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Maximal inequalities for the best approximation operator and Simonenko indices

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Abstract. In an abstract set up, we get strong type inequalities in L^{p+1} by assuming weak or extra-weak inequalities in Orlicz spaces. For some classes of functions, the number p is related to Simonenko indices. We apply the results to get strong inequalities for maximal functions associated to best Φ-approximation operators in an Orlicz space L^{Φ}.

Key Words: Simonenko indices, Maximal Inequalities, Best Approximation.
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1 Introduction

In this paper we denote by $\mathcal{I}$ the set of all non decreasing functions φ defined for all real number $x>0$, such that $\varphi(x)>0$ for all $x>0$, $\varphi(0+)=0$ and $\lim\limits_{x\to\infty}\varphi(x)=\infty$.

We say that a non decreasing function $\varphi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ satisfies the Δ_2 condition, symbolically $\varphi\in\Delta_2$, if there exists a constant $\Lambda_\varphi>0$ such that $\varphi(2x)\le\Lambda_\varphi\varphi(x)$ for all $x\ge 0$.

Now, given $\varphi\in\mathcal{I}$, we consider $\Phi(x)=\int_0^x\varphi(t)dt$. Observe that $\Phi:[0,\infty)\to[0,\infty)$ is a convex function such that $\Phi(x)=0$ if and only if $x=0$. In the literature, a function Φ satisfying the previous conditions is known as a Young function. In addition, as $\varphi\in\mathcal{I}$ we have that Φ is increasing, $\frac{\Phi(x)}{x}\to 0$ as $x\to 0$ and $\frac{\Phi(x)}{x}\to\infty$ as $x\to\infty$. Thus, according to [6], a function Φ with this property is called an N-function.

If $\varphi\in\mathcal{I}$ is a right-continuous function that satisfies the Δ_2 condition, then

$$\frac{1}{2}(\varphi(a)+\varphi(b))\le\varphi(a+b)\le\Lambda_\varphi(\varphi(a)+\varphi(b))$$

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for every $a,b\geq 0$.

Also note that the Δ_2 condition on Φ implies

$$\frac{x}{2\Lambda_\varphi}\varphi(x)\leq\Phi(x)\leq x\varphi(x),$$

for every $x\geq 0$.

If $\varphi\in\mathcal{I}$, we define $L^\varphi(\mathbb{R}^n)$ as the class of all Lebesgue measurable functions f defined on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n}\varphi(t|f|)dx<\infty$ for some $t>0$ and where dx denotes the Lebesgue measure on $\mathbb{R}^n$. For a convex function Φ, $L^\Phi(\mathbb{R}^n)$ is the classic Orlicz space (see [10]). And, if $\Phi\in\Delta_2$ then $L^\Phi(\mathbb{R}^n)$ is the space of all measurable functions f defined on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n}\Phi(|f|)dx<\infty$.

A non decreasing function $\varphi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ satisfies the ∇_2 condition, denoted $\varphi\in\nabla_2$, if there exists a constant $\lambda_\varphi>2$ such that $\varphi(2x)\geq\lambda_\varphi\varphi(x)$ for all $x\geq 0$.

We claim that a non decreasing function $\varphi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ satisfies the Δ' condition, symbolically $\varphi\in\Delta'$, if there exists a constant $K>0$ such that $\varphi(xy)\leq K\varphi(x)\varphi(y)$ for all $x,y\geq x_0\geq 0$. If $x_0=0$ then φ satisfies the Δ' condition globally (denoted $\varphi\in\Delta'$ globally).

With the aim of comparing functions in Orlicz spaces, some partial ordering relations were treated in Chapter II of [10]. In [9] Mazzone and Zó introduce the quasi-increasing function's concept, they define the relation $\prec$ between two non negative functions and they determine some properties of the relation. Later, in [1], it is defined and thoroughly studied another relation $\prec_N$. Both relations are used to obtain strong type inequalities as follows.

Let $\varphi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ be a non decreasing function such that $\varphi(0)=0$ and satisfies a weak type inequality like

$$\mu(\{f>a\})\leq C_w\int_{\{f>a\}}\frac{\varphi(g)}{\varphi(a)}d\mu \text{ for all } a>0,$$

or an extra-weak type inequality like

$$\mu(\{f>a\})\leq 2C_w\int_{\{f>a\}}\varphi\left(\frac{g}{a}\right)d\mu \text{ for all } a>0,$$

where $f,g:\Omega\to\mathbb{R}_0^+$ are two fixed measurable functions. Then, in [9] and [1] it has considered functions $\Psi\in C^1([0,\infty))$, $\Psi(x)=\int_0^x\psi(t)dt$ and $\varphi\prec\psi$ or $\varphi\prec_N\psi$, which allows us to get strong type inequalities like

$$\int_\Omega\Psi(f)d\mu\leq 2C_w\rho\int_\Omega\Psi\left(\frac{2}{c}g\right)d\mu. \tag{1.1}$$

In this paper we set $p\in\mathbb{R}$, very related to Boyd indices of φ, such that $\varphi\prec x^p$ or $\varphi\prec_N x^p$ in order to obtain strong inequalities like

$$\int_\Omega f^{p+1}d\mu\leq\tilde{K}\int_\Omega g^{p+1}d\mu, \tag{1.2}$$

with f and g non negative, measurable functions.

In Section 2, we recall the definitions of the relations $\prec$ and $\prec_N$ and we enumerate some of their properties that will be useful in the searching of the real number p to have (1.2) Then, we determine sufficient conditions on p to have $\varphi \prec x^p$ or $\prec_N x^p$. From such conditions, we generate bounds for p in the case that φ is a non decreasing, differentiable function; next, we extend the results to a class of non decreasing and non differentiable functions.

In Section 3, we estimate p by using a class of Boyd indices in Orlicz spaces called Simonenko indices. These indices were defined by Simonenko in [11] and they were studied in Chapter 11 of [8]. In [3], Simonenko indices are used to get Harnack's type inequalities and regularity conditions for some integral operators. In [12], relationships between Simonenko indices and other indices in Orlicz spaces are established.

In Section 4, we apply the results to a maximal function associated to best Φ-approximation operators in an Orlicz space L^Φ and one-sided operators related to the classical Hardy-Littlewood maximal function.

2 On relations between non negative functions

We begin recalling a concept introduced by Mazzone and Zó in [9].

Definition 2.1. A function $\eta:\mathbb{R}^+\to\mathbb{R}^+$ is quasi-increasing if and only if there exists a constant $\rho>0$ such that

$$\frac{1}{x}\int_0^x \eta(t)\,dt \le \rho\eta(x) \text{ for every } x\in\mathbb{R}^+.$$

Hereinafter, we will call ρ the quasi increasing constant.

In [9], the relation $\prec$ between non negative functions was presented as follows.

Definition 2.2. Let $\varphi,\psi:\mathbb{R}^+\to\mathbb{R}^+$.
$\varphi\prec\psi$ if and only if $\frac{\psi}{\varphi}$ is a quasi-increasing function; that is, if and only if there exists a constant $\rho>0$ such that

$$\frac{1}{x}\int_0^x \frac{\psi(t)}{\varphi(t)}\,dt \le \rho\frac{\psi(x)}{\varphi(x)} \text{ for every } x\in\mathbb{R}^+.$$

In Theorem 2.4 of [9], the authors employ the relation $\prec$ to get a strong type inequality like (1.1). In [1], following an analogous pattern with an extra-weak type inequality as starting point, the relation $\prec_N$ is defined.

Definition 2.3. Let $\varphi,\psi:\mathbb{R}^+\to\mathbb{R}^+$.
$\varphi\prec_N\psi$ if and only if $\{\psi(x)\varphi(\frac{\alpha}{x})\}_{\alpha\in\mathbb{R}^+}$ is a collection of quasi-increasing functions with

the same quasi increasing constant; namely, if and only if there exists a constant $\rho>0$ such that

$$\frac{1}{x}\int_0^x \psi(t)\varphi\left(\frac{\alpha}{t}\right)dt \le \rho\psi(x)\varphi\left(\frac{\alpha}{x}\right),$$

for every $x\in\mathbb{R}^+$ and for every $\alpha\in\mathbb{R}^+$.

Remark 2.1. Note that $\prec$ is a reflexive relation while $\prec_N$ is not (see [1, p. 2183]).

Next, we set conditions to assure $\varphi\prec x^p$ or $\varphi\prec_N x^p$ for some $p\in\mathbb{R}$.

Proposition 2.1. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$.
a) If $\frac{1}{\varphi(x)}$ is a quasi-increasing function, then $\varphi\prec x^p$ for every $p\ge 0$.
b) If $\{\varphi\left(\frac{\alpha}{x}\right)\}_{\alpha\in\mathbb{R}^+}$ is a collection of quasi-increasing functions with the same quasi increasing constant, then $\varphi\prec_N x^p$ for every $p\ge 0$.

Proof. Definition 2.2 and Definition 2.3 imply that $\varphi\prec 1$ and $\varphi\prec_N 1$, respectively. Now, applying Proposition 3.5 of [1] with $M(x)=x^p$ for $p\ge 0$, we obtain $\varphi\prec x^p$ and $\varphi\prec_N x^p$ for every $p\ge 0$, respectively. □

Proposition 2.2. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a non decreasing function.
If $\varphi\in\Delta_2$ with $\Lambda_\varphi<2$, then $\varphi\prec x^p$ and $\varphi\prec_N x^p$ for every $p\ge 0$.

Proof. We take $\psi(x)=x^p$ with $p\ge 0$ in Proposition 3.11 of [1]. □

Example 2.1. Let $\varphi(x)=\ln(\sqrt[3]{x}+1)\in\Delta_2$ with $\Lambda_\varphi<2$, then $\ln(\sqrt[3]{x}+1)\prec x^p$ and $\ln(\sqrt[3]{x}+1)\prec_N x^p$ for every $p\ge 0$.

It is easy to see that every non decreasing function is a quasi-increasing one (see [1, Prop. 3.4]). An immediate consequence of this fact is the following result.

Proposition 2.3. Let $p\in\mathbb{R}$.
a) If $\frac{x^p}{\varphi(x)}$ is a non decreasing function from $\mathbb{R}^+$ into itself, then $\varphi\prec_N x^p$.
b) If $x^p\varphi\left(\frac{\alpha}{x}\right)$ is a non decreasing function from $\mathbb{R}^+$ into itself for every $\alpha>0$, then $\varphi\prec_N x^p$.

Proposition 2.4. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$.
Assume there exists $\tilde{\varphi}:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\tilde{\varphi}(x)\le\varphi(x)\le C\tilde{\varphi}(x) \text{ for every } x>0. \tag{2.1}$$

If there exists $p\in\mathbb{R}$ such that $\tilde{\varphi}\prec x^p$ or $\tilde{\varphi}\prec_N x^p$, then $\varphi\prec x^p$ or $\varphi\prec_N x^p$.

Proof. If there exists $p\in\mathbb{R}$ such that $\tilde{\varphi}\prec x^p$, then there exists $\rho_1>0$ such that

$$\frac{1}{x}\int_0^x \frac{t^p}{\tilde{\varphi}(t)}dt\le\rho_1\frac{x^p}{\tilde{\varphi}(x)} \text{ for every } x>0. \tag{2.2}$$

By (2.1) and (2.2), we get

$$\frac{1}{x}\int_0^x \frac{t^p}{\varphi(t)}dt \le \frac{1}{x}\int_0^x \frac{t^p}{\tilde{\varphi}(t)}dt \le \rho_1 \frac{x^p}{\tilde{\varphi}(x)} \le \rho_1 C \frac{x^p}{\varphi(x)},$$

for every $x>0$. Therefore, from Definition 2.2, we have $\varphi \prec x^p$.
Now, if there exists $p\in\mathbb{R}$ such that $\tilde{\varphi} \prec_N x^p$, then there exists $\rho_2>0$ such that

$$\frac{1}{x}\int_0^x t^p \tilde{\varphi}\left(\frac{\alpha}{t}\right)dt \le \rho_2 x^p \tilde{\varphi}\left(\frac{\alpha}{x}\right), \tag{2.3}$$

for every $x>0$ and for every $\alpha>0$. By (2.1) and (2.3), we obtain

$$\frac{1}{x}\int_0^x t^p \frac{1}{C}\varphi\left(\frac{\alpha}{t}\right)dt \le \frac{1}{x}\int_0^x t^p \tilde{\varphi}\left(\frac{\alpha}{t}\right)dt \le \rho_2 x^p \tilde{\varphi}\left(\frac{\alpha}{x}\right) \le \rho_2 x^p \varphi\left(\frac{\alpha}{x}\right),$$

for every $x>0$ and for every $\alpha>0$. Thus, Definition 2.3 implies that $\varphi \prec_N x^p$. □

Now, given $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ a non decreasing function, we use Proposition 2.3 to set conditions which assure that $\frac{x^p}{\varphi(x)}$ and $x^p\varphi(\frac{\alpha}{x})$ are non decreasing functions. We first consider the case $p>0$.

Theorem 2.1. *Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a non decreasing, differentiable function. If there exists $p>0$ such that $\frac{\varphi'(x)}{\varphi(x)} \le \frac{p}{x}$ for every $x>0$, then $\varphi \prec x^p$ and $\varphi \prec_N x^p$.*

Proof. By elementary algebraic operations and the hypothesis on φ and p, we have

$$\left(\frac{x^p}{\varphi(x)}\right)' = \frac{px^{p-1}\varphi(x) - x^p\varphi'(x)}{\varphi^2(x)} \ge 0 \text{ for every } x>0.$$

Thus $\frac{x^p}{\varphi(x)}$ is a non decreasing function and, by Proposition 2.3, we get $\varphi \prec x^p$.

On the other hand, exchanging x by $\frac{\alpha}{x}>0$ in the hypothesis and operating, we get

$$\left(x^p\varphi\left(\frac{\alpha}{x}\right)\right)' = px^{p-1}\varphi\left(\frac{\alpha}{x}\right) + x^p\varphi'\left(\frac{\alpha}{x}\right)\left(\frac{-\alpha}{x^2}\right) \ge 0,$$

for every $x>0$ and for every $\alpha>0$.
Then, $x^p\varphi\left(\frac{\alpha}{x}\right)$ is a non decreasing function for every $\alpha>0$; and, Proposition 2.3 implies that $\varphi \prec_N x^p$. □

Example 2.2. Let $\varphi(x)=x^q$ for $q>0$. As $\frac{p}{x} \ge \frac{\varphi'(x)}{\varphi(x)}$ for every $x>0$ provided that $p\ge q>0$, then $x^q \prec x^p$ and $x^q \prec_N x^p$ for every $p\ge q>0$.

Example 2.3. Let $\varphi(x)=\ln(x+1)$. Since $\frac{\varphi'(x)}{\varphi(x)} \le \frac{p}{x}$ for every $x>0$ provided that $p\ge 1$, then $\ln(x+1) \prec x^p$ and $\ln(x+1) \prec_N x^p$ for every $p\ge 1$.

Theorem 2.1 allows us to give a necessary condition in order that $\frac{x^p}{\varphi(x)}$ and $x^p\varphi\left(\frac{\alpha}{x}\right)$ are non decreasing functions.

Corollary 2.2. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be non a decreasing, differentiable function.
If $\frac{x^p}{\varphi(x)}$ and $x^p\varphi\left(\frac{\alpha}{x}\right)$ are non decreasing with $p>0$, then $\frac{\varphi'(x)}{\varphi(x)}\to 0$ as $x\to\infty$.

Proof. Since $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is non decreasing and differentiable, then $\frac{x^p}{\varphi(x)}$ and $x^p\varphi\left(\frac{\alpha}{x}\right)$ are non decreasing with $p>0$. Next, elementary calculations as in the proof of Theorem 2.1, give us

$$0\leq\frac{\varphi'(x)}{\varphi(x)}\leq\frac{p}{x}\text{ for } x>0,$$

which implies the wished result. □

Example 2.4. Let $\varphi(x)=e^x-x-1$, then $\frac{\varphi'(x)}{\varphi(x)}\to 1$; thus, there does not exist $p>0$ such that $\frac{x^p}{e^x-x-1}$ and $x^p\left(e^{\frac{\alpha}{x}}-\frac{\alpha}{x}-1\right)$ are non decreasing functions.

In a similar way to that developed in the proof of Theorem 2.1 but requesting that the derivatives of $\frac{x^p}{\varphi(x)}$ and $x^p\varphi\left(\frac{\alpha}{x}\right)$ are negative, we obtain the following result.

Proposition 2.5. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a non decreasing, differentiable function.
If there exists $p>0$ such that $\frac{p}{x}<\frac{\varphi'(x)}{\varphi(x)}$ for every $x>0$, then $\frac{x^p}{\varphi(x)}$ and $x^p\varphi\left(\frac{\alpha}{x}\right)$ are decreasing.

As Lemma 3.1 of [9] establishes the background in which non increasing functions become quasi-increasing, we employ that result together with Proposition 2.5 and we get the following.

Theorem 2.3. *Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a non decreasing, differentiable function.
If there exists $p>0$ such that $\frac{p}{x}<\frac{\varphi'(x)}{\varphi(x)}$ for every $x>0$ and there exists $K_p<2^{p+1}$ such that $\frac{\varphi(x)}{\varphi\left(\frac{x}{2}\right)}\leq K_p$ for every $x>0$, then $\frac{x^p}{\varphi(x)}$ and $x^p\varphi\left(\frac{\alpha}{x}\right)$ are quasi-increasing, i.e. $\varphi\prec x^p$ and $\varphi\prec_N x^p$.*

Example 2.5. Let $\varphi(x)=x^q$ with $q>0$, then $\frac{p}{x}<\frac{\varphi'(x)}{\varphi(x)}$ for every $x>0$ provided that $p<q$ and $\frac{\varphi(x)}{\varphi\left(\frac{x}{2}\right)}<2^{p+1}$ for $p>q-1$.
Thus, $x^q\prec x^p$ and $x^q\prec_N x^p$ in the case of $\max\{q-1,0\}<p<q$.

Remark 2.2. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a non decreasing function. If φ is non differentiable, invoking Proposition 2.4, the above results for the case of differentiable functions can be employed provided that (2.1) holds.

If $p\leq 0$, it is not possible to use Theorem 2.3. However, we can establish some conditions to guarantee a relationship between φ and x^p by $\prec$ and $\prec_N$ for such values of p, as stated below.

Theorem 2.4. *Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a non decreasing function.*
If there exists $p\in(-1,0]$ such that $\frac{\varphi(x)}{\varphi\left(\frac{x}{2}\right)}\le K_p<2^{p+1}$ for every $x>0$, then $\varphi\prec x^p$ and $\varphi\prec_N x^p$.

Proof. As φ is a non decreasing function on $\mathbb{R}^+$ and $p\in(-1,0]$, then $\eta_1(x)=\frac{x^p}{\varphi(x)}$ and $\eta_2(x)=x^p\varphi\left(\frac{\alpha}{x}\right)$ are non increasing functions on $\mathbb{R}^+$. In addition, $\frac{\eta_1\left(\frac{x}{2}\right)}{\eta_1(x)}\le K_p$ and $\frac{\eta_2\left(\frac{x}{2}\right)}{\eta_2(x)}\le K_p$ for every $x>0$ and with $K_p<2^{p+1}<2$. Consequently, by Lemma 3.1 of [9] η_1 and η_2 are quasi-increasing, i.e. $\varphi\prec x^p$ and $\varphi\prec_N x^p$ for $p\in(-1,0]$. □

Remark 2.3. In Theorem 2.4, it is required that $\varphi\in\Delta_2$ with $1\le\Lambda_\varphi<2$ as it has done in Proposition 3.10 of [1]. But in that case, p were positive; while here, p belongs to the set of real negative numbers.

Example 2.6. Let $\varphi(x)=\ln(\sqrt{x}+1)$. Then $\frac{\varphi(x)}{\varphi\left(\frac{x}{2}\right)}\le\sqrt{2}<2^{p+1}$ provided that $p>-\frac{1}{2}$ and as it is necessary that $p\le 0$, then $\ln(\sqrt{x}+1)\prec x^p$ and $\ln(\sqrt{x}+1)\prec_N x^p$ for $p\in\left(-\frac{1}{2},0\right]$.

3 Simonenko indices

Let $\Phi\in\mathcal{I}$ be an increasing, continuous function.
If $h_\Phi(\lambda)=\sup\limits_{t>0}\dfrac{\Phi(\lambda t)}{\Phi(t)}$ with $\lambda>0$, then the numbers

$$i(\Phi)=\lim_{\lambda\to 0^+}\frac{\ln h_\Phi(\lambda)}{\ln\lambda}=\sup_{0<\lambda<1}\frac{\ln h_\Phi(\lambda)}{\ln\lambda}, \tag{3.1}$$

and

$$I(\Phi)=\lim_{\lambda\to\infty}\frac{\ln h_\Phi(\lambda)}{\ln\lambda}=\inf_{1<\lambda<\infty}\frac{\ln h_\Phi(\lambda)}{\ln\lambda}, \tag{3.2}$$

are called lower index of Φ and upper index of Φ respectively, or fundamentals indices of Φ. The numbers $i(\Phi)$ and $I(\Phi)$ are also known as *indices of Orlicz spaces* or *Matuszewska-Orlicz indices*.

From (3.1) and (3.2), it is clear that $1\le i(\Phi)\le I(\Phi)$. And, it is well known that $I(\Phi)<\infty$ if and only if $\Phi\in\Delta_2$ (see [8, Thm. 11.7]).
Moreover, if Φ and Ψ are complementary Young functions, then

- $i(\Phi)>1$ if and only if $\Psi\in\Delta_2$;
- the pairs $i(\Psi)$, $I(\Phi)$, and $i(\Phi)$, $I(\Psi)$ satisfy

$$\frac{1}{i(\Phi)}+\frac{1}{I(\Psi)}=\frac{1}{I(\Phi)}+\frac{1}{i(\Psi)}=1$$

  (see [8, Cor. 11.6]);

- $\Phi \in \Delta_2$ is equivalent to the existence of $p_0, p_1 \in [1,\infty)$, $p_0 \le p_1$ such that

$$C^{-1}\min(\lambda^{p_0},\lambda^{p_1})\Phi(t) \le \Phi(\lambda t) \le C\max(\lambda^{p_0},\lambda^{p_1})\Phi(t), \tag{3.3}$$

  for some constant $C>0$ and for every $\lambda, t \ge 0$; and, $\sup p_0 = i(\Phi)$ and $\inf p_1 = I(\Phi)$ ([4]);

- from (3.3) other formulae for $i(\Phi)$ and $I(\Phi)$ are obtained, i.e.

$$i(\Phi) = \sup\left\{ p : \inf_{\substack{u>0\\ \lambda\ge 1}} \lambda^{-p}\frac{\Phi(\lambda u)}{\Phi(u)} > 0 \right\},$$
$$I(\Phi) = \inf\left\{ p : \sup_{\substack{u>0\\ \lambda\ge 1}} \lambda^{-p}\frac{\Phi(\lambda u)}{\Phi(u)} < \infty \right\}$$

  (c.f. [8, Thm. 11.13]).

In [11] and [8] other indices, related to (3.1) and (3.2), are introduced.

Definition 3.1. Let $\Phi \in \mathcal{I}$ be an increasing, differentiable function and assume that $\Phi = \int_0^x \varphi(t)\,dt$.
If there exist $p, q \in \mathbb{R}$ such that

$$p\Phi(t) \le x\varphi(x) \le q\Phi(x) \text{ for every } x \in \mathbb{R}, \tag{3.4}$$

then the best p and q that verify (3.4) are called Simonenko indices and they satisfy

$$p(\Phi) = \inf_{x>0} \frac{x\varphi(x)}{\Phi(x)} \text{ and } q(\Phi) = \sup_{x>0} \frac{x\varphi(x)}{\Phi(x)}.$$

Remark 3.1. The relationship between Simonenko indices and indices of Orlicz spaces is given by the following inequality

$$p(\Phi) \le i(\Phi) \le I(\Phi) \le q(\Phi),$$

that was proved in Theorem 11.11 of [8].

If Φ and Ψ belong to C^1 and they are complementary N-functions, then

- $\Phi \in \Delta_2$ if and only if $q(\Phi) < \infty$ (c.f. [10, Cor. 4, pp. 22-23]);
- $\Psi \in \Delta_2$ if and only if $1 < p(\Phi)$ (c.f [10, Cor. 4, pp. 22-23]);
- $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < p(\Phi) \le q(\Phi) < \infty$.

- As in the case of indices of Orlicz spaces, Simonenko indices of complementary, N-functions behave as the conjugate exponents of the power functions, i.e.

$$\frac{1}{p(\Phi)}+\frac{1}{q(\Psi)}=1=\frac{1}{p(\Psi)}+\frac{1}{q(\Phi)}$$

([10, Cor. 6, p. 27]).

Now, we get power functions related to some function Φ by $\prec$ or $\prec_N$ employing Simonenko indices.

Theorem 3.1. *Let $\Phi \in C^1$ be an N-function that satisfies the Δ_2 condition.*
If $q(\Phi)$ is the upper Simonenko index, then
1) $\Phi \prec x^{q(\Phi)}$ and $\Phi \prec_N x^{q(\Phi)}$;
2) $\Phi \prec x^q$ and $\Phi \prec_N x^q$ for every $q \geq q(\Phi)$.

Proof. 1) Since Φ is a differentiable, N-function that satisfies the Δ_2 condition, there exists $q(\Phi)<\infty$ such that $\frac{\varphi(x)}{\Phi(x)} \leq \frac{q(\Phi)}{x}$ for every $x>0$. Next, by Theorem 2.1 we have $\Phi \prec x^{q(\Phi)}$ and $\Phi \prec_N x^q(\Phi)$.
2) By the definition of $q(\Phi)$, we have $\frac{\varphi(x)}{\Phi(x)} \leq \frac{q}{t}$ for every $q \in [q(\Phi),\infty)$ and for every $x>0$, then $\Phi \prec x^q$ and $\Phi \prec_N x^q$ for every $q \geq q(\Phi)$. □

Remark 3.2. If $\Phi \in \Delta_2$, then $1 \leq \frac{x\varphi(x)}{\Phi(x)} \leq 2\Lambda_\varphi$ for every $x>0$ and therefore $1 \leq p(\Phi) \leq q(\Phi) < \infty$. If, in addition, $\Phi \in \nabla_2$, then Theorem 3.1 is satisfied with $q(\Phi) \in (1,\infty)$.

Example 3.1. c.f. [8].
If $\Phi(x)=x^p\left(1+\frac{1}{\sqrt{5}}\sin(p\ln x)\right)$ for $x\in(0,\infty)$ and with $p\geq 6$, then Φ is an N-function such that $\Delta_2 \cap \nabla_2$ and $p(\Phi)=\frac{1}{2}p$ y $q(\Phi)=\frac{3}{2}p$. Now, by Theorem 3.1, $\Phi \prec x^{\frac{3}{2}p}$ and $\Phi \prec_N x^{\frac{3}{2}p}$ with $p \geq 6$.

Theorem 3.2. *Let $\Phi \in C^1$ be an N-function that satisfies the Δ_2 condition and let $p(\Phi)$ be the lower Simonenko index.*
1) If there exists $K_{p(\Phi)} < 2^{p(\Phi)+1}$ such that $\frac{\Phi(x)}{\Phi\left(\frac{x}{2}\right)} \leq K_{p(\Phi)}$ for every $x>0$, then $\Phi \prec x^{p(\Phi)}$ and $\Phi \prec_N x^{p(\Phi)}$.
2) If $0<p\leq p(\Phi)$ and there exists $K_p < 2^{p+1}$ such that $\frac{\Phi(x)}{\Phi\left(\frac{x}{2}\right)} \leq K_p$ for every $x>0$, then $\Phi \prec x^p$ and $\Phi \prec_N x^p$.

Proof. 1) If Φ is a differentiable, N-function such that $\Phi \in \Delta_2$, then there exists $0<p(\Phi)<\infty$ such that $\frac{\varphi(x)}{\Phi(x)} \geq \frac{p(\Phi)}{x}$ for every $x>0$. If there also exists $K_{p(\Phi)} < 2^{p(\Phi)+1}$ such that $\frac{\Phi(x)}{\Phi\left(\frac{x}{2}\right)} \leq K_{p(\Phi)}$ for every $x>0$ then, by Theorem 2.3, $\Phi \prec x^{p(\Phi)}$ and $\Phi \prec_N x^{p(\Phi)}$.
2) From the definition of $p(\Phi)$, we have $\frac{\varphi(x)}{\Phi(x)} \geq \frac{p}{x}$ for every $p \in (0,p(\Phi)]$ and for every $t>0$; and, as there exists $K_p < 2^{p+1}$ such that $\frac{\Phi(x)}{\Phi\left(\frac{x}{2}\right)} \leq K_p$ for every $x>0$, then $p \leq p(\Phi)$ and therefore $\Phi \prec x^p$ and $\Phi \prec_N x^p$. □

Remark 3.3. If it is also assumed that $\Phi \in \nabla_2$, then Theorem 3.2 holds for $p(\Phi) \in (1,\infty)$.

Example 3.2. If $\Phi(x) = x^\alpha$ with $\alpha \geq 1$, then $p(\Phi) = q(\Phi) = \alpha$. In addition, if $\Phi \in \Delta_2$ with $\Lambda_\Phi < 2^{\alpha+1}$, then $\Phi \prec \Phi$ and $\Phi \prec_N \Phi$.
Let $0 < p \leq \alpha$ such that $\alpha - 1 < p$, then $\Phi \in \Delta_2$ with $\Lambda_\Phi < 2^{p+1}$ and therefore $x^\alpha \prec x^p$ and $x^\alpha \prec_N x^p$ for $0 < p \leq \alpha$.

4 Main Result

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space.

A subset $\mathcal{L} \subset \mathcal{A}$ is a σ-lattice if and only if $\emptyset, \Omega \in \mathcal{L}$ and $\mathcal{L}$ is closed under countable unions and intersections.

A function $f: \Omega \to \mathbb{R}$ is said to be $\mathcal{L}$-measurable if $\{f > a\} \in \mathcal{L}$ for all $a \in \mathbb{R}$.

We denote by $L^\Phi(\mathcal{L})$ the set of all $\mathcal{L}$-measurable functions in $L^\Phi(\Omega)$.

Definition 4.1. A function $g \in L^\Phi(\mathcal{L})$ is called a best Φ-approximation to $f \in L^\Phi(\Omega)$ if and only if $\int_\Omega \Phi(f-g)\,d\mu = \min\limits_{h \in L^\Phi(\mathcal{L})} \int_\Omega \Phi(f-h)\,d\mu$.

Let $\mu(f,\mathcal{L})$ be the set of all best Φ-approximations to $f \in L^\Phi(\Omega)$. It is well known that for each $f \in L^\Phi(\Omega)$, the set $\mu(f,\mathcal{L})$ is not empty (see [7]).

Suppose that $\mathcal{L}_n$ is an increasing sequence of σ-lattices, i.e. $\mathcal{L}_n \subset \mathcal{L}_{n+1}$ for every $n \in \mathbb{N}$. Let f be a non negative function of $L^\Phi(\Omega)$ and let f_n be any selection of functions of $\mu(f,\mathcal{L}_n)$.

In [9] it is defined the maximal function $f^* = \sup\limits_n f_n$ and the authors obtain strong type inequalities in some L^Ψ spaces.

Next, we will get strong type inequalities for f^* in some L^p spaces employing the results of the previous sections where Simonenko indices are involved.

Theorem 4.1. *Let $\Phi \in C^2([0,\infty))$ and let φ be its derivative. Assume that φ is an N-function such that $\varphi(0) = 0$ and $\varphi \in \Delta'$ globally.*

If $q(\varphi)$ is the upper Simonenko index, then

$$\int_\Omega (f^*)^{q(\varphi)+1}\,d\mu \leq 2KC_q \left(\frac{2}{c}\right)^{q(\varphi)+1} \int_\Omega f^{q(\varphi)+1}\,d\mu, \tag{4.1}$$

for some $c > 0$ and with KC_q independent of f.

If $p(\varphi)$ is the lower Simonenko index and there exists $K_{p(\varphi)} < 2^{p(\varphi)+1}$ such that $\frac{\varphi(x)}{\varphi(\frac{x}{2})} \leq K_{p(\varphi)}$ for every $x > 0$, then

$$\int_\Omega (f^*)^{p(\varphi)+1}\,d\mu \leq 2KC_q \left(\frac{2}{c}\right)^{p(\varphi)+1} \int_\Omega f^{p(\varphi)+1}\,d\mu, \tag{4.2}$$

for some $c > 0$ and with KC_q independent of f.

Proof. As $\varphi(0)=0$ and $\varphi\in\Delta'$ globally, by [AF12, Thm. 4.5] we have

$$\mu(\{f^*>a\})\le K\int_{\{f^*>a\}}\varphi\left(\frac{f}{a}\right)d\mu \text{ for every } a>0, \tag{4.3}$$

with K depending on Λ_Φ.

Since $\varphi\in C^1([0,\infty))$, it is also true that $\varphi(0+)=0$. Now, from [1, Lemma 2.1], there exists $c>0$ such that

$$\mu(\{f^*>a\})\le 2K\int_{\{f>ca\}}\varphi\left(\frac{f}{a}\right)d\mu \text{ for every } a>0. \tag{4.4}$$

On the other hand, by Theorem 3.1 we have $\varphi\prec_N x^{q(\varphi)}$; and, from Theorem 3.2, we get $\varphi\prec_N x^{p(\varphi)}$ provided that there exists $K_{p(\varphi)}<2^{p(\varphi)+1}$ such that $\frac{\varphi(x)}{\varphi(\frac{x}{2})}\le K_{p(\varphi)}$ for every $x>0$. Finally, by (4.4) and [1, Thm. 3.17], we obtain (4.1) and (4.2). □

Example 4.1. Let $\varphi(x)=x^\alpha\left(1+\frac{1}{\sqrt{5}}\sin(\alpha\ln x)\right)$ for $x\in(0,\infty)$ and $\alpha\ge 6$, then $\varphi(0+)=0$ and $\varphi\in\Delta'$. The characteristics of φ guarantee that (4.3) and (4.4) are satisfied.
On the other hand, by Example 3.1 we have $\varphi\prec x^{\frac{3}{2}\alpha}$ and $\varphi\prec_N x^{\frac{3}{2}\alpha}$ with $\alpha\ge 6$ being $p(\varphi)=\frac{1}{2}\alpha$ and $q(\varphi)=\frac{3}{2}\alpha$ the lower and upper Simonenko indices of φ, respectively.
As a consequence,

$$\int_\Omega (f^*)^{\frac{3}{2}\alpha+1}d\mu\le 2KC_q\left(\frac{2}{c}\right)^{\frac{3}{2}\alpha+1}\int_\Omega f^{\frac{3}{3}\alpha+1}d\mu,$$

for $\alpha\ge 6$ and where f is a non negative function in $L^\Phi(\Omega)$.

Remark 4.1. If φ is not an N-function, but φ is the right continuous derivative of a Young function Φ, $\varphi(0)=\varphi(0+)=0$, and $\varphi\in\Delta'$, we have that Proposition 2.1 or Proposition 2.2, provides some $p>-1$ such that $\varphi\prec_N x^p$, depending on the properties of φ. Then,

$$\int_\Omega (f^*)^{p+1}d\mu\le 2KC_q\left(\frac{2}{c}\right)^{p+1}\int_\Omega f^{p+1}d\mu,$$

for every $f\in L^\Phi(\Omega)$.

In [9, Thm. 1.1], it is proved that

$$\mu(\{f^*>a\})\le\frac{C}{\varphi_+(a)}\int_{\{f^*>a\}}\varphi_+(f)\,d\mu \text{ for every } a>0, \tag{4.5}$$

with C depending only on Λ_Φ and where φ_+ is the right derivative of the Young function Φ, $\varphi_+\in\Delta_2$ and $\varphi_+(0)=0$. Then, by a similar procedure to that developed in the proof of Theorem 4.1, strong type inequalities for f^* in some L^p spaces can be obtained by means of the relation $\prec$ instead of $\prec_N$ ([9, Cor. 2.4]). However, we point out that it is not possible to carry out such a procedure assuming only $\varphi_+(0)=\varphi_+(0+)=0$. In fact, it is necessary to ask an additional condition on φ_+, which is the existence of a constant $r\in(0,1)$ such that $\varphi_+(rx)\le\frac{1}{2}\varphi_+(x)$ for every $x>0$ (c.f. [9, Lemma 2.2]).

Example 4.2. Let $\varphi(x)=\ln(x+1)$. Then, $\varphi(0)=\varphi(0+)=0$ and $\varphi\in\Delta'$ globally. Then (4.5) and (4.3) are satisfied and there also exists $c>0$ such that (4.4) holds.
In Example 2.3 we have seen that $\ln(x+1)\prec_N x^p$ for every $p>1$, then we obtain

$$\int_\Omega (f^*)^{p+1}d\mu \le 2KC_q\left(\frac{2}{c}\right)^{p+1}\int_\Omega f^{p+1}d\mu \tag{4.6}$$

for every $f\in L^\Phi(\Omega)$ and with $p>-1$ such that $\varphi_+\prec_N x^p$.
Now, as there does not exist $r\in(0,1)$ such that $\ln(rx+1)\le\frac{1}{2}\ln(x+1)$ for every $x>0$, we cannot get (4.6) having (4.5) as a starting point, because we cannot apply [9, Lemma 2.2], although (4.5) is valid and $\ln(x+1)\prec x^p$ for every $p>1$.

4.1 Final remarks on some one-sided operators

Proposition 5.1 of [1] allows us to obtain valid weak type inequalities on proper subsets of $\mathbb{R}$ for the classical Hardy-Littlewood maximal function M and the one-sided Hardy-Littlewood maximal functions $M^{\pm}$, from weak type inequalities that hold true on the whole $\mathbb{R}^n$. In Theorem 2, Theorem 3 and Theorem 5 of [2] conditions on $\varphi\in\mathcal{I}$ to have weak type inequalities for $M^{\pm}$ on the whole $\mathbb{R}$ were established; next, by Proposition 5.1 of [1], we obtain

$$|\{x\in\mathbb{R}:|M^{\pm}f(x)|>\lambda\}|\le C\int_{\{x:|f(x)|>\frac{\lambda}{2}\}}\varphi\Big(\frac{2Cf(x)}{\lambda}\Big)dx,$$

for all $\lambda>0$ and where the constant C is independent of the function f. Then, (1.1) holds with $f=M^{\pm}$ and for values of p that depend on the characteristics of φ, that is,

$$\int_{\mathbb{R}}(M^{\pm}f)^{p+1}dx\le\tilde{K}\int_{\mathbb{R}}f^{p+1}dx, \tag{4.7}$$

for every non negative function $f\in L^1_{loc}(\mathbb{R})$, with $p>-1$ such that $\varphi\prec_N x^p$ and where $\tilde{K}$ is independent of f.

Remark 4.2. In Theorem 7 of [2] strong type inequalities for $M^{\pm}$ are characterized, as it has done in [5] for the Hardy-Littlewood maximal function M.

One-sided maximal operators $\mathcal{M}^{\pm}$, associated to one-sided best approximation by constants, were defined and studied in [2, pp. 155-160]. The relationship between $\mathcal{M}^{\pm}$ and $M^{\pm}$ is established in Lemma 1 of [2]. Using this relationship, strong type inequalities for the one-sided maximal operators $\mathcal{M}^{\pm}$ like (1.1) can be obtained from (4.7).

A similar situation occurs with the operators $M^{\pm}_p$ (see [2, pp. 160-161]) because they were defined from the one-sided Hardy-Littlewood maximal functions $M^{\pm}$.

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References

[1] S. Acinas and S. Favier, Maximal Inequalities in Orlicz spaces, Int. J. Math. Anal. (Ruse), 6:41-44 (2012), 2179-2198.
[2] S. Acinas and S. Favier, Inequalities for one-sided operators in Orlicz spaces, Actas del XII Congreso Dr. Antonio A. R. Monteiro, 12 (2014), 149-161.
[3] A. Fiorenza and M. Krbec: Indices of Orlicz spaces and some applications, Comment. Math. Univ. Carolin., 38:3 (1997), 433-451.
[4] J. Gustavsson and J. Peetre, Interpolation of Orlicz spaces, Studia Mathematica, 60:1 (1977), 33-59.
[5] V. Kokilashvili, M. Krbec, Weighted Inequalities in Lorentz and Orlicz Spaces, World Scientific, Singapore, 1991.
[6] M. A. Krasnosel'skii and Ya. B. Rutickii, Convex Functions and Orlicz Spaces, P. Noordhoff Ltd., Groningen, The Netherlands, 1961.
[7] D. Landers, L. Rogge, Best approximants in L_Φ-spaces, Z. Wahrsch. Verw. Gabiete, 51 (1980), 215-237.
[8] L. Maligranda, Orlicz Spaces and Interpolation, Seminarios de Matemática, Campinas SP., Brasil, 1989.
[9] F.D. Mazzone, F. Zó, On Maximal Inequalities Arising in Best Approximation, J. Inequal. Pure and Appl. Math., 10:2 (2009), Article 58, 10 pp.
[10] M. Rao, Z. Ren, Theory of Orlicz Spaces, Marcel Dekker Inc., New York, 1991.
[11] I. B. Simonenko, Interpolation and extrapolation of linear operators in Orlicz spaces, Dokl. Akad. Nauk; SSSR, 151 (1963), 1288-1291.
[12] Yakiang Yan, Relationship between Matuszewska-Orlicz, Semenov and Simonenko indices of $\Phi-$ functions, Publ. Inst. Math. (Beograd) (N.S.), 73(87), 2003, 139-147.