] to the mixed context for our purposes.

Lemma 4.2. *Let* $1 \le q \le p \le 2$*. Then we have*

$$\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n)) \le me^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathscr{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'}.$$

Proof. Fix $P \in \mathscr{P}(^m\ell_p^n)$ and $u \in \ell_q^n$. Then, by Lemma 4.1, for any $\mathbf{j} \in \mathscr{J}(m,n)^*$,

$$\left(\sum_{k:\ (\mathbf{j},k)\in\mathscr{J}(m,n)} |c_{(\mathbf{j},k)}(P)|^{p'}\right)^{1/p'} \le \left(\sum_{k=j_{m-1}}^{n} |c_{(\mathbf{j},k)}(P)|^{p'}\right)^{1/p'} \le me^{1+\frac{m-1}{q}} |\mathbf{j}|^{1/p} \|P\|_{\mathscr{P}(^m\ell_p^n)}.$$

Now applying the above inequality, Hölder's inequality (two times) and the multinomial formula we have

$$\begin{aligned}
\sum_{\mathbf{j}\in\mathscr{J}(m,n)} |c_{\mathbf{j}}(P)||u_{\mathbf{j}}| &= \sum_{\mathbf{j}\in\mathscr{J}(m,n)^*} \left(\sum_{k:\ (\mathbf{j},k)\in\mathscr{J}(m,n)} |c_{(\mathbf{j},k)}||u_{\mathbf{j}}||u_k|\right) \\
&\le \sum_{\mathbf{j}\in\mathscr{J}(m,n)^*} |u_{\mathbf{j}}| \left(\sum_{k:\ (\mathbf{j},k)\in\mathscr{J}(m,n)} |c_{(\mathbf{j},k)}|^{q'}\right)^{1/q'} \left(\sum_k |u_k|^q\right)^{1/q} \\
&\le \sum_{\mathbf{j}\in\mathscr{J}(m,n)^*} |u_{\mathbf{j}}| \left(\sum_{k:\ (\mathbf{j},k)\in\mathscr{J}(m,n)} |c_{(\mathbf{j},k)}|^{p'}\right)^{1/p'} \|u\|_q \\
&\le me^{1+\frac{m-1}{p}} \sum_{\mathbf{j}\in\mathscr{J}(m,n)^*} |\mathbf{j}|^{1/p} |u_{\mathbf{j}}| \|u\|_q \|P\|_{\mathscr{P}(^m\ell_p^n)} \\
&\le me^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathscr{J}(m,n)^*} |\mathbf{j}||u_{\mathbf{j}}|^q\right)^{1/q} \left(\sum_{\mathbf{j}\in\mathscr{J}(m,n)^*} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} \|u\|_q \|P\|_{\mathscr{P}(^m\ell_p^n)} \\
&\le me^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathscr{J}(m-1,n)} |\mathbf{j}||u_{\mathbf{j}}|^q\right)^{1/q} \left(\sum_{\mathbf{j}\in\mathscr{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} \|u\|_q \|P\|_{\mathscr{P}(^m\ell_p^n)} \\
&= me^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathscr{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} \|u\|_q^m \|P\|_{\mathscr{P}(^m\ell_p^n)},
\end{aligned}$$

which gives the desired inequality. $\square$

The key to prove the lower bound is to obtain good bounds for the sum on the right hand side of the previous lemma. This will require some hard work.

We define for any $1 \le k \le m$ the k-bounded index set as

$$\Lambda_k(m,n) = \{\alpha \in \Lambda(m,n) : \alpha_i \le k \text{ for all } 1 \le i \le n\}.$$

Let F be the bijective mapping connecting $\Lambda(m,n)$ and $\mathscr{J}(m,n)$ defined as

$$\begin{aligned} F : \mathscr{J}(m,n) &\to \Lambda(m,n) \\ \mathbf{j} &\mapsto \alpha \end{aligned}$$

where $\alpha_i = F(\mathbf{j})_i = \#\{k : \mathbf{j}_k = i\}$ for every $1 \le i \le n$. We denote

$$\mathscr{J}_k(m,n) = F^{-1}(\Lambda_k(m,n)),$$

for the corresponding k-bounded subsets of $\mathscr{J}(m,n)$. Observe that for any $1 \le k \le m$ and $\mathbf{j} \in \mathscr{J}_k(m,n)$ the following hold:

$$|\mathbf{j}| \ge \frac{m!}{k!^{\lceil \frac{m}{k} \rceil}} \ge \frac{m!}{k!^{\frac{m}{k}+1}}, \tag{9}$$

(10)
$$\left(\sum_{\mathbf{j} \in \mathscr{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \le 2^{1/q'} \max \left\{ \left(\sum_{\mathbf{j} \in \mathscr{J}_k(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'}, \left(\sum_{\mathbf{j} \in \mathscr{J}_k^c(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \right\},$$

and finally,

$$\begin{aligned} |\mathscr{J}_k^c(m-1,n)| &\le n |\mathscr{J}(m-k-2,n)| \\ &\le n \binom{n+m-k-3}{m-k-2} \le n \frac{(n+m-k-3)^{m-k-2}}{(m-k-2)!}, \end{aligned} \tag{11}$$

since $\mathbf{j} \in \mathscr{J}_k^c(m-1,n)$ requires that at least one of the variables is at the power of $k+1$. For the particular case $m \le n$ we can extract from inequality (11) the fact that

$$|\mathscr{J}_k^c(m-1,n)| \le 2^m \frac{n^{m-k-1}}{(m-k-2)!}. \tag{12}$$

Note also that,

$$\left(\sum_{\mathbf{j} \in \mathscr{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \le m^{1/q'} \max_{k=1,\dots,m-1} \left\{ \left(\sum_{\mathbf{j} \in \mathscr{J}_k(m-1,n) \cap \mathscr{J}_{k-1}^c(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \right\}, \tag{13}$$

Lemma 4.3. *For $1 < q \le p \le 2$ and for $m, n \in \mathbb{N}$ fulfilling $m \ge \log(n)^{\frac{q'}{p'}}$, it follows*

$$\left(\sum_{\mathbf{j} \in \mathscr{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \le C^m \frac{n^{m/q'}}{\log(n)^{m/p'}}.$$

Proof. For $m \geq \log(n)^{\frac{q'}{p'}}$ we just bound $|\mathbf{j}|^{(1/p-1/q)q'}$ by 1, thus we have by Stirling formula,

$$\begin{aligned}
\left(\sum_{\mathbf{j}\in\mathscr{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} &\leq |\mathscr{J}(m-1,n)|^{1/q'} \\
&= \left(\frac{(n+m-2)!}{(m-1)!(n-1)!}\right)^{1/q'} \\
&\leq \left(c_1^{m-1}\left(1+\frac{n}{m-1}\right)^{m-1}\right)^{1/q'} \\
&\leq C_1^{\frac{m-1}{q'}}\left(1+\frac{n}{\log(n)^{\frac{q'}{p'}}}\right)^{\frac{m-1}{q'}} \\
&\leq C_2^m \frac{n^{\frac{m-1}{q'}}}{\log(n)^{\frac{m-1}{p'}}} \\
&\leq C^m \frac{n^{\frac{m}{q'}}}{\log(n)^{\frac{m}{p'}}}.
\end{aligned}$$

□

Lemma 4.4. *For* $1 < q \leq p \leq 2$ *and for* $m, n \in \mathbb{N}$ *fulfilling* $m \leq \frac{\log(n)}{\log\log(n)\beta}$ *with* $\beta = q'\left(\frac{1}{q}-\frac{1}{p}\right)$ *it follows*

$$\left(\sum_{\mathbf{j}\in\mathscr{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} \leq C^m \frac{n^{m/q'}}{\log(n)^{m/p'}}.$$

Proof. Now let $m \leq \frac{\log(n)}{\log\log(n)\beta}$, we will use inequality (10) for $k=1$. First, being $k=1$, we have

$$\begin{aligned}
\left(\sum_{\mathbf{j}\in\mathscr{J}_1(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} &= \frac{1}{m!^{1/q-1/p}}|\mathscr{J}_1(m-1,n)|^{\frac{1}{q'}} \\
&\leq C^m \frac{n^{(m-1)/q'}}{m^{m/p'}}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\left(\sum_{\mathbf{j}\in\mathscr{J}_1^c(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} &\leq |\mathscr{J}_1^c(m-1,n)|^{1/q'} \\
&\leq C^m \left(\frac{n^{m-2}}{m-2!}\right)^{1/q'} \quad \text{using inequality (12)} \\
&\leq C^m \left(\frac{n^{m-2}}{m!}\right)^{1/q'}.
\end{aligned}$$

Now its enough to prove the bound

$$\left(\frac{n^{m-2}}{m!}\right)^{1/q'} \leq C^m \frac{n^{(m-1)/q'}}{m^{m/p'}},$$

which is equivalent to

$$m^{(1/q-1/p)q'} \leq C n^{1/m},$$

as $m \leq \frac{\log(n)}{\log\log(n)\beta}$ we have, for some $C > 0$

$$\begin{aligned} m^{(1/q-1/p)q'} &\leq \left(\frac{\log(n)}{\log\log(n)\beta}\right)^{\beta} \leq C\log(n)^{\beta} = Ce^{\log\log(n)\beta} \\ &= Ce^{\frac{\log(n)\log\log(n)\beta}{\log(n)}} = Cn^{\frac{\log\log(n)\beta}{\log(n)}} \leq Cn^{1/m}. \end{aligned}$$

Therefore we have, for some $C > 0$,

$$\left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} \leq C^m \frac{n^{(m-1)/q'}}{m^{m/p'}}.$$

To finish the proof, note that using Remark 3.1,

$$\frac{n^{(m-1)/q'}}{m^{m/p'}} = \left[\frac{n^{1/q'}}{m^{1/p'} n^{m/q'}}\right]^m \leq C^m \frac{n^{m/q'}}{\log(n)^{m/p'}}.$$

□

Lemma 4.5. *For* $1 < q \leq p \leq 2$ *and for* $m, n \in \mathbb{N}$ *fulfilling* $\log(n)^{\frac{1}{c}} \leq m \leq \log(n)^c$, *for some* $c > 1$. *Then there exists* $C > 0$ *such that,*

$$\left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} \leq C^m \frac{n^{m/q'}}{\log(n)^{m/p'}}.$$

Proof. By (9) and Stirling formula, we have, for each $1 \leq k \leq m-1$,

$$\begin{aligned} \left(\sum_{\mathbf{j}\in\mathcal{J}_k(m-1,n)\cap\mathcal{J}^c_{k-1}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} &\leq C^m |\mathcal{J}^c_{k-1}(m-1,n)|^{1/q'} \frac{k^{(m+k)(\frac{1}{q}-\frac{1}{p})}}{m^{m(\frac{1}{q}-\frac{1}{p})}} \\ &\leq C^m \left(\frac{n^{m-k}}{(m-k-1)^{m-k-1}}\right)^{\frac{1}{q'}} \frac{k^{(m+k)(\frac{1}{q}-\frac{1}{p})}}{m^{m(\frac{1}{q}-\frac{1}{p})}}. \end{aligned}$$

Thus, by (13), we will prove the lemma if we are able to show that this last expression is $\leq C^m \frac{n^{m/q'}}{\log(n)^{m/p'}}$, for some constant $C > 0$. Therefore, it suffices to prove that, if $\beta := (\frac{1}{q} - \frac{1}{p})q'$,

$$\frac{k^{\beta(m+k)}}{(m-k-1)^{m-k-1} m^{\beta m}} \leq C^m \frac{n^k}{\log(n)^{m(\beta+1)}}. \tag{14}$$

Let us first suppose that $k \geq \min\{m/2, \frac{m}{3\beta}\} = dm$ for some $0 < d < 1$. Then, bounding k by $m-1$, the left hand side is less than or equal to $m^{\beta m}$, which is $\leq \frac{n^{dm}}{\log(n)^{m(\beta+1)}}$ for big enough n. Note that, for $k \leq dm$, (14) is equivalent to

$$\frac{k^{\beta(m+k)}}{m^{m-k}m^{\beta m}} \leq C^m \frac{n^k}{\log(n)^{m(\beta+1)}}, \tag{15}$$

for some constant C. Thus, for $1 < k \leq dm$ (for $k = 1$ (15) is trivially satisfied), since $k^{\beta k} m^k / k^m \leq m^k k^{m/3} / k^m = m^k / k^{2m/3} \ll 1$, it is enough to show that

$$\frac{k^{(\beta+1)m}}{m^{(\beta+1)m}} \leq C^m \frac{n^k}{\log(n)^{m(\beta+1)}},$$

or,

$$(\beta+1)m\log(\frac{k\log(n)}{m}) - k\log(n) \leq Cm. \tag{16}$$

Note that this inequality holds trivially if $k\log(n) \leq m$ or if $k\log(n) \geq (\beta+1)m\log(\frac{k\log(n)}{m})$. Suppose then that $1 < \frac{k\log(n)}{m} < (\beta+1)\log(\frac{k\log(n)}{m}) = f(\frac{k\log(n)}{m})$, where f is the logarithm in base $e^{\frac{1}{\beta+1}}$. Thus by (16), it suffices to see that

$$m \cdot f^{\circ 2}(\frac{k\log(n)}{m}) - k\log(n) \leq Cm, \tag{17}$$

where, $f^{\circ j}$ denotes the function f composed with itself j times. Again, (17) is true if $k\log(n) \geq m f^{\circ 2}(\frac{k\log(n)}{m})$, and if this does not hold, then replacing in (16), it suffices to see that

$$m f^{\circ 3}(\frac{k\log(n)}{m}) - k\log(n) \leq Cm. \tag{18}$$

We can continue this process, and it is enough to prove that, for some j,

$$m f^{\circ j}(\frac{k\log(n)}{m}) - k\log(n) \leq Cm. \tag{19}$$

Finally, note that for some $t_0 = t_0(\beta)$, which may suppose is bigger than 2, $f(t) \leq t^{1/2}$ for every $t \geq t_0$. Let $t = \frac{k\log(n)}{m}$. Then if $\min\{t, f(t), \ldots, f^{\circ i}(t)\} \geq t_0$, we have

$$f^{\circ(i+1)}(t) \leq (f^{\circ i}(t))^{1/2} \leq (f^{\circ(i-1)}(t))^{1/4} \leq \cdots \leq t^{1/2^{i+1}}.$$

Therefore, for some j, we will have that $f^{\circ j}(t) < t_0$ and (19) is fulfilled taking $C = t_0$. □

Proof of the lower bound of the case $1 < q \leq p \leq 2$ *on Theorem 1.2.* Thanks to Lemma 2.2 it is enough to prove that

$$\frac{\log(n)^{1-1/p}}{n^{1-1/q}} \ll \frac{1}{\sup_{m\geq 1} \chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n))^{1/m}} = \inf_{m\geq 1} \frac{1}{\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n))^{1/m}}, \tag{20}$$

which follows by Lemma's 4.2, 4.3, 4.4 and 4.5. □

4.4. **The case $p \geq 2$.** For the remaining cases it will be crucial the monomial convergence point of view from [BDF+17, DMP09].

As a consequence of [DMP09, Example 4.9 (2)] we have that whenever $\frac{1}{r} = \frac{1}{p} + \frac{1}{2}$ and $p \geq 2$ it follows

$$\ell_r \cap B_{\ell_p} \subset \operatorname{dom} H_\infty(B_{\ell_p}).$$

Since $r \leq p$, then $B_{\ell_r} \subset B_{\ell_p}$ and then $B_{\ell_r} \subset \ell_r \cap B_{\ell_p} \subset \operatorname{dom} H_\infty(B_{\ell_p})$. Finally by Lemma 2.4 we have that there is some constant $C = C(p,r) > 0$ such that for every $n \in \mathbb{N}$ and p, r fulfilling the previous conditions it holds

$$\frac{1}{\sup_{m\geq 1}\left(\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_r^n))\right)^{1/m}} \geq C.$$

As $K(B_{\ell_p^n}, B_{\ell_r^n}) \leq 1$ for every $1 \leq p, r \leq \infty$, the previous inequality and Theorem 2.2 lead us to the assertion that for $\frac{1}{r} = \frac{1}{p} + \frac{1}{2}$

$$K(B_{\ell_p^n}, B_{\ell_r^n}) \sim 1. \tag{21}$$

Proof of the case $\frac{1}{2} + \frac{1}{p} \leq \frac{1}{q}$ on Theorem 1.2. Let p, q be such that $\frac{1}{r} := \frac{1}{2} + \frac{1}{p} \leq \frac{1}{q}$, then for any $f(z) = \sum_{m\geq 0} \sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha$, it follows that

$$\begin{aligned}\sup_{z\in K(B_{\ell_p^n}, B_{\ell_r^n})B_{\ell_q^n}} \sum_{m\geq 0} \sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| &\leq \sup_{z\in K(B_{\ell_p^n}, B_{\ell_r^n})B_{\ell_r^n}} \sum_{m\geq 0} \sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| \\ &\leq \|f\|_{H_\infty(B_{\ell_p^n})},\end{aligned}$$

which implies that $K(B_{\ell_p^n}, B_{\ell_q^n}) \geq K(B_{\ell_p^n}, B_{\ell_r^n})$. Therefore by equation (21) we have

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \sim 1.$$

□

For every $z \in \ell_\infty$ we can define $z^* \in \ell_\infty$ the decreasing rearrangement such that $z_n^* \geq z_{n+1}^*$ for every $n \in \mathbb{N}$. In [BDF+17, Theorem 2.2] the authors proved that

$$B_\infty := \left\{ z \in \ell_\infty : \limsup_{n\to\infty} \frac{1}{\sqrt{\log(n)}} \left(\sum_{j=1}^n |z_j^*|^2\right)^{1/2} < 1 \right\} \subset \operatorname{dom} H_\infty(B_{\ell_\infty}).$$

Consider now the Banach sequence space

$$X_\infty = \left\{ z \in \ell_\infty : \sup_{n\geq 2} \frac{1}{\sqrt{\log(n)}} \left(\sum_{j=1}^n |z_j^*|^2\right)^{1/2} < \infty \right\},$$

endowed with the norm $\|z\|_{X_\infty} = \sup_{n\geq 2} \frac{1}{\sqrt{\log(n)}} \left(\sum_{j=1}^{n} |z_j^*|^2 \right)^{1/2}$, observe that

$$B_{X_\infty} \subset B_\infty \subset \operatorname{dom} H_\infty(B_{\ell_\infty}). \tag{22}$$

By Theorem 2.4 and expression (22) we have for some $C = C(p) > 0$

$$\sup_{n\geq 2} \chi_M(\mathscr{P}(^m(X_\infty)_n), \mathscr{P}(^m\ell_p^n)) \leq C^m. \tag{23}$$

Observe that the norm in $(X_\infty)_n$ coincides with the given by

$$\|(z_1,\dots,z_n)\|_{(X_\infty)_n} = \sup_{2\leq k\leq n} \frac{1}{\sqrt{\log(k)}} \left(\sum_{j=1}^{k} |z_j^*|^2 \right)^{1/2}.$$

For any $q \geq 2$ and $z \in \mathbb{C}^n$ it holds

$$\left(\sum_{j=1}^{k} |z_j^*|^2 \right)^{1/2} = \|(z_1^*,\dots,z_k^*)\|_2 \leq k^{\frac{1}{2}-\frac{1}{q}} \|(z_1^*,\dots,z_k^*)\|_q \leq k^{\frac{1}{2}-\frac{1}{q}} \|z\|_q,$$

then $\|z\|_{(X_\infty)_n} \leq \sup_{2\leq k\leq n} \frac{k^{\frac{1}{2}-\frac{1}{q}}}{\sqrt{\log(k)}} \|z\|_q$. As $0 \leq \frac{1}{2} - \frac{1}{q}$ then $\frac{n^{\frac{1}{2}-\frac{1}{q}}}{\sqrt{\log(n)}} \underset{n\to\infty}{\to} \infty$, and there is some $C = C(q)$ such that $\frac{k^{\frac{1}{2}-\frac{1}{q}}}{\sqrt{\log(k)}} \leq C \frac{n^{\frac{1}{2}-\frac{1}{q}}}{\sqrt{\log(n)}}$ for every $2 \leq k \leq n$. Then we have

$$\|id : \ell_q^n \to (X_\infty)_n\| \leq C \frac{n^{\frac{1}{2}-\frac{1}{q}}}{\sqrt{\log(n)}}. \tag{24}$$

Proof of the case $2 \leq q, p$ *on Theorem 1.2.* Thanks to the the fact that $B_{X_\infty} \subset \operatorname{mon} H_\infty(B_{\ell_\infty})$ and Theorem 2.4 we have that there is some $C > 0$ such that for every polynomial $P(z) = \sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha$ and for every $z \in (X_\infty)_n$,

$$\sum_{\alpha\in\Lambda(m,n)} |a_\alpha||z^\alpha| \leq C^m \|z\|_{(X_\infty)_n}^m \|P\|_{\mathscr{P}(^m\ell_\infty^n)}.$$

Thus for every $z \in \mathbb{C}^n$ using (24) we have

$$\sum_{\alpha\in\Lambda(m,n)} |a_\alpha||z^\alpha| \leq C^m \|z\|_{\ell_q^n}^m \left(\frac{1}{\sqrt{\log(n)}} n^{\frac{1}{2}-\frac{1}{q}} \right)^m \|P\|_{\mathscr{P}(^m\ell_\infty^n)}.$$

For any $2 \leq p < \infty$, we have that $\|P\|_{\mathscr{P}(^m\ell_\infty^n)} \leq n^{m/p} \|P\|_{\mathscr{P}(^m\ell_p^n)}$, and then

$$\sum_{\alpha\in\Lambda(m,n)} |a_\alpha||z^\alpha| \leq C^m \|z\|_{\ell_q^n}^m \left(\frac{1}{\sqrt{\log(n)}} n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} \right)^m \|P\|_{\mathscr{P}(^m\ell_p^n)},$$

which implies

$$\chi_M(\mathscr{P}(^m\ell_p^n), \mathscr{P}(^m\ell_q^n))^{1/m} \ll \frac{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}{\sqrt{\log(n)}}.$$

Therefore, by Lemma 2.2

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \gg \frac{\sqrt{\log(n)}}{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}},$$

as we wanted to prove. □

To complete the study of the mixed Bohr radius for $p \geq 2$ it remains to understand the case $\frac{1}{2} \leq \frac{1}{q} \leq \frac{1}{2} + \frac{1}{p}$.

Remark 4.6. For every Reinhardt domain $\mathscr{R} \subset \mathbb{C}^n$, if $P \in \mathscr{P}(^m\mathbb{C}^n)$ and $w \in \mathscr{R}$ if we define $P_w \in \mathscr{P}(^m\mathbb{C}^n)$ as $P_w(z) = P(w \cdot z)$, it follows

$$\|P_w\|_{\mathscr{P}(^m\ell_\infty^n)} \leq \sup_{z\in\mathscr{R}} |P(z)|,$$

and $a_\alpha(P_w) = a_\alpha(P)w^\alpha$.

Proof of the case $\frac{1}{2} \leq \frac{1}{q} < \frac{1}{2} + \frac{1}{p}$ *and* $p \geq 2$ *on Theorem 1.2.* Fix $m \in \mathbb{N}$ and take $P \in \mathscr{P}(^m\mathbb{C}^n)$, $P(z) = \sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha$. By Lemma 2.2, it suffices to show that there exists some $C(p,q) > 0$ such that for every $z \in B_{\ell_q^m}$ it holds

$$\sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| \leq C(p,q)^m \left(\frac{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}{\sqrt{\log(n)}}\right)^m \|P\|_{\mathscr{P}(^m\ell_p^n)}.$$

Consider now $y = (z_1^{\frac{p}{p+2}}, \dots, z_n^{\frac{p}{p+2}})$ and $w = (z_1^{\frac{2}{p+2}}, \dots, z_n^{\frac{2}{p+2}})$. It is easy to see that $z = y\cdot w = (y_1 w_1, \dots, y_n w_n)$, and thus, by (23) and Remark 4.6, we have

$$\begin{aligned}
\sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| &= \sum_{\alpha\in\Lambda(m,n)} |a_\alpha w^\alpha y^\alpha| \\
&\leq C^m \|y\|_{(X_\infty)_n}^m \|P_w\|_{\mathscr{P}(^m\ell_\infty^n)} \\
&\leq C^m \|y\|_{(X_\infty)_n}^m \|w\|_{\ell_p^n}^m \|P\|_{\mathscr{P}(^m\ell_p^n)}.
\end{aligned}$$

It remains to check that

$$\|y\|_{(X_\infty)_n} \|w\|_{\ell_p^n} \leq C(p,q) \frac{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}{\sqrt{\log(n)}}.$$

To start let $1 \leq k \leq n$ then

$$\begin{aligned}
\|(y_1^*, \dots, y_k^*)\|_{\ell_2^k} &= \|(z_1^*, \dots, z_k^*)\|_{\ell^k_{\frac{2p}{p+2}}}^{\frac{p}{p+2}} \\
&\leq \left(\|(z_1^*, \dots, z_k^*)\|_{\ell_q^k} k^{\frac{1}{p}+\frac{1}{2}-\frac{1}{q}}\right)^{\frac{p}{p+2}} \\
&\leq \|z\|_{\ell_q^n}^{\frac{p}{p+2}} \left(k^{\frac{1}{p}+\frac{1}{2}-\frac{1}{q}}\right)^{\frac{p}{p+2}},
\end{aligned}$$

so we have

$$
\begin{aligned}
\|y\|_{(X_\infty)_n} &= \sup_{2\le k\le n} \frac{1}{\sqrt{\log(k)}} \|(y_1^*,\dots,y_k^*)\|_{\ell_2^k} \\
&\le \sup_{2\le k\le n} \frac{1}{\sqrt{\log(k)}} \|z\|_{\ell_q^n}^{\frac{p}{p+2}} \left(k^{\frac{1}{p}+\frac{1}{2}-\frac{1}{q}}\right)^{\frac{p}{p+2}} \\
&\le \sup_{2\le k\le n} \|z\|_{\ell_q^n}^{\frac{p}{p+2}} C(p,q) \frac{1}{\sqrt{\log(n)}} n^{\left(\frac{1}{p}+\frac{1}{2}-\frac{1}{q}\right)\frac{p}{p+2}} \\
& C(p,q)\|z\|_{\ell_q^n}^{\frac{p}{p+2}} \frac{1}{\sqrt{\log(n)}} n^{\frac{1}{2}-\frac{1}{q}\frac{p}{p+2}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|w\|_{\ell_p^n} &= \|z\|_{\ell^n_{\frac{2p}{p+2}}}^{\frac{2}{2+p}} \\
&\le \|z\|_{\ell_q^n}^{\frac{2}{2+p}} n^{\left(\frac{1}{2}+\frac{1}{p}-\frac{1}{q}\right)\frac{2}{2+p}} \\
&= \|z\|_{\ell_q^n}^{\frac{2}{2+p}} n^{\frac{1}{p}-\frac{1}{q}\frac{2}{2+p}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\|y\|_{(X_\infty)_n}\|w\|_{\ell_p^n} &\le C(p,q)\|z\|_{\ell_q^n}^{\frac{p}{p+2}} \frac{1}{\sqrt{\log(n)}} n^{\frac{1}{2}-\frac{1}{q}\frac{p}{p+2}} \|z\|_{\ell_q^n}^{\frac{2}{2+p}} n^{\frac{1}{p}-\frac{1}{q}\frac{2}{2+p}} \\
&= C(p,q)\|z\|_{\ell_q^n} \frac{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}{\sqrt{\log(n)}},
\end{aligned}
$$

as we needed. □

ACKNOWLEDGEMENTS

We thank our friend Pablo Sevilla-Peris for his enormous generosity, the encouragement provided and for the various conversations we had around this topic.

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