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Weak-quasi-Stone algebras

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In this paper we shall introduce the variety $\mathcal{WQS}$ of weak-quasi-Stone algebras as a generalization of the variety $\mathcal{QS}$ of quasi-Stone algebras introduced in [9]. We shall apply the Priestley duality developed in [4] for the variety $\mathcal{N}$ of $\neg$-lattices to give a duality for $\mathcal{WQS}$. We prove that a weak-quasi-Stone algebra is characterized by a property of the set of its regular elements, as well by mean of some principal lattice congruences. We will also determine the simple and subdirectly irreducible algebras.

1 Introduction and preliminaries

The variety $\mathcal{N}$ of distributive lattices with a negation operator was introduced in [4] as a generalization of some known algebraic structures with bounded distributive lattices as reduct and endowed with a unary operation $\neg$, similar to *p*-algebras [2], quasi-Stone algebras [9], and semi-De Morgan algebras [3]. In [4] a topological duality was given for $\neg$-lattices using Priestley spaces with a binary relation. It was also shown that there exists a duality between Priestley spaces endowed with an equivalence relation and quasi-Stone algebras. A similar duality was described by H. Gaitan in [7]. The underlying idea being that the unary operation $\neg$ on bounded distributive lattices behaves like and encompasses both pseudocomplemented distributive lattices and quasi-Stone algebras.

In this paper we shall introduce a new variety, called the variety of weak-quasi-Stone algebras, as a natural generalization of the variety of quasi-Stone algebras. We shall prove that a weak-quasi-Stone algebra is characterized by a property of the set of its regular elements, as well by mean of some principal lattice congruences. We will prove that there exists a duality between Priestley spaces endowed with a serial, transitive and euclidean relation and weak-quasi-Stone algebras in terms of the duality developed in [4]. Using Priestley duality, we shall also characterize the simple and subdirectly irreducible weak-quasi-Stone algebras.

We assume that the reader is familiar with the basic concepts from universal algebra, distributive lattices, Priestley spaces, and in particular, with the duality between the categories of distributive lattices with a negation operator and Priestley relational spaces as it is presented in [4]. Nevertheless, we shall review some terminology and notation.

Definition 1.1 An algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a *distributive lattice with a negation operator* $\neg$ (or a $\neg$-*lattice*), if $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and it satisfies the identities:

N1: $\neg 0 = 1$.

N2: $\neg(a \vee b) = \neg a \wedge \neg b$.

The variety of $\neg$-lattices will be denoted by $\mathcal{N}$.

Let us recall that a quasi-Stone algebra (QS-algebra) is a $\neg$-lattice $\boldsymbol{A}$ satisfying the following equations:

QS1: $a \wedge \neg\neg a = a$.

QS2: $\neg a \vee \neg\neg a = 1$.

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The variety of quasi-Stone algebras will be denote by $\mathcal{QS}$. The following identities hold in every quasi-Stone algebra:

1. $\neg 1 = 0$.
2. $\neg(a \wedge \neg b) = \neg a \vee \neg\neg b$.

Given a poset $\langle X, \leq \rangle$, a set $Y \subseteq X$ is *increasing* if it closed under $\leq$, that is if for every $y \in Y$ and every $x \in X$, if $y \leq x$, then $x \in Y$. The set of all increasing subsets of X will be denoted by $\mathcal{P}_{\mathrm{i}}(X)$, and the power set of X by $\mathcal{P}(X)$. The set of maximal elements of a set $Y \subseteq X$ shall be denoted by $\mathrm{Max}Y$.

If R is a binary relation on a set X and $x \in X$, we define $R(x) = \{y \in X : xRy\}$.

Example 1.2 Let us consider a relational structure $\boldsymbol{F} = \langle X, \leq, R \rangle$, where $\langle X, \leq \rangle$ is a poset, and R is a binary relation on X such that $(\leq \circ R \circ \leq^{-1}) \subseteq R$. Then the set $\mathcal{P}_{\mathrm{i}}(X)$ of all increasing subsets of X is closed under the operation $\neg_R$ defined by

$$\neg_R(U) = \{x \in X : R(x) \cap U = \emptyset\},$$

for all $U \in \mathcal{P}_{\mathrm{i}}(X)$. It is easy to see that the structure

$$\mathcal{F}(\boldsymbol{F}) = \langle \mathcal{P}_{\mathrm{i}}(X), \cup, \cap, \neg_R, \emptyset, X \rangle$$

is an example of $\neg$-lattice.

The set of all prime filters of a bounded distributive lattice $\boldsymbol{L}$ shall be denoted by $X(\boldsymbol{L})$. The filter (ideal) generated by a set $H \subseteq A$ will be denoted by $[H)$ ($(H]$). Given $\boldsymbol{A} \in \mathcal{N}$, $X(\boldsymbol{A})$ will denote the set of all prime filters of its bounded distributive lattice reduct.

Lemma 1.3 [4] *Let $\boldsymbol{A} \in \mathcal{N}$.*

1. *For each $P \in X(\boldsymbol{A})$, the set $\neg^{-1}(P) = \{a \in A \mid \neg a \in P\}$ is an ideal.*
2. *$\neg a \notin P$ iff there is $Q \in X(\boldsymbol{A})$ such that $\neg^{-1}(P) \cap Q = \emptyset$ and $a \in Q$.*

Let $\boldsymbol{A} \in \mathcal{N}$. We shall define a binary relation $R_\neg$ on the set $X(\boldsymbol{A})$ given by

$$(P, Q) \in R_\neg \quad \text{iff} \quad \neg^{-1}(P) \cap Q = \emptyset.$$

It is easy to see that $(\subseteq \circ R_\neg \circ \subseteq^{-1}) \subseteq R_\neg$. If we denote $\mathcal{F}(\boldsymbol{A}) = \langle X(\boldsymbol{A}), \subseteq, R_\neg \rangle$, by Example 1.2 we get that $A(\mathcal{F}(\boldsymbol{A})) = \langle \mathcal{P}_{\mathrm{i}}(X(\boldsymbol{A})), \cup, \cap, \neg_{R_\neg}, \emptyset, X(\boldsymbol{A}) \rangle$ is a $\neg$-lattice.

As in the case of the representation for bounded distributive lattices to obtain the representation theorem for $\neg$-lattices let us consider the family of sets $\sigma(\boldsymbol{A}) = \{\sigma(a) : a \in A\}$, where for each $a \in A$,

$$\sigma(a) = \{P \in X(\boldsymbol{A}) : a \in P\}.$$

Then it is easy to see that $\sigma(\neg a) = \neg_{R_\neg}(\sigma(a))$. Thus the set $\sigma(\boldsymbol{A})$ is closed under the operation $\neg_{R_\neg}$ defined on $\mathcal{P}_{\mathrm{i}}(X(\boldsymbol{A}))$. So, the algebra $\mathrm{D}(X(\boldsymbol{A})) = \langle \sigma(\boldsymbol{A}), \cup, \cap, \neg_R, \emptyset, X(\boldsymbol{A}) \rangle$ is a subalgebra of the algebra $A(\mathcal{F}(\boldsymbol{A}))$. Thus, we have that every $\neg$-lattice $\boldsymbol{A}$ is isomorphic to the $\neg$-lattice of sets $\mathrm{D}(X(\boldsymbol{A})))$, that is, σ is an embedding of $\boldsymbol{A}$ into the algebra $A(\mathcal{F}(\boldsymbol{A}))$.

A *Priestley space* is a triple $X = \langle X, \leq, \tau \rangle$, where $\langle X, \leq \rangle$ is a poset and $\langle X, \tau \rangle$ is a Stone space (compact, Hausdorff and 0-dimensional topological space) that satisfies the Priestley separation axiom: for every $x, y \in X$ such that $x \nleq y$ there is a clopen increasing subset U of X such that $x \in U$ and $y \notin U$. If X is a Priestley space, the set of all clopen increasing subsets of X will be denoted by $D(X)$. Since $D(X)$ is a ring of sets, then $\langle D(X), \cup, \cap, \emptyset, X \rangle$ is a bounded distributive lattice. Let us recall that if X is a Priestley space and Y is a closed subset of X, then $\mathrm{Max}Y \neq \emptyset$, whenever $Y \neq \emptyset$.

Let $\boldsymbol{A} = \langle A, \vee, \wedge, 0, 1 \rangle$ be a bounded distributive lattice. The topology τ_σ on $X(\boldsymbol{A})$ generated by the subbase whose elements are the sets of the form

$$\sigma(a) = \{P \in X(\boldsymbol{A}) : a \in P\} \quad \text{and} \quad \sigma(a)^{\mathrm{c}} = X(\boldsymbol{A}) \setminus \sigma(a),$$

for $a \in A$, gives a Priestley space $\langle X(\boldsymbol{A}), \subseteq, \tau_\sigma \rangle$, and the map $\sigma : A \longrightarrow \mathrm{D}(X(\boldsymbol{A}))$ is a bounded lattice isomorphism.

The duality between bounded distributive lattices and Priestley spaces can be specialized to ¬-lattices. Let us recall that a *¬-space* (see [4]), is a structure $\langle X, \leq, \tau, R\rangle$, where $\langle X, \leq, \tau\rangle$ is a Priestley space and R is a binary relation defined on X such that:

1. For each $x \in X$, $R(x)$ is a closed and decreasing subset of X.
2. For each $U \in D(X)$, $\neg_R(U)$ is an increasing clopen.

Remark 1.4 If $\langle X, \leq, \tau, R\rangle$ is a ¬-space, then $(\leq \circ R \circ \leq^{-1}) \subseteq R$. Indeed, let $x, y, z, w \in X$ such that $x \leq y$, $(y, z) \in R$, and $w \leq z$. Assume that $(x, w) \notin R$. As $R(x)$ is a closed and decreasing subset of X, there exists $U \in D(X)$ such that $R(x) \cap U = \emptyset$ and $w \in U$. It follows that $x \in \neg_R(U)$. So, $y \in \neg_R(U)$, and as $(y, z) \in R$, $z \notin U$, which is a contradiction, because $w \leq z$. Thus, $(\leq \circ R \circ \leq^{-1}) \subseteq R$.

If $\boldsymbol{A}$ is a ¬-lattice, then the structure $X(\boldsymbol{A}) = \langle X(\boldsymbol{A}), \subseteq, \tau_\sigma, R_\neg\rangle$ is a ¬-space such that the mapping $\sigma : A \longrightarrow \mathrm{D}(X(\boldsymbol{A}))$ is an isomorphism of ¬-lattices.

Now, we shall prove that certain identities defined in a ¬-lattice $\boldsymbol{A}$ are characterized by certain conditions defined in the relational structure $\langle X(\boldsymbol{A}), R_\neg\rangle$.

Theorem 1.5 *Let $\boldsymbol{A} \in \mathcal{N}$. Then it holds:*

1. $\boldsymbol{A} \models a = a \wedge \neg\neg a$ *if and only if $R_\neg$ is symmetrical.*
2. $\boldsymbol{A} \models \neg a \vee \neg\neg a = 1$ *if and only if $R_\neg$ is euclidean, i. e., $R_\neg^{-1} \circ R_\neg \subseteq R_\neg$.*
3. $\boldsymbol{A} \models \neg a \wedge a = 0$ *if and only if $R_\neg$ is reflexive.*
4. $\boldsymbol{A} \models \neg 1 = 0$ *if and only if $R_\neg$ is serial, i. e., $R_\neg(P) \neq \emptyset$ for any $P \in X(\boldsymbol{A})$.*

Proof.

1. Suppose that $\boldsymbol{A} \models a = a \wedge \neg\neg a$. Let $(P, Q) \in R_\neg$, i. e., $\neg^{-1}(P) \cap Q \neq \emptyset$. Suppose that $(Q, P) \notin R_\neg$, i. e., $\neg^{-1}(Q) \cap P \neq \emptyset$. Let $a \in A$ such that $\neg a \in Q$ and $a \in P$. So, $\neg\neg a \in P$, this implies that $\neg a \notin Q$ and $\neg a \in \neg^{-1}(P)$, which is a contradiction. Thus, $(Q, P) \in R_\neg$. For the converse, assume that $R_\neg$ is symmetrical and suppose that there exists $a \in A$ such that $a \nleq \neg\neg a$. Then $a \in P$ and $\neg\neg a \notin P$ for some $P \in X(\boldsymbol{A})$. So, there exists $Q \in X(\boldsymbol{A})$ such that $(P, Q) \in R_\neg$ and $\neg a \in Q$, by Lemma 1.3. Since $(Q, P) \in R_\neg$ and $a \in P$, we have $\neg a \notin Q$, which is a contradiction.

2. Suppose that $\boldsymbol{A} \models \neg a \vee \neg\neg a = 1$. Let $P, Q, D \in X(\boldsymbol{A})$ such that $(P, Q) \in R_\neg$ and $(P, D) \in R_\neg$. Let $\neg a \in Q$. Then, $\neg\neg a \notin P$. Since P is prime, $\neg a \in P$. It follows that $a \notin D$. Therefore $(Q, D) \in R_\neg$. For the converse, assume that $R_\neg$ is euclidean and suppose that there exists $a \in A$ such that $\neg a \vee \neg\neg a \neq 1$. Then there exist $P, Q, D \in X(\boldsymbol{A})$ such that $\neg a, \neg\neg a \notin P$, $(P, Q) \in R_\neg$, $a \in Q$, $(P, D) \in R_\neg$ and $\neg a \in D$. Since $R_\neg$ is euclidean, $(D, Q) \in R_\neg$. As $a \in Q$, $\neg a \notin D$, which is a contradiction.

The proofs of the assertions 3. and 4. are similar and left to the reader. □

By the previous result it follows that if $\boldsymbol{A}$ is a quasi-Stone algebra, then the relation $R_\neg$ defined on $X(\boldsymbol{A})$ is symmetrical and euclidean. Since $\boldsymbol{A}$ also satisfies the identity $\neg 1 = 0$, $R_\neg$ is serial. Therefore, $R_\neg$ is an equivalence relation.

2 Weak-quasi-Stone algebras

As it is shown in [4] and in [7], the dual space of a quasi-Stone algebra is a ¬-space $\langle X, R\rangle$, where R is an equivalence relation. It is known that an equivalence relation is also serial, euclidean and transitive, but the converse is not valid. So, we can ask if it is possible to find the class of ¬-lattices such that the relation in its dual spaces are serial, euclidean and transitive. We shall see that the corresponding class of ¬-lattices is a variety which obviously will contain the variety of quasi-Stone algebras.

Definition 2.1 A *weak-quasi-Stone algebra* (*WQS-algebra*) is a ¬-lattice $\boldsymbol{A}$ satisfying the following conditions:

WQS1: $\neg a \wedge \neg\neg a = 0$,

WQS2: $\neg a \vee \neg\neg a = 1$.

The variety of weak-quasi-Stone algebras will be denoted by $\mathcal{WQS}$.

Lemma 2.2 *Let $\boldsymbol{A} \in \mathcal{WQS}$. Then it holds:*

1. $\boldsymbol{A} \models \neg 1 = 0$.
2. $\boldsymbol{A} \models \neg\neg\neg a = \neg a$.

Proof.

1. By WQS2, N1 and N2 we have $\neg 1 = \neg(1 \vee \neg 1) = \neg 1 \wedge \neg\neg 1 = 0$.

2. By WQS1 and WQS2 we have that, if $b \in \{\neg\neg\neg a, \neg a\}$, then $\neg\neg a \wedge b = 0$ and $\neg\neg a \vee b = 1$. By distributivity we have that if an element is complemented, the complement is unique, thus we conclude that $\neg\neg\neg a = \neg a$. □

Remark 2.3 We note that the variety $\mathcal{QS}$ is a proper subvariety of the variety $\mathcal{WQS}$ as it is shown in the following example. Let us consider the Boolean lattice $\mathcal{B} = \{0, a, b, 1\}$, where $\neg a = \neg 0 = 1$ and $\neg b = \neg 1 = 0$. Then, $\mathcal{B} \in \mathcal{WQS} \setminus \mathcal{QS}$, because $a \nleq \neg\neg a = 0$.

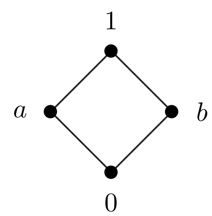

Figure 1.

Example 2.4 In [9] N. Sankappanavar and H. Sankappanavar defined an important class of quasi-Stone algebras, the special QS-algebras. A QS-algebra $\boldsymbol{A}$ is *special* if and only if, for every $a \in A \setminus \{0\}$, $\neg a = 0$. If $\boldsymbol{A}$ is a not trivial special QS-algebra consider the poset

$$A' = A \times \{0, 1\}$$

with the product order, and define $\neg' : A' \longrightarrow A'$ by

$$\neg'(0,0) = \neg'(0,1) = (1,1) \quad \text{and} \quad \neg'(a,\lambda) = (0,0) \text{ in any other case.}$$

It is easy to see that

$$\boldsymbol{A}' = \langle A', \wedge, \vee, \neg', (0,0), (1,1)\rangle \in \mathcal{WQS}.$$

Moreover, $\boldsymbol{A}' \notin \mathcal{QS}$, since $(0,1) \nleq \neg'(\neg'(0,1)) = (0,0)$.

In Section 4 we will prove every subdirectly irreducible but not simple WQS-algebra can be embedded in an algebra $\boldsymbol{A}'$ for some special QS-algebra $\boldsymbol{A}$.

Now we study the representation of the weak-quasi-Stone algebras.

Lemma 2.5 *Let $\boldsymbol{A} \in \mathcal{N}$, satisfying the following equations:*

1. $\neg 1 = 0$.
2. $\neg a \vee \neg\neg a = 1$.

Then $\boldsymbol{A} \models \neg a \wedge \neg\neg a = 0$ if and only if $R_\neg$ is transitive.

Proof. Suppose that $\boldsymbol{A} \models \neg a \wedge \neg\neg a = 0$. Let $P, Q, D \in X(\boldsymbol{A})$ such that $(P,Q), (Q,D) \in R_\neg$. Suppose that $(P,D) \notin R_\neg$, i. e., $\neg^{-1}(P) \cap D \neq \emptyset$. Let $a \in \neg^{-1}(P) \cap D$. Since $\neg a \in P$ and $\neg a \wedge \neg\neg a = 0$, $\neg\neg a \notin P$. As P is a prime filter, by 2. we have that $\neg\neg\neg a \in P$. Thus $\neg\neg a \notin Q$. But since $a \in D$ and $(Q,D) \in R_\neg$, $\neg a \notin Q$. Hence $\neg\neg a \vee \neg a \notin Q$, which is a contradiction. For the converse suppose that there exists $a \in A$ such that $\neg a \wedge \neg\neg a \neq 0$. There exists $P \in X(\boldsymbol{A})$ such that $\neg a, \neg\neg a \in P$. By 1. and Theorem 1.5, we have that $R_\neg$ is serial. Let $Q \in X(\boldsymbol{A})$ such that $(P,Q) \in R_\neg$. Since $\neg\neg a \in P$, $\neg a \notin Q$. Thus there exists $D \in X(\boldsymbol{A})$, such that $a \in D$ and $(Q,D) \in R_\neg$. As $R_\neg$ is transitive, $(P,D) \in R_\neg$. By $\neg a \in P$ we deduce that $a \notin D$, which is a contradiction, and the result follows. □

Theorem 2.6 *Let $\boldsymbol{A} \in \mathcal{N}$. Then $\boldsymbol{A} \in \mathcal{WQS}$ iff the relation $R_\neg$ is transitive, euclidean and serial.*

P r o o f. It follows from Theorem 1.5 and the previous Lemma. □

Remark 2.7 From Theorem 2.6 and the results on Priestley duality for ¬-lattices given in [4], we conclude that the category whose objects are WQS-algebras and whose arrows are the homomorphisms of ¬-lattices is dually equivalent to the full subcategory of ¬-spaces whose objects have transitive, euclidean and serial binary relations.

Example 2.8 If $\boldsymbol{A}$ is a special QS-algebra, then the dual space of $\boldsymbol{A}$ is $\langle X(\boldsymbol{A}), \subseteq, \tau_\sigma, R_\neg\rangle$, where $\langle X(\boldsymbol{A}), \subseteq, \tau_\sigma\rangle$ is the Priestley space of the lattice reduct of $\boldsymbol{A}$ and $R_\neg = X(\boldsymbol{A}) \times X(\boldsymbol{A})$. For this assume that $(P, Q) \in R_\neg$, i. e., $\neg^{-1}(P) \cap Q = \emptyset$. Since $\boldsymbol{A}$ is a special QS-algebra $\neg^{-1}(P) = \{0\}$. Thus, $\neg^{-1}(P) \cap Q = \emptyset$ if and only if $\{0\} \cap Q = \emptyset$ if and only if $(P, Q) \in X(\boldsymbol{A}) \times X(\boldsymbol{A})$.

Now consider the algebra $\boldsymbol{A}'$ defined in Example 2.4. Since the lattice reduct of $\boldsymbol{A}'$ is the lattice product $A \times \{0, 1\}$, it is easy to see that the set of prime filters $X(\boldsymbol{A}')$ is given by

$$X(\boldsymbol{A}') = \{P \times \{0, 1\} : P \in X(\boldsymbol{A})\} \cup \{A \times \{1\}\}.$$

Let us denote $A \times \{1\} = \alpha$. Thus

$$R_{\neg'} = (\{P \times \{0, 1\} : P \in X(\boldsymbol{A})\} \times \{P \times \{0, 1\} : P \in X(\boldsymbol{A})\}) \\ \cup \{(\alpha, P \times \{0, 1\}) : P \in X(\boldsymbol{A})\}.$$

To prove the previous identity, we note that for definition of $\neg'$, if $D \in X(\boldsymbol{A}')$, then $(\neg')^{-1}(D) = \{(0, 0), (0, 1)\}$. Since $\{(0, 0), (0, 1)\} \cap P \times \{0, 1\} = \emptyset$ for every $P \in X(\boldsymbol{A})$ and $(0, 1) \in \alpha = A \times \{1\}$ the result follows. Therefore, up to isomorphism, the dual space of $\boldsymbol{A}'$ is

$$\langle X(\boldsymbol{A}) \cup \{\alpha\}, \leq', \tau, R\rangle,$$

where

$$\alpha \notin X(S),\ \leq' = \ \subseteq \cup \{(\alpha, \alpha)\},\ \tau = \tau_{\boldsymbol{\sigma}} \cup \{U \cup \{\alpha\} : U \in \tau_{\boldsymbol{\sigma}}\},$$

$$R = R_\neg \cup \{(\alpha, P) : P \in X(\boldsymbol{A})\}.$$

3 Algebraic characterization of $\mathcal{WQS}$

Let $\boldsymbol{A} \in \mathcal{N}$. Let us consider the following subset of $\boldsymbol{A}$:

$$B(\boldsymbol{A}) = \{a \in A : \neg\neg a = a\}.$$

If $a \in B(\boldsymbol{A})$, a is called a *regular element* of $\boldsymbol{A}$.

In Theorem 2.6 we gave a characterization of WQS-algebras by means of their dual spaces, now we are going to prove that this variety is characterized by a property of its regular elements. To do so we need the following results.

Lemma 3.1 *Let $\boldsymbol{A} \in \mathcal{WQS}$. Then $\boldsymbol{A} \vDash \neg(a \wedge \neg b) = \neg a \vee \neg\neg b$.*

P r o o f. Suppose that there exist $a, b \in A$ such that $\neg(a \wedge \neg b) \nleq \neg a \vee \neg\neg b$. Then there are prime filters P, Q and D such that

$$\neg(a \wedge \neg b) \in P,\ \neg a \notin P,\ \neg\neg b \notin P,\ (P, Q) \in R_\neg,\ (P, D) \in R_\neg,\ a \in Q,\ \neg b \in D.$$

Since $R_\neg$ is euclidean, $(D, Q) \in R_\neg$, and this implies that $b \notin Q$. From $\neg(a \wedge \neg b) \in P$ and $(P, Q) \in R_\neg$, we get $a \wedge \neg b \notin Q$, i. e., $a \notin Q$ or $\neg b \notin Q$, but as $a \in Q$, $\neg b \notin Q$. By the identity WQS2 it follows that $\neg\neg b \in Q$, but as $(D, Q) \in R_\neg$, $\neg b \notin D$, which is a contradiction. Therefore $\neg(a \wedge \neg b) \leq \neg a \vee \neg\neg b$. We note that for every $x, y \in A$, if $x \leq y$, then $\neg y \leq \neg x$. By this property we have that $\neg a \vee \neg\neg b \leq \neg(a \wedge \neg b)$. Therefore $\neg(a \wedge \neg b) = \neg a \vee \neg\neg b$. □

In [9] it is proved that if $\boldsymbol{A} \in \mathcal{QS}$, $\boldsymbol{B}(\boldsymbol{A}) = \langle B(\boldsymbol{A}), \vee, \wedge, \neg, 0, 1\rangle$ is a subalgebra of $\boldsymbol{A}$ and it is a Boolean algebra. Now we will see that this property determines exactly the variety $\mathcal{WQS}$.

Theorem 3.2 *Let $\boldsymbol{A} \in \mathcal{N}$. Then*

$$\boldsymbol{A} \in \mathcal{WQS} \quad \textit{if and only if} \quad B(\boldsymbol{A}) = \{\neg a : a \in A\},$$

and the structure $\boldsymbol{B}(\boldsymbol{A}) = \langle B(\boldsymbol{A}), \vee, \wedge, \neg, 0, 1\rangle$ is a subalgebra of $\boldsymbol{A}$ and a Boolean algebra.

Proof. $\Rightarrow$) Suppose that $\boldsymbol{A} \in \mathcal{WQS}$. By item 2. of Lemma 2.2 we have that $B(\boldsymbol{A}) = \{\neg a : a \in A\}$. Now, let $a \in B(\boldsymbol{A})$. Then $\neg\neg\neg a = \neg a$. Thus, $\neg a \in B(\boldsymbol{A})$. Let $a, b \in B(\boldsymbol{A})$. Since, using Lemma 3.1

$$\neg\neg(a \vee b) = \neg(\neg a \wedge \neg b) = \neg\neg a \vee \neg\neg b = a \vee b,$$

we have $a \vee b \in B(\boldsymbol{A})$. Again using Lemma 3.1

$$\neg\neg(a \wedge b) = \neg(\neg(a \wedge \neg\neg b)) = \neg(\neg a \vee \neg\neg\neg b) = \neg\neg a \wedge \neg\neg\neg\neg b = a \wedge b.$$

Thus $a \wedge b \in B(\boldsymbol{A})$. It is clear that $0, 1 \in B(\boldsymbol{A})$. Thus, $B(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{A}$. If $a \in B(\boldsymbol{A})$, then

$$0 = \neg\neg a \wedge \neg a = a \wedge \neg a \quad \text{and} \quad 1 = \neg\neg a \vee \neg a = a \vee \neg a.$$

Thus $\boldsymbol{B}(\boldsymbol{A})$ is a Boolean algebra.

$\Leftarrow$) Let $a \in A$. So $\neg a \in B(\boldsymbol{A})$. Since $\boldsymbol{B}(\boldsymbol{A})$ is a Boolean algebra and it is a subalgebra of $\boldsymbol{A}$, $\neg\neg a \wedge \neg a = 0$ and $\neg\neg a \vee \neg a = 1$. Therefore $\boldsymbol{A} \in \mathcal{WQS}$. □

The variety of $\mathcal{WQS}$ can be characterized by means of some principal lattice congruences. To prove that we first need some results.

Given a distributive lattice $\boldsymbol{A}$, it is known that if F is a filter and I an ideal the following relations are lattice congruences:

$$\theta_{\text{lat}}(F) = \{(x, y) \in A \times A : x \wedge f = y \wedge f \text{ for some } f \in F\}$$
$$\theta_{\text{lat}}(I) = \{(x, y) \in A \times A : x \vee i = y \vee i \text{ for some } i \in I\}.$$

The following Lemma is a generalization of Lemma 4.1 of [9].

Lemma 3.3 *Let $\boldsymbol{A} \in \mathcal{WQS}$.*

1. *For each filter F of $\boldsymbol{B}(\boldsymbol{A})$, the relation $\theta_{\text{lat}}([F))$ is a congruence of $\boldsymbol{A}$.*
2. *For each ideal I of $\boldsymbol{B}(\boldsymbol{A})$, the relation $\theta_{\text{lat}}((I])$ is a congruence of $\boldsymbol{A}$.*

Proof.

1. Suppose that $(a, b) \in \theta_{\text{lat}}([F))$. Since $B(\boldsymbol{A}) = \{\neg a : a \in A\}$ and it is a subalgebra of $\boldsymbol{A}$, there exists $c \in A$ such that $\neg c \in F$ and $a \wedge \neg c = b \wedge \neg c$. By the definition of WQS-algebra and Lemma 3.1 we have that

$$\begin{aligned}\neg a \wedge \neg c &= (\neg a \wedge \neg c) \vee (\neg\neg c \wedge \neg c) = (\neg a \vee \neg\neg c) \wedge \neg c \\ &= \neg(a \wedge \neg c) \wedge \neg c = \neg(b \wedge \neg c) \wedge \neg c \\ &= \neg b \wedge \neg c.\end{aligned}$$

Then $(\neg a, \neg b) \in \theta_{\text{lat}}([F))$, and the result follows.

2. Follows in a similar way and is left to the reader. □

Given a $\boldsymbol{A} \in \mathcal{N}$, and $a, b \in A$, we will denote the principal lattice congruence generated by a and b by $\theta_{\text{lat}}(a, b)$.

Theorem 3.4 *Let $\boldsymbol{A} \in \mathcal{N}$. Then $\boldsymbol{A} \in \mathcal{WQS}$ if and only if $\neg 1 = 0$ and for every $a \in A$, $\theta_{\text{lat}}(\neg a, 1)$ and $\theta_{\text{lat}}(0, \neg a)$ are congruences of $\boldsymbol{A}$.*

P r o o f. First note that $\theta_{\rm lat}(\neg a, 1) = \theta_{\rm lat}([\neg a))$ and $\theta_{\rm lat}(0, \neg a) = \theta_{\rm lat}((\neg a])$. Suppose that $\boldsymbol{A} \in \mathcal{WQS}$. From Theorem 3.2, for every $a \in A$, $\neg a \in \boldsymbol{B}(\boldsymbol{A})$. Thus by the previous Lemma, we have that $\theta_{\rm lat}(\neg a, 1)$ and $\theta_{\rm lat}(0, \neg a)$ are congruences of $\boldsymbol{A}$. The fact that $\neg 1 = 0$ was proved in Lemma 2.2. For the only if part, let $a \in A$. By hypothesis, $\theta_{\rm lat}(\neg a, 1) \in \mathrm{Con}(\boldsymbol{A})$ and $\neg 1 = 0$, thus $(\neg\neg a, 0) \in \theta_{\rm lat}(\neg a, 1)$. Therefore we have that

$$(\neg a \wedge \neg\neg a, 0) \in \theta_{\rm lat}(\neg a, 1),$$

i. e.,

$$(\neg a \wedge \neg\neg a) \wedge \neg a = 0 \wedge \neg a.$$

Thus, $\neg a \wedge \neg\neg a = 0$ for each $a \in A$. The proof that $\neg a \vee \neg\neg a = 1$ for each $a \in A$ follows in a similar way using that $\theta_{\rm lat}(0, \neg a) \in \mathrm{Con}(\boldsymbol{A})$. □

Let $\boldsymbol{A} \in \mathcal{N}$. Let us consider the binary relation $R_{\boldsymbol{B}(\boldsymbol{A})}$ defined in $X(\boldsymbol{A})$ as

$$(P, Q) \in R_{\boldsymbol{B}(\boldsymbol{A})} \Leftrightarrow P \cap B(\boldsymbol{A}) \subseteq Q.$$

We note that $R_{\boldsymbol{B}(\boldsymbol{A})}$ is the relation defined by H. Gaitan in [7]. Now, we shall see the connection between the relations $R_\neg$ and $R_{\boldsymbol{B}(\boldsymbol{A})}$ and prove that it is possible to determine QS-algebras using this connection.

Lemma 3.5 *Let $\boldsymbol{A} \in \mathcal{WQS}$. Then, $R_\neg \cup R_\neg^{-1} \subseteq R_{\boldsymbol{B}(\boldsymbol{A})}$. Thus, $\boldsymbol{A} \in \mathcal{QS}$ if and only if $R_{\boldsymbol{B}(\boldsymbol{A})} = R_\neg$.*

P r o o f. Suppose $(P, Q) \in R_\neg$. Let $a \in P \cap B(\boldsymbol{A})$. Since $\neg\neg a = a \in P$, $\neg a \notin Q$, and as $\neg a \vee \neg\neg a = 1$, $a \in Q$. So, $(P, Q) \in R_{\boldsymbol{B}(\boldsymbol{A})}$. Similarly we can prove that if $(P, Q) \in R_\neg^{-1}$, then $(P, Q) \in R_{\boldsymbol{B}(\boldsymbol{A})}$. Thus, $R_\neg \cup R_\neg^{-1} \subseteq R_{\boldsymbol{B}(\boldsymbol{A})}$. Now, if $\boldsymbol{A} \in \mathcal{QS}$, then $R_\neg$ is an equivalence and therefore $R_\neg = R_\neg^{-1} \subseteq R_{\boldsymbol{B}(\boldsymbol{A})}$. Suppose now that there exist $P, Q \in X(\boldsymbol{A})$ such that $(P, Q) \in R_{\boldsymbol{B}(\boldsymbol{A})}$, but $(Q, P) \notin R_\neg$. Then there exists $a \in A$ such that $\neg a \in Q$ and $a \in P$. By WQS1, $\neg\neg a \notin Q$. Since $\boldsymbol{A} \in \mathcal{QS}$, $a \leq \neg\neg a$ and, hence, $\neg\neg a \in P \cap B(\boldsymbol{A}) \subseteq Q$ is a contradiction. Thus $R_{\boldsymbol{B}(\boldsymbol{A})} = R_\neg$. For the converse, if $R_{\boldsymbol{B}(\boldsymbol{A})} = R_\neg$, then $R_\neg$ is an equivalence relation and from the results of [4], we have that $\boldsymbol{A}$ is a quasi-Stone algebra. □

Corollary 3.6 *Let $\boldsymbol{A} \in \mathcal{N}$. Then $\boldsymbol{A} \in \mathcal{QS}$ if and only if $R_{\boldsymbol{B}(\boldsymbol{A})}^{-1} \subseteq R_{\boldsymbol{B}(\boldsymbol{A})}$ and $R_{\boldsymbol{B}(\boldsymbol{A})} = R_\neg$.*

P r o o f. The only if part follows from the previous Lemma. For the if part note that for every $\boldsymbol{A} \in \mathcal{N}$, $R_{\boldsymbol{B}(\boldsymbol{A})}$ is reflexive and transitive. Therefore, if $R_{\boldsymbol{B}(\boldsymbol{A})}^{-1} \subseteq R_{\boldsymbol{B}(\boldsymbol{A})}$ and $R_{\boldsymbol{B}(\boldsymbol{A})} = R_\neg$, then $R_\neg$ is an equivalence relation. In consequence $\boldsymbol{A} \in \mathcal{QS}$. □

4 Simple and subdirectly irreducible algebras

In this section we will determine the simple and subdirectly irreducible algebras of $\mathcal{WQS}$, by some properties of its dual spaces. Using these results we will give an algebraic characterization which will generalize some results of [9].

For that we recall some results on congruences of distributive lattices and negation lattices. For details see [4] and [8].

Let $\langle X, \leq, R\rangle$ be a $\neg$-space. A subset $Y \subseteq X$ is said to be *R-closed* if for all $x \in Y$, $\mathrm{Max}R(x) \subseteq Y$. We shall say that $Y \subseteq X$ is *R-saturated* if it is closed and R-closed. We shall denote by $\mathcal{C}_R(X)$ the lattice of R-saturated subsets of $\langle X, \leq, R\rangle$. For any subset Y of X, $\mathrm{Cl}(Y)$ will denote the topological closure of Y in the Priestley space X.

Let $\boldsymbol{A}$ be a bounded distributive lattice. For each set Y of prime filters of $\boldsymbol{A}$, the relation

$$\theta(Y) = \{(a, b) \in A \times A : \sigma(a) \cap Y = \sigma(b) \cap Y\}$$

is a congruence of the bounded distributive lattice reduct of $\boldsymbol{A}$. The correspondence $Y \longmapsto \theta(Y)$ between subsets of $X(\boldsymbol{A})$ and congruences of the bounded distributive lattice $\boldsymbol{A}$ is onto, because if $q_\theta : \boldsymbol{A} \longrightarrow \boldsymbol{A}/\theta$ denotes the natural projection and $Y = \{q_\theta^{-1}(P) : P \in X(\boldsymbol{A}/\theta)\}$, then $\theta = \theta(Y)$. But in general this correspondence is not one-to-one. On the other hand, for all subsets Y, Z of $X(\boldsymbol{A})$ we have that $\theta(Z) \subseteq \theta(Y)$ iff $Y \subseteq \mathrm{Cl}(Z)$. Thus we have that

$$\theta(Y) = \theta(\mathrm{Cl}(Y)), \text{ and } \theta(Y) = \theta(Z) \text{ if and only if } \mathrm{Cl}(Y) = \mathrm{Cl}(Z).$$

Taking into account that a subset Y of $X(\boldsymbol{A})$ is closed iff $\mathrm{Cl}(Y) = Y$, we conclude that the correspondence $Y \longmapsto \theta(Y)$ establishes an anti-isomorphism from the lattice of closed sets of $X(\boldsymbol{A})$ onto congruences of the bounded distributive lattice $\boldsymbol{A}$.

Theorem 4.1 [4, Theorem 9] *Let $\boldsymbol{A}$ be a $\neg$-lattice. Then the correspondence $Y \longmapsto \theta(Y)$ establishes an anti-isomorphism from $\mathcal{C}_{R_\neg}(X(\boldsymbol{A}))$ onto $\mathrm{Con}(\boldsymbol{A}) = \{\theta \subseteq A \times A : \theta$ is a congruence of the $\neg$-lattice $\boldsymbol{A}\}$.*

Lemma 4.2 *Let $\boldsymbol{A} \in \mathcal{WQS}$. Then the following conditions are equivalent:*

1. $\boldsymbol{B}(\boldsymbol{A}) = \{0, 1\}$.
2. $R_{\boldsymbol{B}(\boldsymbol{A})}(P) = X(\boldsymbol{A})$*, for any* $P \in X(\boldsymbol{A})$.

Proof. $1 \Rightarrow 2$. Suppose that there exists $P \in X(\boldsymbol{A})$ such that $R_{\boldsymbol{B}(\boldsymbol{A})}(P) \neq X(\boldsymbol{A})$. Then there exists $Q \in X(\boldsymbol{A})$ and there is an element $a \in A$ such that $(P, Q) \notin R_{\boldsymbol{B}(\boldsymbol{A})}$, $a \in P \cap \boldsymbol{B}(\boldsymbol{A})$, and $a \notin Q$. But since $a \in \boldsymbol{B}(\boldsymbol{A})$, $a = 1 \in Q$, which is a contradiction. Thus, $R_{\boldsymbol{B}(\boldsymbol{A})}(P) = X(\boldsymbol{A})$.

$2 \Rightarrow 1$. Suppose that there exists $a \in \boldsymbol{B}(\boldsymbol{A}) \setminus \{0, 1\}$. As $a = \neg\neg a \neq 1$, there exists $P \in X(\boldsymbol{A})$ such that $a = \neg\neg a \notin P$. For all $Q \in X(\boldsymbol{A})$ such that $P \cap B(\boldsymbol{A}) = Q \cap B(\boldsymbol{A})$, we get $a \notin Q$. Then $R_{\boldsymbol{B}(\boldsymbol{A})}(P) \cap \sigma(a) = \emptyset$, which is a contradiction, because $R_{\boldsymbol{B}(\boldsymbol{A})}(P) = X(\boldsymbol{A})$ and $\sigma(a) \neq \emptyset$. Therefore, $\boldsymbol{B}(\boldsymbol{A}) = \{0, 1\}$. □

Recall that $\boldsymbol{A}$ is said to be *subdirectly irreducible* if there exists a least not trivial congruence relation of $\boldsymbol{A}$, and $\boldsymbol{A}$ is *simple* if $\mathrm{Con}(\boldsymbol{A})$ has only two elements.

Proposition 4.3 *Let $\boldsymbol{A} \in \mathcal{WQS}$. If $\boldsymbol{A}$ is subdirectly irreducible, then $B(\boldsymbol{A}) = \{0, 1\}$.*

Proof. If $\boldsymbol{A}$ is subdirectly irreducible, then there exists a not trivial minimal congruence θ. Suppose that $B(\boldsymbol{A}) \neq \{0, 1\}$. Then there exists $a \in B(\boldsymbol{A}) \setminus \{0, 1\}$. Let us consider the congruences $\theta_a = \theta_{\mathrm{lat}}([a))$ and $\theta_{\neg a} = \theta_{\mathrm{lat}}([\neg a))$, which by Lemma 3.3 belong to $\mathrm{Con}(\boldsymbol{A})$. So, $\theta \subseteq \theta_a \cap \theta_{\neg a}$. Let $(x, y) \in \theta_a \cap \theta_{\neg a}$. Then

$$x = x \wedge 1 = x \wedge (\neg a \vee \neg\neg a) = x \wedge (\neg a \vee a) = (x \wedge \neg a) \vee (x \wedge a) = (y \wedge \neg a) \vee (y \wedge a) = y,$$

which is a contradiction. Therefore, $B(\boldsymbol{A}) = \{0, 1\}$. □

Lemma 4.4 *Let $\boldsymbol{A} \in \mathcal{N}$. Then $\boldsymbol{A}$ is a special quasi-Stone algebra if and only if $R_\neg(P) = X(\boldsymbol{A})$ for each $P \in X(\boldsymbol{A})$.*

Proof. If $\boldsymbol{A}$ is a special quasi Stone algebra, then $\neg^{-1}(P) = \{0\}$ for each $P \in X(\boldsymbol{A})$. Therefore for every $Q \in X(\boldsymbol{A})$, $\neg^{-1}(P) \cap Q = \emptyset$. Hence $R_\neg(P) = X(\boldsymbol{A})$ for each $P \in X(\boldsymbol{A})$. For the converse let $a \in A$ such that $\neg a \neq 0$. There exists $P \in X(\boldsymbol{A})$ such that $\neg a \in P$, then for each $Q \in R_\neg(P) = X(\boldsymbol{A})$, $a \notin Q$. Therefore $a = 0$. □

In the following Theorem we prove that the simple algebras in the varieties $\mathcal{WQS}$ and $\mathcal{QS}$ are the same.

Theorem 4.5 *Let $\boldsymbol{A} \in \mathcal{WQS}$. Then the following conditions are equivalent:*

1. *$\boldsymbol{A}$ is simple.*
2. $\mathrm{Cl}(\mathrm{Max}R_\neg(P)) = X(\boldsymbol{A})$ *for each* $P \in X(\boldsymbol{A})$.
3. $\mathrm{Cl}(\mathrm{Max}X(\boldsymbol{A})) = X(\boldsymbol{A})$ *and* $R_\neg(P) = X(\boldsymbol{A})$ *for each* $P \in X(\boldsymbol{A})$.
4. *$\boldsymbol{A}$ is a simple quasi-Stone algebra.*

Proof. $1 \Rightarrow 2$. Suppose that $\boldsymbol{A}$ is simple. Let $P \in X(\boldsymbol{A})$. Since $R_\neg(P)$ is a closed decreasing subset of $X(\boldsymbol{A})$ and by the serial property of $R_\neg$ is not empty, we have that $\mathrm{Max}R_\neg(P)$ is not empty.

We will prove that $\mathrm{Max}R_\neg(P)$ is an $R_\neg$-closed subset, i. e., for each $Q \in \mathrm{Max}R_\neg(P)$, we have to prove that $\mathrm{Max}R_\neg(Q) \subseteq \mathrm{Max}R_\neg(P)$. Let $Q \in \mathrm{Max}R_\neg(P)$. Suppose that $\mathrm{Max}R_\neg(Q) \nsubseteq \mathrm{Max}R_\neg(P)$, then there exists $D \in X(\boldsymbol{A})$ such that

(1) $\quad D \in \mathrm{Max}R_\neg(Q)$ and $D \notin \mathrm{Max}R_\neg(P)$.

From (1), $(Q, D) \in R_\neg$, and as $Q \in \mathrm{Max}R_\neg(P)$, we get $(P, Q) \in R_\neg$. Since $R_\neg$ is transitive,

(2) $(P, D) \in R_\neg.$

From (2) and since $D \notin \mathrm{Max}R_\neg(P)$, we have that there exists a prime filter $K \in \mathrm{Max}R_\neg(P)$ such that

$$(P, K) \in R_\neg \text{ and } D \subsetneq K.$$

By the euclidean property and the fact that $(P, K) \in R_\neg$ and $(P, Q) \in R_\neg$, we get $(Q, K) \in R_\neg$, which together with $D \subsetneq K$ contradict the fact that $D \in \mathrm{Max}R_\neg(Q)$. So, $\mathrm{Max}R_\neg(Q) \subseteq \mathrm{Max}R_\neg(P)$. Thus, $\mathrm{Max}R_\neg(P)$ is an $R_\neg$-closed subset of $X(A)$.

Since $\mathrm{Max}R_\neg(P)$ is an $R_\neg$-closed subset of $X(\boldsymbol{A})$, $\theta(\mathrm{Max}R_\neg(P))$ is a congruence of $\boldsymbol{A}$. By the remark above Theorem 4.1,

$$\theta(\mathrm{Max}R_\neg(P)) = \theta(\mathrm{Cl}(\mathrm{Max}R_\neg(P))).$$

Thus, by Theorem 4.1 we have that $\mathrm{Cl}(\mathrm{Max}R_\neg(P))$ is not empty and $R_\neg$-saturated subset of $X(\boldsymbol{A})$. By hypothesis $\boldsymbol{A}$ is simple, thus $\mathrm{Cl}(\mathrm{Max}R_\neg(P)) = X(\boldsymbol{A})$.

$2 \Rightarrow 3$. Since $R_\neg(P)$ is closed for each $P \in X(\boldsymbol{A})$ and $\mathrm{Max}R_\neg(P) \subseteq R_\neg(P)$, we have that

$$\mathrm{Cl}(\mathrm{Max}R_\neg(P)) = X(\boldsymbol{A}) \subseteq R_\neg(P),$$

and the result follows.

$3 \Rightarrow 4$. $\boldsymbol{A}$ is a special quasi-Stone algebra by the previous Lemma. Let $Y \neq \emptyset$ be a $R_\neg$-saturated subset of $X(\boldsymbol{A})$, and let $P \in Y$. Then

$$\mathrm{Cl}(\mathrm{Max}R_\neg(P)) = \mathrm{Cl}(\mathrm{Max}X(\boldsymbol{A})) = X(\boldsymbol{A}) \subseteq Y.$$

Therefore $\mathcal{C}_{R_\neg}(X(\boldsymbol{A})) = \{\emptyset, X(\boldsymbol{A})\}$.

$4 \Rightarrow 1$. It is immediate. □

Remark 4.6 An example of WQS-algebra which is subdirectly irreducible and not simple is the algebra $\boldsymbol{A}$ shown in Figure 1. The set $X(\boldsymbol{A})$ has two prime filters $P_a = \{a, 1\}$ and $P_b = \{b, 1\}$. The relation $R_\neg$ is given by $R_\neg = \{(P_a, P_b), (P_b, P_b)\}$, and the lattice of $R_\neg$-saturated subsets is

$$\mathcal{C}_{R_\neg}(X(\boldsymbol{A})) = \{\emptyset, \{P_b\}, X(\boldsymbol{A})\}.$$

Clearly $\boldsymbol{A}$ is subdirectly irreducible but not simple.

Theorem 4.7 *Let $\boldsymbol{A} \in \mathcal{WQS}$. If there is $Q \in X(\boldsymbol{A}) \setminus \mathrm{Cl}(\mathrm{Max}R_\neg(Q))$ such that*

$$\mathrm{Cl}(\mathrm{Max}R_\neg(Q)) \cup \{Q\} = X(\boldsymbol{A}),$$

then $\boldsymbol{A}$ is subdirectly irreducible, but not simple.

P r o o f. $\Rightarrow$) Suppose that $\boldsymbol{A}$ is subdirectly irreducible but not simple by Theorem 4.1, there exists the greatest element Y of $\mathcal{C}_{R_\neg}(X(\boldsymbol{A}))\setminus\{\emptyset, X(\boldsymbol{A})\}$. As $Y \neq X(\boldsymbol{A})$, there exists $Q \in X(\boldsymbol{A})\setminus Y$. In the proof of Theorem 4.5 we have seen that $\mathrm{Cl}(\mathrm{Max}R_\neg(P))$ is an $R_\neg$-saturated subset of $X(\boldsymbol{A})$, for each $P \in X(\boldsymbol{A})$. Now, it is easy to see that the set

$$Y_Q = \{Q\} \cup \mathrm{Cl}(\mathrm{Max}R_\neg(Q))$$

is an $R_\neg$-saturated subset, and since $Y_Q \nsubseteq Y$, $X(\boldsymbol{A}) = Y_Q$.

$\Leftarrow$) Suppose that there exists $Q \notin \mathrm{Cl}(\mathrm{Max}R_\neg(Q))$ such that $\mathrm{Cl}(\mathrm{Max}R_\neg(Q)) \cup \{Q\} = X(\boldsymbol{A})$. Since $\mathrm{Max}R_\neg(Q)$ is an $R_\neg$-closed subset of $X(\boldsymbol{A})$, the set $Y = \mathrm{Cl}(\mathrm{Max}R_\neg(Q))$ is an $R_\neg$-saturated subset of $X(\boldsymbol{A})$. Let $Y' \in \mathcal{C}_{R_\neg}(X(\boldsymbol{A})) \setminus \{\emptyset, X(\boldsymbol{A})\}$. We prove that $Y' \subseteq Y$. Suppose that $Y' \nsubseteq Y$. Then there exists $D \in Y'$ such that $D \notin Y$. Since $D \in Y' \subset X(\boldsymbol{A}) = \mathrm{Cl}(\mathrm{Max}R_\neg(Q)) \cup \{Q\} = Y \cup \{Q\}$, we have $D = Q$. As $Y' \in \mathcal{C}_{R_\neg}(X(\boldsymbol{A}))$, and $\mathrm{Max}R_\neg(Q) \subseteq Y'$,

$$X(\boldsymbol{A}) = \mathrm{Cl}(\mathrm{Max}R_\neg(Q)) \cup \{Q\} \subseteq Y',$$

which is a contradiction. So $Y' \subseteq Y$, and consequently Y is the greatest element of the lattice $\mathcal{C}_{R_\neg}(X(\boldsymbol{A}))$. Thus, $\boldsymbol{A}$ is subdirectly irreducible. Since $\mathrm{Cl}(\mathrm{Max}R_\neg(Q)) \neq X(\boldsymbol{A})$, we have that $\boldsymbol{A}$ is not simple. □

Now we will see that the example given in Remark 4.6 is a particular case of a general result.

Corollary 4.8 *If $\boldsymbol{A}$ is a simple QS-algebra, then $\boldsymbol{A}'$ is a subdirectly irreducible but not simple WQS-algebra.*

Proof. In Example 2.8 we have seen that the set of prime filters $X(\boldsymbol{A}')$ is isomorphic $X(\boldsymbol{A}) \cup \{\alpha\}$, and that $R(\alpha) = X(\boldsymbol{A})$. Since $\boldsymbol{A}$ is a simple QS-algebra, $\mathrm{Cl}(\mathrm{Max} X(\boldsymbol{A})) = X(\boldsymbol{A})$. Therefore the result follows from Theorem 4.7. □

In fact we will see that any subdirectly irreducible but not-simple WQS-algebra is isomorphic to a subalgebra of an algbra $\boldsymbol{A}'$ for some special QS-algebra. First, we recall the definition of $\neg$-morphisms between $\neg$-spaces (see [4]).

Definition 4.9 Let $\langle X, R_X\rangle$ and $\langle Y, R_Y\rangle$ be $\neg$-spaces. A function $f : X \longrightarrow Y$ is said to be a *$\neg$-morphism* iff f is continuous, monotonic and the following conditions hold:

1. For all $x, y \in X$, if $(x, y) \in R_X$, then $(f(x), f(y)) \in R_Y$.
2. If $(f(x), y) \in R_Y$, then there is an element $z \in X$ such that $(x, z) \in R_X$ and $y \leq f(z)$.

A $\neg$-morphisms $f : X \longrightarrow Y$ is a *$\neg$-isomorphism* if it is an order-isomorphism and a homeomorphism between Priestley spaces.

Corollary 4.10 *If $\boldsymbol{A}$ is a subdirectly irreducible but not simple WQS-algebra, then there exists a simple QS-algebra $\boldsymbol{S}$ such that $\boldsymbol{A}$ is isomorphic to a subalgebra of $\boldsymbol{S}'$.*

Proof. Let $\boldsymbol{A}$ be a subdirectly irreducible but not simple WQS-algebra. By Theorem 4.7, there exists $Q \in X(\boldsymbol{A}) \setminus \mathrm{Cl}(\mathrm{Max} R_\neg(Q))$ such that $\mathrm{Cl}(\mathrm{Max} R_\neg(Q)) \cup \{Q\} = X(\boldsymbol{A})$. Let us consider the set

$$Y = X(\boldsymbol{A}) \setminus \{Q\} = \mathrm{Cl}(\mathrm{Max} R_\neg(Q)).$$

Note that since $\mathrm{Max} R_\neg(Q) \subseteq R_\neg(Q)$, $R_\neg(Q)$ is a closed set $Y \subseteq R_\neg(Q)$. By the euclidean property of $R_\neg$ we obtain that $Y \times Y \subseteq R_\neg$. It is easy to see that the structure

$$\boldsymbol{Y} = \langle Y, \subseteq, \tau_Y, S\rangle,$$

where τ_Y is the induced topology on $X(\boldsymbol{A}) \setminus \{Q\}$ by τ_σ and $S = Y \times Y = R_\neg \cap (Y \times Y)$ is a $\neg$-space. The dual algebra, $\boldsymbol{S} = D(\boldsymbol{Y})$, of $\boldsymbol{Y}$is a simple WQS-algebra. By Theorem 4.5 we have that $\boldsymbol{S}$ is a QS-simple algebra. Now consider the structure

$$\boldsymbol{Z} = \langle X(\boldsymbol{A}), \leq, \tau_\sigma, R\rangle,$$

where

$$P \leq D \quad \text{iff} \quad P = D = Q \text{ or } P \subseteq D \text{ with } P \neq Q \text{ and } D \neq Q,$$

and

$$R = Y \times Y \cup \{(Q, P) : P \in Y\}$$

Then $\boldsymbol{Z}$ is a $\neg$-space and and by Lemma 4.8 we have that $\boldsymbol{Z}$ is isomorphic as $\neg$-space to $X(\boldsymbol{S}')$. So $\boldsymbol{S}'$ is isomorphic as a $\neg$-lattice to $D(\boldsymbol{Z})$.

Clearly the identity function id: $X(\boldsymbol{A}) \longrightarrow X(\boldsymbol{A})$ is an onto continuous monotonic function from $\boldsymbol{Z}$ to $X(\boldsymbol{A})$. Since $Y \subseteq R_\neg(Q)$, we have that $R \subseteq R_\neg$ and then id satisfy the first conditionn of a $\neg$-morphism between $\neg$-spaces of Definition 4.9.

To prove the second one, let $(P, D) = (\mathrm{id}(P), D) \in R_\neg$. Suppose first that $D \neq Q$, then $(P, D) \in R$ and the condition is satisfied. In the case that $D = Q$, we have that $(P, Q) \in R_\neg$. If we suppose that $P \neq Q$, since $(Q, P) \in R_\neg$, by transitivity of R$_\neg$ we obtain that $(Q, Q) \in R_\neg$, clearly the same conclusion follows if $P = Q$. Since $(Q, Q) \in R_\neg$, there exists $K \in \mathrm{Max} R_\neg(Q) \subseteq Y$, such that $Q \subseteq K$. Since $Q \neq K$, otherwise $Q \in (\mathrm{Max} R_\neg(Q) \subseteq Y$ which is a contradiction, clearly $(P, K) \in R$ and $Q \subseteq \mathrm{id}(K)$. Then the condition holds.

Therefore there exists an injective homomorphism of $\neg$-lattices from $\boldsymbol{A}$ to $\boldsymbol{S}'$. □

4.1 Answering a question posed in [9]

In [9] N. Sankappanavar and H. Sankappanavar characterized the simple QS-algebras as the ones that are special and have a secluded lattice reduct.

Let us recall that a lattice $\boldsymbol{A}$ is *secluded* if and only if given $a, b \in A$, if $a < b$, then there exists $c \in A$ such that $a \wedge c = 0$ and $b \wedge c \neq 0$.

Lemma 4.11 *Given a bounded lattice $\boldsymbol{A}$, the following are equivalent:*

1. *$\boldsymbol{A}$ is secluded.*
2. $\mathrm{Max}(X(\boldsymbol{A}))$ *is a dense subset of* $X(\boldsymbol{A})$.
3. *The congruence Φ^* defined by*

$$(a, b) \in \Phi^* \Leftrightarrow \textit{ for every } c \in A, a \wedge c = 0 \textit{ if and only if } b \wedge c = 0,$$

is the identity relation on $\boldsymbol{A}$.

Proof. The equivalence between 1. and 2. can be obtained using the caracterization of simple quasi-Stone algebra given in [9] and Theorem 4.5. This equivalence is also proved too in [5, Lemma 3.5]. The equivalence between 2. and 3. is proved in [1, Lemma 3.2]. Note that in [1] Adams and Beazer prove this result using the Priestley duality for lattices but in terms of prime ideals. □

In [9], they left as an open problem the question: if there exist infinite simple QS-algebras whose underlying lattices are not Boolean? The answer to this question is positive. We can deduce it from the previous Lemma and the examples, given in [1], of infinite lattices with are not boolean lattices and satisfy the identity Φ^*.

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■ Please regard Definition 1.1 and Definition 2.1. There wasn't marked the term, which should be defined. Please check if everything is correct there now.

■ Please regard page 2, line 17 (2 lines over Lemma 1.3). There was introduced the notation $Fi(\boldsymbol{L})$, which has nowhere been used in the article. So I deleted it.

■ Please regard page 3, Theorem 1.5, and page 4, Lemma 2.2. I inserted "it holds" in the first line. Please check if everything is correct there now.

■ Please regard page 5, line 6 in Example 2.8. I inserted "$X(\boldsymbol{A}')$ is given by" at the end of the line. Please check if everything is correct there now.

■ Please regard page 6, line 4 in the Proof of Theorem 3.2. I inserted "we have" at the beginning of the line. Please check if everything is correct there now.

■ Please regard page 7, Lemma 3.5. I substituted "if and only if" for "iff".

In the proof I inserted "is" at the beginning and "then" in the center of line 6. Please check if everything is correct there now.

■ Please regard page 10, line 5 in Definition 4.9. I inserted “it” between if and is. Please check if everything is correct there now.

■ Please regard page 11, line 1. I inserted “WQS-algebra”in the first sentence. Please check if everything is correct there now.

■ You are using the term “not trivial” sometimes in the article. I think it would be better to say nontrivial.

