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# Complexity and disequilibrium as telltales of superconductivity

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## HIGHLIGHTS

- An SU2 x Su2 exactly solvable of superconductivity (SC) is analyzed.
- We focus on quantifiers like the statistical complexity C and the disequilibrium D.
- C detects the SC phase transition by becoming maximal at the constant.
- D becomes maximal at the SC state, indicative of maximum order.
- High temperature SC is detected.

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## ABSTRACT

We show that the statistical complexity $C$ and the disequilibrium $D$ are good indicators of the critical pairing coupling constant's value that signals the onset of superconductivity, a well-known phase transition. This suggests that statistical quantifiers like $C$ and $D$ can be gainfully used in many quantum body problems.



## 1. Introduction

To be cognizant of a system's level of randomness (entropy $S$) does not by itself afford a quite adequate insight into its correlation structures. How could one attain a position from which to discern, via some (to be concocted) specific quantifier, among a system's components, mimicking the way in which $S$ describes disorder? To search for it one could start by considering two opposite situations: (A) perfect order or (B) maximal randomness, for which correlations do not exist [1]. In between (A) and (B), diverse degrees of correlation could exist and our putative quantifier should quantify them. Following current usage, we choose to call this quantifier a "complexity". How is one to represent it? This is a tough question. It is often repeated that Seth Lloyd enumerated about 40 manners of defining this "complexity", none of them optimal.

It is far easier to define its opposite: simplicity. Intuitively, a system is complex, of course, when it does not fit simple patterns, as in the case of either (i) a perfect crystal or (ii) the isolated ideal gas. These are good examples of simplicity, or of situations of zero complexity. In (i) the information stored, or negentropy ($-S$), attains a minimum. A few parameters

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should suffice for a good description. On the other hand, in (ii) we face a completely disordered system. Any of its accessible states possesses the same probability, a situation that yields maximum entropy. Systems (i) and (ii) are extremal in the scale of information/order. We see then that complexity cannot be expressed in terms of neither order or information. A great advance was made in Ref. [1]. Its authors proposed a measure of complexity by recourse to a sort of distance to the maximum entropy situation, named disequilibrium ($D$) [2]. $D$ provides a notion of hierarchy that makes it non-null if there are privileged states among the accessible ones. $D$ would then be maximal for (i) and vanish for (ii). In the case of $S$, matters are exactly reversed. $S$ is minimal for (i), while it attains a maximum for (ii). This line of reasoning motivated L. Ruiz, Mancini, and Calvet (LMC) [1] to propose what constitutes today the standard form for a statistical complexity measure $C$ of the form

$$C = DS, \tag{1}$$

an intriguing functional of the probability distributions (PDs) that indeed, as desired, grasps correlations in the fashion that entropy captures randomness [1]. $D$ indicates (in probability space) the distance from i) the actual probability distribution $p_i$ to ii) the uniform PD. It reveals the extant amount of structural detail [1,3]. In the case of a system of $N$ particles we have

$$D = \sum_{i=1}^{N} \left( p_i - \frac{1}{N} \right)^2. \tag{2}$$

Here $p_1, p_2, \ldots, p_N$ are the individual normalized probabilities ($\sum_{i=1}^{N} p_i = 1$) [1]. $D$ attains the maximum value for a fully ordered state and vanishes in the case of equiprobable states. Moreover, LMC's statistical complexity also individualizes and quantifies what Boltzmann's entropy (or information $S = -\sum_{i=1}^{N} p_i \ln p_i$) conveys.

LMC's proposal has received immense attention (see Refs. [1,3–9] a small sample). It was applied in different scenarios for both the canonical, microcanonical, and grand canonical ensembles. It is obvious that $C$ vanishes in the (opposite) simple cases (i) and (ii) above.

We will here focus attention, for the first time as far as we know, *on the LMC workings in the scenario of superconductivity* (SC). We are speaking here of a quantum phenomenon of vanishingly electrical resistance and expulsion of magnetic flux fields. It occurs in certain materials, the superconductors, when cooled below a characteristic critical temperature. As it is well known, it was discovered by Heike Kamerlingh Onnes in 1911, at the University of Leiden. Superconductivity is a quantum mechanical phenomenon, characterized by the Meissner effect. The occurrence of the Meissner effect signals that it cannot be understood just an idealization of perfect conductivity in classical systems. Bardeen, Cooper, and Schrieffer's (BCS) theory is the standard microscopic theory of superconductivity [10]. It describes superconductivity as a microscopic effect caused by a condensation of Cooper pairs into a boson-like state (see, amongst a plethora of works, for instance Refs. [11,12]). The theory is also used in nuclear physics to describe the pairing interaction between nucleons in an atomic nucleus [13]. The pairing interaction is essential for the description of the shell nature of nuclear structure. Their discoverers, M. Mayer and D. Jensen, were awarded the Nobel Price en 1963.

The superconducting state can be viewed as a specially *ordered* state in which all fermions are paired to spin zero, i.e., we have an ordered system of bosons. Partial order here would mean that an appreciable fraction of all the fermions are thus coupled. Accordingly, the SC-effect is a natural candidate to be examined using the statistical complexity tool-arsenal, our task in this work. It is worth mentioning that a statistical treatment of high temperature superconductivity is that of [12], where, introducing the generalized, non-extensive statistics proposed by Tsallis [14] into the standard $s$-wave pairing BCS theory of superconductivity yields, in two-dimensions, a reasonable description of many of the main properties of high temperature superconductors, provided some allowance is made for non-phonon mediated interactions.

The paper is organized as follows. In Section 2 we introduce relevant concepts related to an exactly solvable two-level model that adequately mimics the essentials of superconductivity. In Section 3 we define the quantities $C$ and $D$ mentioned above. This constitutes a crucial issue. Finally, we draw conclusions in Section 5.

## 2. Exactly solvable model for superconductivity

### 2.1. Preliminaries

The Lipkin Model (LM) [15] has been seen to be extraordinarily useful in theoretical investigations with regards to the validity and usefulness of variegated approaches designed so as to scrutinize the manifold facets of the quantum many body problem. LM's is based on an SU2 algebra corresponding to special operators called quasi-spin ones. It yields easily obtained exact solutions, which, in turn, are to be confronted with results found by appealing to diverse types of theoretical constructs. The relevant Casimir operator [15] has attached several multiplets, and only the one associated to the ground state of the system is usually the focus of concern. Cambiaggio and Plastino (CP) [16] advanced a simple LM-extension to deal with excited multiplets and this allowed for the formulation, in quasi-spin language, of a BCS-like theory which mimics superconductivity, providing us with exact solutions. This was an extension of the LM to the case of a variable particle number, requiring only SU2 x SU2 algebra.

### 2.2. Cambiaggio–Plastino SU2 x SU2 model

The model considers $N$ fermions distributed in two $(2\Omega)$-fold degenerate single-particle levels, separated by an energy gap $\epsilon$. One characterizes the $2\Omega$ lower states by the quantum numbers $p$, $\mu$ with $p = 1, \ldots, 2\Omega$ and $\mu = \pm 1$. One has the usual SU2 quasi-spin operators [15]

$$J_z = (1/2)\sum_{p,\mu} C^+_{p,\mu} C_{p,\mu}, \tag{3}$$

$$J_+ = \sum_p C^+_{p,+} C_{p,-}, \tag{4}$$

$$J_- = \sum_p C^+_{p,-} C_{p,+}, \tag{5}$$

to which CP add the SU2 angular momentum-like "pairing" operators

$$Q_0 = (1/2)\sum_{p,\mu} C^+_{p,\mu} C_{p,\mu} - \Omega^+, \tag{6}$$

$$Q_+ = \sum_p C^+_{p,+} C_{p,-}, \tag{7}$$

$$Q_- = \sum_p C_{p,-} C_{p,+}. \tag{8}$$

Obviously, $Q_+$ creates, and $Q_-$ destroys, two particles which yield zero contribution to the $J_z$-value, and which could then be said to "couple" to $J_z = 0$. Any $Q$-operator commutes with all $J$-operators, and vice versa (SU2 x SU2). Accordingly, one can construct a complete orthonormal basis characterized by the eigenvalues of the operators $J^2, J_z, Q^2, Q_0$, i.e., $|J, Q, J_z, Q_0\rangle$. A "pairing" Hamiltonian which commutes with the number operator would read, mimicking the nuclear pairing one [13], (and with $\epsilon = 1$) as

$$H = J_z - (G/2) Q_+ Q_-, \tag{9}$$

with $G$ the pairing-strength. CP introduce next the quasi-spin seniority number $\nu$

$$\nu = 2(\Omega - Q), \tag{10}$$

which indicates the number of particles not "paired" to $J_z = 0$, this is, $\nu$ is the number of "unpaired" particles in a $Q$-multiplet. We have also [16]

$$J = \nu/2, \tag{11}$$

$$J + Q = \Omega. \tag{12}$$

In the case of the Lipkin model, $N = 2\Omega$, $Q_0 = 0$ [16]. The unperturbed ground state (no interaction) has $J = \Omega$, $J_z = -\Omega$, $Q = Q_0 = 0$ and belongs to the multiplet $J = \Omega$, $Q = Q_0 = 0$. The exact eigenvalues of $H$ are [16]

$$E(J, Q, J_z, Q_0) = J_z - (g/2)\left[Q(Q+1) - Q_0(Q_0 - 1)\right]. \tag{13}$$

The energy of the unperturbed ground state ($\nu = N$, $Q = Q_0$) is [16]

$$E_0 = -\Omega. \tag{14}$$

The state of quasi-spin seniority zero, in which all particles are "paired" to $J_z = 0$ mimics a nuclear "superconducting" state [16,17]. It is characterized by $\nu = 0$ and $Q = \Omega$. In general, the energy is

$$E = -(G/2)\left[\Omega(\Omega+1) - Q_0(Q_0 - 1)\right] = -(G/2)\left[\Omega(N-\nu) + \frac{\nu}{2}\left(\frac{\nu}{2} - 1\right) - \frac{N}{2}\left(\frac{N}{2} - 1\right)\right]. \tag{15}$$

The associated state becomes the superconducting state of the pairing-interacting system for

$$G = G_{crit} \geq (4\Omega/N)\frac{1}{2\Omega + 1 - N/2} = \frac{2}{\Omega + 1}, \tag{16}$$

(the last equality in the r.h.s. holds just for the Lipkin case) while if the coupling constant is smaller than the critical one the system remains in the unperturbed ground state (UGS). A phase-transition ensues at $G = G_{crit}$ from the UGS to the superconducting state, belonging to another multiplet. It is shown in Ref. [16] that, in this model, the BCS solution coincides with the exact one. The preceding considerations were made at temperature zero. We proceed to the statistical treatment for finite $T$'s next.

### 2.3. Statistical treatment

This treatment was devised and implemented in Ref. [17]. For investigating ground states, only the $J + Q = \Omega$ "band" requires consideration. For non-zero temperatures, though, a multitude of states belonging to other bands become "accessible" as one builds up the pertinent statistical ensemble. Defining the degeneracy $Y(J, Q)$ (deduced in Ref. [17])

$$Y(J,Q) = \frac{(2\Omega+2)!(2\Omega)!(2J+1)(2\Omega+1)}{(\Omega+J+Q+2)!(\Omega+J-Q+1)!(\Omega-J+Q+1)!(\Omega-J-Q)!}, \tag{17}$$

the pseudo partition function $Z_M$ is seen to be [17] [see also Eq. (13)]

$$Z_M = \sum_{M=-J}^{M=J} \exp[-\beta(M - \frac{G}{2}Q(Q+1))], \tag{18}$$

and the system's partition function $Z$ reads then

$$Z = \sum_{J,Q} Y(J,Q) Z_M, \tag{19}$$

where $J$ and $Q$ run over all the values permitted by the following restrictions [17]

$$0 \le J \le \Omega, \tag{20}$$

$$0 \le Q \le \Omega, \tag{21}$$

$$0 \le J + Q \le \Omega. \tag{22}$$

Once in possession of $Z$, all statistical quantifiers become available.

## 3. Present formalism for the statistical quantifiers *C* and *D*

We begin here to present our results. We slightly modify (22) in the fashion

$$0 \le J + Q = s \le \Omega. \tag{23}$$

To perform the double sum over $J, Q$, it is better to sum over $J + Q = s$ and over $J$, leaving $Q$ fixed at $Q = s - J$, with

$$0 \le s \le \Omega, \tag{24}$$

and $s = 0, 1, 2, 3, \ldots, \Omega$ and $J = 0, 1, 2, \ldots, S$. Thus,

$$Z = \sum_{J,Q} Y(J,Q) Z_M = \sum_{s=0}^{\Omega} \sum_{J=0}^{s} Y(J,Q) Z_M. \tag{25}$$

Let us insist: once we have $Z$, all statistical quantifiers become available.

As usual [15], we fix the number of particles in the fashion $N = 2\Omega$. The canonical probability distribution (CPD) is now

$$P(M,Q) = \frac{Y(J,Q)\exp[-\beta(M - \frac{G}{2}Q(Q+1))]}{Z}, \tag{26}$$

while the uniform distribution becomes

$$P_{unif} = \frac{1}{V}, \tag{27}$$

with

$$V = \binom{4\Omega}{N}. \tag{28}$$

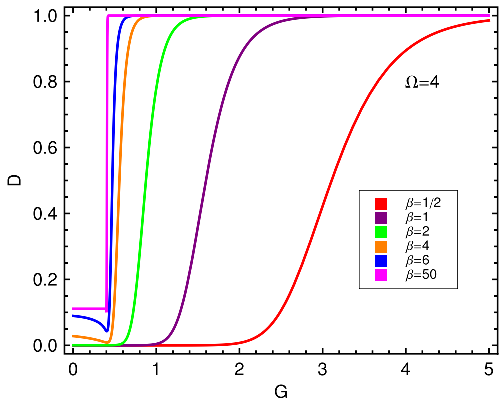


**Fig. 1.** Disequilibrium $D$ versus $G$ for different values of $\beta = 1/2$ (red), 1, 2, 4, 6, 50 (magenta) for $\Omega = 4$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

LMC's disequilibrium is then

$$D = \sum_{J,Q} \sum_{M=-J}^{M=J} [P(M,Q) - P_{unif}]^2, \tag{29}$$

while the entropy $S$ becomes

$$S = -\sum_{J,Q} \sum_{M=-J}^{M=J} P(M,Q) \ln P(M,Q), \tag{30}$$

so that our canonical complexity is $C = SD$.

Two very important quantities are (1) the effective superconductivity index $X$ [Cf. Eq. (10)]

$$X = \sum_{J,Q} \sum_{M=-J}^{M=J} P(M,Q)[(N-\nu)/N], \tag{31}$$

which is unity for a perfect superconductor and vanishes for the unperturbed system (no pairing interaction), and (2) the average quasi-spin seniority $\langle \nu \rangle$

$$\langle \nu \rangle = \sum_{J,Q} \sum_{M=-J}^{M=J} P(M,Q)\nu, \tag{32}$$

which yields the average number of unpaired fermions. Remember that the critical coupling constant is $G_{crit} = 2/(\Omega + 1)$.

## 4. Results

In Fig. 1 we plot, for different $\beta$-values, $D$ vs. $G$ for $\Omega = 4$ ($G_{crit} = 0.4$). At low temperatures $T$ the sharp superconductivity (SC) phase transition (PT) is clearly delineated. As $T$ grows the PT blurs and it takes larger $G$'s to observe it. The important point is that it does take place, leaving thus room to speak of "high temperature SC" [18]. For these same temperatures we depict in Fig. 2 $C$ vs. $G$ for $\Omega = 4$. We clearly appreciate that the statistical complexity becomes maximal at the critical coupling constant, if $T$ is low enough.

The average quasi-seniority $\langle \nu \rangle$ is plotted vs. $G$ for $\Omega = 10$ ($G_{crit} \approx 0.18$) — see Fig. 3. Again, the phase transition is clearly seen. A very important new feature is to be noticed, though. There are excited coupled pairs of fermions for $G < G_{crit}$ (partial order). That they become excited is due to $T$-effects. This is a *new effect* in the sense that here temperature becomes an "ordering" device, contrary to conventional wisdom. It is worth repeating that excited pairs denote order (the SC state is the most ordered one, and for it $D$ becomes maximal). These observations are reinforced by the graph of Fig. 4, that plot the SC index $X$ for the same temperature range.

## 5. Conclusions

We have seen here that, by statistically dealing with an old exactly solvable superconductivity model, rather interesting insights are gained:

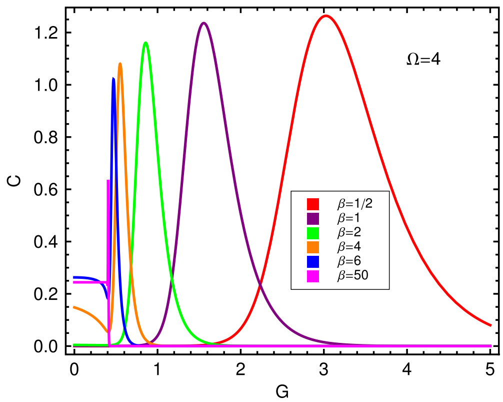


**Fig. 2.** Statistical complexity $C$ versus $G$ for different values of $\beta = 1/2$ (red),1, 2, 4, 6, 50 (magenta) for $\Omega = 4$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

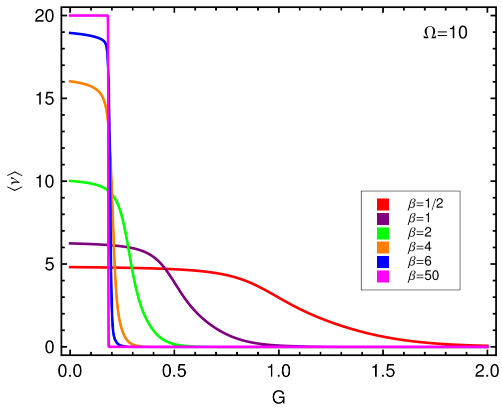


**Fig. 3.** Quasi-spin seniority $\langle \nu \rangle$ versus $G$ for different values of $\beta = 1/2$ (red), 1, 2, 4, 6, 50 (magenta) for $\Omega = 10$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

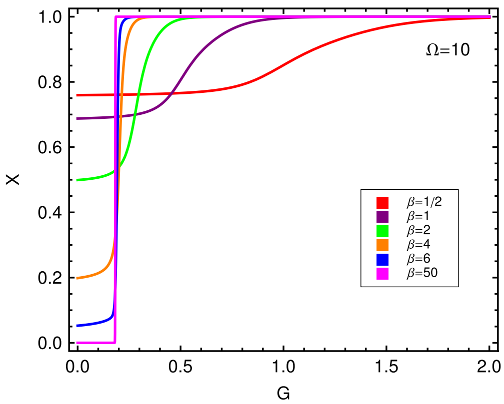


**Fig. 4.** Superconductivity index $X$ versus $G$ for different values of $\beta = 1/2$ (red), 1, 2, 4, 6, 50 (magenta) for $\Omega = 10$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

- The statistical complexity and the disequilibrium are good statistical indicators of the onset of superconductivity (SC).
- $C$ is maximal at the phase transition from the unperturbed ground state to the superconducting one, even at finite temperatures.
- $D$ is maximal for the superconducting state, even at finite temperatures.

- In this model, high $T$-SC naturally emerges in statistical fashion. This might be an artifact of our model or maybe something else.
- We have unexpectedly seen that temperature might be, in special circumstances, an ordering agent, as Fig. 3 clearly indicates.
- Note that there are, in our model, two ways of arriving at high $T$ SC: (1) for $G < G_{crit}$ due to temperature induced fluctuations and (2) for $G > G_{crit}$ due to the persistence of SC at high temperature.

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