HAAR TYPE BASES IN LORENTZ SPACES VIA EXTRAPOLATION

LUIS NOWAK
DEPARTAMENTO DE MATEMÁTICA (FAEA-UNCOMA), NEUQUÉN;
IMAL, CONICET, SANTA FE, ARGENTINA

Abstract. In this note we consider Haar type systems as unconditional bases for Lorentz spaces defined on spaces of homogeneous type. We also give characterizations of these spaces in terms of the Haar coefficients. The basic tools are the Rubio de Francia extrapolation technique and the characterization of weighted Lebesgue spaces with Haar bases.

1. Introduction

The characterization of function spaces via wavelet coefficients as well as the unconditionality of such bases for these spaces are two of the most important properties of wavelets in Euclidean context. The nature of the function spaces and the particular features of wavelets can be very variable. However, in the general context of metric measure spaces (X, d, μ), the Haar functions are the basic example of wavelet orthonormal systems in L^2. Given the discontinuity of Haar functions, these systems can only be bases in the Schauder sense of spaces of functions without regularity in the classical sense. This is the case of Lebesgue spaces. In [2], the authors introduce Haar systems associated with Christ's dyadic cubes [6] and prove that such systems are unconditional bases for weighted Lebesgue spaces. Moreover, they give a characterization of such spaces via Haar coefficients. The aim of this note is to consider the case of Lorentz spaces. Precisely we prove the characterization of $L^{p,q}$ spaces through Haar coefficients and the unconditionality of the Haar system in the Lorentz spaces when $1 < p, q < \infty$. The main tools are, the extrapolation technique introduced by Rubio de Francia as it was generalized in [7], and the results of characterization and unconditionality given in [2] for weighted Lebesgue spaces. We would like to point out that similar results in Euclidean spaces are contained in [12] where a different approach was used.

We present our result in six sections. In Section 2 we briefly review the basic definitions and properties of Lorentz spaces. Section 3 is devoted to introduce the dyadic and Haar systems on a space of homogeneous type. Section 4 and 5 provide the precise statements of the main analytical tools for our results, which are started and proved in Section 6.

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2. Lorentz spaces

In this section we shall briefly recall the basic theory of Lorentz spaces on measure spaces (X,μ) such that μ is a σ-finite measure. We shall restrict our attention only to the scale $L^{p,q}$ with $1<p,q<\infty$. For details we refer the reader to [9] and [10] (see also [8] or [5] for an approach from the point of view of Banach function spaces). Given a measurable real valued function f defined on X, we denote with λ_f the distribution function of f, that is $\lambda_f(s)=\mu(\{x\in X:|f(x)|>s\})$. The non increasing rearrangement of f is the function given by $f^*(t)=inf\{s>0:\lambda_f(s)\leq t\}$, for $t\geq 0$.

For $1<p,q<\infty$, the $L^{p,q}(X,\mu)=L^{p,q}$ space is defined as the linear space of all measurable functions f on X such that $\|f\|^*_{p,q}<\infty$, where

$$\|f\|^*_{p,q}=\left(\frac{q}{p}\int_0^\infty (t^{1/p}f^*(t))^q\frac{dt}{t}\right)^{1/q}.$$

The quantity $\|.\|^*_{p,q}$ is not a norm. However, R. Hunt introduces in [9] a norm on $L^{p,q}$ such that the topology given by $\|.\|^*_{p,q}$ is equivalent to topology induced by the norm. This norm is defined by $\|f\|_{p,q}=\|f^{**}\|^*_{p,q}$, where

$$f^{**}(t)=\begin{cases} \sup\limits_{\mu(E)\geq t}\frac{1}{\mu(E)}\int_E|f(y)|d\mu(y) & \text{if } t\leq\mu(X),\\ \\ \frac{1}{t}\int_X|f(y)|d\mu(y) & \text{if } t>\mu(X).\end{cases}$$

The following statements collect the main properties of Lorentz spaces that we shall use later. Some of them are elementary and some other can be found in [9] and [5]. We shall denote with $\mathbb{V}_X$ the space of all simple functions defined on the measure space (X,μ). That is, $f\in\mathbb{V}_X$ if there exist real numbers $a_i, i=1,...,M$ such that $f=\sum_{i=1}^M a_i\chi_{E_i}$, where each set E_i is measurable. With $\mathcal{V}_{(X)}$ we shall denote the class which coincides with $\mathbb{V}_X$ when $\mu(X)=\infty$ and the class of those functions $f\in\mathbb{V}_X$ such that $\int_X f d\mu=0$ when $\mu(X)<\infty$.

- $(L1)$ $(L^{p,q},\|.\|_{p,q})$ is a Banach spaces and $\|f\|^*_{p,q}\leq\|f\|_{p,q}\leq\frac{p}{p-1}\|f\|^*_{p,q}$ for every function f in $L^{p,q}$.
- $(L2)$ For every measurable set E we have that $\|\chi_E\|^*_{p,q}=\mu(E)^{1/p}$.
- $(L3)$ If f and g are two measurable functions defined on X such that $|f|\leq|g|$ μ-a.e., then $\|f\|_{p,q}\leq\|g\|_{p,q}$.
- $(L4)$ If $(f_n:n\in\mathbb{Z}^+)$ is a sequence of functions in $L^{p,q}$ such that $0\leq f_n\nearrow f$ μ-a.e., then either $f\notin L^{p,q}$ and $\|f_n\|_{p,q}\nearrow\infty$ or $f\in L^{p,q}$ and $\|f_n\|_{p,q}\nearrow\|f\|_{p,q}$.
- $(L5)$ The space $\mathcal{V}_{(X)}$ is dense in $\mathcal{L}^{p,q}$, where $\mathcal{L}^{p,q}$ is $L^{p,q}$ if $\mu(X)=\infty$ and those functions in $L^{p,q}$ with vanishing integral if $\mu(X)<\infty$.
- $(L6)$ (Dominated Convergence Theorem) Let f be a measurable function defined on X. If $(f_n:n\in\mathbb{Z}^+)$ is a sequence of measurable functions defined on X such that $f_n\longrightarrow f$ μ-a.e. and $|f_n|\leq|g|$ for some function g in $L^{p,q}$ and every positive integer n, then $\|f_n-f\|_{p,q}\longrightarrow 0$.
- $(L7)$ (Fatou's lemma) If $(f_n:n\in\mathbb{Z}^+)$ is a sequence of measurable functions defined on X, then $\|\lim\inf_{n\longrightarrow\infty} f_n\|_{p,q}\leq\lim\inf_{n\longrightarrow\infty}\|f_n\|_{p,q}$.
- $(L8)$ (Duality) $(L^{p,q})^*=L^{p',q'}$, where p' and q' are the conjugate Hölder exponents of p and q respectively.

(*L*9) (Hölder inequality) If $f \in L^{p,q}$ and $g \in L^{p',q'}$ where p' and q' are the conjugate Hölder exponents of p and q respectively, then the function $f.g$ belongs to L^1 and $\int_X |f(x)g(x)|d\mu(x) \leq C\|f\|^*_{p,q}\|g\|^*_{p',q'}$.

Some comments are in order. In our general setting, that we shall introduce in the next section, atoms in the measure sense usually appear. In the classical literature on the subject of Lorentz spaces, such as [13], atoms are excluded. As the authors explicitly emphasize, this restriction is assumed only for the sake of simplicity. Nevertheless, in the original paper of Hunt the restriction to non-atomic spaces is shown to be only relevant for the duality when p or q ranks over $(0, 1)$.

3. Haar systems in spaces of homogeneous type

Let us recall the basic properties of the general theory of spaces of homogeneous type. Assume that X is a set, a nonnegative symmetric function d on $X \times X$ is called a quasi-distance if there exists a constant K such that

$$d(x, y) \leq K[d(x, z) + d(z, y)], \tag{3.1}$$

for every $x, y, z \in X$, and $d(x, y) = 0$ if and only if $x = y$.

We shall say that (X, d, μ) is a space of homogeneous type if d is a quasi-distance on X, μ is a positive Borel measure defined on a σ-algebra of subsets of X which contains the balls, and there exists a constant C such that the inequalities

$$0 < \mu(B(x, 2r)) \leq C \mu(B(x, r)) < \infty$$

hold for every $x \in X$ and every $r > 0$.

It is well known that the d-balls are generally not open sets. Moreover, sometimes some balls are not even Borel measurable subsets of X. Nevertheless in [11], R. Macias and C. Segovia prove that if d is a quasi-distance on X, then there exists a distance ρ and a number $\alpha \geq 1$ such that d is equivalent to ρ^α. Hence we shall assume along this paper that (X, d, μ) is a space of homogeneous type with d a distance on X, in other words that $K = 1$ in (3.1). In order to be able to apply Lebesgue Differentiation Theorem we shall also suppose that continuous functions are dense in $L^1(X, \mu)$.

The construction of dyadic type families of subsets in metric or quasi-metric spaces with some inner and outer metric control of the sizes of the dyadic sets is given in [6]. These families satisfy all the relevant properties of the usual dyadic cubes in $\mathbb{R}^n$. Actually, the only properties of Christ's cubes needed in our further analysis are contained in the next definition which we borrow from [4].

Definition 3.1. The class $\mathfrak{D}(\delta)$ of all dyadic families. We say that $\mathcal{D} = \bigcup_{j\in\mathbb{Z}}\mathcal{D}^j$ is a dyadic family on X with parameter $\delta \in (0, 1)$, briefly that $\mathcal{D}$ belongs $\mathfrak{D}(\delta)$, if each $\mathcal{D}^j$ is a family of open subsets Q of X, such that

(d.1) *For every $j \in \mathbb{Z}$ the cubes in $\mathcal{D}^j$ are pairwise disjoint.*
(d.2) *For every $j \in \mathbb{Z}$ the family $\mathcal{D}^j$ covers almost all X in the sense that $\mu(X - \bigcup_{Q\in\mathcal{D}^j} Q) = 0$.*
(d.3) *If $Q \in \mathcal{D}^j$ and $i < j$, then there exists a unique $\tilde{Q} \in \mathcal{D}^i$ such that $Q \subseteq \tilde{Q}$.*
(d.4) *If $Q \in \mathcal{D}^j$ and $\tilde{Q} \in \mathcal{D}^i$ with $i \leq j$, then either $Q \subseteq \tilde{Q}$ or $Q \cap \tilde{Q} = \emptyset$.*
(d.5) *There exist two constants a_1 and a_2 such that for each $Q \in \mathcal{D}^j$ there exists a point $x \in Q$ for which $B(x, a_1\delta^j) \subseteq Q \subseteq B(x, a_2\delta^j)$.*

The main properties of a dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$ are contained in the following result.

Proposition 3.2. *Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. Then*

- *(d.6) There exists a positive integer N depending only on the doubling constant such that for every $j \in \mathbb{Z}$ and all $Q \in \mathcal{D}^j$ the inequalities $1 \leq \#(\mathcal{O}(Q)) \leq N$ hold, where $\mathcal{O}(Q) = \{Q' \in \mathcal{D}^{j+1} : Q' \subseteq Q\}$.*
- *(d.7) X is bounded if and only if there exists a dyadic cube Q in $\mathcal{D}$ such that $X = Q$.*
- *(d.8) The families $\tilde{\mathcal{D}}^j = \{Q \in \mathcal{D}^j : \#(\{Q' \in \mathcal{D}^{j+1} : Q' \subseteq Q\}) > 1\}$, $j \in \mathbb{Z}$ are pairwise disjoint.*

For a given dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$, arguing as in [1] (see also [2]), we always can construct Haar type bases $\mathcal{H}$, of Borel measurable simple real functions h, satisfying the following properties.

- *(h.1) For each $h \in \mathcal{H}$ there exists a unique $j \in \mathbb{Z}$ and a cube $Q = Q(h) \in \tilde{\mathcal{D}}^j$ such that $\{x \in X : h(x) \neq 0\} \subseteq Q$, and this property does not hold for any cube in $\mathcal{D}^{j+1}$.*
- *(h.2) For every $Q \in \tilde{\mathcal{D}} = \bigcup_{j\in\mathbb{Z}} \tilde{\mathcal{D}}^j$ there exist exactly $M_Q = \#(\mathcal{O}(Q)) - 1 \geq 1$ functions $h \in \mathcal{H}$ such that (h.1) holds. We shall write $\mathcal{H}_Q$ to denote the set of all these functions h.*
- *(h.3) For each $h \in \mathcal{H}$ we have that $\int_X h d\mu = 0$.*
- *(h.4) For each $Q \in \tilde{\mathcal{D}}$ let V_Q denote the vector space of all functions on Q which are constant on each $Q' \in \mathcal{O}(Q)$. Then the system $\{\frac{\chi_Q}{(\mu(Q))^{1/2}}\} \bigcup \mathcal{H}_Q$ is an orthonormal basis for V_Q.*

The following result is an easy consequence of *(h.1)* to *(h.4)*. We shall denote with $\mathcal{L}^p(X,\mu) = \mathcal{L}^p, (p \geq 1)$ the space L^p of all measurable functions f such that $\|f\|_{L^p} < \infty$ when $\mu(X) = \infty$ and the space $L^p_0 = \{f \in L^p(X,\mu) : \int_X f d\mu = 0\}$ if $\mu(X) < \infty$. Here, as usual, for a measurable function f, $\|f\|^p_{L^p} = \int_X |f|^p d\mu$ if $1 \leq p < \infty$ and $\|f\|_{L^\infty} = sup\ ess|f|$.

Theorem 3.3. *Let $\mathcal{D}$ be a dyadic family on X such that $\mathcal{D}$ belongs to class $\mathfrak{D}(\delta)$. Then every Haar type system $\mathcal{H}$ associated to $\mathcal{D}$ is an orthonormal basis in $\mathcal{L}^2(X,\mu)$.*

4. Characterization of weighted Lebesgue spaces

Let us start this section introducing the basic tools of dyadic analysis on spaces of homogeneous type. When a dyadic family $\mathcal{D}$ is given we define, as usual, the class of Muckenhoupt type dyadic weight functions associated to $\mathcal{D}$. A non-negative, measurable and locally integrable function w defined on the space of homogeneous type (X,d,μ), is said to be a Muckenhoupt dyadic weight of class $A^{\mathcal{D}}_p$, $1 < p < \infty$ if the inequality

$$\left(\frac{1}{\mu(Q)}\int_Q w(x)d\mu(x)\right)\left(\frac{1}{\mu(Q)}\int_Q w(x)^{\frac{-1}{p-1}}d\mu(x)\right)^{p-1} \leq C, \tag{4.1}$$

holds for some constant C and every dyadic set $Q \in \mathcal{D}$.

For $p = 1$ we say that $w \in A^{\mathcal{D}}_1$ if there is a constant C such that the inequality

$$\frac{w(Q)}{\mu(Q)} \leq Cw(x) \tag{4.2}$$

holds for almost every point $x \in Q$ and for every dyadic cube $Q \in \mathcal{D}$. The class $A_\infty^{\mathcal{D}}$ is defined as

$$A_\infty^{\mathcal{D}} = \bigcup_{p\geq 1} A_p^{\mathcal{D}}. \tag{4.3}$$

Associated to a dyadic system $\mathcal{D}$ in $\mathfrak{D}(\delta)$ the dyadic Hardy-Litllewood maximal operator is given by

$$M_{\mathcal{D}}f(x) \;=\; \sup_Q \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y), \tag{4.4}$$

where the supremum is taken over the family of dyadic cubes Q in $\mathcal{D}$ containing x. Since from *(d.2)* we have that $E = \bigcup_{Q\in\mathcal{D}} \partial(Q)$ has zero measure, we may think that $M_{\mathcal{D}}f(x)$ is defined to be zero when $x \in E$.

The following result can be proved as in [2] where the authors prove that Haar systems associated to Christ's dyadic cubes are unconditional bases of the spaces $\mathcal{L}^p_w$ with $w \in A_p^{\mathcal{D}}$ and $1 < p < \infty$. As before, the space $\mathcal{L}^p_w$ is the space $L^p_w = \{f : (\int_X |f|^p d\mu)^{1/p} < \infty\}$ if $\mu(X) = \infty$ and is the space of those functions in $L^p_w(X,\mu)$ with vanishing integral if $\mu(X) < \infty$. In the sequel we shall use the following notation,

$$T_F(f) = \sum_{h\in F} < f, h > h = \sum_{h\in F} \left(\int_X fh \; d\mu \right) h,$$

where F is a finite subset of $\mathcal{H}$ and

$$\mathcal{S}(f)(x) = \left(\sum_{h\in\mathcal{H}} | < f, h > |^2 |h(x)|^2 \right)^{1/2}.$$

Theorem 4.1. *Let (X, d, μ) be a space of homogeneous type and let $\mathcal{H}$ be a Haar system associated to a dyadic family $\mathcal{D} \in \mathfrak{D}(\delta)$. If $1 < p < \infty$ and $w \in A_p^{\mathcal{D}}$ then*

(1) There exist two positive constants C_1 and C_2 such that for all $f \in \mathcal{L}^p_w(X,\mu)$ have that

$$C_1 \|f\|_{L^p_w} \;\leq \|\mathcal{S}(f)\|_{L^p_w} \;\leq\; C_2 \|f\|_{L^p_w};$$

(2) $\mathcal{H}$ is an unconditional basis for $\mathcal{L}^p_w(X,\mu)$ in the sense that

- *(2.1) The operators T_F are uniformly bounded on $\mathcal{L}^p_w$ with F varying on the finite subsets of $\mathcal{H}$,*
- *(2.2) for each $h \in \mathcal{H}$ the functional $< f, h >= \int_X fh \; d\mu$, is linear and continuous for $f \in \mathcal{L}^p_w$,*
- *(2.3) the linear span of $\mathcal{H}$ is dense in $\mathcal{L}^p_w$.*

5. Extrapolation: from weighted Lebesgue spaces to Lorentz spaces

This section is devoted to introduces the extrapolation technique of Rubio de Francia. Such extrapolation result provides boundedness in the $L^{p,q}$-norm from boundedness in dyadic weighted Lebesgue spaces. The precise result of extrapolation that we shall use is a slight modification of the generalized technique of Rubio de Francia given in Theroem 3.5 in [7]. Let us start by introducing briefly the basic notions of Banach function spaces which are needed to state precisely Theorem 3.5 in [7]. We refer to [5] for complete details. Let (X,μ) be a σ-finite measure space. We shall write $\mathcal{M}_\mu$ and $\mathcal{M}_\mu^+$ to denote the set of all μ-measurable functions $f : X \longrightarrow [-\infty, +\infty]$ and the subset of $\mathcal{M}_\mu$ whose values lie in $[0,\infty]$ respectively.

A function norm is a mapping $\rho : \mathcal{M}_\mu^+ \longrightarrow [0,\infty]$ such that for all f, g and f_n (with $n \in \mathbb{Z}^+$) in $\mathcal{M}_\mu^+$ the following statements hold

- $(B1)$ $\rho(f) = 0$ if and only if $f = 0$ μ-a.e.,
- $(B2)$ for all $a > 0$ we have that $\rho(af) = a\rho(f)$,
- $(B3)$ $\rho(f + g) \leq \rho(f) + \rho(g)$,
- $(B4)$ if $0 \leq g \leq f$ μ-a.e., then $\rho(g) \leq \rho(f)$,
- $(B5)$ if $0 \leq f_n \nearrow f$ μ-a.e., then $\rho(f_n) \nearrow \rho(f)$,
- $(B6)$ if $E \subseteq X$ with $\mu(E) < \infty$, then $\rho(\chi_E) < \infty$,
- $(B7)$ for each $E \subseteq X$ with $\mu(E) < \infty$, there exists a positive and finite constant C such that $\int_E f d\mu \leq C\rho(f)$.

The space $\mathbb{B} = \{f \in \mathcal{M}_\mu : \|f\|_{\mathbb{B}} < \infty\}$ is a normed Banach space with norm given by $\|f\|_{\mathbb{B}} = \rho(|f|)$. We shall say that $\mathbb{B}$ is a Banach function space.

Given a Banach function space $\mathbb{B}$, we define the scale of Banach function spaces $\mathbb{B}^r, 1 \leq r < \infty$, by $\mathbb{B}^r = \{f \in \mathcal{M}_\mu : |f|^r \in \mathbb{B}\}$ with norm $\|f\|_{\mathbb{B}^r} = \||f|^r\|_{\mathbb{B}}^{1/r}$. The associated space to $\mathbb{B}$, $\mathbb{B}^{'}$, is the space of all functions $f \in \mathcal{M}_\mu$ such that $\|f\|_{\mathbb{B}'} < \infty$, where

$$\|f\|_{\mathbb{B}'} = sup\left\{\int_X |f(x)g(x)|d\mu(x) : g \in \mathbb{B}, \|g\|_{\mathbb{B}} \leq 1\right\}.$$

These space $\mathbb{B}^{'}$ is a Banach function space and the following generalized Hölder inequality holds: for all $f \in \mathbb{B}$ and every $g \in \mathbb{B}^{'}$,

$$\int_X |f(x)g(x)|d\mu(x) \leq \|f\|_{\mathbb{B}}\|g\|_{\mathbb{B}'}.$$

Also, since $(\mathbb{B}^{'})^{'} = \mathbb{B}$, we obtain the following fundamental identity

$$\|f\|_{\mathbb{B}} = sup\left\{\int_X |f(x)g(x)|d\mu(x) : g \in \mathbb{B}^{'}, \|g\|_{\mathbb{B}'} \leq 1\right\}. \tag{5.1}$$

Given a family $\mathfrak{B}$ of open sets in (X, d, μ) we define, for $1 \leq p \leq \infty$, the Muckenhoupt classes $A_p^{\mathfrak{B}}$ as the family of all locally integrable functions w such that the inequalities (4.1), (4.2) and (4.3) respectively, hold with $\mathfrak{B}$ instead $\mathcal{D}$. Also, we define the operator $M_{\mathfrak{B}}$ via (4.4) with the supremum taking over the family $\mathfrak{B}$. As in [7], we shall say that $\mathfrak{B}$ is a Muckenhoupt basis if for every $1 < p < \infty$, $w \in A_p^{\mathfrak{B}}$ is a sufficient condition for the L^p_w boundedness of $M_{\mathfrak{B}}$. On the other hand, if $\mathcal{F}$ is a family of ordered pairs (f, g) of non negative and measurable functions on X, we shall say that $\mathcal{F}$ is $\mathbb{B}$-admissible if for each f such that for some g $(f, g) \in \mathcal{F}$, then both

(a) $\int_X f^p w d\mu < \infty$ for every $1 < p < \infty$ and every $w \in A_p^{\mathcal{D}}$, and
(b) $\|f\|_{\mathbb{B}} < \infty$.

The precise statement of extrapolation given in [7] is contained in the following result.

Theorem 5.1. *(Theorem 3.5 in [7]) Let $\mathfrak{B}$ be a Muckenhoupt basis and let $\mathbb{B}$ be a Banach function space. Let $\mathcal{F}$ be a $\mathbb{B}$-admissible family of pairs (f, g). Suppose that for some p_0, $0 < p_0 < \infty$, and every $w \in A_1^{\mathfrak{B}}$,*

$$\int_X f(x)^p w(x)d\mu(x) \leq C\int_X g(x)^p w(x)d\mu(x). \tag{5.2}$$

If there exists q_0, $p_0 \leq q_0 < \infty$, such that $\mathbb{B}^{1/q_0}$ is a Banach function space and $M_{\mathfrak{B}}$ is bounded on $(\mathbb{B}^{1/q_0})'$, then

$$\|f\|_{\mathbb{B}} \leq C\|g\|_{\mathbb{B}}. \tag{5.3}$$

Next we shall state the extrapolation result that we shall use in the sequel, which, as mentioned, is a slight variant of the above theorem for the particular context of Lorentz spaces on general measure spaces.

Theorem 5.2. *Let $1 < p, q < \infty$ be given. Let $\mathcal{F}$ be an $L^{p,q}$-admissible family of ordered pair (f, g). Assume that for every r, $1 < r < \infty$, there exists a positive constant $C = C(r)$ such that the inequality*

$$\int_X f(x)^r w(x) d\mu(x) \leq C \int_X g(x)^r w(x) d\mu(x). \tag{5.4}$$

holds for every $(f, g) \in \mathcal{F}$ and every $w \in A_1^{\mathcal{D}}$. Then, for some constant C we have

$$\|f\|_{p,q} \leq C\|g\|_{p,q}, \tag{5.5}$$

for every $(f, g) \in \mathcal{F}$.

In [3] the authors proved that the dyadic maximal operator $M_{\mathcal{D}}$ is bounded in L^p_w for $w \in A_p^{\mathcal{D}}$ and $1 < p < \infty$. Then, since each dyadic cube $Q \in \mathcal{D}$ is an open set, we get that every dyadic family $\mathcal{D}$ is a Muckenhoupt basis. Moreover, from interpolation (see for example [13] page 197) we get the following result.

Theorem 5.3. *Let (X, d, μ) be a space of homogeneous type. If $1 < p, q < \infty$, then there exists a positive constant C such that $\|M_{\mathcal{D}} f\|_{p,q} \leq C\|f\|_{p,q}$ for every function f.*

Even when, at first, Theorem 5.2 looks like a special case of Theorem 5.1, this is not the case in our general geometric setting. In fact, in (X, d, μ) atoms are allowed. Hence, as shown [5], it could happen that the spaces $(L^{p,q})'$ and $(L^{p,q})^*$ does not coincide. On the other hand, since $M_{\mathcal{D}}$ is bounded as an operator in $L^{p',q'} = (L^{p,q})^*$, Theorem 5.2 has to be proved as Theorem 5.1 (Theorem 3.5 in [7]) after changing the boundedness hypothesis of $M_{\mathcal{D}}$ in $(L^{p,q})'$ by its boundedness in $(L^{p,q})^*$.

Proof of Theorem 5.2. Let us start choose $p_0 > 1$ such that $1 < \frac{p}{p_0}, \frac{q}{p_0} < \infty$. Then, since $\mathcal{F}$ is $L^{p,q}$-admissible, there exists a positive constant C such that

$$\int_X f(x)^{p_0} w(x) d\mu(x) \leq C \int_X g(x)^{p_0} w(x) d\mu(x), \tag{5.6}$$

for every $(f, g) \in \mathcal{F}$ and each $w \in A_{p_0}^{\mathcal{D}}$.
On the other hand, from Theorem 5.3, the dyadic maximal operator $M_{\mathcal{D}}$ is bounded on the space $(L^{\frac{p}{p_0},\frac{q}{p_0}})^* = (L^{\frac{p}{p_0},\frac{q}{p_0}}(X,\mu))^* = L^{(\frac{p}{p_0})',(\frac{q}{p_0})'}(X,\mu)$, where $(\frac{p}{p_0})'$ and $(\frac{q}{p_0})'$ are the conjugate Hölder exponents of $(\frac{p}{p_0})$ and $(\frac{q}{p_0})$ respectively. So, we can define the iteration algorithm of Rubio de Francia for each function h in $(L^{\frac{p}{p_0},\frac{q}{p_0}})^*$ by

$$Rh(x) = \sum_{k=0}^{\infty} \frac{M_{\mathcal{D}}^k h(x)}{2^k \|M_{\mathcal{D}}\|^k},$$

where with $\|M_{\mathcal{D}}\|$ we denote the operator norm $M_{\mathcal{D}}$. It is easy see that

(RF1) $h(x) \leq Rh(x)$,

(RF2) $\|Rh\|_{(L^{\frac{p}{p_0},\frac{q}{p_0}})^*} \leq 2\|h\|_{(L^{\frac{p}{p_0},\frac{q}{p_0}})^*}$,

(RF3) $M(Rh)(x) \leq 2\|M_{\mathcal{D}}\|Rh(x)$.

Now, let $(f,g) \in \mathcal{F}$. Since $L^{\frac{p}{p_0},\frac{q}{p_0}}$ is a Banach function space, we get that

$$\|f\|_{p,q}^{p_0} = \|f^{p_0}\|_{\frac{p}{p_0},\frac{q}{p_0}} = \sup\left\{\int_X |f(x)|^{p_0}|h(x)|d\mu(x) : \|h\|_{(\frac{p}{p_0})',(\frac{q}{p_0})'} \leq 1\right\}.$$

Notice that, since f is non negative, we may restrict the supremum to non negative h. Therefore, it will suffice prove that there exists a positive constant C such that the inequality

$$\int_X f(x)^{p_0}h(x)d\mu(x) \leq C\|g\|_{p,q}^{p_0}, \tag{5.7}$$

holds for each non negative function h with $\|h\|_{(\frac{p}{p_0})',(\frac{q}{p_0})'} \leq 1$.

To this end, note first that, from (RF1), the generalized Hölder inequality and (RF2) we get that

$$\begin{aligned}\int_X f(x)^{p_0}h(x)d\mu(x) &\leq \int_X f(x)^{p_0}Rh(x)d\mu(x)\\ &\leq \|f^{p_0}\|_{\frac{p}{p_0},\frac{q}{p_0}}\|Rh\|_{(\frac{p}{p_0})',(\frac{q}{p_0})'}\\ &\leq \|f^{p_0}\|_{\frac{p}{p_0},\frac{q}{p_0}}\|h\|_{(\frac{p}{p_0})',(\frac{q}{p_0})'} < \infty.\end{aligned}$$

Thus, since from (RF3) we have that $Rh \in A_1^{\mathcal{D}} \subseteq A_{p_0}^{\mathcal{D}}$, then from (RF1), (5.6), the generalized Hölder inequality and (RF2) we get that

$$\begin{aligned}\int_X f(x)^{p_0}h(x)d\mu(x) &\leq \int_X f(x)^{p_0}Rh(x)d\mu(x)\\ &\leq \int_X g(x)^{p_0}Rh(x)d\mu(x)\\ &\leq C\|g^{p_0}\|_{\frac{p}{p_0},\frac{q}{p_0}}\|Rh\|_{(\frac{p}{p_0})',(\frac{q}{p_0})'}\\ &\leq C\|g^{p_0}\|_{\frac{p}{p_0},\frac{q}{p_0}}\|h\|_{(\frac{p}{p_0})',(\frac{q}{p_0})'} < \infty\\ &\leq C\|g\|_{p,q}^{p_0},\end{aligned}$$

with C independent of h. Hence,

$$\|f\|_{p,q}^{p_0} \leq C\|g\|_{p,q}^{p_0}.$$

□

6. The main result

In this section we shall prove the following result that is the analogous of Theorem 4.1 for Lorentz spaces.

Theorem 6.1. *Let (X,d,μ) be a space of homogeneous type and let $\mathcal{H}$ be a Haar system associated to a dyadic family $\mathcal{D}$ in $\mathfrak{D}(\delta)$. If $1 < p,q < \infty$, then*

(I) there exist two positive constants C_1 and C_2 such that for all $f \in \mathcal{L}^{p,q}$ we have that

$$C_1\|f\|_{p,q} \leq \left\|\left(\sum_{h\in\mathcal{H}} |< f,h >|^2|h|^2\right)^{1/2}\right\|_{p,q} \leq C_2\|f\|_{p,q}.$$

(*II*) *$\mathcal{H}$ is an unconditional basis for $\mathcal{L}^{p,q}(X,\mu)$ in the sense that:*
(II.1) the operators T_F are uniformly bounded on $\mathcal{L}^{p,q}$ with F varying on the finite subsets of $\mathcal{H}$,
(II.2) for each $h \in \mathcal{H}$ the functional $< f, h >= \int_X fh\ d\mu$, is linear and continuous for $f \in \mathcal{L}^{p,q}$,
(II.3) the linear span of $\mathcal{H}$ is dense in $\mathcal{L}^{p,q}$.

In order to prove (I) and (II), we shall apply the extrapolation result in Theorem 5.2 to admissible classes which are given in terms of the operators T_F and $\mathcal{S}$. The next two propositions shall be the central tools for the proof of Theorem 6.1.

Proposition 6.2. *The operators T_F are uniformly bounded in $\mathcal{L}^{p,q}$, $1 < p, q < \infty$. That is, there exists a positive constant C such that the inequality*

$$\|T_F(f)\|_{p,q} \leq C\|f\|_{p,q},$$

holds for every function $f \in \mathcal{L}^{p,q}$ and all finite subset $F \subseteq \mathcal{H}$.

Proof. First notice that, since the functions $h \in \mathcal{H}$ are simple, for each finite set $F \subseteq \mathcal{H}$ and each function $f \in \mathcal{L}^{p,q}$ we get that $T_F(f) \in \mathbb{V}_X$. Therefore, $T_F(f) \in L^r_w \cap L^{p,q}$ for all $w \in A^{\mathcal{D}}_r$ with $1 < r < \infty$ and every $f \in \mathcal{L}^{p,q}$. Set $\mathcal{F} = \{(T_F(f), f) : f \in \mathcal{L}^{p,q}, F \subseteq \mathcal{H}, \#(F) < \infty\}$, where $\#(F)$ denotes the number of elements of the set F. Then $\mathcal{F}$ is an $L^{p,q}$-admissible family. From Theorem 4.1, $\mathcal{H}$ is an unconditional basis for $\mathcal{L}^r_w(X,\mu)$ for every $w \in A^{\mathcal{D}}_r$ with $1 < r < \infty$. Then, for some constant C we get that

$$\|T_F(f)\|_{L^r_w} \leq C\|f\|_{L^r_w},$$

holds for every $w \in A^{\mathcal{D}}_r$, any $1 < r < \infty$ and for all finite $F \subseteq \mathcal{H}$. Since $A^{\mathcal{D}}_1 \subseteq A^{\mathcal{D}}_r$ for all $1 < r < \infty$, the proposition follows from Theorem 5.2. □

Proposition 6.3. *There exist two positive constants C_1 and C_2 such that the inequalities*

$$C_1\|f\|_{p,q} \leq \|\mathcal{S}(f)\|_{p,q} \leq C_2\|f\|_{p,q}, \tag{6.1}$$

hold for every function $f \in \mathcal{V}_{(X)}$

Proof. The left inequality in (6.1) follows directly from Theorem 5.2 taking $\mathcal{F} = \{(f, \mathcal{S}(f)) : f \in \mathcal{V}_{(X)}\}$ and using Theorem 4.1. To prove the right hand side inequality in (6.1), we begin applying Theorem 5.2 with the family $\mathcal{F} = \{(S_F(f), f) : f \in \mathcal{V}_{(X)}, F \subseteq \mathcal{H}, \#(F) < \infty\}$, where for each finite set $F \subseteq \mathcal{H}$, $\mathcal{S}_F(f) = \left(\sum_{h\in F} |< f, h >|^2 |h|^2\right)^{1/2}$. In fact, it is easy to see that $\mathcal{S}_F(f) \in \mathbb{V}_X$ and therefore $\mathcal{F}$ is an $L^{p,q}$-admissible family. Applying Theorem 5.2, since from Theorem 4.1 the $\mathcal{L}^p_w$-boundedness of $\mathcal{S}_{\mathcal{F}}$ is uniform in F, we get

$$\|\mathcal{S}_F(f)\|_{p,q} \leq C\|f\|_{p,q}, \tag{6.2}$$

for all $f \in \mathcal{V}_{(X)}$ and every finite subset F of $\mathcal{H}$.

Now we shall show that (6.2) holds also for $\mathcal{S}(f)$. Take a sequence $(F_n : n \in \mathbb{Z}^+)$ of subsets of $\mathcal{H}$ such that $\#(F_n) < \infty$, $F_n \subseteq F_{n+1}$ for each positive integer n and $\bigcup_n F_n = \mathcal{H}$. Then $\mathcal{S}_{F_n}(f)(x) \nearrow \mathcal{S}(f)(x)$ for all $x \in X$ and every $f \in \mathcal{V}_{(X)}$. Hence, from $(L4)$ and (6.2) we get that $\mathcal{S}(f) \in L^{p,q}$ and (6.2) holds for $\mathcal{S}(f)$ □

Proof of Theorem 6.1. We first prove (I). Let us start by showing that

$$\|\mathcal{S}(f)\|_{p,q} \leq C\|f\|_{p,q}, \tag{6.3}$$

for some positive constant C and every function f in $\mathcal{L}^{p,q}$. Let $f \in \mathcal{L}^{p,q}$ be given. Thus, from $(L5)$, there exits a sequence $(f_k : k \in \mathbb{Z}^+)$ of functions $f_k \in \mathcal{V}_{(X)}$ such that

$$\|f_k - f\|_{p,q} \underset{k\to\infty}{\longrightarrow} 0. \tag{6.4}$$

Notice that for such a sequence and each function $h \in \mathcal{H}$ we get that

$$< f_k, h > \underset{k\to\infty}{\longrightarrow} < f, h > \tag{6.5}$$

In fact. From *(h.1)* to *(h.4)* we have, for each $h \in \mathcal{H}$, that

$$|h(x)| \leq \mu(Q(h))^{-1/2}\chi_{Q(h)}(x).$$

Therefore, from $(L9)$, $(L2)$, $(L1)$ and (6.4) we get that

$$\begin{aligned} |< f_k, h > - < f, h >| &\leq \int_X |f_k - f||h| d\mu \\ &\leq C\mu(Q(h))^{-1/2}\|f_k - f\|^*_{p,q}\|\chi_{Q(h)}\|^*_{p',q'} \\ &\leq C\mu(Q(h))^{-1/2+1/p'}\|f_k - f\|_{p,q} \longrightarrow 0. \end{aligned}$$

It is easy to see, using a discrete version of Fatou Lemma, that

$$\mathcal{S}(f)(x) \leq \underset{k\longrightarrow\infty}{lim\ inf} \mathcal{S}(f_k)(x).$$

Thus, from $(L7)$ and Proposition 6.3 we have that

$$\begin{aligned} \|\mathcal{S}(f)\|_{p,q} &\leq \underset{k\longrightarrow\infty}{lim\ inf}\|\mathcal{S}(f_k)\|_{p,q} \\ &\leq C\ \underset{k\longrightarrow\infty}{lim\ inf}\|f_k\|_{p,q} = C\ \|f\|_{p,q}. \end{aligned}$$

Now we shall prove that there exists a positive constant C such that $\|f\|_{p,q} \leq C\|\mathcal{S}(f)\|_{p,q}$ for every function $f \in \mathcal{L}^{p,q}$. Notice that if f belongs to $\mathcal{L}^{p,q}$ and $(f_k : k \in \mathbb{Z}^+)$ is a sequence of functions in $\mathcal{V}_{(X)}$ as in (6.4), then

$$\mathcal{S}(f)(x) \leq 2\left[\mathcal{S}(f - f_k)(x) + \mathcal{S}(f_k)(x)\right]. \tag{6.6}$$

Thus, from Proposition 6.3, (6.6), $(L3)$, $(L1)$ and (6.3) we get that

$$\begin{aligned} \|f\|_{p,q} &= \underset{k\longrightarrow\infty}{lim}\|f_k\|_{p,q} \\ &\leq C\ \underset{k\longrightarrow\infty}{lim\ inf}\|\mathcal{S}(f_k)\|_{p,q} \\ &\leq 2C\left(\underset{k\longrightarrow\infty}{lim\ inf}\|\mathcal{S}(f_k - f)\|_{p,q} + \|\mathcal{S}(f)\|_{p,q}\right) \\ &\leq 2C\left(\underset{k\longrightarrow\infty}{lim\ inf}\|f_k - f\|_{p,q} + \|\mathcal{S}(f)\|_{p,q}\right) \\ &= 2C\|\mathcal{S}(f)\|_{p,q}, \end{aligned}$$

which finishes the proof of (I). Now, we shall prove (II). First notice that $(II.1)$ is the Proposition 6.2. Therefore, we only need to show $(II.2)$ and $(II.3)$. Since each function $h \in \mathcal{H}$ belong to $L^\infty(X, \mu)$, from $(L9)$, $(L2)$ and $(L1)$ we get

$$\begin{aligned} |< h, f >| &= \left|\int_X h(x)f(x)d\mu(x)\right| \\ &\leq \|h\|_\infty \int_{Q(h)} |f(x)|d\mu(x) \end{aligned}$$

$$\leq \quad C\|h\|_\infty \|\chi_{Q(h)}\|^*_{p',q'} \|f\|^*_{p,q}$$
$$\leq \quad C\|h\|_\infty \mu(Q(h))^{1/p} \|f\|_{p,q},$$

for each $h \in \mathcal{H}$ and all function $f \in L^{p,q}$. Then $(II.2)$ holds. Let us finally show $(II.3)$. Set $f \in L^{p,q}$. Take a sequence $(\mathcal{H}_n : n \in \mathbb{Z}^+)$ of subsets of $\mathcal{H}$ such that $\bigcup_n \mathcal{H}_n = \mathcal{H}$, $\#(\mathcal{H}_n) < \infty$ and $\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$. For each positive integer n we write

$$T_{\mathcal{H}_n}(f)(x) = \sum_{h \in \mathcal{H}_n} < f, h > h(x).$$

Thus, from the orthogonality of the Haar system $\mathcal{H}$ and the linearity of the operators $T_{\mathcal{H}_n}$, we get that

$$\mathcal{S}(f - T_{\mathcal{H}_n}(f))(x) = \left(\sum_{h \in \mathcal{H} \setminus \mathcal{H}_n} | < f, h > |^2 |h(x)|^2 \right)^{1/2}.$$

Then $\mathcal{S}(f - T_{\mathcal{H}_n}(f))(x) \longrightarrow 0$ μ-a.e., and $\mathcal{S}(f - T_{\mathcal{H}_n}(f))(x) \leq \mathcal{S}(f)(x)$. Hence, from (I) and $(L6)$ we have that

$$\|f - T_{\mathcal{H}_n}(f)\|_{p,q} \leq C \|\mathcal{S}(f - T_{\mathcal{H}_n}(f))\|_{p,q} \longrightarrow 0,$$

when $n \longrightarrow \infty$ and $(II.3)$ is proved. □

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Luis Nowak: Departamento de Matemática (FaEA-UNComa), Neuquén; IMAL, CONICET, Santa Fe, Argentina
E-mail address: luisenlitoral@yahoo.com.ar