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Symmetry enhancement interpolation, non-commutativity and Double Field Theory

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Abstract: We present a moduli dependent target space effective field theory action for (truncated) heterotic string toroidal compactifications. When moving continuously along moduli space, the stringy gauge symmetry enhancement-breaking effects, which occur at particular points of moduli space, are reproduced.

Besides the expected fields, originating in the ten dimensional low energy effective theory, new vector and scalar fields are included. These fields depend on "double periodic coordinates" as usually introduced in Double Field Theory. Their mode expansion encodes information about string states, carrying winding and KK momenta, associated to gauge symmetry enhancements. It is found that a non-commutative product, which introduces an intrinsic non-commutativity on the compact target space, is required in order to make contact with string theory amplitude results.

Keywords: Duality in Gauge Field Theories, Flux compactifications, String Duality, Superstrings and Heterotic Strings

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1 Introduction

In this article we propose a target space effective field theory description of string theory interactions. Clearly the subject is not new. Indeed, the conventional low energy effective action for given values of moduli fields can be found in string books [1–3]. However, several works point towards a richer structure with some intrinsic compact target space non-commutativity. Among them, there are recent analyses [4–8] performed from the perspective of Double Field Theory (DFT)[1] aiming at the inclusion of gauge symmetry enhancement aspects in the field theory description, as well as recent (and not so recent) proposals about non-commutativity of string zero modes [22–24].

A key guide in our analysis is the field theory description of gauge symmetry enhancement on toroidal compactifications. Gauge symmetry enhancement is a very stringy

[1] See [9–15] for some original references on DFT and for instance [16, 17] for reviews. DFT approaches to heterotic string can be found in [18–21].

phenomenon associated to the fact that the string is an extended object and, therefore, it can wind around non-contractible cycles. At certain moduli points (i.e., fixed points of T-duality transformations) vector boson states, associated to definite values of windings and compact momenta become massless. These vectors, combined with massless vectors inherited from the metric and antisymmetric tensor fields, give rise to an enhanced gauge symmetry group G_1 (see for instance [25–27]). Further displacements on moduli space can lead to a different fixed point where, generically, other vectors associated to different values of winding and momenta will become massless leading to a different enhanced gauge group G_2, etc. At generic points only a $\mathrm{U}(1)_L^{r+16} \times \mathrm{U}(1)_R^r$ symmetry exists. Here, r is the number of compactified dimensions associated to the KK zero modes of the metric and antisymmetric fields and the 16 comes from Cartan generators of the ten dimensional gauge group, in the heterotic string case. The low energy effective theory, *at a given moduli point*, where massive states are neglected, can be described by a usual gauge field theory Lagrangian coupled to gravity with no explicit reference to any windings. By slightly moving away from this fixed moduli point, gauge symmetry gets broken. The symmetry breaking can be understood as a conventional higgsing mechanism and also, as found from a DFT approach [5, 7], as associated to a dependence on moduli fields of the "will-be structure constants" fluxes.

The main aim of the present work is to write down a lower dimensional field theory able to provide a description of the enhancement phenomena occurring on toroidally compactified heterotic string. This action depends on moduli fields expectation values such that the different low energy effective field theories, associated to heterotic enhancement situations, can be reached by varying such values. Our construction is restricted to fields corresponding to low string oscillator number and includes the fields that are involved in the enhancement phenomena. Clearly a full, consistent description of the string theory would require the introduction of an infinite number of fields of all possible spins. We comment on a possible step by step completion of our construction, going beyond low energy, at the end of the article.

Very schematically, the idea is to incorporate a vector boson field $A_\mu(x, \mathbb{Y})$ and a scalar $M_{\bar{I}}(x, \mathbb{Y})$ into the action, in addition to the fields inherited from the usual ten dimensional metric, the dilaton and the Kalb-Ramond B_2. All fields must depend on both d spacetime x^μ coordinates as well as on internal compact toroidal $\check{\mathbb{Y}} \equiv (y^I, y^m, \tilde{y}_m)$ coordinates. Namely, besides the y^I coordinates associated to the heterotic string degrees of freedom, $2r$ double coordinates $(y^m, \tilde{y}_m)$, conjugate to momenta and windings modes $(p_m, \tilde{p}^m)$, for each of the r compact dimension are considered in the spirit of DFT. A generalized mode expansion (GKK) in periodic internal coordinates would produce d dimensional fields $A_\nu^{(\mathbb{L})}(x)$ (and $M_{\bar{I}}^{(\mathbb{L})}(x)$) with $\mathbb{L}$ labeling modes, depending on windings and KK momenta. As mentioned before, for certain moduli values some of these modes become massless and, when combined with KK zero modes coming from metric and B field (as well as heterotic Cartan fields) they enhance the gauge symmetry. The other modes, not participating in the enhancement process, remain very massive (with masses of the order of string mass α'^{-1}) and do not contribute to the low energy effective theory.

The resulting action, in terms of the "uplifted" $A_\mu(x, \mathbb{Y})$ and $M_{\bar{I}}(x, \mathbb{Y})$ fields, appears to require a non-commutativity on fields introduced through a non-commutative $\star$-product in the compact space [22, 23]. At the neighborhood of each specific moduli fixed point and when only the slightly massive modes that become massless at this point are kept, the usual, commutative, effective gauge theory action is recovered after integrating over the internal coordinates. The gauge symmetry gets enhanced exactly at the fixed point.

Therefore, the action provides an effective interpolation among theories at different points. It is worth mentioning that enhancement can be described in DFT constructions as an enlargement of the compactification tangent space [4–8] at a fixed point. Here, however, the compact manifold is an r dimensional double torus and we find that this enlargement is effectively provided by Fourier modes associated to fields that "will-be massless at such point". Interestingly enough, the mentioned non-commutativity can be traced back to cocycle factors in string vertices. These factors were first mentioned in [15] but did not manifest in previous DFT constructions due to the considered level matching conditions and to the fact that calculations were performed up to third order terms in the fields.

We organize the article as follows: in section 2, we introduce the proposed action in $D = d + 2r$ dimensions. In section 3 we perform the mode expansion and analyze the different contributions. Section 4 deals with the physical content of the action, like vector and scalar masses, Goldstone bosons, enhancement-breaking of gauge symmetries, etc. An illustrative torus compactification ($r = 2$) example is briefly discussed. A summary and a discussion of the limitations and possible extensions of the present work are presented in section 5. Notation and technical aspects are reserved to the appendices. A more detailed description of the $\star$-product is extended to incorporate the heterotic string gauge modes.

2 The effective action

In this section we present a moduli dependent field theory effective action that captures essential features of symmetry enhancement in toroidal compactification of heterotic string. The basic ingredients and notation conventions are introduced here. The reader is referred to the appendices for details.

Let us denote by $\Phi \equiv (g, b, A)$ a moduli point encoding the background metric g, the b field and Wilson line values. At a given fixed point Φ_0 on moduli space the heterotic gauge group is of the form $G_L \times \mathrm{U}(1)^r_R$. The rank of G_L is $r_L = r + 16 = 26 - d$ originating in the 16 Cartan generators of the ten dimensional heterotic gauge group plus the $r = 10 - d$ vector bosons coming from Left combinations of the KK reductions of the metric and the antisymmetric tensor. Therefore, the dimension of the gauge group is $\dim G_L = n_c + r_L$ where n_c denotes the number of charged generators. These generators correspond to string vertex operators containing KK momenta and windings associated, generically, with massive fields that become massless at the fixed point. These fields will play a central role in our construction. Let us stress that n_c depends on the moduli point and that, at generic points, there is no enhancement at all ($n_c = 0$) and the generic gauge group is $\mathrm{U}(1)^{r_L}_L \times \mathrm{U}(1)^r_R$. The low energy effective action for the bosonic sector of heterotic string, at a fixed point Φ_0 with $G_L \times \mathrm{U}(1)^r_R$ gauge group and up to third order in the fields, reads

$$\begin{aligned}S_{\text{eff}}(\Phi_0) = &\int d^dx\sqrt{g}\Big[e^{-2\varphi}\left(R+4\partial^\mu\varphi\partial_\mu\varphi-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right)\\&-\frac{1}{4}\left(F_{A\mu\nu}F^A_{\mu\nu}+F^{\mu\nu}_{\bar J}F^{\bar J}_{\mu\nu}-2g_d\sqrt{\alpha'}M_{A\bar I}F^A_{\mu\nu}F^{\bar I\mu\nu}\right)\\&-\frac{1}{4}D_\mu M_{A\bar I}D_\nu M^{A\bar I}g^{\mu\nu}+\mathcal{O}(M^4)\Big]\end{aligned} \tag{2.1}$$

where $\bar I$ Right indices correspond to the Abelian Right group $\mathrm{U}(1)^r_R$ and A indices label Left G_L (generically non Abelian) group. We have

$$F^B = dA^B + \frac{g_d}{2}f_{CD}{}^B A^C\wedge A^D, \qquad F^{\bar I} = dA^{\bar I} \tag{2.2}$$

$$D_\mu M_{A\bar I} = \partial_\mu M_{A\bar I} + g_d f^K{}_{SA}A^S_{L\mu}M_{K\bar I} \tag{2.3}$$

where scalar fields $M_{A\bar I}$ live in the $(\mathbf{dimG_L})_{\bar{\mathbf{q}}=\mathbf{0}}$ adjoint representation of G_L and carry zero vector charge $\bar{\mathbf{q}} = (\bar q_1,\ldots,\bar q_r) = 0$ with respect to $\mathrm{U}(1)^r_R$ Abelian Right group. H is the B field strength (with Chern-Simons interactions) defined as

$$H = dB + F^B\wedge A_B, \tag{2.4}$$

φ is the dilaton and R the scalar curvature.

As mentioned above, the terms in this expression corresponding to fields originating in reductions of the 10D fields will be always present, whereas terms associated with charged fields will change when moving on moduli space. In our construction it proves convenient to separate these contributions and rewrite the above action (2.1) as

$$\begin{aligned}S_{\text{eff}}(\Phi_0) = &\int d^dx\sqrt{g}e^{-2\varphi}\Big[\left(R+4\partial^\mu\varphi\partial_\mu\varphi-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right)\\&-\frac{1}{4}\left(\delta_{\hat I\hat J}F^{\hat I\mu\nu}F^{\hat J}_{\mu\nu}+\delta_{\bar I\bar J}F^{\bar I\mu\nu}F^{\bar I}_{\mu\nu}-2g_d\sqrt{\alpha'}M_{\hat I\bar I}F^{\hat I}_{\mu\nu}F^{\bar I\mu\nu}+D_\mu M_{\hat I\bar I}D_\nu M^{\hat I\bar I}g^{\mu\nu}\right)\\&-\frac{1}{4}\left(F_{\alpha\mu\nu}F^\alpha_{\mu\nu}+D_\mu M_{\alpha\bar I}D_\nu M^{\alpha\bar I}g^{\mu\nu}-2g_d\sqrt{\alpha'}M_{\alpha\bar I}F^\alpha_{\mu\nu}F^{\bar I\mu\nu}\right)+\mathcal{O}(M^4)\Big].\end{aligned} \tag{2.5}$$

Several indices are introduced:

- $\bar I = 1,\ldots r$ are Right indices that label the Abelian group $\mathrm{U}(1)^{\bar I}$ associated to Right vector bosons $A^{\bar I}_\mu$
- The Left index A has been conveniently splitted as $A = (\hat I,\alpha)$ where:

  $\alpha = 1,\ldots,n_c$ label the Left gauge group charged generators with vector bosons A^α_μ. They correspond to roots of the algebra in a Cartan-Weyl basis.

  $\hat I \equiv (I,m)$ are Left indices splitted in terms of $m = 1\ldots r$ compact Left indices and $I = 1\ldots 16$ heterotic indices. $A^{\hat I}_\mu$ correspond to the Left Cartan vector boson fields.

The field strengths introduced in (2.2) are now splitted as

$$F^{\hat I}_{\mu\nu} = 2\partial_{[\mu}A^{\hat I}_{\nu]} + ig\textstyle\sum_\alpha f_{\alpha\beta}{}^{\hat I}A^\alpha_{[\mu}A^\beta_{\nu]} \qquad F^{\bar J}_{\mu\nu} = 2\partial_{[\mu}A^{\bar J}_{\nu]} \tag{2.6}$$

for Cartan fields whereas

$$F^{\alpha}_{\mu\nu} = 2\partial_{[\mu}A^{\alpha}_{\nu]} + igf^{\alpha}{}_{\beta\gamma}A^{\beta}_{[\mu}A^{\gamma}_{\nu]} + igf^{\alpha}{}_{\beta\hat{I}}A^{\beta}_{[\mu}A^{\hat{I}}_{\nu]} \tag{2.7}$$

are the field strengths for charged vectors.[2] Similarly, for scalar fields we have

$$D_\mu M_{\hat{I}\bar{J}} = \partial_\mu M_{\hat{I}\bar{J}} + ig_d f_{\beta\hat{I}}{}^{\alpha} A^{\beta}_{\mu} M_{\alpha\bar{J}} \tag{2.8}$$

$$D_\mu M_{\alpha\bar{J}} = \partial_\mu M_{\alpha\bar{J}} + ig_d f_{\alpha\beta}{}^{\hat{I}} M_{\hat{I}\bar{J}} A^{\beta}_{\mu} + ig_d f_{\alpha\beta}{}^{\lambda} M_{\lambda\bar{J}} A^{\beta}_{\mu} + ig_d f_{\alpha\hat{I}}{}^{\beta} M_{\beta\bar{J}} A^{\hat{I}}_{\mu} \tag{2.9}$$

where a sum over repeated root indices is implicit. We are using a Cartan-Weyl basis such that $f_{\alpha\beta}{}^{\gamma} = f_{\alpha\beta(-\gamma)} = 1$ (with $\gamma = \alpha + \beta$) and $f_{\beta\alpha}{}^{\hat{I}} = f_{-\alpha\alpha}{}^{\hat{I}} = f_{\alpha-\alpha\hat{I}} = \alpha^{\hat{I}}$ (no sum on α here) etc. Also, charged indices are contracted with the corresponding Cartan-Killing form whereas Cartan indices contract with a delta function.

Finally, it proves useful to perform a further rewriting of the above action by collecting Left and Right "Cartan" indices into a unique generalized $\mathcal{I} = (\hat{I}, \bar{I})$ index spanning the vector representation of $O(r_l, r)$ duality group. The $\mathcal{I}$ indices are contracted with an $O(r_L, r)$ invariant metric that we will generically express in the L-R basis (also called $C-$basis) as

$$\eta^{\mathcal{I}\mathcal{J}}_{C} = \begin{pmatrix} 1_{16+r} & 0 \\ 0 & -1_r \end{pmatrix}. \tag{2.10}$$

In order to have a covariantly looking form in this basis we introduce the generalized vector $A^{\mathcal{I}}_{\mu} = (A^{\hat{I}}_{\mu}, A^{\bar{I}}_{\mu})$ that incorporates the Left and Right Cartan fields respectively and define the scalars $M_{\alpha\mathcal{J}} = (0, M_{\alpha\bar{J}})$ where Left components are projected out. We discuss this projection in (2.25) below. Also, inspired by DFT constructions [7] we introduce a generalized $O(r_l, r)$ metric $\mathcal{H}^{\mathcal{I}\mathcal{J}}$ and we expand on fluctuations around a flat background as

$$\mathcal{H}^{\mathcal{I}\mathcal{J}} = \delta^{\mathcal{I}\mathcal{J}} + \mathcal{H}^{(1)\mathcal{I}\mathcal{J}} + \frac{1}{2}\mathcal{H}^{(2)\mathcal{I}\mathcal{J}} + \dots \tag{2.11}$$

where matrix elements vanish unless

$$\begin{aligned} \mathcal{H}^{(1)}_{\hat{I}\bar{J}} &= -M_{\hat{I}\bar{J}}, & \mathcal{H}^{(1)}_{\bar{J}\hat{I}} &= -M^{T}_{\hat{I}\bar{J}} \\ \mathcal{H}^{(2)}_{\hat{I}\hat{J}} &= (MM^T)_{\hat{I}\hat{J}}, & \mathcal{H}^{(2)}_{\bar{I}\bar{J}} &= (M^TM)_{\bar{I}\bar{J}}\,. \end{aligned} \tag{2.12}$$

In terms of this metric and keeping terms up to second order in fluctuations (assuming vector fields are first order) the action[3] can be re-expressed as

$$\begin{aligned} S_{\text{eff}}(\Phi_0) = \int d^dx\sqrt{g}e^{-2\varphi}\bigg[&\left(R + 4\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) \\ &-\frac{1}{4}\mathcal{H}_{\mathcal{I}\mathcal{J}}F^{\mathcal{I}\mu\nu}F^{\mathcal{J}}_{\mu\nu} + \frac{1}{8}D_\mu\mathcal{H}_{\mathcal{I}\mathcal{J}}D^\mu\mathcal{H}^{\mathcal{I}\mathcal{J}} \\ &-\frac{1}{4}F_{\alpha\mu\nu}F^{\alpha\mu\nu} + \frac{1}{4}D_\mu M_{\alpha\mathcal{I}}D^\mu M^{\alpha\mathcal{I}} - \frac{1}{2}g_d\sqrt{\alpha'}M_{\alpha\mathcal{I}}F^{\alpha}_{\mu\nu}\bar{F}^{\mathcal{I}\mu\nu}\bigg]. \end{aligned} \tag{2.13}$$

[2]Here we use the convention $2A_{[\mu}B_{\nu]} = A_\mu B_\nu - B_\nu A_\mu$.

[3]Notice that there is no scalar potential. Scalar interactions appear at fourth order in the fields when only massless states are considered.

Field Modes	$\mathbb{L}^2$	N	Vertex Operators
$g_{\mu\nu}^{(\mathbb{L})}, b_{\mu\nu}^{(\mathbb{L})}, \phi^{(\mathbb{L})}$	0	1	$\partial_z X^\mu \tilde{\psi}^\nu(z) e^{i\mathbb{L}.\mathbb{Y}(z)} e^{iK.X(z)}$
$A_\mu^{\bar{I}(\mathbb{L})}$	0	1	$\partial_z X^\mu \tilde{\psi}_{\bar{I}}(z) e^{i\mathbb{L}.\mathbb{Y}(z)} e^{iK.X(z)}$
$A_\mu^{\hat{I}(\mathbb{L})}$	0	1	$\partial_z Y_I \tilde{\psi}^\mu(z) e^{i\mathbb{L}.\mathbb{Y}(z)} e^{iK.X(z)}$ $\partial_z Y_m \tilde{\psi}^\mu(z) e^{i\mathbb{L}.\mathbb{Y}(z)} e^{iK.X(z)}$
$M_{\hat{I}\bar{I}}^{(\mathbb{L})}$	0	1	$\partial_z Y^I \tilde{\psi}^{\bar{I}}(z) e^{i\mathbb{L}.\mathbb{Y}(z)} e^{iK.X(z)}$ $\partial_z Y^m \tilde{\psi}^{\bar{I}}(z) e^{i\mathbb{L}.\mathbb{Y}(z)} e^{iK.X(z)}$
$A_\mu^{(\mathbb{L})}$	2	0	$\tilde{\psi}^\mu(z) e^{i\mathbb{L}.\mathbb{Y}(z)} e^{iK.X(z)}$
$M_{\bar{I}}^{(\mathbb{L})}$	2	0	$\tilde{\psi}^{\bar{I}}(z) e^{i\mathbb{L}.\mathbb{Y}(z)} e^{iK.X(z)}$

Table 1. Field Modes, LMC, oscillator number and corresponding string vertex operator. In all cases considered here $\bar{N} = \bar{N}_F + \bar{N}_B - \frac{1}{2} = 0$.

The different contributions to the action above are read out from 3-point amplitudes of massless heterotic string vertices. Vertex operators for vector bosons and scalars considered here are collected in table 1 and explained below.

String vertex operators generically contain an internal factor (see appendix A for notation)

$$e^{i\mathbb{L}^{(\check{\mathbb{P}})}(\Phi).\mathbb{Y}(z)} = e^{il_L^{(\check{\mathbb{P}})}.y_L(z)+il_R^{(\check{\mathbb{P}})}.y_R(z)} \tag{2.14}$$

where $\mathbb{L}^{(\check{\mathbb{P}})}(\Phi) = (l_L^{(\check{\mathbb{P}})}(\Phi), l_R^{(\check{\mathbb{P}})}(\Phi))$ is the generalized momentum[4] (see (A.1)) that depends on windings $\tilde{p}^m$, KK momenta p_m and Λ_{16} weights P^I that we organize into the generalized Kaluza-Klein (GKK) momenta

$$\check{\mathbb{P}} \equiv (P^I, p_m, \tilde{p}^m) \tag{2.15}$$

and on moduli field values Φ . Generalized momenta are constrained to satisfy the Level Matching Condition (LMC)

$$\frac{1}{2}\mathbb{L}^2 = \frac{1}{2}l_L^2 - \frac{1}{2}l_R^2 = \tilde{p}.p + \frac{1}{2}P^2 = (1 - N + \bar{N}). \tag{2.16}$$

In all the cases considered here $\bar{N} = 0$ and $N = 0, 1$. Vertex operators with $N = 1$ correspond to KK reductions of the metric, B-field, dilaton field and heterotic vector fields in 10 dimensions.

For instance, the Cartan vectors $A_\mu^{\hat{I}}$ do originate in string vertex operators coming from KK reductions of the metric and antisymmetric field of the form

$$V(\hat{I}, \mathbb{L}) \propto A_\mu^{\hat{I}(\mathbb{L})}(K)\, \partial_z Y_{\hat{I}} \tilde{\psi}^\mu e^{i\mathbb{L}^{(\check{\mathbb{P}})}(\Phi).\mathbb{Y}(z)} e^{iK.X(z)} \tag{2.17}$$

[4]We will generally write $\mathbb{L} \equiv \mathbb{L}^{(\check{\mathbb{P}})}(\Phi)$ and omit the explicit writing of the dependence on $\check{\mathbb{P}}$ and Φ to lighten the notation.

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where K^μ is the space time momentum. Due to the presence of oscillators $\partial_z Y^{\hat{I}}$, $N = 1$ and therefore LMC (2.16) reads

$$\mathbb{L}^2 = 0. \tag{2.18}$$

This requirement is trivially satisfied by massless states that correspond to $\mathbb{L} \equiv l_L = l_R = 0$ (with null windings and KK momenta), as it is indeed the case for Cartan vectors $A_\mu^{\hat{I}} \equiv A_{\mu\hat{I}}^{(0)}$.

On the other hand, the left handed charged vector bosons arise from vertices

$$V(\mathbb{L}) \propto A_\mu^{(\mathbb{L})}(K)\, \tilde{\psi}^\mu(z) e^{i\mathbb{L}^{(\check{\mathbb{P}})}(\Phi).\mathbb{Y}(z)} e^{iK.X(z)} \tag{2.19}$$

with LMC

$$\frac{1}{2}\mathbb{L}^2 = 1 \tag{2.20}$$

since $N = 0$. The other cases in the table 1 are understood in a similar way. Let us stress that ghost factors as well as cocycle factors must be included.

At a fixed point Φ_0 and for specific values of windings and momenta (i.e., for specific values of $\check{\mathbb{P}}$)

$$l_R^{(\check{\mathbb{P}})}(\Phi_0) = 0 \qquad\qquad l_L^{(\check{\mathbb{P}})}(\Phi_0) = \alpha^{(\check{\mathbb{P}})} \qquad \text{with} \qquad \frac{1}{2}\alpha^{(\check{\mathbb{P}})2} = 1 \tag{2.21}$$

the states become massless (see (A.6)) and $l_L^{(\check{\mathbb{P}})}(\Phi_0)$ become the roots $\alpha^{(\check{\mathbb{P}})}$ of the enhanced gauge group algebra charged generators.[5] Generically, at a different fixed point, other set of $\check{\mathbb{P}}'$s will ensure (2.21), leading to a different enhanced gauge group. We will denote this set of n_c GKK modes, satisfying (2.20), by

$$\check{G}(\Phi_0)_{n_c} = \{\check{\mathbb{P}} \equiv (P^I, p_m, \tilde{p}^m) : l_R^{(\check{\mathbb{P}})}(\Phi_0) = 0 \,(\text{thus} \quad l_L^{(\check{\mathbb{P}})}(\Phi_0) = \alpha^{(\check{\mathbb{P}})}, m^2 = 0)\}. \tag{2.22}$$

Namely, $\check{G}(\Phi_0)_{n_c}$ encodes the n_c "will-be massless charged fields at fixed point Φ_0". At Φ_0, and for $\check{\mathbb{P}} \in \check{G}(\Phi_0)_{n_c}$ the $A_\mu^{(\check{\mathbb{P}})}(K)$ modes give rise to charged vector field $A_\mu^\alpha(x)$ in the action above (similarly with charged scalars).

As stated in the Introduction the main aim of our work is to provide a unified field theory description such that at given fixed points the different effective gauge theories are reproduced. Following the suggestions in [5] we propose to consider a sort of generalized Kaluza-Klein expansion on generalized momenta $\mathbb{L}$ of the different fields coming into play in the enhancement process. The GKK modes in this expansion are identified with a corresponding polarization of a vertex operator. For instance, in order to describe charged vector bosons we introduce the expansion

$$A_\mu(x, \mathbb{Y}) = \sum_{\check{\mathbb{P}}}{}' A_\mu^{(\mathbb{L})}(x) e^{i\mathbb{L}_\mathcal{I}\mathbb{Y}^\mathcal{I}} = \sum_{\mathbb{L}} A_\mu^{(\mathbb{L})}(x) e^{il_L\cdot y_L + il_R\cdot y_R}\, \delta\left(\frac{1}{2}\mathbb{L}^2, 1\right) \tag{2.23}$$

where $A_\mu^{(\mathbb{L})}(x)$ correspond to polarization modes in (2.19). The prime in the sum indicates that LMC (2.20) must be imposed (with an abuse of notation we indicate the sum on mode

[5] As string theory operators, $e^{il_L^{(\check{\mathbb{P}})}.Y(z)} \to J_{\alpha^{(\check{\mathbb{P}})}}$ are the charged generators of the algebra.

index $\check{\mathbb{P}}$ by $\mathbb{L}$). Recall that generically the sum contains an infinite number of terms even though the LMC is a severe constraint.

Generically, if the mass of the GKK components $A_\mu^{(\mathbb{L})}(x)$ were given by the string mass formula (A.6), as we will show to be the case, these modes would be massive. However, when moving continuously along the moduli space, for specific values $\check{\mathbb{P}} \in \check{G}_{n_c}(\Phi_0)$, n_c vector fields $A_\mu^{(\mathbb{L})}(x) \equiv A_\mu^{\alpha(\check{\mathbb{P}})}(x)$ would become massless and would lead to the enhanced G_L gauge group.[6] In a similar way we introduce the GKK expansion for scalar fields by associating the fields $M_{\alpha\bar{I}}(x)$, coming from string vertex operators modes $M_{\bar{I}}^{(\mathbb{L})}(K)$ (see table 1 above) to modes in a GKK expansion

$$M_{\bar{I}}(x,\mathbb{Y}) = \sum_{\mathbb{L}} M_{\bar{I}}^{(\mathbb{L})}(x) e^{i\mathbb{L}_{\mathcal{I}}\mathbb{Y}^{\mathcal{I}}}\,\delta\left(\frac{1}{2}\mathbb{L}^2,1\right). \tag{2.24}$$

Before addressing the other mode expansions let us note that the Right fields $M_{\bar{I}}$ can be embedded into constrained fields $M_{\mathcal{I}} = (M_{\hat{I}}, M_{\bar{I}})$ with $O(r_L, r)$ indices. Namely, they are defined as

$$\begin{aligned} P_{\mathcal{I}}{}^{\mathcal{J}} M_{\mathcal{J}} &= 0 & \qquad P &= \frac{1}{2}(\eta+\mathcal{H}) \\ \bar{P}_{\mathcal{I}}{}^{\mathcal{J}} M_{\mathcal{J}} &= M_{\mathcal{I}} & \qquad \bar{P} &= \frac{1}{2}(\eta-\mathcal{H}) \end{aligned} \tag{2.25}$$

where $\mathcal{H}$ is the generalized metric satisfying $\mathcal{H}\eta\mathcal{H} = \eta$ and $P, \bar{P}$ are projectors [18–21] that eliminate r_L degrees of freedom. From the first equation we obtain, by plugging in the generalized metric expansion (2.12),

$$PM = \frac{1}{2}(\eta+\mathcal{H})M = \begin{pmatrix} (1+MM^T)_{\hat{I}\hat{J}} + \dots & -M_{\hat{I}\bar{J}} + \dots \\ -(M^T)_{\bar{I}\hat{J}} + \dots & -(M^T M)_{\bar{I}\bar{J}} + \dots \end{pmatrix} \begin{pmatrix} M_{\hat{J}} \\ M_{\bar{J}} \end{pmatrix} = 0 \tag{2.26}$$

where $\dots$ indicate higher order terms in fluctuations. Therefore $M_{\hat{I}} = -M_{\hat{I}\bar{J}}M_{\bar{J}} + \dots$. We see that $M_{\hat{I}}$ degrees of freedom are not independent and contribute at order two or higher in fluctuations. As we show below, these will give rise to terms of order four in the action. For this reason we can set $M_{\mathcal{J}} = (0, M_{\bar{J}})$.

Expansions of fields originating in the $D = 10$ metric, B field and the Cartan generators of the heterotic gauge group, namely, $G_{\mu\nu}(x,\mathbb{Y}), B_{\mu\nu}(x,\mathbb{Y})$, $A_\mu^{\mathcal{I}}(x,\mathbb{Y}), M^{\hat{I}\bar{J}}(x,\mathbb{Y})$ must also be considered. Now, since the corresponding modes (first four rows in table 1) must satisfy LMC (2.18) we restrict the sum to modes obeying this constraint. For instance

$$A_\mu^{\mathcal{I}}(x,\mathbb{Y}) = \sum_{\mathbb{L}} A_\mu^{\mathcal{I}(\mathbb{L})}(x) e^{i\mathbb{L}_{\mathcal{I}}\mathbb{Y}^{\mathcal{I}}}\,\delta(\mathbb{L}^2,0). \tag{2.27}$$

Recall that these modes correspond to $N = 1$ and, therefore, only the zero mode $A_\mu^{\mathcal{I}(0)}(x) = A_\mu^{\mathcal{I}}(x) = (A_\mu^{\hat{I}}(x), A_\mu^{\bar{I}}(x))$ would correspond to massless fields. These are the vector fields of the $\mathrm{U}(1)_L^{r_L} \times \mathrm{U}(1)_R^r$ gauge group for a generic point in moduli space. The same considerations are valid for the other $N = 1$ fields. Thus, for example, $G_{\mu\nu}(x)^{(0)} =$

[6] A reality condition $A_\mu^{(\mathbb{L})*} = A_\mu^{(-\mathbb{L})}$ must be imposed.

$G_{\mu\nu}(x)$ is the d dimensional metric field whereas non-zero modes would describe massive gravitons, etc. In most of the considerations below only these zero modes will be needed.

Notice that a generic, moduli dependent, field $\phi(x,\mathbb{Y}) \equiv A_\mu(x,\mathbb{Y}), G_{\mu\nu}(x,\mathbb{Y}), \ldots$ could be interpreted as an uplifting of d dimensional fields to $d+r+r_l$ dimensions with $r+r_l$ periodic.[7] A Lagrangian $\mathcal{L}(x,\mathbb{Y})$ in terms of these fields, when integrated over the $d+r+r_L$ dimensions $\int d^dx\, d\mathbb{Y}\, \mathcal{L}(x,\mathbb{Y}) = \int d^dx\, dy_L\, dy_R\, \mathcal{L}(x,\mathbb{Y})$ will lead to an action in d space-time dimensions after periodic coordinates are integrated out, where the physical fields will be the $\phi(x)^{(\mathbb{L})}$ GKK modes. Our expectation is that such action includes the effective low energy heterotic effective action (2.13) for different fixed moduli point Φ_0.

A crucial point is how to generate a non-Abelian structure out of these fields in order to give rise to enhancements at fixed points. We will see that the job is accomplished by a new so called "star product" [22, 23], which we denote by $\star$, accounting for non-commutativity.

In the following we present the action and subsequently we discuss its particular features. Let us assume that we are able to write down a full field theory action $S_{het}(\Phi)$ by computing all possible heterotic string theory amplitudes. This action should include an infinite number of fields, let us call them $\Phi_{\mu_1\mu_2\ldots;\bar{N},N}(x,\mathbb{Y})$, of all possible spins and oscillator numbers $\bar{N}, N$ that must be mode expanded with the corresponding level matching condition $\frac{1}{2}\mathbb{L}^2 = 1-(N-\bar{N})$. Among all these contributions we isolate the action piece, that we call $S_{\rm enh}(\Phi)$, containing up to third order terms (and some fourth order as we discuss below) and involving fields coming form $10D$ KK reductions $G_{\mu\nu}, B_{\mu\nu}, A_\mu^{\bar{J}}, \varphi, M_{\hat{I},\bar{J}}, A_\mu^{\hat{J}}$ and the extra fields $A_\mu, M_{\bar{I}}$. These fields are associated to oscillator numbers $N=0,1$, respectively, and $\bar{N}=0$. Their corresponding modes are collected in table 1 as well as their associated string vertex operators.

Therefore, we split the full action into

$$S_{\text{full het}}(\Phi) = S_{\rm enh}(\Phi) + S'(\Phi) \tag{2.28}$$

where the term S' encodes all other (infinite) contributions that we are not explicitly considering here. These include higher spin fields, fields associated to oscillator numbers $N>1$, higher order terms in fluctuations, etc. $S_{\rm enh}(\Phi)$ is the action we are going to deal with, given by

$$\begin{aligned} S_{\rm enh}(\Phi) = \int d^dx d\mathbb{Y}\sqrt{g}e^{-2\varphi}\bigg[& \left(R + 4\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) \\ & - \frac{1}{4}\mathcal{H}_{\mathcal{I}\mathcal{J}} \star F^{\mathcal{I}}_{\mu\nu} \star F^{\mathcal{J}\mu\nu} + \frac{1}{8}\mathcal{D}_\mu\mathcal{H}^{\mathcal{I}\mathcal{J}} \star \mathcal{D}^\mu\mathcal{H}_{\mathcal{I}\mathcal{J}} \\ & - \frac{1}{4}F_{\mu\nu} \star F^{\mu\nu} + \frac{1}{4}\mathcal{D}_\mu M^{\mathcal{I}} \star \mathcal{D}^\mu M_{\mathcal{I}} - \frac{1}{2}M_{\mathcal{I}} \star F^{\mu\nu} \star F^{\mathcal{I}}_{\mu\nu} \\ & - \frac{1}{4}\partial_{\mathcal{J}}M^{\mathcal{I}} \star \partial_{\mathcal{K}}M_{\mathcal{I}}(H^{\mathcal{J}\mathcal{K}} - \eta^{\mathcal{J}\mathcal{K}}) + i\frac{1}{2}\partial_{\mathcal{I}}M^{\mathcal{J}} \star M_{\mathcal{J}} \star M^{\mathcal{I}}\bigg]. \end{aligned} \tag{2.29}$$

As we have already emphasized the different terms in the action are expressed in terms of the fields introduced above. These fields depend on the compact coordinate $\mathbb{Y}$ and can

[7] Recall that Heterotic coordinates can be thought of as coordinates on a 16 dimensional torus with a chiral projection.

therefore be mode expanded. Integration over $\mathbb{Y}$ will produce an effective action in d space-time dimensions. In the next section we perform the mode expansions and integrate over internal coordinates in order to obtain a d-dimensional space time action. Before presenting these computations let us first discuss the general structure and the kind of information we expect this action to contain.

Fields originating in $D = 10$ KK reductions, i.e. $G_{\mu\nu}, B_{\mu\nu}, \varphi, M_{\hat{I},\bar{J}}, A^{\mathcal{J}} \equiv (A_\mu^{\hat{J}}, A_\mu^{\bar{J}})$, require a mode expansion with the constraint $\mathbb{L}^2 = 0$ whereas fields $A_\mu, M_{\mathcal{I}}$, associated to enhancements, require $\frac{1}{2}\mathbb{L}^2 = 1$. It appears somewhat unnatural to indicate what kind of constrained mode expansion must be performed in each case. However, these LMC constraints might be implemented in the Lagrangian through, for instance, Lagrange multipliers. Thus, if we indicate by $\phi_N(x, \mathbb{Y})$ a field such that its mode expansion must be restricted to $\frac{1}{2}\mathbb{L}^2 = 1 - N$, in a DFT language we would require

$$-\frac{1}{2}\partial_{\mathcal{I}}\partial^{\mathcal{I}}\phi_N(x, \mathbb{Y}) = -\frac{1}{2}(\partial_L^2 - \partial_R^2)\phi_N(x, \mathbb{Y}) = 1 - N. \tag{2.30}$$

In the cases considered here $N = 0, 1$ label the number of Left indices. Clearly, the so called *strong constraint* of DFT (see for instance [15, 16]) cannot be satisfied if enhancement phenomena are included.

The term $\frac{1}{2}(\eta^{\mathcal{J}\mathcal{I}} - H^{\mathcal{J}\mathcal{I}})$ acts as a covariant $O(r_l, r)$ projector.

If just the zero modes are kept we notice that the first two rows in (2.13) are formally reproduced with $g_{\mu\nu} = G^{(0)}_{\mu\nu}, M_{\hat{I}\bar{J}} = M^{(0)}_{\hat{I}\bar{J}}$, etc. However, a non trivial action of the $\star$-product arises whenever non zero modes come into play as it happens, for instance, in products of fields associated to enhancements (and thus requiring expansions with $\delta(\frac{1}{2}\mathbb{L}^2, 1)$ constraint). We provide a more detailed discussion of this situation in the next section. Also, the different terms in the action are now defined as

$$\begin{aligned} F_{\mu\nu} &= 2\partial_{[\mu}A_{\nu]} + igA_\mu \star A_\nu + 2gA^{\mathcal{I}}_{[\mu} \star \partial_{\mathcal{I}}A_{\nu]} \\ F^{\mathcal{I}}_{\mu\nu} &= 2\partial_{[\mu}A^{\mathcal{I}}_{\nu]} + g\partial^{\mathcal{I}}A_\mu \star A_\nu + 4igA_{[\mu} \star A^{\mathcal{I}}_{\nu]} \end{aligned} \tag{2.31}$$

$$\begin{aligned} \mathcal{D}_\mu M^{\mathcal{I}} &= \partial_\mu M^{\mathcal{I}} + igA_\mu \star M^{\mathcal{I}} + gA^{\mathcal{J}}_\mu \star \partial_{\mathcal{J}}M^{\mathcal{I}} - g\partial_{\mathcal{J}}A_\mu \star \left(H^{\mathcal{J}\mathcal{I}} - \eta^{\mathcal{J}\mathcal{I}}\right) \\ \mathcal{D}_\mu \mathcal{H}^{\mathcal{I}\mathcal{J}} &= \partial_\mu \mathcal{H}^{\mathcal{I}\mathcal{J}} + g\partial^{\mathcal{I}}A_\mu \star M^{\mathcal{J}} + 2igA_\mu \star \mathcal{H}^{\mathcal{I}\mathcal{J}} + igA^{\mathcal{K}}_\mu \partial_{\mathcal{K}}\mathcal{H}^{\mathcal{I}\mathcal{J}} \end{aligned} \tag{2.32}$$

(where $g = \frac{1}{\sqrt{\alpha'}}$) by generalizing (2.7).

Finally, the three form H is defined as

$$H = dB + F^{\mathcal{I}} \star \wedge A_{\mathcal{I}} + F \star \wedge A. \tag{2.33}$$

Whenever a product of two fields appears a $\star$-product must be used. For instance, the generalized metric must be expressed in terms of fluctuations as in (2.11) but with a $\star$ replacing the ordinary product.

All the fields that we are considering contain modes that are massless at some specific values of moduli Φ. This is always the case for the zero modes $G^{(0)}_{\mu\nu}(x), B^{(0)}_{\mu\nu}(x), \varphi^{(0)}(x)$, $M^{(0)}_{\hat{I},\bar{J}}(x)$ that are massless everywhere in moduli space, whereas n_c modes $A^{(\check{\mathbb{P}})}_\mu(x), M^{(\check{\mathbb{P}})}_{\bar{I}}(x)$ become massless at a point Φ_0 for momenta in $\check{G}(\Phi_0)_{n_c}$ (see (2.22)). These are the modes that participate in the enhancement phenomena. When approaching a point Φ_0 in moduli space the light spectrum will contain the zero mode massless fields plus the n_c slightly massive modes in $\check{G}(\Phi_0)_{n_c}$, all other fields having masses of the order of the string mass. When moving to some other fixed point Φ_1 other set of modes (intersections can occur) in $\check{G}(\Phi_1)_{n_c}$ will become light.[8] Therefore, the action (2.29) can be splitted as

$$S_{\text{enh}}(\Phi) = S_{\text{light at }\Phi_0}(\Phi) + S_{\text{heavy at }\Phi_0}(\Phi) = S_{\text{light at }\Phi_1}(\Phi) + S_{\text{heavy at }\Phi_1}(\Phi) = \dots \tag{2.34}$$

The first (second) splitting is convenient when Φ is close to Φ_0 (Φ_1). In this case, at low energies, the second term in the action (and also S' above), containing heavy states (of order α'^{-1}) and light states in interaction with them, does not contribute. We will be left with the effective $S_{\text{eff}}(\Phi \approx \Phi_0)$ low energy action

$$S_{\text{eff}}(\Phi \approx \Phi_0) = S_{\text{enh}}(\Phi \approx \Phi_0) = S_{\text{light at }\Phi_0}(\Phi \approx \Phi_0) \tag{2.35}$$

and similarly for $\Phi \approx \Phi_1$ etc. At $\Phi = \Phi_0$ all fields in $S_{\text{eff}}(\Phi_0)$ become massless and the effective action should reproduce (2.5) with gauge group $G_L \times \mathrm{U}(1)^r_R$. The $\star$-product plays a crucial role in reproducing the non-Abelian group structure. When slightly moving away from Φ_0, the gauge symmetry should break, generically, to $\mathrm{U}(1)^{r+16}_L \times \mathrm{U}(1)^r_R$ and $S_{\text{eff}}(\Phi \approx \Phi_0)$ should contain massless and massive physical states correctly transforming under the Abelian groups.

Besides these features addressed in the next section when mode expansions are performed, we stress that $S_{\text{enh}}(\Phi)$ appears to encode some relevant information about very massive states as discussed in an explicit example in 4.5.

3 The action for GKK modes

In this section we perform the expansion of the fields in the above action in terms of GKK modes, compute the $\star$-products for these modes and finally integrate over the internal coordinates $\mathbb{Y}$ in order to obtain the moduli dependent d dimensional effective action. In particular, we will show that after integrating out the massive modes, the massless GKK modes at a self-dual point, eq. (2.29) give rise to the gauge enhanced action (2.13).

The particular $\star$-product we consider here is a generalization of the one proposed in [22, 23] to the case of the heterotic string. It is described in appendix B. For two mode expanded fields it reads

$$(\phi_{N_1} \star \psi_{N_2})(x, \mathbb{Y}) = \sum_{\mathbb{L}_1,\mathbb{L}_2}{}' e^{i\pi l_1\cdot \tilde{l}_2} \phi_{N_1}^{(\mathbb{L}_1)}(x)\psi_{N_2}^{(\mathbb{L}_2)}(x) e^{i(\mathbb{L}_1+\mathbb{L}_2).\mathbb{Y}} \tag{3.1}$$

where a phase $l_1 \cdot \tilde{l}_2 = p_{1m}\tilde{p}^{2m} + p_{1I}\tilde{p}^I_2$ dependent on the KK momenta l_1 of the first field and the windings $\tilde{l}_2$ of second mode is generated (see (B.4)). The first term corresponds to a sum over the internal compactification lattice indices. The sum over heterotic directions

[8] There will always be modes that remain very massive, as for instance $G^{(\mathbb{L})}_{\mu\nu}(x)$ with $\mathbb{L} \neq 0$.

is constrained by a chiral projection that eliminates Right heterotic momenta. It can be expressed as

$$p_{1I}\tilde{p}_2^I = \frac{1}{2}P_1 E P_2 \tag{3.2}$$

in terms of Spin(32),[9] weights and $E_{IJ} = G_{IJ} + B_{IJ}$ (see appendix B). The prime in the sum indicates that the constraint $\frac{1}{2}\mathbb{L}_i^2 = 1 - N_i$ $(i = 1, 2)$ must be imposed for the field with subindex N_i. By using that

$$l_1 \cdot \tilde{l}_2 + l_2 \cdot \tilde{l}_1 = \mathbb{L}_1\mathbb{L}_2 = \frac{1}{2}(\mathbb{L}_1 + \mathbb{L}_2)^2 - \frac{1}{2}\mathbb{L}_1^2 - \frac{1}{2}\mathbb{L}_2^2 \tag{3.3}$$

by recalling that the exponents are just integer multiples of π and by using LMC we can rewrite the above product as

$$(\phi_{N_1} \star \psi_{N_2})(x, \mathbb{Y}) = \sum_{\mathbb{L}_1,\mathbb{L}_2}{}' e^{i\pi(\frac{1}{2}\mathbb{L}^2 - N_1 - N_2)} e^{i\pi l_2\cdot\tilde{l}_1} \cdot \phi_{N_1}^{(\mathbb{L}_1)}(x)\psi_{N_2}^{(\mathbb{L}_2)}(x) e^{i(\mathbb{L}_1+\mathbb{L}_2).\mathbb{Y}} \tag{3.4}$$

where $\mathbb{L} = \mathbb{L}_1 + \mathbb{L}_2$. By comparing to (3.1) we see that the $\star$-product is non commutative unless the phase $e^{i\pi(\frac{1}{2}\mathbb{L}^2 - N_1 - N_2)} = 1$. For instance, if $\mathbb{L} = 0$, as we would find if we integrated on $\mathbb{Y}$, we find

$$\int d\mathbb{Y}\, \phi_{N_1} \star \psi_{N_2} = e^{i\pi(N_1+N_2)} \int d\mathbb{Y}\, \psi_{N_2} \star \phi_{N_1}. \tag{3.5}$$

For the cases we are considering here we notice that fields with similar LMC commute whereas for $N_1 = 0, N_2 = 1$ (or viceversa) they anticommute.

Let us proceed to consider the product of three fields. In this case, by using associativity (see (B.3)), we have

$$(\phi_{N_1} \star \psi_{N_2} \star \lambda_{N_3})(x, \mathbb{Y}) = \sum_{\mathbb{L}_1,\mathbb{L}_2,\mathbb{L}_3}{}' \tilde{f}_{\mathbb{L}_1\mathbb{L}_2\mathbb{L}_3} \phi_{N_1}^{(\mathbb{L}_1)}(x)\psi_{N_2}^{(\mathbb{L}_2)}(x)\lambda_{N_3}^{(\mathbb{L}_3)}(x) e^{i(\mathbb{L}_1+\mathbb{L}_2+\mathbb{L}_3).\mathbb{Y}} \tag{3.6}$$

where

$$\tilde{f}_{\mathbb{L}_1\mathbb{L}_2\mathbb{L}_3} = e^{i\pi l_1\cdot\tilde{l}_2} e^{i\pi(l_1+l_2)\cdot\tilde{l}_3} \equiv \pm 1. \tag{3.7}$$

If we integrate over internal coordinates, due to momentum conservation $\mathbb{L} = \mathbb{L}_1 + \mathbb{L}_2 + \mathbb{L}_3 = 0$, the second phase becomes $e^{-i\pi\frac{1}{2}\mathbb{L}_3^2}$ and therefore

$$\tilde{f}_{\mathbb{L}_1\mathbb{L}_2\mathbb{L}_3} = e^{i\pi l_1\cdot\tilde{l}_2} e^{-i\pi\frac{1}{2}\mathbb{L}_3^2} \tag{3.8}$$

In this case,

$$\begin{aligned}\tilde{f}_{\mathbb{L}_1\mathbb{L}_2\mathbb{L}_3} &= e^{i\pi l_1\cdot\tilde{l}_2} e^{-i\pi\frac{1}{2}\mathbb{L}_3^2} = e^{i\pi(l_1\cdot\tilde{l}_2 + l_2\cdot\tilde{l}_1)} e^{i\pi l_2\cdot\tilde{l}_1} e^{-i\pi\frac{1}{2}\mathbb{L}_3^2} \\ &= e^{i\pi(\frac{1}{2}\mathbb{L}_3^2 - \frac{1}{2}\mathbb{L}_2^2 - \frac{1}{2}\mathbb{L}_1^2)} e^{i\pi l_2\cdot\tilde{l}_1} e^{-i\pi\frac{1}{2}\mathbb{L}_3^2} = -e^{i\pi(N_1+N_2+N_3)} \tilde{f}_{\mathbb{L}_2\mathbb{L}_1\mathbb{L}_3}\end{aligned} \tag{3.9}$$

where we have used (3.3) above with $\mathbb{L}_1 + \mathbb{L}_2 = -\mathbb{L}_3$. A similar phase is obtained if $2 \leftrightarrow 3$.

[9] We will mainly refer to Spin(32) but results are valid for $E_8 \times E_8$ as well.

We conclude that the product of three fields with $N_i = 0$, as it is the case for charged fields participating in the enhancements, the phases $\tilde{f}_{\mathbb{L}_1\mathbb{L}_2\mathbb{L}_3}$ are completely antisymmetric under index permutation. This result is valid both for massless and massive states. On the contrary, for modes originating in 10D fields, $N_i = 1$, and the phase becomes irrelevant so the $\star$-product reduces to just the ordinary product.

In the following subsections we analyze the different contributions in the action (2.29) in terms of their mode expansions.

3.1 The vectors kinetic term

Let us analyze first $\int d^dx d\mathbb{Y}\; F_{\mu\nu} \star F^{\mu\nu}$. In order to do it let us consider the Fourier[10] component (see (2.31)) $F^{(\mathbb{L})}_{\mu\nu}$ as follows

$$F^{(\mathbb{L})}_{\mu\nu} = \int d\mathbb{Y} F_{\mu\nu} e^{-i\mathbb{L}\cdot\mathbb{Y}}, \tag{3.10}$$

where:

$$F^{(\mathbb{L})}_{\mu\nu} = 2\partial_{[\mu}A^{(\mathbb{L})}_{\nu]} + ig\sum_{\mathbb{L}_2}{}' \tilde{f}_{\mathbb{L}\mathbb{L}_2\mathbb{L}_3} A^{(\mathbb{L}_2)}_{\mu} A^{(\mathbb{L}_3)}_{\nu} + 2ig\sum_{\mathbb{L}_2}{}' f_{\mathbb{L}_3-\mathbb{L}_3\mathcal{I}} A^{\mathcal{I}(\mathbb{L}_2)}_{[\mu} A^{(\mathbb{L}_3)}_{\nu]}. \tag{3.11}$$

Here, $\mathbb{L}_3 = \mathbb{L} - \mathbb{L}_2$, and we have defined $\tilde{f}_{\mathbb{L}-\mathbb{L}\mathcal{I}}$ as

$$\tilde{f}_{\mathbb{L}-\mathbb{L}\hat{I}} = l_{\hat{I}}(\Phi), \qquad \tilde{f}_{\mathbb{L}-\mathbb{L}\bar{I}} = l_{R,\bar{I}}(\Phi). \tag{3.12}$$

Recall that $F^{(\mathbb{L})}_{\mu\nu}$ depends on moduli point.

Let us look at the contributions, at low energy, at self-dual points. On the one hand, from the last sum we keep the zero mode contribution $A^{\mathcal{I}(0)}_{\mu} \equiv A^{\mathcal{I}}_{\mu}(x)$ giving rise to Cartan vector fields. On the other hand the light modes on the first sum correspond to $\check{\mathbb{P}} \in \check{G}_{n_c}(\Phi_0)$. Thus, when sliding to $\Phi = \Phi_0$ (3.11) will reduce to (2.7) as long as we identify

$$A^{(\mathbb{L}_i)}_{\mu} \leftrightarrow A^{\alpha^{(\mathbb{L}_i)}}_{\mu}, \quad -A^{(-\mathbb{L}_i)}_{\mu} \leftrightarrow A^{\alpha^{(-\mathbb{L}_i)}}_{\mu}, \tag{3.13}$$

where $\mathbb{L}_i$ is in a one to one correspondence with the positive roots $\alpha_i \equiv \alpha^{(\mathbb{L}_i)}$, $i = 1, \ldots, \frac{n_c}{2}$, (and $-\mathbb{L}_i$ with $-\alpha_i \equiv \alpha^{(-\mathbb{L}_i)}$) of the enhanced group (the underlying reason for this identification is $f_{\mathbb{L}\mathbb{L}_2\mathbb{L}_3}$ is invariant under $\mathbb{L}_i \to -\mathbb{L}_i$). Thus, for each root $\alpha^{(\check{\mathbb{P}})} = l^{(\check{\mathbb{P}})}_L(\Phi_0)$ (see (2.21))

$$\begin{aligned}
F^{\alpha^{(\check{\mathbb{P}})}}_{\mu\nu} &= 2\partial_{[\mu}A^{\alpha^{(\check{\mathbb{P}})}}_{\nu]} + ig\sum_{\check{\mathbb{P}}_2}{}' f_{\alpha^{(\check{\mathbb{P}}_2)}\alpha^{(\check{\mathbb{P}}_3)}\alpha^{(-\check{\mathbb{P}})}} A^{\alpha^{(\check{\mathbb{P}}_2)}}_{\mu} A^{\alpha^{(\check{\mathbb{P}}_3)}}_{\nu} + 2ig f_{\hat{I}\alpha^{(\check{\mathbb{P}})}\alpha^{(-\check{\mathbb{P}})}} A^{\hat{I}}_{[\mu} A^{\alpha^{(\check{\mathbb{P}})}}_{\nu]} \\
F^{\alpha^{(-\check{\mathbb{P}})}}_{\mu\nu} &= 2\partial_{[\mu}A^{\alpha^{(-\check{\mathbb{P}})}}_{\nu]} + ig\sum_{\check{\mathbb{P}}_2}{}' f_{\alpha^{(-\check{\mathbb{P}}_2)}\alpha^{(-\check{\mathbb{P}}_3)}\alpha^{(\check{\mathbb{P}})}} A^{\alpha^{(-\check{\mathbb{P}}_2)}}_{\mu} A^{\alpha^{(-\check{\mathbb{P}}_3)}}_{\nu} + 2ig f_{\hat{I}\alpha^{(-\check{\mathbb{P}})}\alpha^{(\check{\mathbb{P}})}} A^{\hat{I}}_{[\mu} A^{\alpha^{(-\check{\mathbb{P}})}}_{\nu]}
\end{aligned} \tag{3.14}$$

where we have identified $f_{\alpha^{(-\check{\mathbb{P}}_1)}\alpha^{(\check{\mathbb{P}}_2)}\alpha^{(\check{\mathbb{P}}_3)}} = f_{-\alpha_1\alpha_2\alpha_3} = -f_{\alpha_1-\alpha_2-\alpha_3}$ as the algebra structure constants for charged generators. Therefore, (3.14) becomes the field strength for the

[10] We normalize the integration variables so as to have a unit volume factor. Also, we use that $\int d^{2n}\mathbb{Y} e^{i(\mathbb{P}_M+\mathbb{Q}_M)\mathbb{Y}^M} = \delta^{2n}(\mathbb{P}_M + \mathbb{Q}_M)$.

charged fields of the corresponding gauge theory. Then, up to third order in fluctuations we can write

$$\int dx d\mathbb{Y}\ F_{\mu\nu} \star F^{\mu\nu} = -\sum_{\mathbb{L}}{}' \int dx F^{(\mathbb{L})}_{\mu\nu} F^{(-\mathbb{L})\mu\nu}$$
$$= \int dx F_{\alpha\mu\nu} F^{\alpha\mu\nu}. \tag{3.15}$$

We have thus matched the first term of the third row of action (2.13). The second term of the same row is reproduced by $\mathcal{H}_{\mathcal{I}\mathcal{J}} \star F^{\mathcal{I}}_{\mu\nu} \star F^{\mathcal{J}\mu\nu}$ in (2.29) since, when focusing only on massless GKK modes,

$$F^{\mathcal{I}}_{\mu\nu} = 2\partial_{[\mu} A^{\mathcal{I}}_{\nu]} + 2ig \sum_{\check{\mathbb{P}}} f_{\alpha^{(\check{\mathbb{P}})}\alpha^{(-\check{\mathbb{P}})}}{}^{\mathcal{I}} A^{\alpha^{(\check{\mathbb{P}})}}_{[\mu} A^{\alpha^{(-\check{\mathbb{P}})}}_{\nu]} \tag{3.16}$$

(remember $f_{\alpha^{(\check{\mathbb{P}})}\alpha^{(-\check{\mathbb{P}})}}{}^{\bar{I}} = 0$ for massless GKK modes).

3.1.1 $D = 10$ heterotic string action

It is interesting to consider the $D = 10$ theory. In this case $\mathbb{L} \equiv l_L$ is always Left handed where $l^I = P^I$ are just the components of the Spin(32) roots $P = (\underline{\pm 1, \pm 1, 0, \ldots})$ (underlining denoting permutation). Therefore, (3.15) becomes the Spin(32) gauge kinetic term for charged fields. On the other hand eq. (3.16) with $\mathcal{I} = 1, \ldots 16$ provides the field strength for the Cartan components and, therefore, when combined with the other terms in second and first rows in (2.29) the low energy $D = 10$ heterotic effective action is recovered. Recall that none of the other terms are present since there is no compactification at all.

3.2 Scalars kinetic term

Following similar steps as above we can write

$$\int dx d\mathbb{Y}\ \mathcal{D}_\mu M_{\mathcal{J}} \star \mathcal{D}^\mu M^{\mathcal{J}} = -\int dx\ \sum_{\mathbb{L}}{}' \mathcal{D}_\mu M^{(\mathbb{L})}_{\bar{J}} \mathcal{D}^\mu M^{(-\mathbb{L})\bar{J}}$$

where

$$\mathcal{D}_\mu M^{(\mathbb{L})}_{\bar{J}} = \partial_\mu M^{(\mathbb{L})}_{\bar{J}} + ig \sum_{\mathbb{L}_2}{}' \tilde{f}_{\mathbb{L}\mathbb{L}_2\mathbb{L}_3} M^{(\mathbb{L}_2)}_{\bar{J}} A^{(\mathbb{L}_3)}_\mu + ig \tilde{f}_{\mathbb{L}-\mathbb{L}I} M^{(\mathbb{L})}_{\bar{J}} A^{I(0)}_\mu \tag{3.17}$$
$$+ig \sum_{\mathbb{L}_2}{}' \tilde{f}_{\mathbb{L}\mathbb{L}_2\mathbb{L}_3} M^{(\mathbb{L}_2)}_{\hat{I}\bar{J}} \mathbb{L}_3^{\hat{I}} A^{(\mathbb{L}_3)}_\mu + igl_{R\bar{I}} A^{\bar{I}(0)}_\mu M^{(\mathbb{L})}_{\bar{J}} - 2igl_{R\bar{J}} A^{(\mathbb{L})}_\mu + \ldots$$

is the Fourier transform of first equation in (2.32) with $\mathbb{L}_3 = \mathbb{L} - \mathbb{L}_2$. We have used that $(\bar{H}_{\mathcal{I}\mathcal{J}} - \eta_{\mathcal{I}\mathcal{J}})\mathbb{L}^{\mathcal{J}} = (0, 2\, l_{R\bar{I}})$. The dots amount for terms containing massive modes $A^{I(\mathbb{L}_3)}_\mu$. The usual covariant derivative for charged vectors of the gauge group G_L (first term of the second row of (2.13)) is reproduced at enhancement point Φ_0, for $\check{\mathbb{P}} \in \check{G}_{n_c}(\Phi_0)$ with the identifications

$$M^{(\mathbb{L}_i)}_{\bar{J}} \leftrightarrow M^{\alpha^{(\mathbb{L}_i)}}_{\bar{J}}, \quad -M^{(-\mathbb{L}_i)}_{\bar{J}} \leftrightarrow M^{\alpha^{(-\mathbb{L}_i)}}_{\bar{J}}. \tag{3.18}$$

Covariant derivative of scalar modes $M^{(\mathbb{L}_i)}_{\hat{I}\bar{J}}$ arise from the term $\frac{1}{8}\mathcal{D}_\mu\mathcal{H}^{\mathcal{I}\mathcal{J}}\star\mathcal{D}^\mu\mathcal{H}_{\mathcal{I}\mathcal{J}}$ in (2.13) and read

$$\begin{aligned}\mathcal{D}_\mu M^{(\mathbb{L})}_{\hat{I}\bar{J}} &= \partial_\mu M^{(\mathbb{L})}_{\hat{I}\bar{J}} + ig\sum_{\mathbb{L}_2}{}' \tilde{f}_{\mathbb{L}\mathbb{L}_2\mathbb{L}_3} A^{(\mathbb{L}_2)}_\mu \mathbb{L}_{2\hat{I}} M^{(\mathbb{L}_3)}_{\bar{J}} + 2ig\sum_{l}{}' \tilde{f}_{\mathbb{L}\mathbb{L}_2\mathbb{L}_3} A^{(\mathbb{L}_2)}_\mu M^{(\mathbb{L}_3)}_{\hat{I}\bar{J}}\\ &\quad + ig\mathbb{L}_{\hat{K}} A^{\hat{K}(0)}_\mu M^{(\mathbb{L})}_{\hat{I}\bar{J}} + igl_{R\bar{I}} A^{\bar{I}(0)}_\mu M^{(\mathbb{L})}_{\hat{I}\bar{J}} + \dots \end{aligned} \tag{3.19}$$

The massless scalars are provided by the zero modes $M^{(0)}_{\hat{I}\bar{J}}$. In this case, the last two terms drop out and (2.8) is reproduced.

3.3 Scalar potential and other couplings

The scalar potential is such that it vanishes (up to third order in fluctuations) for massless states, it is $O(r_L, r)$ invariant and it reproduces the scalar potential away from the enhancing point for scalars that would be massless at such point, as computed in [7]. It appears that the most general form is

$$\frac{1}{4}\partial_{\mathcal{J}}M^{\mathcal{I}}\star\partial_{\mathcal{K}}M_{\mathcal{I}}\left(\mathcal{H}^{\mathcal{J}\mathcal{K}}-\eta^{\mathcal{J}\mathcal{K}}\right)+i\frac{1}{2}\partial_{\mathcal{I}}M^{\mathcal{J}}\star M_{\mathcal{J}}\star M^{\mathcal{I}}+\mathcal{O}(M^4). \tag{3.20}$$

It is worth noticing that the first term above, when opened up in terms of fluctuations, contains a fourth-order term (at the enhancement point) of the following schematic form: $\sum_\alpha f_{I\alpha-\alpha}f_{J-\alpha\alpha}M_{I\bar{I}}M_{J\bar{I}}M_{\alpha\bar{K}}M_{-\alpha\bar{K}}$ and is part of the fourth order scalar potential [8]. To complete the full fourth-order terms in the potential more terms are needed. For instance we would need an extra term of the form $\partial_{\mathcal{I}}\phi_{\bar{I}}\star\partial^{\mathcal{I}}\phi_{\bar{I}}\star\phi_{\bar{J}}\star\phi_{\bar{J}}$ among others. We leave the analysis of these extra terms for future work.

Finally our action (2.29) contains a last term, $M_{\mathcal{I}}\star F^{\mu\nu}\star F^{\mathcal{I}}_{\mu\nu}$, which gives rise to the last term of (2.13) at the enhancement point (actually this term is always present and away from the self-dual point it gives rise to the adequate coupling between a massive scalar, a massive vector and a massless $\mathrm{U}(1)_R$ vector).

4 Breaking and enhancement of gauge symmetry along moduli space

We have shown the explicit mode expansions for some of the terms appearing in the action. Computation of other terms proceed by following similar steps.

Several interesting results like vector and scalar masses, presence of would-be Goldstone bosons, etc. can be straightforwardly read out from these expansions. We discuss some of these issues below.

4.1 Vector masses

Vector boson masses can be extracted from the fourth term in the covariant derivative-like term (3.17)

$$\begin{aligned}\int dxd\mathbb{Y}\, \frac{1}{4}D_\mu M_{\mathcal{J}} \star D^\mu M^{\mathcal{J}} &= \sum_{\mathbb{L}}{}' -\frac{1}{4}D_\mu M^{(\mathbb{L})}_{\bar{J}} D^\mu M^{(-\mathbb{L})\bar{J}} = \\ &= \ldots + \sum_{\mathbb{L}}{}' \frac{1}{4} g^2 2\big[\mathbb{L}\cdot\big(\bar{H}-\eta\big)\cdot\mathbb{L}\big] A^{(\mathbb{L})}_\mu A^{(-\mathbb{L})}_\mu \\ &= \cdots + \sum_{\mathbb{L}}{}' \frac{m_A^2}{2} A^{(\mathbb{L})}_\mu A^{(-\mathbb{L})}_\mu\end{aligned} \tag{4.1}$$

where we have used the string theory result $m_A^2 = 2l_R^2 = \mathbb{L}\cdot\big(\bar{H}-\eta\big)\cdot\mathbb{L}$ for the masses of the charged vector fields. We also observe that there are no mass terms for Cartan vectors $A^{\mathcal{I}}_\mu$ so, generically, the gauge group is $\mathrm{U}(1)^{r_L}\times\mathrm{U}(1)^r$. At $\Phi=\Phi_0$ and for $\check{\mathbb{P}}\in\check{G}_{n_c}(\Phi_0)$ vector bosons become massless leading to gauge symmetry enhancement.

Finally we check the normalizations. Since the kinetic term of the vectors reads:

$$-\int dxd\mathbb{Y}\, \frac{1}{4}F_{\mu\nu}\star F^{\mu\nu} = \sum_{\mathbb{L}}{}' \frac{1}{4}\big(\partial_\mu A^{(\mathbb{L})}_\nu - \partial_\nu A^{(\mathbb{L})}_\mu\big)\big(\partial_\mu A^{(-\mathbb{L})}_\nu - \partial_\nu A^{(-\mathbb{L})}_\mu\big) + \ldots \tag{4.2}$$

when adding (3.17) we find

$$\sum_{\mathbb{L}}{}' \frac{1}{4}\big(\partial_\mu A^{(\mathbb{L})}_\nu - \partial_\nu A^{(\mathbb{L})}_\mu\big)\big(\partial_\mu A^{(-\mathbb{L})}_\nu - \partial_\nu A^{(-\mathbb{L})}_\mu\big) + \frac{m_A^2}{2} A^{(\mathbb{L})}_\mu A^{(-\mathbb{L})}_\mu \tag{4.3}$$

which is the Proca Lagrangian with signature $(-+++\ldots)$ (with a global different normalization) of a vector with mass m_A.

4.2 Goldstone bosons

From the same scalars kinetic factors above we find the terms

$$\begin{aligned}+\sum_{\mathbb{L}}{}' \frac{1}{4}D_\mu M^{(\mathbb{L})}_{\bar{J}} D^\mu M^{(-\mathbb{L})\bar{J}} &= \ldots + \sum_{\mathbb{L}}{}' 2\frac{1}{4}\mathbb{L}^{\bar{J}}\partial_\mu M^{(\mathbb{L})}_{\bar{J}} A^{(-\mathbb{L})}_\mu \\ &= \ldots + \sum_{\mathbb{L}}{}' \frac{1}{2} l_R^{\bar{J}}\partial_\mu M^{(-\mathbb{L})}_{\bar{J}} A^{(\mathbb{L})}_\mu .\end{aligned} \tag{4.4}$$

As also discussed in [7] this coupling indicates that, for a given vector boson $A^{(\mathbb{L})}_\mu$, there exists a combination of $\bar{I}=1,\ldots,r=10-d$ of would-be Goldstone bosons $l_R^{\bar{J}}\partial_\mu M^{(-\mathbb{L})}_{\bar{J}}$ (this is exactly the Goldstone combination which can be read from the vertex operators in string theory [4]). In fact, at enhancement point Φ_0 and for $\check{\mathbb{P}}\in\check{G}_{n_c}(\Phi_0)$ these terms are not present since $l_R=0$. However, when sliding away from Φ_0, $l_R\neq 0$, and these n_c combinations provide the longitudinal components of the n_c corresponding $A_\mu(\mathbb{L})$ massive vector bosons. Namely,

$$A^{(\mathbb{L})'}_\mu = A^{(\mathbb{L})}_\mu + l_R^{\bar{J}}\partial_\mu M^{(\mathbb{L})}_{\bar{J}} . \tag{4.5}$$

4.3 Scalar masses

The masses of the scalar fields can be read from the quadratic terms in the scalar potential (3.20)

$$\frac{1}{4}\int dxd\mathbb{Y}(\mathcal{H}^{\mathcal{I}\mathcal{J}}-\eta^{\mathcal{I}\mathcal{J}})\partial_{\mathcal{I}}M^{\mathcal{K}}\star\partial_{\mathcal{J}}M_{\mathcal{K}}={\sum_{\mathbb{L}}}'\frac{1}{4}[\mathbb{L}\cdot(\bar{H}-\eta)\cdot\mathbb{L}]M^{(\mathbb{L})\mathcal{K}}M_{\mathcal{K}}^{(-\mathbb{L})}+\dots$$
$$={\sum_{\mathbb{L}}}'\frac{1}{4}m^2_{M^{(\mathbb{L})}}M^{(\mathbb{L})\bar{K}}M_{\bar{K}}^{(-\mathbb{L})}+\dots \tag{4.6}$$

with

$$m^2_{M^{(\mathbb{L})}}=2l_R^2=\mathbb{L}\cdot(\bar{H}-\eta)\cdot\mathbb{L}, \tag{4.7}$$

as expected from string theory. They coincide with the masses of the corresponding vector boson modes.

As in the vector case, it is still necessary to check for the relative coefficients, so we must compare the above terms with the scalar kinetic term

$$\int d\mathbb{Y}\,\frac{1}{4}D_\mu M_{\mathcal{J}}\star D^\mu M^{\mathcal{J}}=-{\sum_{\mathbb{L}}}'\frac{1}{4}\partial_\mu M^{(\mathbb{L})\mathcal{J}}\partial^\mu M_{\mathcal{J}}^{(-\mathbb{L})}+\dots \tag{4.8}$$

Altogether we have

$$-{\sum_{\mathbb{L}}}'\left[\frac{1}{4}\partial_\mu M^{(\mathbb{L})\mathcal{J}}\partial^\mu M_{\mathcal{J}}^{(-\mathbb{L})}+\frac{1}{4}m^2_{M_{\bar{J}}^{(\mathbb{L})}}M^{(\mathbb{L})\mathcal{J}}M_{\mathcal{J}}^{(-\mathbb{L})}\right] \tag{4.9}$$

which is the Lagrangian of a massive scalar field (with a global normalization) on the signature $(-+++\dots)$ with mass $m_{M_{\bar{J}}^{(\mathbb{L})}}$.

4.4 Duality and gauge invariance

Let us close this section by commenting on T-duality and gauge invariance.

We notice that, even if the different terms in the action (2.13) are written in an $O(r_L,r)$ invariant way, by index contraction, the effect of the $\star$-product on T-duality is not transparent. On the one hand we expect that, whenever windings and momenta are included, the symmetry group becomes $O(r_L,r,\mathbb{Z})$. Consider for instance (3.4). Each term on the expansion contains a Fourier mode labeled by $\check{\mathbb{P}}$, encoding momenta and winding modes (2.15), as well as an exponential term $e^{i\mathbb{L}.\mathbb{Y}}$ where the exponent is explicitly $O(r_L,r,\mathbb{Z})$ covariant. On the other hand, a $h\in O(r_L,r,\mathbb{Z})$ rotation leads to $\check{\mathbb{P}}\to\check{\mathbb{P}}'=h\check{\mathbb{P}}$ but also to $e^{i\pi\tilde{l}_1l_2}\phi^{(\check{\mathbb{P}}_1)}(x)\psi^{(\check{\mathbb{P}}_2)}(x)\to e^{i\pi\tilde{l}'_1l'_2}\phi^{(\check{\mathbb{P}}'_1)}(x)\psi^{(\check{\mathbb{P}}'_2)}(x)$, where the heterotic part is expressed in terms of 16 windings and momenta as discussed in (B.4). However, if $\check{\mathbb{P}}$ satisfies LMC so does $\check{\mathbb{P}}'$ and, since we are summing over all modes satisfying LMC, the sum will be invariant.

Notice that, if we restricted our analysis to a set of GKK momenta in $\check{G}(\Phi_0)_{n_c}$, the above transformations will take us out of this set. Namely, the $\check{\mathbb{P}}'$ will not become massless at Φ_0. However, $\check{\mathbb{P}}'\in\check{G}(h\Phi_0)_{n_c}$, and therefore their associated fields will become massless

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at the transformed moduli point $h\Phi_0$ (note that the mass terms are invariant when transforming both the background and momenta and windings). We will illustrate this fact in an example below. Let us stress that if any of the fields contained an $O(r_L, r)$ index, as for example $M_{\mathcal{I}}^{(\mathbb{P})}$, it must appear contracted in an invariant way as it indeed happens in the action.

The action we are dealing with contains massive and massless states. At a $\mathrm{U}(1)^{r_L} \times \mathrm{U}(1)^r$ generic points, besides the $2r + 16$ Abelian vectors and the gravity sector fields, all other vector and scalar fields will be massive. Recall that a field $\Phi(x)^{(\mathbb{L})}$ carries charge $(\mathbb{L})_{\mathcal{I}} = \tilde{f}_{\mathbb{L}-\mathbb{L}\mathcal{I}}$ with respect to the Abelian factor $A_\mu^{(\mathcal{I})}$. Therefore, under a $U^{\mathcal{I}}(1)$ gauge transformation

$$
\begin{aligned}
\Phi(x)^{(\mathbb{L})} &\rightarrow e^{ig\mathbb{L}_{\mathcal{I}}\alpha^{\mathcal{I}}(x)}\Phi(x)^{(\mathbb{L})} \\
A_\mu^{\mathcal{I}} &\rightarrow A_\mu^{\mathcal{I}} - \partial_\mu \alpha^{\mathcal{I}}
\end{aligned}
\tag{4.10}
$$

and therefore, gauge invariance should be ensured by a derivative of the form $D_\mu\Phi(x)^{(\mathbb{L})} = \partial_\mu\Phi(x)^{(\mathbb{L})} + i(\mathbb{L})_{\mathcal{I}}A_\mu^{(\mathcal{I})}\Phi(x)^{(\mathbb{L})}$. In fact, it can be checked that this is indeed the case for the covariant derivative of scalars in eq. (3.17) as well as for the derivative of the massive vectors as given in eq. (3.11). For instance, in the latter case we have that

$$
\begin{aligned}
F_{\mu\nu}^{(\mathbb{L})} \rightarrow\; & e^{ig\mathbb{L}_{\mathcal{I}}\alpha^{\mathcal{I}}(x)} 2\partial_{[\mu}A_{\nu]}^{(\mathbb{L})} + 2ige^{ig\mathbb{L}_{\mathcal{I}}\alpha^{\mathcal{I}}(x)}\mathbb{L}_{\mathcal{I}}A_{[\nu}\partial_{\mu]}\alpha^{\mathcal{I}} + \\
& + ige^{ig\mathbb{L}_{\mathcal{I}}\alpha^{\mathcal{I}}(x)} \sum_{\mathbb{L}_2}{}' \tilde{f}_{\mathbb{L}\mathbb{L}_2\mathbb{L}_3} A_\mu^{(\mathbb{L}_2)} A_\nu^{(\mathbb{L}_3)} + 2ige^{ig\mathbb{L}_{\mathcal{I}}\alpha^{\mathcal{I}}(x)} \tilde{f}_{\mathbb{L}-\mathbb{L}\mathcal{I}} (A_{[\mu}^{\mathcal{I}} - \partial_{[\mu}\alpha^{\mathcal{I}})\mathbb{L}_{\mathcal{I}} A_{\nu]}^{(\mathbb{L})} \\
=\; & e^{ig\mathbb{L}_{\mathcal{I}}\alpha^{\mathcal{I}}(x)} F_{\mu\nu}^{(\mathbb{L})}
\end{aligned}
\tag{4.11}
$$

where we have used momentum conservation $\mathbb{L} = \mathbb{L}_2 + \mathbb{L}_3$. Therefore $F_{\mu\nu}^{(\mathbb{L})} F^{\mu\nu(-\mathbb{L})}$ terms in the action are U(1) invariant.

On the other hand, at a given fixed point the Abelian gauge group is enhanced to some non-Abelian gauge group G, and all fields in the theory, massless and massive, should organize into G irreducible representations. We have shown that, at a fixed point Φ_0 or close to it, after very massive states are integrated out (see discussion around (2.35), a well defined low energy effective gauge theory is obtained. The light modes that define this theory are the zero modes coming from 10D fields KK reductions plus modes in $\check{G}(\Phi_0)_{n_c}$.

However, if we were to consider the other (very massive) modes, as the ones appearing in the mode summations we are dealing with, we expect to run into trouble. Indeed, generically, these massive modes will fill gauge multiplets that will contain modes associated to higher oscillator numbers. This appears as a limitation of our construction restricted to $N = 0, 1$ modes.

Indeed, assume that $\mathbb{K} = (k_L, k_R)$ with $k_R = 0$, $k_L^2 = 2$ encode the charged gauge vector boson modes $A_\mu^{(\mathbb{K})}$ of the group G and let us call the currents associated to these vectors $J^{(\mathbb{K})}$. From a string theory analysis, if we start with some massive field $\Phi(x)^{(\mathbb{L})}$ with mass $m^{(\mathbb{L})}$ and $\frac{1}{2}\mathbb{L}^2 = 1$, its OPE with the current should lead to another field $\Phi(x)^{(\mathbb{S})}$ with $\mathbb{S} = (s_L, s_R) = \mathbb{L} + \mathbb{K}$ and the same mass, in order to be part of the same multiplet. Thus, by using (A.2) and (A.7) we find that $\Phi(x)^{(\mathbb{S})}$ mode should have $\bar{N}_s = \bar{N}_B + \bar{N}_F + \tilde{E}_0 = 0$ and a left oscillator mode N_s such that

$$\frac{\alpha'}{2} m_L^2 = \frac{1}{2} l_L^2 - 1 = \frac{1}{2} s_L^2 + N_s - 1 = \frac{1}{2}(\mathbb{L} + \mathbb{K})_L^2 + N_s - 1 = \frac{1}{2} l_L^2 + l_L k_L + N_s .$$

Namely

$$l_L k_L = -1 - N_s \tag{4.12}$$

or, equivalently, $\frac{1}{2}\mathbb{S}^2 = 1 - N_s$. Therefore, even if we started with a field corresponding to zero oscillator number we conclude that other values must be generically included. Presumably full consistency would be attained if $\delta(\frac{1}{2}\mathbb{S}^2, 1 - N)$ LMC is allowed in for all possible values of N. However, this would imply introducing higher spin fields, as expected form string theory.

Interestingly enough, it appears that consistency (at tree level) can be achieved up to first mass level, with $N = 0, 1$ as we are indeed considering here. We discuss this issue in the example below.

Finally recall that several consistency conditions are expected to be satisfied by physical states. For instance, physical massive vectors must satisfy $\partial^\mu A^B_\mu = 0$, etc. In string theory such conditions arise from conformal invariance. Namely, physical fields must satisfy the adequate OPE with the stress energy tensor. It was shown in ref. [28], in the case of the bosonic string and for some specific fields, that these conditions can be understood from generalized diffeomorphism invariance. However, as mentioned above, when level matching conditions as $\frac{1}{2}\mathbb{L}^2 = 1$ (or $\frac{1}{2}\mathbb{L}^2 = 0$ for non zero modes) are considered our analysis points towards a modification of the generalized diffeomorphism algebra in order to incorporate the $\star$-product. Therefore consistency conditions expected from diffeomorphism invariance need further investigation in these cases.

In what follows we illustrate some of the issues discussed above in an explicit example for the torus case.

4.5 SU(3) example

We consider the 2-torus compactification case in order to provide a specific example of the issues presented above. The generalized momentum encoding KK and winding modes is $\check{\mathbb{P}} = (P^I, p_1, p_2; \tilde{p}^1, \tilde{p}^2)$. At a generic moduli point $\Phi = (g, b, A)$ non zero momenta lead to massive states. The massless vectors arise from zero modes $A_\mu^{\hat{I}(0)} \equiv A_\mu^{I(0)}, A_\mu^{1(0)}, A_\mu^{2(0)}$ and lead to the generic group $\mathrm{U}(1)^2_R \times \mathrm{U}(1)^{16}$ gauge group. Enhancements will occur at specific moduli. As an example, let us look at moduli point $\Phi \equiv (g, b, 0)$ with turned off Wilson lines. The set of momenta that would lead to massless states at this point-recall (2.22)-is $\check{G}(\Phi)_{480} = \{\check{\mathbb{P}} \equiv (\alpha; 0, 0; 0, 0)\}$ where $\alpha \equiv (\underline{\pm 1, \pm 1, 0, \dots})$ are the SO(32) roots. Thus, when sliding to this moduli point these sates become massless and, together with the Cartan vectors should lead to an enhancement to $\mathrm{U}(1)^2_R \times \mathrm{SO}(32)$ gauge group. Actually, we see from (3.1) that $f_{\mathbb{L} - \mathbb{L} I} \equiv f_{\alpha - \alpha I} = \alpha^I$ providing the right structure constants involving charged fields and Cartans. Moreover, the phases arising from the $\star$-product-see for instance (3.11) for the field strength modes-now read from (3.9) and (3.2)

$$\tilde{f}_{\mathbb{L}\mathbb{L}_2\mathbb{L}_3} \equiv \tilde{f}_{\alpha\alpha_2\alpha-\alpha_2} = -e^{i\frac{1}{2}P_1 E P_2} \tag{4.13}$$

where $\frac{1}{2}E_{IJ} = \frac{1}{2}(G_{IJ}+B_{IJ})$ is the SO(32) Cartan-Weyl metric for $I \geq J$ and zero otherwise. These values exactly correspond to structure constants involving three charged fields (see for instance [1]). We conclude that $S_{\rm eff}(\Phi \equiv (g, b, 0))$ corresponds to a well defined $\mathrm{U}(1)_R^2 \times \mathrm{SO}(32)$ gauge theory.

Moduli points $\Phi \equiv (g, b, 0)$ can lead to further enhancements for specific values of g and b on the compactification 2-torus. It proves convenient to rewrite this point as $\Phi = (T, U, 0)$ where $U = U_1 + iU_2$ and $T = T_1 + iT_2$ are the complex and the Kähler structure of the torus respectively. They are defined in terms of the metric and b field as $U_1 = \frac{g_{12}}{g_{22}}$, $U_2 = \frac{\sqrt{\det g}}{g_{22}}$, $2T_1 = b_{12}$, $2T_2 = \sqrt{\det g}$. An $\mathrm{SU}(3)_L$ gauge symmetry enhancement occurs at point $\Phi_0 = (-\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, 0)$ which is obtained from the SU(3) Cartan matrix and b field

$$g_{mn} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad b_{mn} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.14}$$

whereas at $\Phi_1 = (i, i, 0)$, associated to

$$g_{mn} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{4.15}$$

and $b = 0$, an enhancement to $(\mathrm{SU}(2) \times \mathrm{SU}(2))_L$ group occurs.

At the SU(3) point (some basic information and notation is presented in appendix C), Left and Right momenta (A.2) become

$$l_L^1 = \frac{1}{3}(2p_1 + p_2 + \tilde{p}^1 + \tilde{p}^2), \qquad l_L^2 = \frac{1}{3}(p_1 + 2p_2 - \tilde{p}^1 + 2\tilde{p}^2), \tag{4.16}$$

$$l_R^1 = \frac{1}{3}(2p_1 + p_2 - 2\tilde{p}^1 + \tilde{p}^2), \qquad l_R^2 = \frac{1}{3}(p_1 + 2p_2 - \tilde{p}^1 - \tilde{p}^2), \tag{4.17}$$

whereas at $\mathrm{SU}(2) \times \mathrm{SU}(2)$ we have

$$l_L^1 = \frac{1}{2}(p_1 + \tilde{p}^1), \qquad l_L^2 = \frac{1}{2}(p_2 + \tilde{p}^2), \tag{4.18}$$

$$l_R^1 = \frac{1}{2}(p_1 - \tilde{p}^1), \qquad l_R^2 = \frac{1}{2}(p_2 - \tilde{p}^2), \tag{4.19}$$

where we have set $\alpha' = 1$.

It is easy to check that the six weights

$$\check{\mathbb{P}}_0 = \pm(0; 0, 1; 1, 1), \pm(0; 1, 0; 1, 0), \pm(0; -1, 1; 0, 1) \tag{4.20}$$

lead to $l_R^m = 0$ and to $(l_L^1, l_L^2) = \pm(1,1), \pm(1,0), \pm(0,1)$. The latter are the coordinates[11] in the simple root lattice base $\alpha_1 = (\sqrt{2}; 0)$, $\alpha_2 = (-1/\sqrt{2}; \sqrt{3/2})$ and satisfy (see (4.14))

[11] Recall that l_L^m are coordinates of the weight vectors of the representation in the simple root lattice, namely $\Lambda = l_L^m \alpha_m$ with α_m the simple roots, whereas $l_{Lm} = g_{mn} l_L^m$ correspond to coordinates (Dynkin labels) in the dual (weight) lattice.

$l_L^2 = l_L^m g_{mn} l_L^m = 2\tilde{p}^m p_m = 2$. They label the six massless charged vectors of SU(3). We collected this information in table 2. In particular $(1,1)$ corresponds to the highest weight of the adjoint **8** representation. Together with the SO(32) modes above the set $\check{G}(\Phi_0)_{480+6} = \{(\alpha;0,0;0,0); \pm(0;0,1;1,1), \pm(0;1,0;1,0), \pm(0;-1,1;0,1)\}$ defines, by similar considerations as above, a heterotic low energy effective theory $S_{\text{eff}}(\Phi_0)$ with gauge group $\mathrm{U}(1)_R^2 \times \mathrm{SU}(3) \times \mathrm{SO}(32)$.

On the other hand, the generalized momenta $\check{\mathbb{P}}_1 = \pm(0;1,0;1,0), \pm(0;0,1;0,1)$ provide the charged massless vectors of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ at Φ_1. We notice that states (vectors and scalars) associated to modes $\pm(0;1,0,1,0)$ are massless at both points whereas for $\pm(0;0,1;0,1)$ we have, at Φ_0 point, $l_L = \pm\left(\frac{2}{3},\frac{4}{3}\right)$, $l_R = \pm\left(\frac{2}{3} \pm \frac{1}{3}\right)$ that satisfy $l_L^2 - l_R^2 = 2$ and correspond to very massive states with $\alpha' m^2 = \frac{4}{3}$. That is why it drops out from the effective low energy theory at Φ_0 point. Thus, the low energy theory at Φ_1 corresponds to $S_{\text{eff}}(\Phi_1)$ with gauge group $\mathrm{U}(1)_R^2 \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(32)$ where the charged vectors arise from $\check{\mathbb{P}}_1$ modes together with SO(32) modes above, namely from states in $\check{G}(\Phi_1)_{480+4} = \{(\alpha;0,0;0,0); \pm(0;0,1;1,1), \pm(0;1,0;1,0), \pm(0;-1,1;0,1)\}$.

Having discussed the low energy effective action arising from action (2.29) at different moduli points, we propose to explore the contributions from massive states. Let us concentrate in the Φ_0 point.

For instance, the massive state $l_L = \left(-\frac{2}{3},-\frac{4}{3}\right)$, not contributing to the low energy theory, must now be considered. Interestingly enough, this state corresponds to the lowest weight of the symmetric **6** representation (whereas $\left(\frac{2}{3},\frac{4}{3}\right)$ corresponds to the highest weight of $\bar{\mathbf{6}}$). Indeed, it is easy to see that when shifting with $\check{\mathbb{P}}_0$ vectors we obtain the mode vectors (Λ_2, q_R) with $\Lambda_s = (l_{L_s}^1, l_{L_s}^2)$, $q_R = \left(-\frac{2}{3},-\frac{1}{3}\right)$, with $s = 1,\dots 6$, where

$$\Lambda_s \equiv \left(\frac{4}{3},\frac{2}{3}\right), \left(\frac{1}{3},\frac{2}{3}\right), \left(-\frac{2}{3},\frac{2}{3}\right), \left(\frac{1}{3},-\frac{1}{3}\right), \left(-\frac{2}{3},-\frac{1}{3}\right), \left(-\frac{2}{3},-\frac{4}{3}\right). \tag{4.21}$$

These modes fill the $\mathbf{6}_{\left(-\frac{2}{3},-\frac{1}{3}\right)}$ representation of $\mathrm{U}_R(1)^2 \times \mathrm{SU}(3)$ (and similarly $\bar{\mathbf{6}}_{\left(\frac{2}{3},\frac{1}{3}\right)}$).

We notice that all states have $l_R^2 = \frac{2}{3}$ but $l_L^2 = \frac{8}{3}$ for $s = 1,3,6$ whereas $l_L^2 = \frac{2}{3}$ for $s = 2,4,5$. Indeed, they satisfy

$$\frac{1}{2} l_{Ls}^2 - \frac{1}{2} l_{Rs}^2 = \frac{1}{2} l_{Ls}^2 - \frac{1}{3} = \tilde{p}_s . p_s = 1 - N_s \tag{4.22}$$

with $N_s = 0$ for $s = 1,3,6$ and $N_s = 1$ for $s = 2,4,5$. Moreover, for these values of N_s, all states have the same mass $\alpha' m^2 = \frac{4}{3}$, as it must be if they all belong to the same multiplet. Thus we conclude that states with oscillator numbers $N = 1,0$ must be mixed in order to build up the **6** massive symmetric representation. These results are collected in table 3.

These states, even though they are very massive, are indeed present in our construction. As an illustration let us consider the massive scalar fields with mass $\alpha' m^2 = \frac{4}{3}$. The $N = 0$ states correspond to the modes $M_{\bar{J}}(x)^{(\Lambda_s, q_R)}$ with $r = 1,3,6$ in the GKK expansion of $M_{\bar{J}}(x,\mathbb{Y})$ whereas states with $N = 1$, $M_{\bar{J}m}(x)^{(\Lambda_s,q_R)}$, with $s = 2,4,5$, are contained in $M_{\bar{J}m}(x,\mathbb{Y})$ expansion. It is worth noticing that in the $N = 1$ case there are two states

Dynkin label	$(p_1, p_2; \tilde{p}^1, \tilde{p}^2)$	(l_L^1, l_L^2)	N
$\pm(1,1)$	$\pm(0,1;1,1)$	$\pm(1,1) \equiv \pm\alpha_3$	0
$\pm(2,-1)$	$\pm(1,0,;1,0)$	$\pm(1,0) \equiv \pm\alpha_1$	0
$\pm(-1,2)$	$\pm(-1,1;0,1)$	$\pm(0,1) \equiv \pm\alpha_2$	0
$2 \times (0,0)$	$(0,0;0,0)$	$(0,0)$	1

Table 2. $\mathbf{8}_{(0,0)}$.

r≡Dynkin label	$(p_1, p_2; \tilde{p}^1, \tilde{p}^2)$	(l_L^1, l_L^2)	N
$1 \equiv (2,0)$	$(0,1;2,1)$	$(\frac{4}{3}, \frac{2}{3})$	0
$2 \equiv (0,1)$	(-1,1,;1,1)	$(\frac{1}{3}, \frac{2}{3})$	1
$3 \equiv (-2,2)$	$(-2,1;0,1)$	$(-\frac{2}{3}, \frac{2}{3})$	0
$4 \equiv (1,-1)$	$(0,0;1,0)$	$(\frac{1}{3}, -\frac{1}{3})$	1
$5 \equiv (-1,0)$	(-1,0;0,0)	$(-\frac{2}{3}, -\frac{1}{3})$	1
$6 \equiv (0,-2)$	(0,-1;0,-1)	$(-\frac{2}{3}, -\frac{4}{3})$	0

Table 3. $\mathbf{6}_{(\frac{2}{3},\frac{1}{3})}$ representation.

Some relevant data for $\mathbf{6}_{(\frac{2}{3},\frac{1}{3})}$ symmetric and $\mathbf{8}_{(0,0)}$ adjoint representations of $\mathrm{U}(1)^2 \times \mathrm{SU}(3)$ is provided. In the first column a number is associated to each pair of Dynkin coordinates in weight space. The second column presents their corresponding KK momenta and windings whereas in the third column the weights are given in the root basis. The last column indicates the oscillator number required by level matching.

$m = 1, 2$ for each mode Λ_s so we expect that only a combination of them enters to complete the representation. We discuss this issue below.[12]

It is enlightening to look at the "covariant" derivative for the modes Λ_s above. For $N = 0$ modes, $s = 1, 3, 6$, the expression (3.17) must be considered. For these states it can be expressed as

$$\begin{aligned}\mathcal{D}_\mu M_{\bar{J}}^{(\Lambda_s)} &= \partial_\mu M_{\bar{J}}^{(\Lambda_s)} + ig {\sum_l}' \tilde{f}_{\Lambda_s \alpha_l \Lambda_s - \alpha_l} A_\mu^{(\alpha_l)} M_{\bar{J}}^{(\Lambda_s - \alpha_l)} \\ &+ig {\sum_l}' \tilde{f}_{\Lambda_s \alpha_l \Lambda_s - \alpha_l} A_\mu^{(\alpha_l)} \alpha_l^m M_{m\bar{J}}^{(\Lambda_s - \alpha_l)} + ig\Lambda_{sm} A_\mu^{m(0)} M_{\bar{J}}^{(\Lambda_s)} + igq_{R\bar{I}} A_\mu^{\bar{I}(0)} M_{\bar{J}}^{(\Lambda_s)} + \dots\end{aligned} \tag{4.23}$$

where we have just shown the terms that couple to $\mathrm{U}_R(1)^2 \times \mathrm{SU}(3)$ gauge vectors, the ...encoding all the rest. On the other hand, for $N = 1$ states, $s = 2, 4, 5$, the derivative (3.19) reads

$$\begin{aligned}\mathcal{D}_\mu M_{m\bar{J}}^{(\Lambda_s)} &= \partial_\mu M_{m\bar{J}}^{(\Lambda_s)} + ig {\sum_l}' \tilde{f}_{\Lambda_s \alpha_l \Lambda_s - \alpha_l} A_\mu^{(\alpha_l)} \alpha_l^m M_{\bar{J}}^{(\Lambda_s - \alpha_l)} \\ &+2ig {\sum_l}' \tilde{f}_{\Lambda_s \alpha_l \Lambda_s - \alpha_l} A_\mu^{(\alpha_l)} M_{m\bar{J}}^{(\Lambda_s - \alpha_l)} + ig\Lambda_{sk} A_\mu^{k(0)} M_{m\bar{J}}^{(\Lambda_s)} + igq_{R\bar{I}} A_\mu^{\bar{I}(0)} M_{m\bar{J}}^{(\Lambda_s)} + \dots\end{aligned} \tag{4.24}$$

The last term is just the expected coupling to $\mathrm{U}(1)_R^2$ vectors. The α_l index labels the 6 charged vectors of SU(3) in correspondence with SU(3) roots where $\alpha_4 = -\alpha_1, \alpha_5 = -\alpha_2, \alpha_6 = -\alpha_3$ (see table 2). $A_\mu^{m(0)}$ are the Cartan gauge vector fields.

We observe that in the first equation (4.23) linear combinations of $N = 1$ modes $\alpha_l^m M_{m\bar{J}}^{(\Lambda_s - \alpha_l)}$ do appear. Indeed, as mentioned above, we expect linear combinations to provide the physical degrees of freedom entering in the $\mathbf{6}$ multiplet.

[12] In what follows, in order to lighten the notation, we avoid indicating the charge q_R of each state, being the same for all states in the multiplet.

Before presenting some explicit examples recall that a well defined covariant derivative on fields Φ^s in this multiplet must read

$$\mathcal{D}_\mu \Phi^s = \partial_\mu \Phi^s + ig(T_{\alpha_l})_{sr} A_\mu^{(\alpha_l)} \Phi^r + ig(T_I)_{sr} A_\mu^I \Phi^r \tag{4.25}$$

where T_{α_l} and T_I are the matrices corresponding to SU(3) charged and Cartan generators, respectively, in the **6** representation. They are collected (in Cartan-Weyl basis) in appendix C.

Let us consider the derivative (4.23) for the state $M_{\bar{J}}^{(\Lambda_1)}$. It reads

$$\begin{aligned}\mathcal{D}_\mu M_{\bar{J}}^{(\Lambda_1)} = \partial_\mu M_{\bar{J}}^{(\Lambda_1)} + ig\tilde{f}_{1\alpha_3 4} A_\mu^{(\alpha_3)} \alpha_3^m M_{i\bar{J}}^{(\Lambda_4)} + ig\tilde{f}_{1\alpha_1 2} A_\mu^{(\alpha_1)} \alpha_1^m M_{i\bar{J}}^{(\Lambda_2)} \\ + ig(\sqrt{2} A_\mu^{1(0)} + \sqrt{\frac{2}{3}} A_\mu^{2(0)}) M_{\bar{J}}^{(\Lambda_1)}\end{aligned} \tag{4.26}$$

where we have used that $\Lambda_1 = \frac{4}{3}\alpha_1 + \frac{2}{3}\alpha_2 = \left(\sqrt{2}, \sqrt{\frac{2}{3}}\right)$. Also we denote $\tilde{f}_{\Lambda_s \alpha_j \Lambda_r} = \tilde{f}_{s\alpha_j r}$. By using that $\tilde{f}_{1\alpha_1 2} = \tilde{f}_{1\alpha_3 4} = 1$ and by defining

$$\Phi_{\bar{J}}^1 = M_{\bar{J}}^{(\Lambda_1)}, \qquad \Phi_{\bar{J}}^2 = \frac{1}{\sqrt{2}} \alpha_1^m M_{m\bar{J}}^{(\Lambda_2)}, \qquad \Phi_{\bar{J}}^4 = -\frac{1}{\sqrt{2}} \alpha_3^m M_{m\bar{J}}^{(\Lambda_4)} \tag{4.27}$$

this derivative can be recast in the form (4.25) with $(T_{\alpha_1})_{12} = (T_{\alpha_3})_{14} = \sqrt{2}$ and $(T_1)_{11} = \sqrt{2}, (T_2)_{11} = \sqrt{\frac{2}{3}}$ in exact correspondence with (C.5). Nevertheless, we should check that these definitions are consistent for the six states. Consider, for instance, the derivative of $M_{i\bar{J}}^{(\Lambda_2)}$ field. We have noticed above that this field appears contracted with root α_1 in order to define the field $\Phi_{\bar{J}}^2$ with the correct transformation properties. This indicates that we must actually compute the derivative of $\Phi_{\bar{J}}^2$. Thus, by projecting in (3.19) we find

$$\begin{aligned}\mathcal{D}_\mu \Phi_{\bar{J}}^2 = \partial_\mu \Phi_{\bar{J}}^2 + ig\frac{1}{\sqrt{2}} \tilde{f}_{2(-\alpha_1)1} A_\mu^{(-\alpha_1)} (-\alpha_1^2) M_{\bar{J}}^{(\Lambda_1)} + ig\tilde{f}_{2\alpha_3 5} A_\mu^{(\alpha_3)} \frac{2}{\sqrt{2}} \alpha_1^m M_{m\bar{J}}^{(5)} \\ + ig\tilde{f}_{2\alpha_2 4} A_\mu^{(\alpha_2)} \frac{2}{\sqrt{2}} \alpha_1^m M_{m\bar{J}}^{(4)} + ig\tilde{f}_{2\alpha_1 3} A_\mu^{(\alpha_1)} \frac{2}{\sqrt{2}} \alpha_1^m M_{m\bar{J}}^{(3)} + ig\sqrt{\frac{2}{3}} A_\mu^{2(0)} \Phi_{\bar{J}}^2\end{aligned} \tag{4.28}$$

Interestingly enough we see that $(T_1)_{22} = 0, (T_2)_{22} = \sqrt{\frac{2}{3}}$ as expected from (C.5). Also, since $\tilde{f}_{2(-\alpha_1)1} = -\tilde{f}_{2\alpha_3 5} = \tilde{f}_{2\alpha_2 4} = \tilde{f}_{2\alpha_1 3} = 1$, consistency requires the extra definitions

$$\Phi_{\bar{J}}^3 = M_{\bar{J}}^{(\Lambda_3)}, \qquad \Phi_{\bar{J}}^4 = -\frac{2}{\sqrt{2}} \alpha_1^m M_{m\bar{J}}^{(\Lambda_4)}, \qquad \Phi_{\bar{J}}^5 = \frac{2}{\sqrt{2}} \alpha_1^m M_{m\bar{J}}^{(\Lambda_5)} \tag{4.29}$$

in order to have $(T_{-\alpha_1})_{21} = \sqrt{2}$ and $(T_{\alpha_2})_{24} = (T_{\alpha_3})_{25} = 1$ (see (C.5)). However, we have already defined $\Phi_{\bar{J}}^4$ in (4.27), so the only way to obtain a consistent description is to have

$$(2\alpha_1 - \alpha_3)^m M_{m\bar{J}}^{(\Lambda_4)} = (\alpha_1 - \alpha_2)^m M_{m\bar{J}}^{(\Lambda_4)} = \Lambda_4^m M_{m\bar{J}}^{(\Lambda_4)} = 0. \tag{4.30}$$

Therefore, we observe that the two fields $(m = 1, 2)$ $M_{m\bar{J}}^{(\Lambda_4)}$ must combine into the physical state $\alpha_3^m M_{m\bar{J}}^{(\Lambda_4)}$ and the orthogonal state $(\alpha_1 - \alpha_2)^m M_{m\bar{J}}^{(\Lambda_4)}$ (notice that $\alpha_3(\alpha_1 - \alpha_2) = 0$)

that must decouple. Indeed, by completing the computation of the derivatives for the rest of the states it can be checked that complete derivative (4.25) with T^a given in (C.5) is reproduced if the condition

$$\Lambda_s^m M_{m\bar{J}}^{(\Lambda_s)} = 0 \tag{4.31}$$

is imposed for $s = 2, 4, 5$. The normalized fields are defined as

$$\begin{aligned}
\Phi_{\bar{J}}^s &= M_{\bar{J}}^{(\Lambda_s)}, \qquad\qquad s = 1, 3, 6 \\
\Phi_{\bar{J}}^2 &= \frac{1}{\sqrt{2}}\alpha_1^m M_{m\bar{J}}^{(\Lambda_2)}, \quad \Phi_{\bar{J}}^4 = -\frac{1}{\sqrt{2}}\alpha_3^m M_{m\bar{J}}^{(\Lambda_4)}, \quad \Phi_{\bar{J}}^5 = -\frac{1}{\sqrt{2}}\alpha_2^m M_{m\bar{J}}^{(\Lambda_5)}.
\end{aligned} \tag{4.32}$$

The α_l appearing in the contraction with $M_{m\bar{J}}^{(\Lambda_s)}$ satisfy $\alpha_l.\Lambda_s = 0$. Namely, they select the DOF of $M_{m\bar{J}}^{(\Lambda_s)}$ orthogonal to Λ_s as the physical one.

Since massive vector bosons have the same weights as the scalars, the same line of reasoning leads to a consistent covariant derivative of the massive vector fields, from expressions (3.11) and (3.16). The physical vector bosons are obtained from (4.33) just by replacing $M_{\bar{J}}^{(\Lambda_s)} \to A_\mu^{(\Lambda_s)}$ and $M_{m\bar{J}}^{(\Lambda_s)} \to A_\mu^{m(\Lambda_s)}$. Namely, $A_\mu^s = A_\mu^{(\Lambda_s)}$ for $s = 1, 3, 6$ and $A_\mu^s = \frac{1}{\sqrt{2}}\alpha_l^m A_{m\mu}^{(\Lambda_s)}$ for $s = 2, 4, 5$ where $\alpha_l.\Lambda_s = 0$. The physical vectors satisfy

$$\Lambda_s^m A_{m\mu}^{(\Lambda_s)} = 0. \tag{4.33}$$

The physical degrees of freedom (DOF) conditions above can be interpreted from different perspectives. From a string theory point of view this requirement arises from conformal invariance. Namely, by looking at the OPE of the stress energy tensor with the vertex operators associated to different $N = 1$ modes above ($r = 2, 4, 5$) an anomalous term $\frac{1}{z^3}\Lambda_s^m M_{m\bar{J}}^{(\Lambda_s)}$ is generated for scalars and $\frac{1}{z^3}\Lambda_s^m A_{m\mu}^{(\Lambda_s)}$ for vectors. Absence of anomalies leads to the physical state condition. Even if our whole construction emerges from string theory we would like to deal with consistency conditions contained in the proposed action without introducing external information.

The DOF conditions can also be interpreted as inherited from consistency in 10 dimensions. Schematically, $N = 1$ modes can be interpreted as KK reductions of a generalized metric $\mathcal{H}^{\mathcal{M}\mathcal{N}}$ with $\mathcal{M}, \mathcal{N} = 1, \ldots 10$, encoding the massless metric $g_{\mathcal{M}\mathcal{N}}$ and anti-symmetric field $b_{\mathcal{M}\mathcal{N}}$, satisfying the gauge condition $\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{M}\mathcal{N}} = 0$. By splitting indices into space time and compactification indices, namely $\mathcal{M} \equiv \{\mu, m\}$, this condition gets splitted as

$$\partial_\mu \mathcal{H}^{\mu\nu}(x, \mathbb{Y}) + \partial_m \mathcal{H}^{m\nu}(x, \mathbb{Y}) = 0 \to \partial_\mu \mathcal{H}^{\mu\nu}(x)^{(\mathbb{L})} + \mathbb{L}_m \mathcal{H}^{m\nu}(x)^{(\mathbb{L})} = 0 \tag{4.34}$$

$$\partial_\mu \mathcal{H}^{\mu n}(x, \mathbb{Y}) + \partial_m \mathcal{H}^{mn}(x, \mathbb{Y}) = 0 \to \partial_\mu \mathcal{H}^{\mu n}(x)^{(\mathbb{L})} + \mathbb{L}_m \mathcal{H}^{mn}(x)^{(\mathbb{L})} = 0. \tag{4.35}$$

For massless modes, corresponding to $\mathbb{L} = 0$, we recover the expected gauge conditions for the $g_{\mu\nu}$, the B field contained in $\mathcal{H}^{\mu\nu}(x)^{(0)}$ and the gauge vectors $A_{m\mu} \equiv \mathcal{H}_{\mu n}^{(0)}(x)$. However, for massive modes, a consistency requirements $\partial_\mu \mathcal{H}^{\mu n}(x)^{(\mathbb{L})} = 0$ is needed for a Proca field to have the right number of degrees of freedom. Similarly $\partial_\mu \mathcal{H}^{\mu\nu}(x)^{(\mathbb{L})} = 0$ for massive symmetric and anti-symmetric tensors. We conclude that

$$\mathbb{L}_m \mathcal{H}^{mn}(x)^{(\mathbb{L})} = \mathbb{L}_m \mathcal{H}^{m\nu}(x)^{(\mathbb{L})} = 0 \tag{4.36}$$

which correspond to physical state conditions (4.31) and (4.33) respectively.

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Finally let us provide a third way of looking at physical DOF conditions. Given the two $N = 1$ fields, $m = 1, 2$ we have seen that we can combine them into two (orthogonal) linear independent combinations. For instance, in the case of $M^{(\Lambda_4)}_{m\bar{J}}$ fields above (4.30), we can consider the l.i. combinations $(\alpha_1 - \alpha_2)^m M^{(\Lambda_4)}_{m\bar{J}} = \Lambda_4^m M^{(\Lambda_4)}_{m\bar{J}}$ and $(\alpha_1 + \alpha_2)^m M^{(\Lambda_4)}_{m\bar{J}}$. The second one provides the $\Phi^4_{\bar{J}}$ physical DOF whereas the first term should not be present in the spectrum. Similarly, the vector boson combination $\Lambda_4^m A^{(\Lambda_4)}_{m\mu}$ must decouple. The same reasoning holds for $s = 2, 4, 5$.

Now, recall that the first two rows in (2.5) contain massive field modes corresponding to graviton and Kalb-Ramond $g^{(\mathbb{L})}_{\mu\nu}, B^{(\mathbb{L})}_{\mu\nu}$. These modes satisfy $\mathbb{L}^2 = l_l^2 - l_R^2 = 0$ level matching condition. It is easy to see that the lowest massive levels have mass $\alpha' m^2 = \frac{4}{3}$ and correspond to weights $\pm\Lambda_s$ with $s = 2, 4, 5$ and $\mathrm{U}(1)^2_R$ charges $q_R = \pm\left(-\frac{2}{3}, -\frac{1}{3}\right)$ as the vector and scalars states discussed above.

With this observation in mind, the decoupling can be understood (see [28] for a related discussion in a DFT context) as follows: the scalars $\Lambda_s^m M^{(\Lambda_s)}_{m\bar{J}} = \Lambda_s \cdot M^{(\Lambda_s)}_{\bar{J}}$ are "eaten" by the corresponding vector boson to become massive. At the same time a massive graviton mode $g^{(\Lambda_s)}_{\mu\nu}$ "eats" the vector boson to become a massive graviton with the correct degrees of freedom. Indeed, this can be explicitly shown by following the construction in ref. [28]. Namely, we can write write the massive vector meson as

$$A'^{(\Lambda_s)}_{m\mu} = A^{(\Lambda_s)}_{\mu} - \frac{1}{m^2}\partial_\mu(\Lambda_s \cdot M^{(\Lambda_s)}_{\bar{J}} M^{(\Lambda_s)\bar{J}}_{m}) \tag{4.37}$$

such that the $\Lambda_s \cdot A'^{(\Lambda_s)}_{\mu}$ projection is "eaten" by the massive graviton

$$g^s_{\mu\nu} = g^{(\Lambda_s)}_{\mu\nu} - \frac{1}{m^2}\Lambda_s \cdot \partial_{(\nu} A'^{(\Lambda_s)}_{\mu)} \tag{4.38}$$

and the remaining physical states $A^s \propto \alpha_l . A'^{(\Lambda_s)}_{m\mu}$, with $\alpha_l \cdot \Lambda_s = 0$ satisfy $\partial_\mu A'^{\mu s} = 0$.

Moreover, by noticing that the weights in the fundamental representation of SU(3) correspond to modes $\Lambda_s \equiv (0, 1), (1, -1), (0, -1)$, we see that gravitons organize into multiplets of $(\mathbf{3})_{\left(-\frac{2}{3}, -\frac{1}{3}\right)}$ (and $\bar{\mathbf{3}}\left(\frac{2}{3}, \frac{1}{3}\right)$) of the gauge group $\mathrm{U}(1)^2_R \times \mathrm{SU}(3)$.

Let us close this section by stressing that the action (2.29) appears to contain very non trivial information even for states with masses of the order of the string mass. This is what the analysis of the covariant derivative of the massive symmetric representation in the above example indicates. Again, going to higher massive states would require the introduction of $N > 1$ and call for further investigation.

5 Summary and outlook

A striking and distinctive feature of string compactifications is that, at certain values of the compactification background -namely a point in moduli space- compact momenta and winding modes can combine to generate new (let us say n_c) massless vector bosons leading to an enhancement of the gauge symmetry group. Different enhancement can occur for other values of moduli and, generically, for other values of winding and momenta. In the notation presented above (2.22) we would say that, for a given number of compact dimensions r, several sets $\check{G}(\Phi_i)_{n_c^i}$ of generalized momenta $\check{\mathbb{P}}$ could exist. These lead to enhancement at moduli point Φ_i where n_c^i vector bosons and scalars become massless. The structure gets richer for lower space-time dimensions.

We have shown that the heterotic low energy effective theory at each Φ_i is obtained by considering fields associated to $\check{\mathbb{P}} \in \check{G}(\Phi_i)_{n_c^i}$ modes and the zero modes arising from fields in the gravity sector whereas all other, very massive modes, are integrated out. Slight displacements from Φ_i can be interpreted as a Higgs mechanism. Actually, when moving along moduli space some (or all) of these fields become massive whereas other fields become lighter at a different point. Therefore, a moduli dependent description able to account for these different enhancements implies handling an infinite number of fields. In this work we were able to identify some guiding lines towards this description, which is encoded in a moduli dependent effective action where a non-commutative $\star$-product plays a central role.

The proposed action, written in d space time dimensions, contains a generically infinite number of fields labeled by allowed momenta and winding modes. In principle, this action could have been obtained by carefully looking at string 3-point amplitudes of vertex operators associated with these modes. We have shown that these infinite fields in d dimensions can be understood as modes of a GKK expansion in the internal double torus and heterotic coordinates $\mathbb{Y} \equiv (y^I, y_L^m, y_R^m)$, providing an uplifting to higher dimensions. In this sense the action can be seen as a Kaluza-Klein inspired rewriting of a double field theory (see for instance ref. [32]), where coordinates are split into space-time coordinates (that could be formally doubled) and internal double coordinates. However, once compact coordinates come into play we noticed that a $\star$-product that introduces a non-commutativity in the target compact space is called for. Indeed, it is this non-commutativity that leads to the adequate factors to reproduce the structure constants. As we have shown in an example, this non-commutativity also appears to have the right features to reproduce the generator matrix elements in higher order representations where massive states live, as required by gauge invariance. It would be interesting to trace the origin of this product for the heterotic string case [33] back. In the context of bosonic string it was shown in [22, 23] to be associated to non-commutativity of string coordinate zero modes.

An interesting result of the construction is that, close to a given enhancement point Φ_0, by keeping just the n_c slightly massive fields, the Higgs mechanism can be cast in terms of $\tilde{f}$ moduli dependent "structure like constants" that become the enhanced group structure constants at Φ_0. This description provides a field theory stringy version of the gauge symmetry breaking-enhancement mechanism. This fact was already addressed in the context of DFT in [5, 7] where it was shown that constants $\tilde{f}(\Phi)$ can be interpreted as DFT Scherk -Schwarz [16, 29–31] compactifications generalized fluxes. These fluxes can be read from the DFT generalized diffeomorphism algebra. Actually, it is worth noticing that these fluxes were explicitly constructed from a generalized frame only in the circle case where a SU(2) enhancing at the self dual radio $R = \tilde{R} = \sqrt{\alpha'}$ occurs [4–7]. Interestingly enough, the SU(2) case is the only situation where the $\star$-product is not needed (essentially due to the absence of a b field). Difficulties in going beyond this case were mentioned in [5, 6]. In refs. [6, 8] a connection among these difficulties and vertex operators cocycle factors was suggested. The non-commutative product could provide a solution for this problem since the $\star$ appears as a manifestation of the cocycle factors in the DFT context. Let us stress that the $\star$-product is not needed, at third order in fluctuations, if fields satisfy $\mathbb{L}^2 = 0$ level matching condition. This is why it did not manifest in original DFT constructions but would be required in a DFT formulation including four (or higher) order terms in the fields, where cocycle factors would be required, as it stressed in [15].

Actually, the problem already arises at third order when the Lie algebra of three charged fields with $\frac{1}{2}\mathbb{L}^2 = 1$ LMC is considered, which is just the situation where the $\star$-product phase is relevant. Moreover, since $\star$-product is providing cocycle factors, four order terms (or higher) could be consistently considered in DFT. In fact, we have shown that this appears to be the case in a partial computation of the fourth order scalar potential.

As mentioned above, a modified version of generalized diffeomorphisms is called for to handle these cases. The detailed construction is left for future investigation.

In our construction we started by proposing mode expansions restricted by the level matching constraint $\frac{1}{2}\mathbb{L}^2 = 1$ (corresponding to $N = 0$ oscillators) necessary to contain massless vectors at the enhancement point. Even if it effectively interpolates among different enhancement points, we stressed that new ingredients must be incorporated. In particular, at first mass level, we noticed that for massive states to organize into multiplets of the enhanced group G, $N = 1$ oscillator number is also required. Since we had already included the $N = 1$ case, to tackle the gravity sector, we showed in an example that indeed massive vector and scalar states nicely fill G multiplets for first massive level. This happens to be the case also for gravity sector massive modes. However, for higher masses, other oscillator numbers are expected (this was was noticed in [28]). Namely, if we consider next to first massive level, in order to complete a G multiplet, a level matching condition with $N > 1$ is required.[13] We see that a simple gauge symmetry consistency check points towards the necessity of including massive higher spin fields and higher derivative terms in the action (see [34, 35] for a discussion from another perspective), as is in fact expected from string theory. Let us stress that gauge invariance underscores the limitations of the construction but at the same time it is a guide for consistent extensions. Indeed, gauge invariance provides a tool to systematically include higher spin modes and α' corrections by looking for consistency all the way from the very first massive levels up to the highest ones.

Throughout our construction we have made intensive use of DFT tools. In particular, before mode expanding, all fields are expressed in terms of higher dimensional coordinates. However, a fully higher dimensional version is still lacking in the sense that fields are written here in terms of space time d dimensional indices. Formally it appears rather straightforward. On the one hand, the new fields we are introducing here associated to $N = 0$, can be cast in terms of a D dimensional "charged vector" field $A^{\mathcal{M}}(x, \mathbb{Y}) \equiv (A^\mu(x, \mathbb{Y}), M^{\mathcal{I}}(x, \mathbb{Y}))$ and, on the other hand, the sectors originating in the generalized metric in 10-dimensions were already addressed in [28] (up to a third order expansion) in terms of a generalized metric. However, it appears that the latter must be modified by the presence of the new fields, as required by gauge invariance. Moreover, the form of generalized diffeomorphism and $\star$-product should be understood.

[13]These observations are based on the 2-torus example of 4.5 with Wilson lines turned off. The general situation with WL and/or higher dimensional torus needs further investigation.

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Finally let us mention that even if we have restricted our analysis to the bosonic sector of the heterotic string, the inclusion of fermions could also be addressed by invoking supersymmetry, generalizing the discussion in [7] (see also [36–38]) where "will-be massless fermions at a fixed point", specifically for modes in $\breve{G}_{n_c}$, were considered. From a duality invariant field theory point of view, an uplift including fermions would require an analysis from an Extended Field Theory (EFT) [39, 40] in order to include magnetic modes. The recent work in in ref. [41] might be helpful in this direction.

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A Some heterotic string basics

We summarize here some string theory ingredients (that can be found in string books) needed in the body of the article. We mainly concentrate in the SO(32) string.

For a heterotic string compactified to d space-time dimensions, Left and Right momenta are encoded in momentum

$$\mathbb{L} = (l_L, l_R), \tag{A.1}$$

defined on a self-dual lattice $\Gamma_{26-d,10-d}$ of signature $(26-d, 10-d)$. By writing $l_L^{\hat{I}} = (L_L^I, p_L^m)$ with $I = 1, \ldots, 16$ and $m = 1, \ldots 10 - d = r$, the moduli dependent momenta, read

$$\begin{aligned} L_L^I &= P^I + R A_n^I \tilde{p}^n \\ l_L^m &= \frac{\sqrt{\alpha'}}{2}\left[\frac{\tilde{p}^m}{\tilde{R}} + 2g^{mn}\left(\frac{p_n}{R} - \frac{1}{2}B_{nr}\frac{\tilde{p}^r}{\tilde{R}}\right) - P^I A_I^m - \frac{R}{2}A_I^m A_n^I \tilde{p}^n\right] \\ l_R^m &= \frac{\sqrt{\alpha'}}{2}\left[-\frac{\tilde{p}^m}{\tilde{R}} + 2g^{mn}\left(\frac{p_n}{R} - \frac{1}{2}B_{nr}\frac{\tilde{p}^r}{\tilde{R}}\right) - P^I A_I^m - \frac{R}{2}A_I^m A_n^I \tilde{p}^n\right], \end{aligned} \tag{A.2}$$

where g_{mn}, B_{mn} are internal metric and antisymmetric tensor components, A_m are Wilson lines and p_n and $\tilde{p}^n$ are integers corresponding to KK momenta and windings, respectively. P_I are Spin(32) weight components.

More schematically, by defining the vector $\check{\mathbb{P}} = (P_I, p_n, \tilde{p}^n)$ and $\mathbb{L} = (L_L^I, l_L^m, l_R^m)$ we can write

$$\mathbb{L} = \mathcal{R}(\Phi)\check{\mathbb{P}}, \tag{A.3}$$

where

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & RA \\ -\frac{\sqrt{\alpha'}}{2}A & \frac{\sqrt{\alpha'}}{R}g^{-1} & \frac{\sqrt{\alpha'}}{2\tilde{R}}\left(1 - g^{-1}B - \frac{1}{2}A.A\alpha'\right) \\ -\frac{\sqrt{\alpha'}}{2}A & \frac{\sqrt{\alpha'}}{R}g^{-1} & \frac{\sqrt{\alpha'}}{2\tilde{R}}\left(1 - g^{-1}B - \frac{1}{2}A.A\alpha'\right) \end{pmatrix} \tag{A.4}$$

performs the change of basis. It also rotates the coordinates $\check{\mathbb{Y}} = (y^I, y_m, \tilde{y}^m)$ to $\mathbb{Y} = (Y^I, y_L^m, y_R^m)$. In particular it transforms the $O(16+r, r)$ metric η_C defined in (2.10) to

$$\eta^{\mathcal{I}\mathcal{J}} = \begin{pmatrix} 1_{16} & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix} . \tag{A.5}$$

Notice that $\mathcal{R}(\Phi)$ encodes the dependence on moduli.

The mass formulas for string states are (we mainly use the notation in [3])

$$\begin{aligned} \frac{\alpha'}{2} m_L^2 &= \frac{1}{2} l_L^2 + (N-1) \\ \frac{\alpha'}{2} m_R^2 &= \frac{1}{2} l_R^2 + \bar{N} \, , \end{aligned} \tag{A.6}$$

where $N = N_B, \bar{N} = \bar{N}_B + \bar{N}_F + \bar{E}_0$ where N_B, $\bar{N}_B$ are the bosonic L and R-oscillator numbers, $\bar{N}_F$ is the R fermion oscillator number and $\bar{E}_0 = -\frac{1}{2}(0)$ for NS (R) sector. The level matching condition is $\frac{1}{2}m_L^2 - \frac{1}{2}m_R^2 = 0$ or, in terms of above notation

$$\frac{1}{2}\mathbb{L}^2 = \tilde{p}.p + \frac{1}{2}P^2 = (1 - N + \bar{N}). \tag{A.7}$$

In our discussion we restrict to $\bar{N}_B = 0, N_F = \frac{1}{2}$, namely $\bar{N} = 0$. The "charged vectors" sector, corresponds to $N = 0$, i.e. $\mathbb{L}^2 = 2$. Massless vectors are a particular case with $\frac{1}{2}l_L^2 = 1$, $l_R = 0$.[14]

As is well known, there are $10 - d + 16$ Left gauge bosons corresponding to 16 Cartan generators $\partial_z Y^I \tilde{\psi}^\mu$ of the original gauge algebra as well as $10 - d$ KK Left gauge bosons coming from a Left combination of the metric and antisymmetric field $\partial_z Y^m \tilde{\psi}^\mu$. The $10 - d$ Right combinations $\partial_z X^\mu \tilde{\psi}^m$ with $m = 1, \ldots 10 - d$ generate the Right Abelian group. These states have $l_R = 0$ and $l_L = 0$, with vanishing winding and KK momenta.

B The $\star$-product

A $\star$-product, was proposed in [22, 23] in order to incorporate, in a "Double Field theory" description, information about bosonic string vertex cocycle factors. If $\check{\mathbb{P}} \equiv (p_m, \tilde{p}^m)$ is an $O(n, n)$ vector encoding information about winding numbers $\tilde{p}^m$ and Kaluza-Klein (KK) compact momenta p_m, then for two fields depending on the compact double coordinate $\check{\mathbb{Y}} \equiv (y^m, \tilde{y}_m)$ their proposed $\star$-product reads

$$\begin{aligned} (\phi_1 \star \phi_2)(x, \mathbb{Y}) &= \sum_{\mathbb{P}_1, \mathbb{P}_2} e^{i\pi p_1 \cdot \tilde{p}_2} \phi_1^{(\mathbb{P}_1)}(x) \phi_2^{(\mathbb{P}_2)}(x) e^{i(\mathbb{P}_1 + \mathbb{P}_2).\mathbb{Y}} \\ &= \sum_{\mathbb{L}} \left[\sum_{\mathbb{P}_1} e^{i\pi p_1 \cdot (\tilde{l} - \tilde{p}_1)} \phi_1^{(\mathbb{P}_1)}(x) \phi_2^{(\mathbb{L} - \mathbb{P}_1)}(x) \right] e^{i\mathbb{L}.\mathbb{Y}} \\ &= \sum_{\mathbb{L}} (\phi_1 \star \phi_2)^{(\mathbb{L})}(x) e^{i\mathbb{L}.\mathbb{Y}} \end{aligned} \tag{B.1}$$

[14]The normalizations are chosen such that, $\frac{l_L^m}{\sqrt{\alpha'}}$, for an enhancement point, correspond to the coordinates of the weight vectors of a representation in the lattice span by simple roots α_m with $\alpha_m^2 = 2$.

where

$$(\phi_1 \star \phi_2)^{(\mathbb{L})}(x) = \sum_{\mathbb{P}_1} e^{i\pi p_1\cdot(\tilde{l}-\tilde{p}_1)}\phi_1^{(\mathbb{P}_1)}(x)\phi_2^{(\mathbb{L}-\mathbb{P}_1)}(x) \tag{B.2}$$

is the Fourier mode of the star product.

It is straightforward to show that the $\star$-product is indeed associative. Namely

$$\begin{aligned}
(\phi_1 \star \phi_2)(x,\mathbb{Y}) \star \phi_3(x,\mathbb{Y}) &= \sum_{\mathbb{L}}\sum_{\mathbb{P}_3} e^{i\pi l\cdot\tilde{p}_3}(\phi_1\star\phi_2)^{(\mathbb{L})}(x)\phi_3^{(\mathbb{P}_3)}(x)e^{i(\mathbb{L}+\mathbb{P}_3).\mathbb{Y}} \\
&= \sum_{\mathbb{P}_1,\mathbb{P}_2,\mathbb{P}_3} e^{i\pi(p_1+p_2)\cdot\tilde{p}_3}e^{i\pi p_1\cdot\tilde{p}_2}\phi_1^{(\mathbb{P}_1)}(x)\phi_2^{(\mathbb{P}_2)}(x)\phi_3^{(\mathbb{P}_3)}(x)e^{i(\mathbb{P}_1+\mathbb{P}_2+\mathbb{P}_3).\mathbb{Y}} \\
&= \sum_{\mathbb{P}_1,\mathbb{P}_2,\mathbb{P}_3} e^{i\pi p_1\cdot(\tilde{p}_2+\tilde{p}_3)}e^{i\pi p_2\cdot\tilde{p}_3}\phi_1^{(\mathbb{P}_1)}(x)\phi_2^{(\mathbb{P}_2)}(x)\phi_3^{(\mathbb{P}_3)}(x)e^{i(\mathbb{P}_1+\mathbb{P}_2+\mathbb{P}_3).\mathbb{Y}} \\
&= \sum_{\mathbb{P}_1,\mathbb{L}} e^{i\pi p_1\cdot\tilde{l}}\phi_1^{(\mathbb{P}_1)}(x)\left[\sum_{\mathbb{P}_2} e^{i\pi p_2\cdot(\tilde{l}-\tilde{p}_2)}\phi_2^{(\mathbb{P}_2)}(x)\phi_3^{(\mathbb{L}-\mathbb{P}_2)}(x)\right]e^{i(\mathbb{L}+\mathbb{P}_1).\mathbb{Y}} \\
&= \phi_1(x,\mathbb{Y}) \star (\phi_2\star\phi_3)(x,\mathbb{Y})\,.
\end{aligned} \tag{B.3}$$

Interestingly enough, the appearance of the phases can be traced back as a non-commutativity of the string compact coordinates zero modes (see [24]).

If the sum over $\phi_i^{(\mathbb{P}_i)}$ modes is constrained by LMC's, namely to modes satisfying $\delta(\frac{1}{2}\mathbb{P}_i^2, 1-N_i)$ then the same proof goes through if we define

$$(\phi_1 \star \phi_2)^{(\mathbb{L})}(x) = \sum_{\mathbb{P}_1} e^{i\pi p_1\cdot(\tilde{l}-\tilde{p}_1)}\phi_1^{(\mathbb{P}_1)}(x)\phi_2^{(\mathbb{L}-\mathbb{P}_1)}(x)\delta(\frac{1}{2}\mathbb{P}_1^2, 1-N_1)\delta(\frac{1}{2}(\mathbb{L}-\mathbb{P}_1)^2, 1-N_2).$$

Here we just extend this product to account for heterotic string degrees of freedom in a $O(r_L, r)$ context. Actually, since it is possible to interpret the heterotic string momenta P^I as originating in a 16 dimensional torus [27] with some winding and momenta $(\tilde{p}^I, p_I)$ (with $I = 1,\dots 16$) we can generalize above expression by including a phase that contains not only the compactified winding and momenta but also the gauge ones. More concretely, P_L^I, P_R^I can be computed using similar expressions as (A.2) above (no Wilson lines) but by imposing $P_R^I = 0$. Then, $P^I \equiv P_L^I$ root vectors are obtained with, G_{IJ} the Cartan Weyl metric of Spin(32) and $B_{IJ} = G_{IJ} = -B_{JI}$ for $I > J$. It is possible to check then that for two vectors P_1, P_2 we have $\tilde{p}_1^I p_{2I} = \frac{1}{2}P_1^I E_{IJ} P_{2J}$ where $E_{IJ} = G_{IJ} + B_{IJ}$. Therefore, for the heterotic string we would have (see (A.1) above) $\mathbb{L} = (l_L, l_R) \equiv (L_L^I, l_L^m, l_R^m)$ and $\mathbb{Y} = (y_l, y_R) \equiv (y^I, y_L^m, y_R^m)$ and using that

$$l_1 \cdot \tilde{l}_2 = p_{1m}\tilde{p}_2^m + p_{1I}\tilde{p}_2^I = p_1.\tilde{p}_2 + \frac{1}{2}P_1 E P_2, \tag{B.4}$$

we recover the expression in (3.4). Notice that the phase $\epsilon(P_1, P_2) = e^{i\pi\frac{1}{2}P_1 E P_2}$ introduces a notion of ordering for Spin(32) roots. For two adjacent roots in the corresponding Dynkin diagram $E_{IJ} = -1$ for $I > J$ and vanishes otherwise. This provides an adequate representation of structure constants for Spin(32) charged operator algebra. Namely $[E_{P_1}, E_{P_2}] = \epsilon(P_1, P_2)\, E_{P_3}$ (see e.g. the construction in [1]).

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The same reasoning holds for the full enhanced group. At the enhancement point Φ_0 with $\check{\mathbb{P}} \in \check{G}_{n_c}(\Phi_0)$, $p_r = 0$, $l_{\hat{I}}$ become the roots of the gauge group and from equations (A.2) above we can express windings and momenta in terms of the l_L such that

$$l_1 \cdot \tilde{l}_2 = p_{1m}\tilde{p}_2^m + \frac{1}{2}P_1 E P_2 = l_{1L}\mathcal{E} l_{2L}, \tag{B.5}$$

with [27]

$$\mathcal{E} = \begin{pmatrix} \left(B + g + \frac{\alpha'}{2}A_I A^I\right)_{nm} & \sqrt{\alpha'} g_{nm} A_I^m \\ 0 & (G+B)_{IJ} \end{pmatrix}. \tag{B.6}$$

C Some useful SU(3) expressions

Here we collect some relevant SU(3) conventions used in the example in 4. The eight SU(3) generators are denoted by the Cartan generators T_1, T_2 and the step raising (lowering) generators T_α ($T_{-\alpha}$) with $\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}$, ($\alpha_3 = \alpha_1 + \alpha_2$). In particular, they must satisfy

$$[T_i, T_\alpha] = \alpha^m T_\alpha \tag{C.1}$$

$$[T_\alpha, T_{-\alpha}] = \alpha^m T_m. \tag{C.2}$$

Here we choose a simple root α_1, α_2 basis with R^2 coordinates

$$\alpha_1 = \left(\sqrt{2}; 0\right), \qquad \alpha_2 = \left(-\frac{1}{\sqrt{2}}; \sqrt{\frac{3}{2}}\right) \tag{C.3}$$

$$\omega_1 = \left(\frac{1}{\sqrt{2}}; \frac{1}{2}\sqrt{\frac{2}{3}}\right), \qquad \omega_2 = \left(0; \frac{1}{2}\sqrt{\frac{2}{3}}\right) \tag{C.4}$$

where ω_j are the fundamental weights, i.e., the dual basis to the roots: $\omega_j . \alpha^m = \delta_j^m$. A weight vector can be expressed in either root or fundamental weight basis as $\Lambda = a_1\omega_1 + a_2\omega_2 = \lambda_1\alpha_1 + \lambda_2\alpha_2$ where a_i are the Dynkin labels. The fundamental **3** representation corresponds to

Dynkin		λ_1	λ_2	original basis	
0	1	$\frac{1}{3}$	$\frac{2}{3}$	0	$\sqrt{\frac{2}{3}}$
1	-1	$\frac{4}{3}$	$\frac{2}{3}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$
-1	0	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$

The weights for the **8** adjoint representation and **6** symmetric representation are given in tables 2 and 3 respectively. In the basis of table 3, the six dimensional **6** representation of

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SU(3) generators read

$$T_1 = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\sqrt{\frac{2}{3}} \end{pmatrix} \tag{C.5}$$

$$T_{\alpha_1} = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad T_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad T_{\alpha_3} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $T_{-\alpha} = (T_\alpha)^t$.

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