Finite growth representations of conformal Lie algebras that contain a Virasoro subalgebra.

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Abstract

In the present paper we classify all finite growth representations of all infinite rank conformal subalgebras of gc_N that contain a Virasoro subalgebra.

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1 Introduction

Since the pioneering papers [1, 3], there has been a great deal of work towards understanding of the algebraic structure underlying the notion of the operator product expansion (OPE) of chiral fields of a conformal field theory. The singular part of the OPE encodes the commutation relations of fields, which leads to the notion of a Lie conformal algebra [8]. In the past few years a structure theory [16], representation theory [17, 18] and comohology theory [4] of finite Lie conformal algebras has been developed. The associative conformal algebra Cend_N and the corresponding general Lie conformal algebra gc_N are the most important examples of simple conformal algebras which are not finite ([8], Section 2.10).

Recall an *associative conformal algebra* R is defined as a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map,

$$R \otimes R \to \mathbb{C}[\lambda\,] \otimes R, \quad a \otimes b \to a_\lambda b,$$

called the λ-product , and satisfying the following axioms $(a, b, c \in R)$,

(A1) $(\partial a)_\lambda b = -\lambda a_\lambda b$, $a_\lambda(\partial b) = (\lambda + \partial) a_\lambda b$,

(A2) $a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c$.

A *conformal Lie algebra* R is a $\mathbb{C}[\partial\,]$-module endowed with a $\mathbb{C}$-linear map $R \otimes R \to \mathbb{C}[\lambda\,] \otimes R$, $a \otimes b \to [a_\lambda b]$ called the λ-bracket , and satisfying the following axioms $(a, b, c \in R)$,

(C1) $[(\partial a)_\lambda b\,] = -\lambda[a_\lambda b\,]$,

(C2) $[a_\lambda b\,] = -[b_{-\partial-\lambda} a\,]$,

(C3) $[a_\lambda [b_\mu c]\,] = [[a_\lambda b]_{\lambda+\mu} c\,] + [b_\mu [a_\lambda c]\,]$.

In general, given any associative conformal algebra R with λ-product $a_\lambda b$, the λ-bracket defined by

$$[a_\lambda b] := a_\lambda b - b_{-\partial-\lambda} a$$

makes R a Lie conformal algebra.

A *module* M over a conformal Lie algebra R is a $\mathbb{C}[\partial\,]$-module endowed with a $\mathbb{C}$-linear map $R \otimes M \to \mathbb{C}[[\lambda\,]] \otimes M$, $a \otimes v \to a_\lambda^M v$, satisfying the properties $(a, b \in R, v \in M)$,

(M1) $(\partial a)_\lambda^M\, v = -\lambda\, a_\lambda^M v$, (M2) $[a_\lambda^M, b_\mu^M]\, v := [a_\lambda b]_{\lambda+\mu}^M\, v = a_\lambda^M (b_\mu^M v) - b_\mu^M (a_\lambda^M v)$.

A module M over a conformal Lie algebra R is called a *conformal module* if $a_\lambda^M\, v \in R \otimes \mathbb{C}[\lambda]$ for all $a \in R$, $v \in M$ and it called *finite*, if it has a finite rank as a $\mathbb{C}[\partial]$-module.

Remark. (a) If R is a conformal Lie algebra, we have that the λ-bracket is of the form $[a_\lambda b] = \sum_{n\in\mathbb{Z}_+} \lambda^{(n)} a_{(n)} b$ for all $a, b \in R$, where $a_{(n)}b$ is called the n-product such that $a_{(n)}b = 0$, $n \gg 0$ and $\lambda^{(n)} = \lambda^n/n!$ Therefore, we can define a conformal Lie algebra R giving $\mathbb{C}$-bilineal products $a_{(n)}b$ for all $n \in \mathbb{Z}_+$, $a, b \in R$, such that satisfy equivalent axioms to $(C1) - (C3)$ (see [8]).

(b) Similarly, a conformal module M over a conformal algebra R, can be defined giving $\mathbb{C}$-bilineal actions $a_{(n)}v$ for all $n \in \mathbb{Z}_+$, $a \in R$, $v \in M$ such that, $a_{(n)}v = 0$, $n \gg 0$ that satisfy equivalent axioms to $(M1) - (M3)$.

Given two $\mathbb{C}[\partial]$-modules M and N, a *conformal linear map* from M to N is a $\mathbb{C}$-linear map $\tau : M \to \mathbb{C}[\lambda] \otimes_{\mathbb{C}} N$, denoted by $v \to \tau_\lambda(v)$, such that $\tau_\lambda(\partial^M v) = (\lambda + \partial^N)\tau_\lambda(v)$. The vector space of all such maps, denoted by $\mathrm{Chom}(M,\, N)$, is a $\mathbb{C}[\partial]$-module with $(\partial\tau)_\lambda(v) := -\lambda\tau_\lambda(v)$. Now, we consider $\mathrm{Cend}\, M := \mathrm{Chom}(M,\, M)$, and provided that M is a finite $\mathbb{C}[\partial]$-module, $\mathrm{Cend}\, M$ has a canonical structure of an associative conformal algebra defined by

$$(\tau_\lambda \sigma)_\mu v = \tau_\lambda(\sigma_{\mu-\lambda}\, v), \qquad \tau, \sigma \in \mathrm{Cend}\, M,\ v \in M.$$

The Lie conformal algebra associated to $\mathrm{Cend}\, M$ is called the *general conformal Lie algebra* and denoted by $\mathrm{gc}\, M$.

Remark. Observe that, by definition, a structure of conformal module over an associative conformal algebra in a finite $\mathbb{C}[\partial\,]$-module V is the same as a homomorphism of R to the associative conformal algebra $\mathrm{Cend}\, V$.

For any positive integer N, we set $\mathrm{Cend}_N := \mathrm{Cend}\, \mathbb{C}[\partial\,]^N$.

Cend_N can also be viewed as the associative conformal algebra associated to the associative algebra $\mathcal{D}_{as}^N$ of all $N \times N$ matrix valued regular differential operators on the circle (see[8], Section 2.10.). That is, we consider the conformal algebra of $\mathcal{D}_{as}^N$,

$$\mathrm{Conf}(\mathcal{D}_{as}^N) := \oplus_{n\in\mathbb{Z}_+} \mathbb{C}[\partial] J^n \otimes \mathrm{Mat}_N\, \mathbb{C}$$

with λ-product given by

$$J_A^k{}_\lambda J_B^l = \sum_{j=0}^{k} \binom{k}{j} (\lambda + \partial)^j\, J_{AB}^{k+l-j},$$

where $J^k_A = J^k \otimes A := \sum_{n\in\mathbb{Z}} t^n(-\frac{d}{dt})^k z^{-n-1} \otimes A$, with $A \in \mathrm{Mat}_N\,\mathbb{C}$.

Given $\alpha \in \mathbb{C}$, the natural representation of $\mathcal{D}^N_{as}$ on $\mathrm{e}^{-\alpha t}\mathbb{C}^N[t, t^{-1}]$ gives rise a conformal module structure on $\mathbb{C}[\partial]^N$ over $\mathrm{Conf}(\mathcal{D}^N_{as})$, with λ-action

$$J^m_{A\,\lambda} v = (\lambda + \partial + \alpha)^m\, Av, \qquad m \in \mathbb{Z}_+,\, v \in \mathbb{C}^N. \tag{1}$$

Now, using the Remark above, we obtain a natural homomorphism of conformal associative algebras from $\mathrm{Conf}(\mathcal{D}^N_{as})$ to Cend_N, wich turns out to be an isomorphism ([8] Proposition 2.10), where the functor Conf was introduced in [8], Charter 2 to associate an associative conformal algebra to a given associative algebra.

Similarly, the general conformal Lie algebra gc_N associated to Cend_N can also be viewed as the conformal Lie algebra associated to the Lie algebra $\mathcal{D}^N$, where $\mathcal{D}^N$ is the Lie algebra associated to the associative algebra $\mathcal{D}^N_{as}$.

Also gc_N can be identified by $\mathrm{Mat}_N\mathbb{C}[\partial, x]$, with λ-bracket given by (see Refs. [5] and [8])

$$[A(\partial, x)_\lambda B(\partial, x)] = A(-\lambda, x+\lambda+\partial)B(\lambda+\partial, x) - B(\lambda+\partial, -\lambda+x)A(-\lambda, x).$$

Recall that the *Virasoro conformal algebra* is defined as the free $\mathbb{C}[\partial]$-module of rank 1 generated by an element L, with λ-bracket defined by

$$[L_\lambda L] = (2\lambda + \partial)L,$$

and extended to $\mathbb{C}[\partial]L$ using sesquilinearity. Observe that all Virasoro subalgebras of gc_N are generated by

$$L = (x + \alpha\partial)I, \alpha \in \mathbb{C} \text{ and } I \text{ the } N \times N \text{ identity matrix}$$

The complete list of infinite rank proper subalgebras of gc_N that contain a Virasoro subalgebra is (see Remark 6.5 in Ref. [5] and Remark 3.10 in Ref. [6])

$$gc_{N,xI} = xI\,\mathrm{Mat}_N\mathbb{C}[\partial, x],$$
$$oc_N = \{A(\partial, x) - A(\partial, -\partial - x) :\, A(\partial, x) \in \mathrm{Mat}_N\mathbb{C}[\partial, x]\},$$
$$spc_{N,xI} = \{xI[A(\partial, x) + A(\partial, -\partial - x)] :\, A(\partial, x) \in \mathrm{Mat}_N\mathbb{C}[\partial, x]\},$$

where the Virasoro element is $L = (x + \alpha\partial)I$ with $\alpha = 0, \frac{1}{2}, 0$, respectively. To study the finite growth representations over these algebras, we used the following results, which relate modules over a conformal Lie algebra and modules over its annihilation Lie algebra. The *affinization* of a conformal Lie algebra R is the conformal algebra

$$\widetilde{R} = R[t,\, t^{-1}] := R \otimes \mathbb{C}[t, t^{-1}]$$

with $\widetilde{\partial} = \partial \otimes 1 + 1 \otimes \partial_t$ and n-product is defined by $(a, b \in R,\ f, g \in \mathbb{C}[t, t^{-1}], n \in \mathbb{Z}_+)$ (cf. [8])

$$(a \otimes f)_{(n)}(b \otimes g) = \sum_{j \in \mathbb{Z}_+} a_{(n+j)} b \otimes ((\partial^t f) g). \tag{2}$$

Letting $a_n = a \otimes t^n$, formula (2) becomes $(m, n \in \mathbb{Z})$

$$(a_m)_{(k)}(b_m) = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(k+j)} b)_{m+n-j}. \tag{3}$$

Letting

$$\mathrm{Lie}\, R = \widetilde{R} / \widetilde{\partial} \widetilde{R}$$

with the bracket induced by the 0-product on $\widetilde{R}$, (and keeping the notation a_n for its image in $\mathrm{Lie}\, R$) we obtain the *Lie algebra* associated to the conformal algebra R.

Remark. It is clear from (3), that $-1 \otimes \partial_t$ is a derivation of the 0-product of the conformal algebra $\widetilde{R}$. Since this operator commutes with $\widetilde{\partial}$, it induces a derivation T of the Lie algebra $\mathrm{Lie}\, R$, given by the formula

$$T(a_n) = -n a_{n-1}.$$

From the definition of Lie bracket on $\widetilde{R}$ is follows that

$$(\mathrm{Lie}\, R)_- = span\{a_n : a \in R,\ n \in \mathbb{Z}_+\},$$

is a Lie subalgebra of Lie R, this is called the *annihilation algebra.* Is clear that $(\mathrm{Lie}\, R)_-$ is T-invariant, then we can consider the direct sum $(\mathrm{Lie}\, R)^- = \mathbb{C}T \oplus (\mathrm{Lie}\, R)_-$, which is a Lie algebra called the *extended annihilation algebra.*

Then we have the following result (cf. [8], Remark 2.9a), a module M over a conformal algebra R is the same as a module over the extended annihilation algebra. This R-module is conformal iff the following property holds:

$$a_n v = 0, \quad a \in R, v \in M,\ n \gg 0.$$

Therefore our problem reduces to the study of finite growth representations of the corresponding extended annihilation algebras, which are certain subalgebras of $\mathcal{D}^N$ (see Ref. [5]). The main tools used here are the results (Refs. [11], [12],[13] and [14]) on the classification of quasifinite highest weight modules over the central extension of $\mathcal{D}^N$ and some of its subalgebras. The paper is organized as follows, in Sec. 2 we describe the infinite rank Lie algebra $\widehat{g\ell}_\infty^{[m]}$ and its classical subalgebras, and discuss their representation theory. In Secs. 3-6, we obtain the classification of all finite growth representations of gc_N, $gc_{N,xI}$, oc_N, and $spc_{N,xI}$ respectively.

2 Lie algebra $\widehat{g\ell}_\infty^{[m]}$ and its classical subalgebras

2.1 Lie algebra $\widehat{g\ell}_\infty^{[m]}$

Let $\mathbb{C}^{+\infty}$ be set of all sequences $\lambda = (\lambda_1, \lambda_2, \cdots)$ for which all but a finite number of $\lambda_i{}'$ s are zero, and $d(\lambda)$ the number of nonzero $\lambda_i{}'$ s and $|\lambda|$ be their sum. Denote by Par^+ the subset of $\mathbb{C}^{+\infty}$ consisting of nonincreasing sequences of non-negative integers and denote by $\mathrm{g}\ell_{+\infty}$ the Lie algebra of all matrices $(a_{i,j})_{i,j=1}^{+\infty}$ with a finite number of nonzero entries $a_{i,j} \in \mathbb{C}$. Given $\lambda \in \mathbb{C}^{+\infty}$, there exists a unique irreducible $\mathrm{g}\ell_{+\infty}$-module $L^+(\lambda)$, also denoted by $L(\mathrm{g}\ell_{+\infty}, \lambda)$, which admits a nonzero vector v_λ such that

$$E_{i,j} v_\lambda = 0 \quad \text{for} \quad i < j \quad \text{and} \quad E_{i,i} v_\lambda = \lambda_i v_\lambda. \tag{4}$$

Here and further $E_{i,j}$ denotes, as usual, the matrix whose (i,j)-entry is 1 and all other entries are 0. Each $L^+(\lambda)$ has a unique $\mathbb{Z}_+$-gradation. $L^+(\lambda) = \bigoplus_{j \in \mathbb{Z}_+} L^+(\lambda)_j$, called its *principal gradation*, which satisfies the properties

$$L^+(\lambda)_0 = \mathbb{C} v_\lambda, \quad E_{i,j} L^+(\lambda)_k \subset L^+(\lambda)_{k+i-j}.$$

Since $\lambda \in \mathbb{C}^{+\infty}$, it is easy to see that $\mathrm{dim} L^+(\lambda)_j < \infty$, hence we can define the q-character

$$ch_q L^+(\lambda) = \sum_{j \in \mathbb{Z}_+} (\mathrm{dim} L^+(\lambda)_j) q^j.$$

For $\lambda \in \mathrm{Par}^+$, let $d = d(\lambda)$ and $\bar{\lambda} = (\lambda_1, \cdots, \lambda_d)$. Let $g\ell_d$ be the Lie algebra of all $d \times d$ matrices $(a_{i,j})_{i,j=1}^d$; it may be viewed as subalgebra of $\mathrm{g}\ell_{+\infty}$ in a natural way. Denote by $\bar{L^+}(\bar{\lambda})$ the (irreducible) $g\ell_d$-submodule of $L^+(\lambda)$ generated by v_λ. It is, of course, isomorphic to the finite-dimensional irreducible $g\ell_d$-module associated to $\bar{\lambda}$, so that its q-character is a (well-known) polynomial in q.

Lemma 1. *Let $\lambda \in Par^+$, $d = d(\lambda)$. Then*

$$ch_q L^+(\lambda) = ch_q \bar{L^+}(\bar{\lambda}) / \prod_{j=1}^{d} (1 - q^j)_q^{\lambda_d - j + 1},$$

where $(1-a)_q^m = (1-a)(1-qa)\cdots(1-q^{m-1}a)$.

Proof. See Lemma 2.1 in [15]. ☐

Recall that given a vector space V with an increasing filtration by finite-dimensional subspaces $V_{[j]}$, the *growth* of V is defined by (Cf. [15])

$$\text{growth}V = \overline{\lim}_{j\to\infty}(\log \dim V_{[j]})/\log j.$$

We define the growth of $L^+(\lambda)$ using its filtration $L^+(\lambda)_{[j]} = \bigoplus_{i\leq j} L^+(\lambda)_i$ associated to the principal gradation. In Theorem 2.2 [15], it was used the Lemma above to prove the following Theorem,

Theorem 1. *(a) If $\lambda \in Par^+$, then*

$$growth L^+(\lambda) = |\lambda|.$$

(b)If $\lambda \in \mathbb{C}^{+\infty} \setminus Par^+$, then $growth L^+(\lambda) = \infty$.

In a similar fashion one may consider the Lie algebra $\mathrm{g}\ell_{-\infty}$ of all matrices $(a_{i,j})_{i,j=0}^{-\infty}$ with a finite number of nonzero entries and the irreducible $\mathrm{g}\ell_{-\infty}$-modules $L^-(\lambda)$, also denoted by $L(\mathrm{g}\ell_{-\infty};\lambda)$, parameterized by the set $\mathbb{C}^{-\infty}$ of sequences $\lambda = (\cdots, \lambda_{-1}, \lambda_0)$ with finitely many nonzero entries. Results similar to Lemma 1 and Theorem 1 hold for the subset $\mathrm{Par}^- \subset \mathbb{C}^{-\infty}$ consisting of nondecreasing sequences of (nonpositive) integers. Let $\widetilde{\mathrm{g}\ell}_\infty$ denote the Lie algebra of all matrices $(a_{i,j})_{i,j\in\mathbb{Z}}$ such that $a_{i,j} = 0$ if $|i-j| \gg 0$. Denote by $\widetilde{\mathrm{g}\ell}_{+\infty}$ (respectively $\widetilde{\mathrm{g}\ell}_{-\infty}$) the subalgebra of $\widetilde{\mathrm{g}\ell}_\infty$ consisting of matrices with $a_{i,j} = 0$ for i or $j \leq 0$ (respectively, i or $j > 0$). Note that these two subalgebras commute and that $\widetilde{\mathrm{g}\ell}_{\pm\infty}$ contains $\mathrm{g}\ell_{\pm\infty}$ as a subalgebra. Note also $\mathrm{g}\ell_{\pm\infty}$-modules $L^\pm(\lambda)$ extended uniquely to $\widetilde{\mathrm{g}\ell}_{\pm\infty}$. The Lie algebra $\widetilde{\mathrm{g}\ell}_\infty$ has a well-known central extension $\widehat{\mathrm{g}\ell}_\infty = \widetilde{\mathrm{g}\ell}_\infty + \mathbb{C}C$ by $\mathbb{C}$ defined by the cocycle

$$\alpha(A,B) = tr[J,A]B, \quad \text{where } J = \sum_{i\leq 0} E_{i,i}. \tag{5}$$

The restriction of this cocycle to $\widetilde{\mathrm{g}\ell}_{+\infty}$ and to $\widetilde{\mathrm{g}\ell}_{-\infty}$ is zero. We will also need briefly the Lie algebra $\widehat{g\ell}_\infty^{[m]}$ defined for each $m \in \mathbb{Z}_+$ by replacing $\mathbb{C}$ by $R_m = \mathbb{C}[u]/u^{m+1}$. That is, $\widehat{g\ell}_\infty^{[m]} = \widetilde{g\ell}_\infty^{[m]} \oplus R_m$ is the central extension of $\widetilde{g\ell}_\infty^{[m]}$ by the 2-cocycle (5) with values in R_m, where $\widetilde{g\ell}_\infty^{[m]}$ is the Lie algebra of infinite matrices with finitely many nonzero diagonals with entries in R_m. The principal $\mathbb{Z}$-gradation of all above Lie algebras are defined by letting

$$\deg E_{i,j} = i - j \tag{6}$$

(in the case of $\widehat{g\ell}_\infty^{[m]}$ we also let deg$R_m = 0$). This give us a triangular decomposition

$$\widehat{g\ell}_\infty^{[m]} = (\widehat{g\ell}_\infty^{[m]})_+ \oplus (\widehat{g\ell}_\infty^{[m]})_0 \oplus (\widehat{g\ell}_\infty^{[m]})_- ,$$

where

$$(\widehat{g\ell}_\infty^{[m]})_\pm = \oplus_{j\in\mathbb{N}} (\widehat{g\ell}_\infty^{[m]})_{\pm j}.$$

The Lie algebra $\widehat{g\ell}_\infty$ has a family of modules $L(\widehat{g\ell}_\infty;\lambda,c)$, parameterized by $\lambda \in \mathbb{C}^\infty = \{(\lambda_i)_{i\in\mathbb{Z}} : \text{all but finitely many of } \lambda_i \text{ are } 0\}$ and $c \in \mathbb{C}$ defined by (4) and $Cv_\lambda = cv_\lambda$. Similarly $\widehat{g\ell}_\infty^{[m]}$ has a family of modules $L(\widehat{g\ell}_\infty^{[m]};\vec{\lambda},\vec{c})$ where $\vec{\lambda} \in (\mathbb{C}^\infty)^{m+1}$, $c \in \mathbb{C}^{m+1}$, defined in a similar fashion. That is, the highest weight $\widehat{g\ell}_\infty^{[m]}$-module $L(\widehat{g\ell}_\infty^{[m]};\Lambda)$, with highest weight $\Lambda \in (\widehat{g\ell}_\infty^{[m]})_0^*$ that is determined by its *labels* $\vec{\lambda}_i^{(j)} = \Lambda(u^j E_{i,i})$ and the *central charges* $\vec{c_j} = \Lambda(u^j)$. The gradation (6) is obviously consistent with the principal gradation of $L^\pm(\lambda)$ and of $L(\widehat{g\ell}_\infty;\lambda,c)$.

2.2 Lie algebras $b_\infty^{[m]}$, $c_\infty^{[m]}$ and $d_\infty^{[m]}$

The Lie algebra $\widetilde{g\ell}_\infty^{[m]}$ acts on the vector space $R_m[t,t^{-1}]$ via the usual formula

$$E_{i,j}v_k = \delta_{j,k}v_i,$$

where $v_i = t^{-i}$, $i \in \mathbb{Z}$ is a basis of $R_m[t,t^{-1}]$ over R_m. Now consider the following $\mathbb{C}$-bilinear forms on this spac:

$$\begin{aligned} B(u^{\widetilde{m}}v_i, u^n v_j) &= u^{\widetilde{m}}(-u)^n\,\delta_{i,-j}, \\ C(u^{\widetilde{m}}v_i, u^n v_j) &= u^{\widetilde{m}}(-u)^n(-1)^i\delta_{i,1-j}, \\ D(u^{\widetilde{m}}v_i, u^n v_j) &= u^{\widetilde{m}}(-u)^n\,\delta_{i,1-j}. \end{aligned} \tag{7}$$

Denote by $\bar{b}_\infty^{[m]}$ (respectively $\bar{c}_\infty^{[m]}$, and $\bar{d}_\infty^{[m]}$) the Lie subalgebra of $\widetilde{g\ell}_\infty^{[m]}$ which preserves the bilinear form B(respectively C and D). We have

$$\bar{b}_\infty^{[m]} = \left\{(a_{i,j}(u))_{i,j\in\mathbb{Z}} \in \widetilde{g\ell}_\infty^{[m]} \,:\, a_{i,j}(u) = -a_{-j,-i}(-u)\right\},$$

$$\bar{c}_\infty^{[m]} = \{(a_{i,j}(u))_{i,j\in\mathbb{Z}} \in \widetilde{\mathrm{g}\ell}_\infty^{[m]} \,|\, a_{i,j}(u) = (-1)^{i+j+1}a_{1-j,\,1-i}(-u)\},$$

$$\bar{d}_\infty^{[m]} = \left\{(a_{i,j}(u))_{i,j\in\mathbb{Z}} \in \widetilde{g\ell}_\infty^{[m]} \,:\, a_{i,j}(u) = -a_{1-j,\,1-i}(-u)\right\}.$$

Denote by $b_\infty^{[m]} = \bar{b}_\infty^{[m]} \oplus R_m$ (respectively, $c_\infty^{[m]} = \bar{c}_\infty^{[m]} \oplus R_m$ and $d_\infty^{[m]} = \bar{d}_\infty^{[m]} \oplus R_m$) the central extension of $\bar{b}_\infty$ (respectively, $\bar{c}^{[m]}$ and $\bar{d}_\infty^{[m]}$) given by the 2-cocycle defined in $\widetilde{g\ell}_\infty^{[m]}$. Both subalgebras inherit the form $\widehat{g\ell}_\infty^{[m]}$ the principal $\mathbb{Z}$-gradation and the triangular decomposition, (see Refs. for notation [8] and [19]).

$$
\begin{aligned}
b_\infty^{[m]} &= \oplus_{j\in\mathbb{Z}}(b_\infty^{[m]})_j, \quad b_\infty^{[m]} = (b_\infty^{[m]})_+ \oplus (b_\infty^{[m]})_0 \oplus (b_\infty^{[m]})_-, \\
c_\infty^{[m]} &= \oplus_{j\in\mathbb{Z}}(c_\infty^{[m]})_j \qquad c_\infty^{[m]} = (c_\infty^{[m]})_+ \oplus (c_\infty^{[m]})_0 \oplus (c_\infty^{[m]})_-, \\
d_\infty^{[m]} &= \oplus_{j\in\mathbb{Z}}(d_\infty^{[m]})_j, \quad d_\infty^{[m]} = (d_\infty^{[m]})_+ \oplus (d_\infty^{[m]})_0 \oplus (d_\infty^{[m]})_-.
\end{aligned}
$$

In particular when $m = 0$, we have the usual Lie subalgebras of $\widehat{g\ell}_\infty$, denoted by b_∞ (respectively, c_∞ and d_∞). Denote by $L(b_\infty^{[m]};\lambda)$ [respectively, $L(c_\infty^{[m]};\lambda)$ and $L(d_\infty^{[m]};\lambda)$] the highest weight module over $b_\infty^{[m]}$ (respectively $c_\infty^{[m]}$ and $d_\infty^{[m]}$) with highest weight $\lambda \in (b_\infty^{[m]})_0^*$ (respectively $\lambda \in (c_\infty^{[m]})_0^*$ and $\lambda \in (d_\infty^{[m]})_0^*$) parameterized by ${}^b\vec{\lambda} \in (\mathbb{C}^\infty)^{m+1}$, $\vec{c} \in \mathbb{C}^{m+1}$, with

$$\vec{c_i} = \lambda(u^i),$$

$${}^b\vec{\lambda}_j^{(i)} = \lambda(u^i E_{j,j} - (-u)^i E_{-j,-j}),$$

[respectively ${}^c\vec{\lambda} \in (\mathbb{C}^\infty)^{m+1}$ ${}^c\lambda_j^{(i)} = \lambda(u^i E_{j,j} - (-u)^i) E_{1-j,1-j}$ and ${}^d\vec{\lambda} \in (\mathbb{C}^\infty)^{m+1}$, ${}^d\vec{\lambda}_j^{(i)} = \lambda(u^i E_{j,j} - (-u)^i E_{1-j,1-j})$]. The superscripts b, c and d here mean B, C and D type respectively. The ${}^b\vec{\lambda}_j^{(i)}$ (respectively ${}^c\vec{\lambda}_j^{(i)}$ and ${}^d\vec{\lambda}_j^{(i)}$) are called the labels and $\vec{c_j}$ the central charges of $L(b_\infty^{[m]};\lambda)$ [respectively, $L(c_\infty^{[m]};\lambda)$ and $L(d_\infty^{[m]};\lambda)$].

All these modules will appear in Sec.V. In Theorems 2.4 and 2.6 in [15], it was proved the following result. To do this, they used Lemmas 2.3 and 2.5, in [15] about the q-character of each one of the subalgebras of type B, C and D.

Theorem 2. *All non-trivial modules $L(\mathfrak{g}^{[m]};\lambda)$ have infinite growth, where $\mathfrak{g}^{[m]}$ can be $b_\infty^{[m]}$, $c_\infty^{[m]}$ or $d_\infty^{[m]}$.*

3 Irreducible finite growth gc_N-modules

Let $\mathcal{D}_-^N$ be the Lie algebra of matrix differential operators on $\mathbb{C}$. It consists of linear combinations of matrix differential operators of the form

$f(t)\left(\frac{d}{dt}\right)^m e_{i,j}$, where f is a polynomial, $m \in \mathbb{Z}_+$ and $e_{i,j}$ is the standard basis of $\mathrm{Mat}_N\,\mathbb{C}$, with $i, j \in \{1, \cdots, N\}$. In particular, $De_{i,j} := \left(t\frac{d}{dt}\right) e_{i,j} \in \mathcal{D}_-^N$. The principal $\mathbb{Z}$-gradation $\mathcal{D}_-^N = \oplus_{q\in\mathbb{Z}} \left(\mathcal{D}_-^N\right)_q$ is defined by letting

$$\deg t = -N, \;\; \deg \frac{d}{dt} = N, \;\text{ and } \deg e_{i,j} = j - i\,. \tag{8}$$

Given $\vec{\Delta} = \{\vec{\Delta}_n\}_{n\in\mathbb{Z}_+}$ with $\vec{\Delta}_n \in \mathbb{C}^N$ for all $n \in \mathbb{Z}_+$, we consider the highest weight module $L(\vec{\Delta}; \mathcal{D}_-^N)$ over $\mathcal{D}_-^N$ as the (unique) irreducible module that has a non-zero vector $v_{\vec{\Delta}}$ with the following properties:

$$\left(\mathcal{D}_-^N\right)_p v_{\vec{\Delta}} = 0 \text{ for } p < 0, \quad D^n e_{i,i} v_{\vec{\Delta}} = \Delta_n^i v_{\vec{\Delta}} \;\text{ for } n \in \mathbb{Z}_+,\, i = 1, \cdots, N\,.$$

The principal gradation of $\mathcal{D}_-^N$ induces the principal gradation $L(\vec{\Delta}; \mathcal{D}_-^N) = \oplus_{q\in\mathbb{Z}_+} L_q$ such that $L_0 = \mathbb{C}v_{\vec{\Delta}}$. The module $L(\vec{\Delta}; \mathcal{D}_-^N)$ is called *quasifinite* if $\dim L_q < \infty$ for all $q \in \mathbb{Z}_+$.

Quasifinite modules over $\mathcal{D}_-^N$ can be constructed as follows. Consider the natural action of $\mathcal{D}_-^N$ on $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}^N$, and the action of $\widetilde{g\ell}_\infty$ on $\mathbb{C}[t, t^{-1}]$ given by $E_{i,j} v_k = \delta_{j,k} v_i$, where $v_j = t^{-j}$ $(j \in \mathbb{Z})$ is a base of Laurent polynomials. Let $\varphi : \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}^N \to \mathbb{C}[t, t^{-1}]$ be the isomorphism defined by $e_i t^j \to t^{jN+i-1}$, where e_i with $i = 1, \cdots, N$ is the standard base of $\mathbb{C}^N$ (cf [11].) This gives an embedding of $\mathcal{D}_-^N$ in $\widetilde{g\ell}_\infty$. Since $\mathbb{C}[t] \otimes \mathbb{C}^N$ is $\mathcal{D}_-^N$-invariant, we get $\mathcal{D}_-^N$-modules $(\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}^N)/(\mathbb{C}[t] \otimes \mathbb{C}^N)$ and $\mathbb{C}[t] \otimes \mathbb{C}^N$, which gives us an embedding of $\mathcal{D}_-^N$ in $\widetilde{g\ell}_{+\infty}$ and $\widetilde{g\ell}_{-\infty}$ respectively, hence an embedding of $\mathcal{D}_-^N$ in $\widetilde{g\ell}_{+\infty} \oplus \widetilde{g\ell}_{-\infty}$. All these embeddings respect the principal gradations.

Here and further we will denote $\mathbb{C}^N[t, t^{-1}] := \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}^N$ and $\mathbb{C}^N[t] := \mathbb{C}[t] \otimes \mathbb{C}^N$.

Now take $\lambda^\pm \in \mathbb{C}^{\pm\infty}$ and consider the $\widetilde{g\ell}_{+\infty} \oplus \widetilde{g\ell}_{-\infty}$-module $L^+(\lambda^+) \otimes L^-(\lambda^-)$. The same argument as in [10], gives us the following.

Lemma 2. *When restricted to $\mathcal{D}_-^N$, the module $L^+(\lambda^+) \otimes L^-(\lambda^-)$ remains irreducible.*

It follows immediately that $L^+(\lambda^+) \otimes L^-(\lambda^-)$ is an irreducible highest weight module over $\mathcal{D}_-^N$, which is obviously quasifinite. It is easy to see that we have:

$$\Delta_n^i \;=\; \sum_{j\geq 1} (-j)^n \lambda^+_{jN-i+1} + \sum_{j\leq 0} (-j)^n \lambda^-_{jN-i+1}$$

so that

$$\Delta_i(x) : \;=\; \sum_{n\geq 0}\Delta_n x^n/n! = \sum_{j\geq 1}\lambda^+_{jN-i+1}e^{-jx} + \sum_{j\leq 0}\lambda^-_{jN-i+1}e^{-jx}\,.$$

with $i = 1, \cdots, N$. It is also clear that for $\lambda^\pm \in \mathrm{Par}^\pm$ we have (cf. Theorem 1(a)):

$$\mathrm{growth}\, L^+(\lambda^+)\otimes L^-(\lambda^-) = |\lambda^+| + |\lambda^-|\,.$$

We shall prove the following theorem.

Theorem 3. *The $\mathcal{D}^N_-$-modules $L^+(\lambda^+)\otimes L^-(\lambda^-)$, where $\lambda^\pm \in \mathrm{Par}^\pm$, exhaust all quasifinite irreducible highest weight $\mathcal{D}^N_-$-modules that have finite growth.*

Let $\mathcal{D}^N$ denote the Lie algebra of all matrix differential operators on $\mathbb{C}^\times$. The Lie algebra $\mathcal{D}^N$ is the linear span of matrix differential operators $f(t)\left(\frac{d}{dt}\right)^k A$, where $f(t) \in \mathbb{C}[t,t^{-1}]$, $k \in \mathbb{Z}_+$ and $A \in \mathrm{Mat}_N\mathbb{C}$, or equivalently of operators $t^k f(D)e_{i,j}$, where $f(D) \in \mathbb{C}[D]$, $k \in \mathbb{Z}$ and $e_{i,j}$ is the standard basis of $\mathrm{Mat}_N\mathbb{C}$, with $i,j \in \{1,\cdots,N\}$. Obviously, $\mathcal{D}^N_-$ is a subalgebra of $\mathcal{D}^N$, and the principal gradation extends from $\mathcal{D}^N_-$ to $\mathcal{D}^N$ in the obvious way.

The basic idea of the proof of Theorem 3 is the same as in [15]: to reduce the problem to the well developed (in [11]) representation theory of the universal central extension $\widehat{\mathcal{D}}^N$ of $\mathcal{D}^N$. Recall that the central extension $\widehat{\mathcal{D}}^N = \mathcal{D}^N \oplus \mathbb{C}C$ is defined by the cocycle [10].

$$\Psi\left(f(t)\left(\frac{d}{dt}\right)^m A,\, g(t)\left(\frac{d}{dt}\right)^n B\right) = \mathrm{Res}_0\,\frac{\mathrm{Tr}(AB)m!n!}{(m+n+1)!}f^{(n+1)}(t)g^{(m)}(t)dt, \tag{9}$$

where Tr is the usual trace. The principal gradation of $\mathcal{D}^N$ lifts to $\widehat{\mathcal{D}}^N$ by letting $\deg C = 0$. Note also that the restriction of the cocycle Ψ to $\mathcal{D}^N_-$ is zero.

Consider again $\varphi : \mathbb{C}^N[t,t^{-1}] \to \mathbb{C}[t,t^{-1}]$ be the isomorphism defined by $e_i t^j \to t^{jN+i-1}$. For each $s \in \mathbb{C}$ one defines a Lie algebra homomorphism $\varphi_s : \mathcal{D}^N \to \widetilde{\mathrm{g}\ell}_\infty$ (via the action of $\mathcal{D}^N$ on $t^s\,\mathbb{C}^N[t,t^{-1}]$) by

$$\varphi_s(t^k f(D)e_{i,j}) = \sum_{l\in\mathbb{Z}} f(-l+s)E_{(l-k)N-i+1,\,lN-j+1}\,. \tag{10}$$

This homomorphism lifts to a homomorphism of central extension $\widehat{\varphi}_s : \widehat{\mathcal{D}}^N \to \widehat{g\ell}_\infty$ by

$$\widehat{\varphi}_s|_{(\widehat{\mathcal{D}}^N)_j} = \varphi_s|_{(\widehat{\mathcal{D}}^N)_j} \text{ if } j \neq 0\,,$$

$$\widehat{\varphi}_s(e^x D e_{i,i}) = \varphi_s(e^{xD} e_{i,i}) - \frac{e^{sx}-1}{e^x-1}\,,$$
$$\widehat{\varphi}_s(C) = C \tag{11}$$

More generally, for each $m \in \mathbb{Z}_+$ one defines a homomorphism $\varphi_s^{[m]} : \mathcal{D}^N \to \widetilde{\mathrm{g}\ell}_\infty^{[m]}$ by

$$\varphi_s^{[m]}(t^k f(D) e_{i,j}) = \sum_{l\in\mathbb{Z}} f(-l+s+u) E_{(l-k)N-i+1,\, lN-j+1}\,, \tag{12}$$

which lifts to $\widehat{\varphi}_s^{[m]} : \widehat{\mathcal{D}}^N \to \widehat{g\ell}_\infty^{[m]}$ in a similar way,

$$\widehat{\varphi}_s^{[m]}|_{\left(\widehat{\mathcal{D}}^N\right)_j} = \varphi_s^{[m]}|_{\left(\widehat{\mathcal{D}}^N\right)_j} \text{ if } j \neq 0\,,$$

$$\widehat{\varphi}_s^{[m]}(e^{xD} e_{i,i}) = \varphi_s^{[m]}(e^{xD} e_{i,i}) - \frac{e^{sx}-1}{e^x-1} - \sum_{j=1}^{m} \frac{x^j e^{sx}}{e^x-1} t^j/j!,$$
$$\widehat{\varphi}_s^{[m]}(C) = C \tag{13}$$

One of the main results of [11] is the following.

Lemma 3. *For each $i = 1, \ldots, r$, pick a collection $m_i \in \mathbb{Z}_+$, $s_i \in \mathbb{C}$, $\vec{\lambda}_i \in (\mathbb{C}^\infty)^{m_i+1}$, $\vec{c}_i \in \mathbb{C}^{m_i+1}$, such that $s_i - s_j \notin \mathbb{Z}$ for $i \neq j$. Then the $\oplus_{i=1}^r \widehat{g\ell}_\infty^{[m_i]}$-module $\otimes_{i=1}^r L^{[m_i]}(\vec{\lambda}_i, \vec{c}_i)$ remains irreducible when restricted to $\widehat{\mathcal{D}}^N$ via the embedding $\oplus_{i=1}^r \widehat{\varphi}_{s_i}^{[m_i]} : \widehat{\mathcal{D}}^N \to \oplus_{i=1}^r \widehat{g\ell}_\infty^{[m_i]}$. All irreducible quasifinite highest weight $\widehat{\mathcal{D}}^N$-modules are obtained in this way.*

Proof of Theorem 3. Note that for $p \geq 1$ there exists a positive integer k such that $p = kN + r = (k+1)N - (N-r)$ with $0 \leqslant r \leqslant N-1$ One has:

$$\begin{aligned}\left(\mathcal{D}_-^N\right)_p = \{&t^{-k} f(D) e_{i,\,i+r} \;:\; f(0) = f(1) = \ldots = f(k-1) = 0,\\ &\qquad\qquad\qquad\qquad i = 1, \cdots, N-r\}\\ \bigcup (1-\delta_{r,0})\{&t^{-(k+1)} g(D) e_{i,\,i-N+r} \;:\; g(0) = g(1) = \ldots = g(k) = 0\\ &\qquad\qquad\qquad\qquad i = N-r+1, \cdots, N\}.\end{aligned} \tag{14}$$

Hence $\left(\mathcal{D}_-^N\right)_p$ has finite codimension in $\mathcal{D}_p^N$ and therefore the quasifiniteness of a $\mathcal{D}_-^N$-module $L(\vec{\Delta}; \mathcal{D}_-^N)$ implies the quasifiniteness of any of the $\widehat{\mathcal{D}}^N$-modules $L(\vec{\Delta}, c; \widehat{\mathcal{D}}^N)$. Due to Lemma 3 , $L(\vec{\Delta}, c; \widehat{\mathcal{D}}^N)$ is a tensor product of the $\widehat{g\ell}_\infty^{[m]}$-modules $L^{[m]}(\vec{\lambda},\vec{c})$ on which $\widehat{\mathcal{D}}^N$ acts via the embedding $\widehat{\varphi}_s^{[m]}$ defined by (12) and (13).

It is clear from Theorem 1 that all non-trivial modules $L^{[m]}(\vec{\lambda}_i,\vec{c}_i)$ have infinite growth (by choosing an appropriate subalgebra isomorphic to $\mathrm{g}\ell_{+\infty}$ in $\mathrm{g}\ell_\infty$).

Recall that for any quasifinite $\widehat{\mathcal{D}}^N$-module one can extend the action of $\left(\widehat{\mathcal{D}}^N\right)_p$ for $p \neq 0$ to $\left(\widehat{\mathcal{D}}^{N\mathcal{O}}\right)_p$, where $\mathcal{O}$ is the algebra of all holomorphic functions on $\mathbb{C}$ (see [11]), in other words, in (12) and in the central extension of (13) one can take any $f \in \mathcal{O}$ if $p \neq 0$. The same holds for $\mathcal{D}_-^N$, except that for $p \geq 1$, f must obey conditions in (14). We apply this to the $\widehat{\mathcal{D}}^N$-module $L^{[m]}(\vec{\lambda},\vec{c})$ on which $\widehat{\mathcal{D}}^N$ acts via $\widehat{\varphi}_s^{[m]}$. Choosing $f_1, f_2 \in \mathcal{O}$ such that if $q \in \mathbb{Z}$ and satisfies

(a) $q = k_1 N + r$, with $k_1 \in \mathbb{Z}$ and $0 < r \leq N-1$, then

$$f_1(-l+s) = \delta_{l-1,\,k_1},\ f_1^{(n)}(-l+s) = 0 \text{ if } n = 1,\ldots,m\,,$$

(b) $q = k_1 N$, with $k_1 \in \mathbb{Z}$, then

$$f_2(-l+s) = \delta_{l,\,k_1},\ f_2^{(n)}(-l+s) = 0 \text{ if } n = 1,\ldots,m\,,$$

we see from (12) that all operators $E_{q+1,q}$ lie in the image of $\widehat{\varphi}_s^{[m]}(\mathcal{D}_-^{N\mathcal{O}})$, except for $E_{1,0}$ when $s = 0$ (here we use (14) for $p = 1$). Hence, when restricted to $\mathcal{D}_-^N$, the module $L^{[m]}(\vec{\lambda},\vec{c})$ remains irreducible, provided that $s \neq 0$. Thus, if $L(\vec{\Delta};\mathcal{D}_-^N)$ has finite growth, then $L(\vec{\Delta};\widehat{\mathcal{D}}^N) = L^{[m]}(\vec{\lambda},\vec{c})$ on which $\widehat{\mathcal{D}}^N$ acts via the embedding $\widehat{\varphi}_0^{[m]}$.

Let q as in (a), choosing $f_1 \in \mathcal{O}$ to vanish in all $l \in \mathbb{Z}$ up to m^{th} derivative except for i^{th} derivative ($0 < i \leq m$) at $l = k_1+1$, and if q as in (b) choosing $f_2 \in \mathcal{O}$ to vanish in all $l \in \mathbb{Z}$ up to m^{th} derivative except for i^{th} derivative ($0 < i \leq m$) at $l = k_1$ we see that all operators $u^i E_{q+1,q}$ with $0 < i \leq m$ lie in the image of $\hat{\varphi}_s^{[m]}(\mathcal{D}_-^{N\mathcal{O}})$.

Suppose that the m^{th} coordinate of $\vec{\lambda}_q$ is non-zero, and that $m > 0$. Then $v := (u^m E_{q+1,q})^n v_{\vec{\lambda}} \neq 0$ for all $n > 0$. But

$$E_{q,q}v = (-N+\lambda_q^0)v,\ E_{q+1,\,q+1}v = (N+\lambda_{q+1}^0)v.$$

Therefore, restricting to the subalgebra of $\mathrm{g}\ell_\infty$ consisting of matrices $(a_{i,j})_{i,j\leq q}$ or $(a_{i,j})_{i,j\geq q+1}$ we conclude by Theorem 1, that $L^{[m]}(\vec{\lambda},\vec{c})$ is either trivial or is of infinite growth.

Thus, the only possibility that remains is $s = m = 0$. As has been already shown, the image of $\widehat{\varphi}_s(\mathcal{D}_-^{N\mathcal{O}})$ contains all $E_{q+1,q}$ except for $E_{1,0}$, hence it contains all operators from $g\ell_{-\infty} \oplus g\ell_{+\infty}$. Therefore, by Theorem 1, the highest weight of a finite growth $\mathcal{D}_-^N$-module must be the same as one of the $\mathcal{D}_-^N$-modules $L^+(\lambda^+) \otimes L^-(\lambda^-)$ with $\lambda^\pm \in \mathrm{Par}^\pm$.

□

Given two partitions $\lambda^\pm \in \mathrm{Par}^\pm$, we denote by $L(\lambda^+, \lambda^-)$ the $\mathcal{D}_-^N$-module, obtained by restriction via φ_0 from the $\widetilde{g\ell}_{+\infty} \oplus \widetilde{g\ell}_{-\infty}$-module $L^+(\lambda^+) \otimes L^-(\lambda^-)$. Now we shall construct the $\mathcal{D}_-^N$-modules $L(\lambda^+, \lambda^-)$ explicitly.

Consider the $\mathcal{D}_-^N$-module $\mathbb{C}^N[t, t^{-1}]$. Then $\mathbb{C}^N[t\,]$ is the maximal submodule (which is irreducible). Hence the $\mathcal{D}_-^N$-module

$$V := \mathbb{C}^N[t, t^{-1}]/\mathbb{C}^N[t\,] \tag{15}$$

is irreducible. It is clear that this is the highest weight $\mathcal{D}_-^N$-module of growth 1 with a highest weight vector $(t^{-1} + \mathbb{C}[t])e_N$, where e_N is a vector in $\mathbb{C}^N$ which has 1 in the N-entry and zero in the other entries . It is immediate to deduce that V is isomorphic to $L(\omega_1, 0)$ where $\omega_1 \in \mathrm{Par}^+$, such that $\omega_1^i = 0$, for $i \neq 1$ and $\omega_1^1 = 1$.

Likewise, the $\mathcal{D}_-^N$-module $\mathbb{C}^N[t\,]^* := (\mathbb{C}[t\,] \otimes \mathbb{C}^N)^* = \oplus_{j\in\mathbb{Z}_+}(\mathbb{C}t^j \otimes \mathbb{C}^N)^*$ is an irreducible highest weight module of growth 1 with a highest weight vector $(1 \otimes e_1)^*$, where e_1 is a vector in $\mathbb{C}^N$ which has 1 in the entry one and zero in the other entries. This module is isomorphic to $L(0, \omega_{-1})$, where $\omega_{-1} = (\dots, 0, -1) \in \mathrm{Par}^-$. We denote this $\mathcal{D}_-^N$-module by V'.

As in the Schur-Weyl theory, the $\mathcal{D}_-^N$-module $T^M(V) \otimes T^N(V')$ has a natural decomposition as $(\mathcal{D}_-^N, S_M \times S_N)$-modules:

$$T^M(V) \otimes T^N(V') = \oplus_{\substack{\lambda^\pm\in\mathrm{Par}^\pm \\ |\lambda^+|=M \\ |\lambda^-|=N}} (V_{\lambda^+} \otimes V'_{\lambda^-}) \otimes (U_{\lambda^+} \otimes U_{\lambda^-})$$

where $U_{\lambda^+\,(\text{resp. }\lambda^-)}$ denotes the irreducible $S_{M\,(\text{resp. }N)}$-module corresponding to the partition λ^+ (resp. λ^-).

Lemma 4. *The $\mathcal{D}_-^N$-modules $V_{\lambda^+} \otimes V'_{\lambda^-}$ are irreducible.*

Proof. As in the proof of Theorem 3, we extend the action of $\mathcal{D}_-^N$ on $V_{\lambda^+} \otimes V'_{\lambda^-}$ to $\left(\mathcal{D}_-^{N\mathcal{O}}\right)_j$ for each $j \neq 0$, to obtain that any $\mathcal{D}_-^N$-submodule of $V_{\lambda^+} \otimes V'_{\lambda^-}$ is a submodule of $g\ell_{+\infty} \oplus g\ell_{-\infty}$. But, by Schur–Weyl theory, the $g\ell_{+\infty} \oplus g\ell_{-\infty}$-module $V_{\lambda^+} \otimes V'_{\lambda^-}$ is irreducible, which completes the proof. □

Thus, we have proved

Theorem 4. *The $\mathcal{D}^N_-$-module $L(\lambda^+,\lambda^-)$ is isomorphic to $V_{\lambda^+}\otimes V'_{\lambda^-}$ for any pair $\lambda^\pm \in \mathrm{Par}^\pm$.*

Remark. Considering $\lambda = (\lambda^-,\lambda^+) \in \mathbb{C}^\infty$ we may say that irreducible highest weight $\mathcal{D}^N_-$-modules of finite growth are parameterized by non-increasing sequences of integers $(\lambda_j)_{j\in\mathbb{Z}} \in \mathbb{C}^\infty$ with the exception that $\lambda_0 \le \lambda_1$. Equivalently, letting $m_i = \lambda_i - \lambda_{i+1}$ we may say that these modules are parameterized by sequences of non-negative integers $(m_i)_{i\in\mathbb{Z}\setminus\{0\}}$, all but finite numbers of which are zero.

Recall that the extended annihilation algebra $\mathrm{Lie}^-(\mathrm{gc}_N)$ for gc_N is isomorphic to the direct sum of the Lie algebra $\mathcal{D}^N_-$ and the N-dimensional Lie algebra $\mathbb{C}^N(\partial + \frac{d}{dt})$ and that conformal modules for a Lie conformal algebra coincide with the conformal modules over the associated extended annihilation algebra [7].

Given a module M over a Lie conformal algebra R and $\alpha\in\mathbb{C}$, we may construct the α-twisted module M_α by replacing ∂ by $\partial+\alpha$ in the formulas for action of R on M. Theorems 3 and 4 and the above remarks imply

Theorem 5. *The gc_N-modules $L(\lambda^+,\lambda^-)_\alpha$, where $\lambda^\pm \in \mathrm{Par}^\pm$, $\alpha\in\mathbb{C}$, exhaust all irreducible conformal gc_N-modules of finite growth.*

Corollary. *The gc_N-modules $\mathbb{C}^N[\partial]_\alpha$ and $\mathbb{C}^N[\partial]^*_\alpha$, where $\alpha\in\mathbb{C}$, exhaust all finite irreducible gc_N-modules.*

4 Irreducible finite growth $\mathrm{gc}_{N,xI}$-modules

Let $\mathcal{D}^N_0$ (respectively $\mathcal{D}^N_{0,-}$) be the Lie subalgebra of $\mathcal{D}^N$ (respectively $\mathcal{D}^N_-$) of all matrix regular differential operator on $\mathbb{C}^\times$ (respectively, $\mathbb{C}$) that kill constants. That is $\mathcal{D}^N_0$ consists of linear combinations of elements of the form $t^kDf(D)e_{i,j}$, where f is a polynomial, $i,j\in\{1,\cdots,N\}$, $k\in\mathbb{Z}_{\ge 0}$ and $e_{i,j}$ is the standard basis of $\mathrm{Mat}_N(\mathbb{C})$. Denote by $\widehat{\mathcal{D}}^N_0$ the corresponding central extension. These algebras inherit the $\mathbb{Z}$-gradation from $\widehat{\mathcal{D}}^N$. In this section, we will need the representation theory of the Lie algebra $\widehat{\mathcal{D}}^N_{0,-}$.

Given $\vec{\Delta} = \{\vec{\Delta}_n\}_{n\in\mathbb{Z}_+}$ with $\vec{\Delta}_n\in\mathbb{C}^N$ for all $n\in\mathbb{Z}_+$ we consider the highest weight module $L(\vec{\Delta},\mathcal{D}^N_{0,-})$ over $\mathcal{D}^N_{0,-}$ as the (unique) irreducible module that has a nonzero vector $v_{\vec{\Delta}}$ with the following properties:

$$(\mathcal{D}^N_{0,-})_p v_{\vec{\Delta}} = 0 \,\text{for}\, p<0, \quad D^n e_{i\,i} v_{\vec{\Delta}} = \Delta^i_n v_{\vec{\Delta}} \quad \text{for}\, n\in\mathbb{N},\ i=1,\cdots,N.$$

The principal gradation of $\mathcal{D}^N_{0,-}$ induces the principal gradation of $L(\vec{\Delta};\mathcal{D}^N_{0,-})$. Quasifinite modules over $\mathcal{D}^N_{0,-}$ can be constructed as follows. The $\mathcal{D}^N_{0,-}$-modules $\mathbb{C}^N[t,t^{-1}]/\mathbb{C}^N[t]$ and $\mathbb{C}^N[t]/\mathbb{C}^N$ give us an embedding of $\mathcal{D}^N_{0,-}$ in $\widetilde{\mathrm{g}\ell}_{+\infty}$ and $\widetilde{\mathrm{g}\ell}_{-\infty}$ respectively, hence an embedding of $\mathcal{D}^N_{0,-}$ in $\widetilde{\mathrm{g}\ell}_{+\infty} \oplus \widetilde{\mathrm{g}\ell}_{-\infty}$. All these embedding respect the principal gradation. Now take $\lambda^{\pm} \in \mathbb{C}^{\pm\infty}$ and consider the $\widetilde{\mathrm{g}\ell}_{+\infty} \oplus \widetilde{\mathrm{g}\ell}_{-\infty}$-module $L^+(\lambda^+) \otimes L^-(\lambda^-)$.

The same argument as in [10], gives us the following.

Lemma 5. *When restricted to $\mathcal{D}^N_{0,-}$, the module $L^+(\lambda^+) \otimes L^-(\lambda^-)$ remains irreducible.*

It follows immediately that $L^+(\lambda^+) \otimes L^-(\lambda^-)$ is an irreducible highest weight module over $\mathcal{D}^N_{0,-}$, which is obviously quasifinite.

We have the following theorem.

Theorem 6. *The $\mathcal{D}^N_{0,-}$-modules $L^+(\lambda^+) \otimes L^-(\lambda^-)$, where $\lambda^{\pm} \in \mathrm{Par}^{\pm}$, exhaust all quasifinite irreducible highest weight $\mathcal{D}^N_{0,-}$-modules that have finite growth.*

The proof of Theorem 6 is the same as Theorem 3, but in this case we reduce the problem to the representation theory of the universal central extension $\widehat{\mathcal{D}}^N_0$ of $\mathcal{D}^N_0$ that was developed in Refs. [9] and [13].

Recall that the homomorphism $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^N \to \widehat{g\ell}^{[m]}_{\infty}$ defined in (13) lifts to a homomorphism $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^{N\mathcal{O}} \to \widehat{g\ell}^{[m]}_{\infty}$. Now, the restriction $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^{N\mathcal{O}}_0 \to \widehat{g\ell}^{[m]}_{\infty}$ to $\widehat{\mathcal{D}}^{N\mathcal{O}}_0$ is surjective iff $s \notin \mathbb{Z}$. If $s \in \mathbb{Z}$, $m \neq 0$ we denote by $\widehat{g\ell}^{[m]}_{\infty,s}$ the Lie subalgebra of $\widehat{g\ell}^{[m]}_{\infty}$ where we remove all the elements $\{E_{i,\,sN-j+1} : i \in \mathbb{Z},\, j = 1, \cdots, N\}$. The homomorphism $\hat{\varphi}^{[m]}_s$ defined on (13), restricted to $\widehat{\mathcal{D}}^{N\mathcal{O}}_0$ is an epimorphism over $\widehat{g\ell}^{[m]}_{\infty,s}$. If $s = 0 = m$ we redefine $\widehat{g\ell}_{\infty,0}$ as the Lie subalgebra of $\widehat{g\ell}_{\infty}$ generated by C and $\{E_{i,j} :\ i \neq 0, j \neq 0\}$ and $\widehat{\varphi}_0$ by the homomorphism $p_0 \circ \widehat{\varphi}_0 : \widehat{\mathcal{D}}^N_0 \to \widehat{g\ell}_{\infty}$ where $p_0 : \widehat{g\ell}_{\infty} \to \widehat{g\ell}_{\infty,0}$ is the projection map. Observe that $\widehat{g\ell}_{\infty,0}$ is naturally isomorphic to $\widehat{g\ell}_{\infty}$. Then $\widehat{\varphi}_0 : \widehat{\mathcal{D}}^N_0 \to \widehat{g\ell}_{\infty,0} \simeq \widehat{g\ell}_{\infty}$ is a surjective homomorphism.

Now, let us consider the restriction to $\widehat{\mathcal{D}}^{N\mathcal{O}}_{0,-}$. Since the constrains given by (14) do not affect the case $s \neq 0$, we still have that $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^{N\mathcal{O}}_{0,-} \to \widehat{g\ell}^{[m]}_{\infty}$ $(s \notin \mathbb{Z})$ is surjective.

Remark. The description of the image of the homomorphism $\hat{\varphi}^{[m]}_s$, with $s \in \mathbb{Z}$ and $m \neq 0$ in Ref. [13] (pag. 9), as the Lie subalgebra of $\widehat{g\ell}^{[m]}_{\infty}$ from which we remove the elements $\{E_{sN-i+1,\,sN-j+1}:,\, i,j=1,\cdots,N\}$, should be replaced by $\widehat{g\ell}^{[m]}_{\infty,s}$ the Lie subalgebra of $\widehat{g\ell}^{[m]}_{\infty}$ where all the elements $\{E_{i,\,sN-j+1}: i\in\mathbb{Z},\, j=1,\cdots,N\}$ were removed.

One of the results of Ref. [13] (see also Ref. [9]) is the following:

Lemma 6. *For each $i=1,\ldots,r$, pick a collection $m_i\in\mathbb{Z}_+$, $s_i\in\mathbb{C}$, $\vec{\lambda}_i\in(\mathbb{C}^\infty)^{m_i+1}$, $\vec{c}_i\in\mathbb{C}^{m_i+1}$, such that $s_i-s_j\notin\mathbb{Z}$ for $i\neq j$. Then the $\oplus_{i=1}^r g^{[m_i]}$-module $\otimes_{i=1}^r L^{[m_i]}(\vec{\lambda}_i,\vec{c}_i)$ remains irreducible when restricted to $\widehat{\mathcal{D}}_0^N$ via the embedding $\oplus_{i=1}^r\widehat{\varphi}_{s_i}^{[m_i]}:\widehat{\mathcal{D}}_0^N\to\oplus_{i=1}^r g^{[m_i]}$, where $g^{[m_i]}=\widehat{g\ell}^{[m_i]}_\infty$ (respectively, $\widehat{g\ell}^{[m_i]}_{\infty,s_i}$) if $s_i\notin\mathbb{Z}$ (respectively $s_i\in\mathbb{Z}$). All irreducible quasifinite highest weight $\widehat{\mathcal{D}}_0^N$-modules are obtained in this way.*

Proof of Theorem 6. The proof is the same as Theorem 3 but use Lemma 6, but in the case $s=0, m\neq 0$ we do no get the operators $E_{q+1,\,q}$ with $q=-N+1,\cdots,0$ since $\mathrm{Im}\varphi_0^{[m]}=\widehat{g\ell}^{[m]}_{\infty,0}$. Then the argument is the same that Theorem 3 but in the case $s=0, m\neq 0$ restrict to the subalgebra of $\widehat{g\ell}^{[m]}_{\infty,0}$ instead of $\widehat{g\ell}^{[m]}_{\infty}$ consisting the matrices $(a_{i,j})_{i,j\leq r}$ or $(a_{i,j})_{i,j\geq r+1}$ in the case $s=0=m$ redefine φ_0 as $p_0\circ\widehat{\varphi}_0$.

□

Given two partitions $\lambda^\pm\in\mathrm{Par}^\pm$, the $\mathcal{D}_-^N$-module $L(\lambda^+,\lambda^-)$, that is obtained by restriction via φ_0 from the $\widetilde{\mathrm{g}\ell}_{+\infty}\oplus\widetilde{\mathrm{g}\ell}_{-\infty}$-module $L^+(\lambda^+)\otimes L^-(\lambda^-)$ remains irreducible as a $\mathcal{D}_{0,-}^N$-modules. The construction of the $\mathcal{D}_{0,-}^N$-module $L(\lambda^+,\,\lambda^-)$ is the same as before and Lemma 4 and Theorem 4 hold for $\mathcal{D}_{0,-}^N$. In this case, the extended annihilation algebra $\mathrm{Lie}(\mathrm{gc}_{N,xI})$ for $\mathrm{gc}_{N,xI}$ is isomorphic to the direct sum of the Lie algebra $\mathcal{D}_{0,-}^N$ and the N-dimensional algebra $\mathbb{C}^N[\partial+(d/dt)]$. Theorems 4 and 6 and the above remarks imply the following.

Theorem 7. *The $\mathrm{gc}_{N,xI}$-modules $L(\lambda^+,\lambda^-)_\alpha$, where $\lambda^\pm\in\mathrm{Par}^\pm$, $\alpha\in\mathbb{C}$, exhaust all irreducible conformal $\mathrm{gc}_{N,xI}$-modules of finite growth.*

Corollary. *The $\mathrm{gc}_{N,xI}$-modules $\mathbb{C}^N[\partial]_\alpha$ and $\mathbb{C}^N[\partial]^*_\alpha$, where $\alpha\in\mathbb{C}$, exhaust all finite irreducible $\mathrm{gc}_{N,xI}$-modules.*

5 Irreducible finite growth oc_N-modules.

For any $A\in\mathrm{Mat}_N\,\mathbb{C}$ we define $(A)^\dagger_{i\,j}=A_{N+1-j\,N+1-i}$. Consider the anti-involution on $\mathcal{D}=\mathcal{D}^1$, introduced in [19],

$$\tau_{+,-1}(t^kf(D))=t^kf(-D-k-1).$$

We extend $\tau_{+,-1}$ to a map on $\mathrm{Mat}_N\mathcal{D} = \mathcal{D} \otimes \mathrm{Mat}_N\,\mathbb{C}$ by letting $[\tau_{+,-1}(A)]_{i\,j} = \tau_{+,-1}(A_{i,j})$. Now, consider the anti-involution σ in $\mathcal{D}^N$ defined by

$$\sigma\left(t^k f(D)A\right) = \sigma_{+,-1}\left(t^k f(D)A^\dagger\right). \tag{16}$$

We denote by $\mathcal{D}^N_\sigma$ the Lie subalgebra of $\mathcal{D}^N$ given by $-\sigma$-fixed points in $\mathcal{D}^N$. This subalgebra corresponds to the Lie algebra denoted by $\mathcal{D}^N_o$ in [12]. Let $\widehat{\mathcal{D}}^N_\sigma = \mathcal{D}^N_\sigma \oplus \mathbb{C}C$ denote the central extension given by the restriction of the cocycle (9) on $\mathcal{D}^N$.

We are interested in the representation theory of the Lie algebra $\mathcal{D}^N_{\sigma,-} = \widehat{\mathcal{D}}^N_\sigma \bigcap \mathcal{D}^N_-$ of matrix regular differential operators on $\mathbb{C}$ that are invariant by $-\sigma$. Both subalgebras inherit a $\mathbb{Z}$-gradation from $\mathcal{D}^N$, since σ preserve the principal $\mathbb{Z}$-gradation of $\mathcal{D}^N$, and we have $\mathcal{D}^N_\sigma = \oplus_{p\in\mathbb{Z}}\left(\mathcal{D}^N_\sigma\right)_p$ where, if $p = kN + r$, with $k \in \mathbb{N}$ and $0 \le r \le N-1$,

$$\begin{aligned}(\mathcal{D}^N_\sigma)_p = &\Big\{t^{-k}(f(D_{-(k+1)})e_{i,\,i+r} - f(-D_{-(k+1)}e_{N+1-r-i,\,N+1-i}),\\ &\qquad\qquad\qquad\qquad 1 \le i \le [N+1-r/2]\Big\}\\ \bigcup&\Big\{t^{(-k+1)}(g(D_{-(k+2)})e_{i,\,i-N+r} - g_{-(k+2)}e_{2N+1-i-r,\,N+1-i}),\\ &\qquad\qquad\qquad\qquad N-r+1 \le i \le [2N+1-r/2]\Big\}\end{aligned} \tag{17}$$

where here and further $D_k = D + k/2$ and $[x]$, $x \in \mathbb{R}$ is the integer less or equal than x. In the case of $(\mathcal{D}_{\sigma,-})_p$, we need to add condition (14) for $p > 0$. As before, we have the corresponding subalgebras of $\mathcal{D}^{N\mathcal{O}}$, denoted by $\mathcal{D}^{N\mathcal{O}}_\sigma$ and $\mathcal{D}^{N\mathcal{O}}_{\sigma,-}$. As in the case of $\mathcal{D}^N_-$, given $\vec{\Delta} = \{\vec{\Delta}_n\}$ with $\vec{\Delta}_n \in \mathbb{C}^{\left[\frac{N}{2}\right]+\delta_{N\,odd}}$ we consider the highest weight module $L(\vec{\Delta};\mathcal{D}^N_{\sigma,-})$ over $\mathcal{D}^N_{\sigma,-}$ as the (unique) irreducible module that has a non-zero vector $v_{\vec{\Delta}}$ with the following properties:

$$\left(\mathcal{D}^N_{\sigma,-}\right)_p v_{\vec{\Delta}} = 0 \text{ for } p<0, \quad ((D_1)^n e_{i,i} - (-D_1)^n e_{N+1-i,\,N+1-i})v_{\vec{\Delta}} = \Delta^i_n v_{\vec{\Delta}}$$

for $n \in \mathbb{Z}_+$ $i = 1, \cdots, \left[\frac{N}{2}\right] + \delta_{N,odd}$. The principal gradation of $\mathcal{D}^N_{\sigma,-}$ induces the principal gradation $L(\vec{\Delta};\mathcal{D}^N_{\sigma,-}) = \oplus_{p\in\mathbb{Z}_+} L_p$ such that $L_0 = \mathbb{C}v_{\vec{\Delta}}$. The module $L(\vec{\Delta};\mathcal{D}^N_{\sigma,-})$ is called *quasifinite* if $\dim L_p < \infty$ for all $p \in \mathbb{Z}_+$.

Quasifinite modules over $\mathcal{D}^N_{\sigma,-}$ can be constructed as follows. The $\mathcal{D}^N_{\sigma,-}$-modulo $\mathbb{C}^N[t,t^{-1}]/\mathbb{C}^N$ gives us an embedding of $\mathcal{D}^N_{\sigma,-}$ in $\widetilde{g\ell}_{+\infty}$. This embedding respect the principal gradations.

Now take $\lambda^+ \in \mathbb{C}^{+\infty}$ and consider the $\widetilde{g\ell}_{+\infty}$-module $L^+(\lambda^+)$. The same argument as in [10], gives us the following.

Lemma 7. *When restricted to $\mathcal{D}^N_{\sigma,-}$, the module $L^+(\lambda^+)$ remains irreducible.*

It follows immediately that $L^+(\lambda^+)$ is an irreducible highest weight module over $\mathcal{D}^N_{\sigma,-}$, which is obviously quasifinite. It is easy to see that we have:

$$\Delta^i_n = \sum_{j\geq 1}(-j+1/2)^n\lambda^+_{jN-i+1} - (j-1/2)^n\lambda^-_{(j-1)N+i}$$

so that

$$\Delta_i(x) := \sum_{n\geq 0}\Delta^i_n x^n/n! = \sum_{j\geq 1} e^{(-j+1/2)x}\,\lambda^+_{jN-i+1} + e^{(j-1/2)x}\,\lambda^-_{(j-1)N+i}\,.$$

with $i = 1, \cdots, \left[\frac{N}{2}\right] + \delta_{N,odd}$. We shall prove the following theorem.

Theorem 8. *The $\mathcal{D}^N_{\sigma,-}$-modules $L^+(\lambda^+)$, where $\lambda^+ \in \mathrm{Par}^+$, exhaust all quasifinite irreducible highest weight $\mathcal{D}^N_{\sigma,-}$-modules that have finite growth.*

The basic idea of the proof of Theorem 8 is the same as in Theorem 3: to reduce the problem to the well developed (in [12]) representation theory of the universal central extension $\widehat{\mathcal{D}}^N_\sigma$.

Recall that the homomorphism $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^N \to \widehat{g\ell}^{[m]}_\infty$ defined in (13) lift to a homomorphism $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^{N\mathcal{O}} \to \widehat{g\ell}^{[m]}_\infty$. Now, the restriction $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^{N\mathcal{O}}_\sigma \to \widehat{g\ell}^{[m]}_\infty$ to $\widehat{\mathcal{D}}^{N\mathcal{O}}_\sigma$ is surjective iff $s \notin \mathbb{Z}/2$, and in the other cases, using (21), we have that (see Ref. [12] for details)

$$\widehat{\varphi}^{[m]}_0 : \widehat{\mathcal{D}}^{N\mathcal{O}}_\sigma \to d^{[m]}_\infty; \qquad \widehat{\varphi}^{[m]}_{1/2} : \widehat{\mathcal{D}}^{N\mathcal{O}}_\sigma \to d^{[m]}_\infty \quad if\, N even,$$

$$\widehat{\varphi}^{[m]}_{1/2} : \widehat{\mathcal{D}}^{N\mathcal{O}}_\sigma \to b^{[m]}_\infty \quad if\, N odd. \tag{18}$$

are surjective homomorphism. Now, let us consider the restriction to $\widehat{\mathcal{D}}^{N\mathcal{O}}_{\sigma,-}$. Since the constrains given by (14) do not affect the case $s \neq 0$, we still have that $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^{N\mathcal{O}}_{\sigma,-} \to \widehat{g\ell}^{[m]}_\infty$ $(s \notin \mathbb{Z}/2)$ and $\widehat{\varphi}^{[m]}_{1/2}$ are surjective. One of the main results of Ref [12] is the following.

Lemma 8. *For each $i = 1, \dots, r$, pick a collection $m_i \in \mathbb{Z}_+$, $s_i \in \mathbb{C}$, $\vec{\lambda}_i \in (\mathbb{C}^\infty)^{m_i+1}$, $\vec{c}_i \in \mathbb{C}^{m_i+1}$, such that $s_i \in \mathbb{Z}$ implies $s_i = 0$, $s_i \in \frac{1}{2} + \mathbb{Z}$ implies $s_i = \frac{1}{2}$, and $s_i - s_j \notin \mathbb{Z}$ for $i \neq j$. Then the $\oplus_{i=1}^r g^{[m_i]}$-module $\otimes_{i=1}^r L^{[m_i]}(\vec{\lambda}_i, \vec{c}_i)$ remains irreducible when restricted to $\widehat{\mathcal{D}_\sigma^N}$ via the embedding $\oplus_{i=1}^r \widehat{\varphi}_{s_i}^{[m_i]} : \widehat{\mathcal{D}_\sigma^N} \to \oplus_{i=1}^r g^{[m_i]}$, where $g^{[mi]} = \widehat{g\ell}_\infty^{[m_i]}$ (respectively $b_\infty^{[m_i]}$ or $d_\infty^{[m_i]}$) if $s_i \notin \mathbb{Z}/2$ (respectively , $si = 1/2, N$ odd or $s_i = 0$ or $s_i = 1/2, N$ even.) All irreducible quasifinite highest weight $\widehat{\mathcal{D}}_\sigma^N$-modules are obtained in this way.*

Proof of Theorem 8. The proof is similar to that of Theorem 3 Due to Lemma 8, Theorem 2 and (16), it is easy to see that if $L(\vec{\Delta}, \mathcal{D}_{\sigma,-}^N)$ has finite growth, then $L(\vec{\Delta}, \mathcal{D}_{\sigma,-}^N) = L(d_\infty^{[m]}; \vec{\lambda}, \vec{c})$ on which $\widehat{\mathcal{D}}_\sigma^N$ acts via the embedding $\widehat{\varphi}_0^{[m]}$. Now consider $q \in \mathbb{Z}$ such that,

(a) if $q = k_1 N + r$, with $k_1 \in \mathbb{Z}$ and $1 \le r \le N - 1$, choosing $f_1 \in \mathcal{O}$ to vanish in all $l \in \mathbb{Z}$ up to mth derivative except for ith derivative $(0 < i \le m)$ at $l = k_1 + 1$.

(b) If $q = k_1 N$, with $k_1 \in \mathbb{Z}$ and choosing $f_2 \in \mathcal{O}$ to vanish in all $l \in \mathbb{Z}$ up to mth derivative except for ith derivative $(0 < i \le m)$ at $l = k_1$

we see that all operators $u^i E_{q+1,\,q} - (-u)^i E_{-q+1,\,-q}$, with $0 < i \le m$ lie in the image of $\widehat{\varphi}_s^{[m]}(\mathcal{D}_{\sigma,-}^{N\mathcal{O}})$.

Suppose that the mth coordinate of $\vec{\lambda}_q$ is non-zero, and that $m > 0$. Then $v := (u^m E_{q+1\,q} - (-u)^i E_{-q+1\,-q})^n v_{\vec{\lambda}} \neq 0$ for all $n > 0$. But

$$(E_{q+1,q+1} - E_{-q,-q})v = (-N + \lambda_{q+1}^0)v.$$

As in Theorem 3, restricting to the subalgebra of $d_\infty^{[m]}$ isomorphic to $\mathrm{g}\ell_{+\infty}$consisting of matrices $(a_{i,j} - a_{1-j,1-i})_{i,j \ge q+1}$ we conclude by Theorem 2, that $L^{[m]}(d_\infty^{[m]}; \vec{\lambda}, \vec{c})$ is either trivial or is of infinite growth.

Thus, the only possibility that remains is $s = m = 0$. As has been already shown, the image of $\widehat{\varphi}_s(\mathcal{D}_{\sigma,-}^{N\mathcal{O}})$ contains all $E_{q+1,q} - E_{1-q,-q}$ except for $q \neq 0$, hence it contains all operators from $d_\infty^{[m]} \cap \mathrm{g}\ell_{-\infty} \oplus \mathrm{g}\ell_{+\infty} \simeq \mathrm{g}\ell_{+\infty}$. Therefore, by Theorem 2 , the highest weight of a finite growth $\mathcal{D}_{\sigma,-}^N$-module must be the same as one of the $\mathcal{D}_{\sigma,-}^N$-modules $L^+(\lambda^+)$ with $\lambda^+ \in \mathrm{Par}^+$.

□

Now we shall construct the $\mathcal{D}_{\sigma,-}^N$-modules $L(\lambda^+)$ explicitly. The $\mathcal{D}_-^N$-module $V = \mathbb{C}^N[t, t^{-1}]/\mathbb{C}^N[t]$ defined in (14), viewed as a $\mathcal{D}_{\sigma,-}^N$-module, remains irreducible. This is the highest weight $\mathcal{D}_{\sigma,-}^N$-module of growth 1 isomorphic to $L^+(\omega_1)$ where $\omega_1 \in \mathrm{Par}^+$, such that $\omega_1^i = 0$, for $i \neq 1$ and $\omega_1^1 = 1$. Observe that the $\mathcal{D}^N_{\sigma,-}$-module $\mathbb{C}^N[t]^* = \oplus_{j\in\mathbb{Z}_+}(\mathbb{C}^N t^j)^*$ is isomorphic to $L^+(\omega_1)$. As in the Schur-Weyl theory, the $\mathcal{D}^N_{\sigma,-}$-module $T^M(V)$ has a natural decomposition as $(\mathcal{D}^N_{\sigma,-}, S_M)$-modules:

$$T^M(V) = \oplus_{\substack{\lambda^+\in\mathrm{Par}^+\\ |\lambda^+|=M}} V_{\lambda^+} \otimes U_{\lambda^+}$$

where U_{λ^+} denotes the irreducible S_M-module corresponding to the partition λ^+.

Lemma 9. *The $\mathcal{D}^N_{\sigma,-}$-modules V_{λ^+} are irreducible.*

Proof. As in the proof of Theorem 8, we extend the action of $\mathcal{D}^N_{\sigma,-}$ on V_{λ^+} to $\left(\mathcal{D}^{N\mathcal{O}}_{\sigma,-}\right)_j$ for each $j \neq 0$, to obtain that any $\mathcal{D}^N_{\sigma,-}$-submodule of V_{λ^+} is a submodule of $g\ell_{+\infty} \simeq d_\infty \cap g\ell_{+\infty} \oplus g\ell_{-\infty}$. But, by Schur–Weyl theory, the $g\ell_{+\infty}$-module V_{λ^+} is irreducible, which completes the proof. □

Thus, we have proved

Theorem 9. *The $\mathcal{D}^N_{\sigma,-}$-module $T^M(V)$ has the following decomposition as $(\mathcal{D}^N_{\sigma,-}, S_M)$-modules:*

$$T^M(V) = \oplus_{\substack{\lambda^+\in\mathrm{Par}^+\\ |\lambda^+|=M}} V_{\lambda^+} \otimes U_{\lambda^+}$$

where U_{λ^+} denotes the irreducible S_M-module corresponding to the partition λ^+.

Remark. Considering $\lambda^+ \in \mathbb{C}^{+\infty}$ we may say that irreducible highest weight $\mathcal{D}^N_{\sigma,-}$-modules of finite growth are parameterized by non-increasing sequences of integers $(\lambda_j)_{j\in\mathbb{Z}} \in \mathbb{C}^\infty$ with the exception that $\lambda_0 \leq \lambda_1$. Equivalently, letting $m_i = \lambda_i - \lambda_{i+1}$ we may say that these modules are parameterized by sequences of non-negative integers $(m_i)_{i\in\mathbb{Z}\setminus\{0\}}$, all but finite numbers of which are zero.

Recall that the extended annihilation algebra $\mathrm{Lie}^-(oc_N)$ for oc_N is isomorphic to the direct sum of the Lie algebra $\mathcal{D}^N_{\sigma,-}$ and the N-dimensional Lie algebra $\mathbb{C}^N(\partial + \frac{d}{dt})$ and that conformal modules for a Lie conformal algebra coincide with the conformal modules over the associated extended annihilation algebra [7].

Theorem 8 and the above remarks imply the following

Theorem 10. *The oc_N-modules $L(\lambda^+)_\alpha$, where $\lambda^+ \in \mathrm{Par}^+$, $\alpha \in \mathbb{C}$, exhaust all irreducible conformal oc_N-modules of finite growth.*

Corollary. *The oc_N-modules $\mathbb{C}^N[\partial]_\alpha$ $\alpha \in \mathbb{C}$, exhaust all finite irreducible oc_N-modules.*

6 Irreducible finite growth $spc_{N,xI}$-modules.

Now, consider

$$\widetilde{\sigma}(t^k f(D) De_{i,j}) = -t^k f(-D-k) De_{j,i},$$

the anti-involution on $\mathcal{D}_0^N$, corresponding to those that defines the symplectic type conformal subalgebra in $\mathrm{gc}_{N,x}$ (cf.[5] pag.56). Observe that this coincide with Bloch's anti-involution for $N = 1$ (cf. [2]). This anti-involution does not preserve the principal gradation of $\mathcal{D}^N$. However it is conjugated by the automorphism $\tau(t^k f(D) De_{i,j}) = t^k f(D) De_{i,\, N+1-j}$, to the following anti-involution

$$\bar{\sigma}(t^k f(D) De_{i,j}) = -t^k f(-D-k) De_{N+1-j,\, N+1-i}, \tag{19}$$

where $k \in \mathbb{Z}$. Denote by $\mathcal{D}_{0,\bar{\sigma}}^N$ the Lie subalgebra of $\mathcal{D}_0^N$ fixed by $-\bar{\sigma}$. Let $\widehat{\mathcal{D}}_{0,\bar{\sigma}}^N = \mathcal{D}_{0,\bar{\sigma}}^N \oplus \mathbb{C}C$ denote the central extension given by the restriction of the cocycle on $\mathcal{D}^N$.

We are interested in the representation theory of the Lie subalgebra $\mathcal{D}_{0,\bar{\sigma},-}^N = \mathcal{D}_-^N \cap \widehat{\mathcal{D}}_{0,\bar{\sigma}}^N$ of matrix regular differential operators on $\mathbb{C}$ that kills constants and are invariant by $-\bar{\sigma}$. Both subalgebras inherit a $\mathbb{Z}$-gradation from $\mathcal{D}_0^N$, since $\bar{\sigma}$ preserves the principal $\mathbb{Z}$-gradation of $\mathcal{D}_0^N$, $\mathcal{D}_{0,\bar{\sigma}}^N = \oplus_{p\in\mathbb{Z}} (\mathcal{D}_{0,\bar{\sigma}}^N)_p$, where if $p = kN + r$, with $k \in \mathbb{N}$ and $0 \le r \le N-1$,

$$\begin{aligned}(\mathcal{D}_{0,\bar{\sigma}}^N)_p = \Big\{ & t^{-k}(f(D_{-k}) e_{i,\, i+r} + f(-D_{-k}) e_{N+1-r-i,\, N+1-i}), \\ & \qquad\qquad\qquad 1 \le i \le [N+1-r/2] \Big\} \\ \bigcup \Big\{ & t^{(-k+1)}(g(D_{-(k+1)}) e_{i\,, i-N+r} + g(-D_{-(k+1)}) e_{2N+1-i-r,\, N+1-i}), \\ & \qquad\qquad\qquad N-r+1 \le i \le [2N+1-r/2] \Big\}\end{aligned} \tag{20}$$

In the case of $(\mathcal{D}_{0,\bar{\sigma},-}^N)_p$, we need to add condition (14) for $p > 0$. Similarly, we have the corresponding subalgebras of $\mathcal{D}^{N\mathcal{O}}$, denoted by $\mathcal{D}_{0,\bar{\sigma}}^{N\mathcal{O}}$ and $\mathcal{D}_{0,\bar{\sigma},-}^{N\mathcal{O}}$.

As in the case of $\mathcal{D}^N_-$, given $\vec{\Delta} = \{\vec{\Delta}_n\}$ with $\vec{\Delta}_n \in \mathbb{C}^{[\frac{N}{2}]+\delta_{N,odd}}$ for all $n \in \mathbb{Z}_+$, we consider the highest weight module $L(\vec{\Delta}; \mathcal{D}^N_{0,\bar{\sigma},-})$ over $\mathcal{D}^N_{0,\bar{\sigma},-}$ as the (unique) irreducible module that has a non-zero vector $v_{\vec{\Delta}}$ with the following properties:

$$\left(\mathcal{D}^N_{0,\bar{\sigma},-}\right)_p v_{\vec{\Delta}} = 0 \text{ for } p < 0, \quad (D^n e_{i,i} + (-D)^n e_{N+1-i,N+1-i}) v_{\vec{\Delta}} = \Delta^i_n v_{\vec{\Delta}}$$

for $n \in \mathbb{Z}_+ \; i = 1, \cdots, \left[\frac{N}{2}\right] + \delta_{N,\text{odd}} \cdot$

The principal gradation of $\mathcal{D}^N_{0,\bar{\sigma},-}$ induces the principal gradation $L(\vec{\Delta}; \mathcal{D}^N_{0,\bar{\sigma},-}) = \oplus_{p\in\mathbb{Z}_+} L_p$ such that $L_0 = \mathbb{C}v_{\vec{\Delta}}$. The module $L(\vec{\Delta}; \mathcal{D}^N_{0,\bar{\sigma},-})$ is called *quasifinite* if $\dim L_p < \infty$ for all $p \in \mathbb{Z}_+$.

As in the preceding section, the $\mathcal{D}^N_{0,\bar{\sigma},-}$-module $\mathbb{C}^N[t, t^{-1}]/\mathbb{C}^N$ gives us an embedding of $\mathcal{D}^N_{0,\bar{\sigma},-}$ in $\widetilde{\mathrm{g}\ell}_{+\infty}$. This embedding respect the principal gradations.

Now take $\lambda^+ \in \mathbb{C}^{+\infty}$ and consider the $\widetilde{\mathrm{g}\ell}_{+\infty}$-module $L^+(\lambda^+)$. The same argument as in [10], gives us the following.

Lemma 10. *When restricted to $\mathcal{D}^N_{0,\bar{\sigma},-}$, the module $L^+(\lambda^+)$ remains irreducible.*

Theorem 11. *The $\mathcal{D}^N_{0,\bar{\sigma},-}$-modules $L^+(\lambda^+)$, where $\lambda^+ \in \mathrm{Par}^+$, exhaust all quasifinite irreducible highest weight $\mathcal{D}^N_{0,\bar{\sigma},-}$-modules that have finite growth.*

The basic idea of the proof of Theorem 11 is the same as in Theorem 3: to reduce the problem to the well developed (in [14]) representation theory of the universal central extension $\widehat{\mathcal{D}}^N_{0,\bar{\sigma}}$.

Recall that the homomorphism $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^N \to \widehat{g\ell}^{[m]}_\infty$ defined in (13) lift to a homomorphism $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^{N\mathcal{O}} \to \widehat{g\ell}^{[m]}_\infty$. Now, the restriction $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^{N\mathcal{O}}_{0,\bar{\sigma}} \to \widehat{g\ell}^{[m]}_\infty$ to $\widehat{\mathcal{D}}^{N\mathcal{O}}_{0,\bar{\sigma}}$ is surjective iff $s \notin \mathbb{Z}/2$, and in the other cases, using (20), we have that (see Ref. for details [14])

$$\widehat{\varphi}^{[m]}_{\frac{1}{2}} : \widehat{\mathcal{D}}^{N\mathcal{O}}_{0,\bar{\sigma}} \to c^{[m]}_\infty, \tag{21}$$

$$\widehat{\varphi}_0 : \widehat{\mathcal{D}}^{N\mathcal{O}}_{0,\bar{\sigma}} \to c_\infty, \tag{22}$$

And

$$\widehat{\varphi}^{[m]}_0 : \widehat{\mathcal{D}}^{N\mathcal{O}}_{0,\bar{\sigma}} \to \widetilde{g\ell}^{[m]}_{\infty,0}, \quad \text{if} \quad m \neq 0, \tag{23}$$

with $\widetilde{g\ell}^{[m]}_{\infty,0}$ is the subalgebra of $g\ell^{[m]}_\infty$ generated by

$$\{(u^k - (\widetilde{m}+1)u^{k-1})E_{ij} - ((-u)^k - (n+1)(-u)^{k-1})E_{-N+1-j,-N+1-i}\}$$

with $1 \geq k$, $i = nN + q$; $j = \widetilde{m}N + \bar{q}$ and $1 \leq q, \bar{q} \leq N$, are surjective homomorphism. Now, let us consider the restriction to $\widehat{\mathcal{D}}^{N\mathcal{O}}_{0,\bar{\sigma},-}$. Since the constrains given by (14) do not affect the case $s \neq 0$, we still have that $\widehat{\varphi}^{[m]}_s : \widehat{\mathcal{D}}^{N\mathcal{O}}_{0,\bar{\sigma},-} \to \widehat{g\ell}^{[m]}_\infty$ $(s \notin \mathbb{Z}/2)$ and $\widehat{\varphi}^{[m]}_{1/2} : \widehat{\mathcal{D}}^{N\mathcal{O}}_{0,\bar{\sigma}} \to c^{[m]}_\infty$ are surjective.

Remark. The description of the image of the homomorphism $\hat{\varphi}^{[m]}_0$, with $m \neq 0$ in Ref. [14] (Proposition 5.3), as the Lie subalgebra $c^{[m]}_\infty$ should be replaced by $\widetilde{g\ell}^{[m]}_{\infty,0}$ the Lie subalgebra of $\widehat{g\ell}^{[m]}_\infty$ generated by

$$\{(u^k - (\widetilde{m}+1)u^{k-1})E_{ij} - ((-u)^k - (n+1)(-u)^{k-1})E_{-N+1-j,-N+1-i}\}$$

with $1 \geq k$, $i = nN + q$; $j = \widetilde{m}N + \bar{q}$ and $1 \leq q, \bar{q} \leq N$.

One of the main results of Ref [14] is the following.

Lemma 11. *For each $i = 1, \ldots, r$, pick a collection $m_i \in \mathbb{Z}_+$, $s_i \in \mathbb{C}$, $\vec{c}_i \in (\mathbb{C}^\infty)^{m_i+1}$, $\vec{c}_i \in \mathbb{C}^{m_i+1}$, such that $s_i \in \mathbb{Z}$ implies $s_i = 0$, $s_i \in \frac{1}{2} + \mathbb{Z}$ implies $s_i = \frac{1}{2}$, and $s_i - s_j \notin \mathbb{Z}$ for $i \neq j$. Then the $\oplus^r_{i=1} g^{[m_i]}$-module $\otimes^r_{i=1} L(g^{[m_i]}, \vec{\lambda}_i, \vec{c}_i)$ remains irreducible when restricted to $\widehat{\mathcal{D}}^N_{0,\bar{\sigma}}$ via the embedding $\oplus^r_{i=1}\widehat{\varphi}^{[m_i]}_{s_i} : \widehat{\mathcal{D}}^N_{0,\bar{\sigma}} \to \oplus^r_{i=1} g^{[m_i]}$, where $g^{[mi]} = \widehat{g\ell}^{[m_i]}_\infty$ (respectively $c^{[m_i]}_\infty, g\ell^{[m]}_{\infty,s_i}$) if $s_i \notin \mathbb{Z}/2$ (respectively , $si = 1/2$, or $s_i = 0$.) All irreducible quasifinite highest weight $\widehat{\mathcal{D}}^N_{0,\bar{\sigma}}$-modules are obtained in this way.*

Proof of Theorem 11. The proof is similar to that of Theorem 3 Due to Lemma 11, Theorem 2 and (21)-(23), it is easy to see that if $L(\vec{\Delta}, \mathcal{D}^N_{0,\bar{\sigma},-})$ has finite growth, then $L(\vec{\Delta}, \mathcal{D}^N_{0,\bar{\sigma},-}) = L(c^{[m]}_\infty; \vec{\lambda}, \vec{c})$ on which $\widehat{\mathcal{D}}^N_\sigma$ acts via the embedding $\widehat{\varphi}^{[m]}_0$. Now consider $q \in \mathbb{Z}$ such that,

(a) if $q = k_1N + r$, with $k_1 \in \mathbb{Z}$, $k_1 \neq -1$ and $1 \leq r \leq N-1$ choosing $f_1 \in \mathcal{O}$ such that $f_1(x)x$ to vanish in all $l \in \mathbb{Z}$ up to mth derivative except for ith derivative $(0 < i \leq m)$ at $l = k_1 + 1$. If $k_1 = -1$ choosing $f_1 \in \mathcal{O}$ such to vanish in all $l \in \mathbb{Z}$ up to mth derivative except for $(i-1)$th derivative $(0 < i \leq m)$ at $l = k_1 + 1$.

(b) If $q = k_1N$, with $k_1 \in \mathbb{Z}, k_1 \neq 0$ choosing $f_2 \in \mathcal{O}$ such that $f_2(x)x$ to vanish in all $l \in \mathbb{Z}$ up to mth derivative except for ith derivative $(0 < i \leq m)$ at $l = k_1$.

we see that all operators $u^iE_{q+1,q} + (-u)^iE_{-q+1-N,-q-N}$, with $0 < i \leq m$ lie in the image of $\widehat{\varphi}^{[m]}_s(\mathcal{D}^{N\mathcal{O}}_{0,\bar{\sigma},-}), q \neq 0$.

Suppose that the mth coordinate of $\vec{\lambda}_q$ is non-zero, and that $m > 0$. Then $v := (u^m E_{q+1,q} + (-u)^i E_{-q+1-N,\,-q-N})^n v_{\vec{\lambda}} \neq 0$ for all $n > 0$. But

$$(E_{q,q} - E_{-q-N,-q-N})v = (N + \lambda_q^0)v.$$

for all $q \neq \{-N+1, \cdots, 0\}$. As in Theorem 3, restricting to the subalgebra of $g\ell_{\infty,s}^{[m]}$ isomorphic to $\mathrm{g}\ell_{+\infty}$ consisting of matrices $(a_{ij} - (-1)^{i+j} a_{-N-j,-N-i})_{i,j \geq q+1}$ we conclude by Theorem 2, that $L^{[m]}(c_\infty^{[m]}; \vec{\lambda}, \vec{c})$ is either trivial or is of infinite growth.

Thus, the only possibility that remains is $s = m = 0$. As has been already shown, the image of $\widehat{\varphi}_s(\mathcal{D}_{0,\bar{\sigma},-}^{N\mathcal{O}})$ contains all $E_{q+1,q} + E_{1-q,-q}$ except for $q \neq 0$, hence it contains all operators from $c_\infty^{[m]} \cap \mathrm{g}\ell_{-\infty} \oplus \mathrm{g}\ell_{+\infty} \simeq \mathrm{g}\ell_{+\infty}$. Therefore, by Theorem 3 , the highest weight of a finite growth $\mathcal{D}_{-}^{N}$-module must be the same as one of the $\mathcal{D}_{0,\bar{\sigma},-}^{N}$-modules $L^+(\lambda^+)$ with $\lambda^+ \in \mathrm{Par}^+$. □

As the preceding section, we can construct the $\mathcal{D}_{0,\bar{\sigma},-}^{N}$-modules $L(\lambda^+)$ explicitly. The $\mathcal{D}_{-}^{N}$-module $V = \mathbb{C}^N[t, t^{-1}]/\mathbb{C}^N[t]$ defined in (14), viewed as a $\mathcal{D}_{\sigma,-}^{N}$-module, remains irreducible. This is the highest weight $\mathcal{D}_{0,\bar{\sigma},-}^{N}$-module of growth 1 isomorphic to $L^+(\omega_1)$ where $\omega_1 \in \mathrm{Par}^+$, such that $\omega_1^i = 0$, for $i \neq N$ and $\omega_1^N = 1$.

As in the Schur-Weyl theory, the $\mathcal{D}_{0,\bar{\sigma},-}^{N}$-module $T^M(V)$ has a natural decomposition as $(\mathcal{D}_{0,\bar{\sigma},-}^{N}, S_M)$-modules:

$$T^M(V) = \oplus_{\substack{\lambda^+ \in \mathrm{Par}^+ \\ |\lambda^+| = M}} V_{\lambda^+} \otimes U_{\lambda^+}$$

where U_{λ^+} denotes the irreducible S_M-module corresponding to the partition λ^+.

Lemma 12. *The $\mathcal{D}_{0,\bar{\sigma},-}^{N}$-modules V_{λ^+} are irreducible.*

Proof. As in the proof of Theorem 8, we extend the action of $\mathcal{D}_{0,\bar{\sigma},-}^{N}$ on V_{λ^+} to $\left(\mathcal{D}_{0,\bar{\sigma},-}^{N\mathcal{O}}\right)_j$ for each $j \neq 0$, to obtain that any $\mathcal{D}_{0,\bar{\sigma},-}^{N}$-submodule of V_{λ^+} is a submodule of $\mathrm{g}\ell_{+\infty} \simeq c_\infty \cap \mathrm{g}\ell_{+\infty} \oplus \mathrm{g}\ell_{-\infty}$. But, by Schur–Weyl theory, the $\mathrm{g}\ell_{+\infty}$-module V_{λ^+} is irreducible, which completes the proof. □

Theorem 12. *The $\mathcal{D}_{0,\bar{\sigma},-}^{N}$-module $T^M(V)$ has the following decomposition as $(\mathcal{D}_{0,\bar{\sigma},-}^{N}, S_M)$-modules:*

$$T^M(V) = \oplus_{\substack{\lambda^+ \in \mathrm{Par}^+ \\ |\lambda^+| = M}} V_{\lambda^+} \otimes U_{\lambda^+}$$

where U_{λ^+} denotes the irreducible S_M-module corresponding to the partition λ^+.

Recall that the extended annihilation algebra $\mathrm{Lie}^-(spc_{N,xI})$ for $spc_{N,xI}$ is isomorphic to the direct sum of the Lie algebra $\mathcal{D}^N_{0,\bar{\sigma},-}$ and the N-dimensional Lie algebra $\mathbb{C}^N(\partial + \frac{d}{dt})$ and that conformal modules for a Lie conformal algebra coincide with the conformal modules over the associated extended annihilation algebra [7].

Theorem 11 and the above remarks imply the following

Theorem 13. *The $spc_{N,xI}$-modules $L(\lambda^+)_\alpha$, where $\lambda^+ \in \mathrm{Par}^+$, $\alpha \in \mathbb{C}$, exhaust all irreducible conformal $spc_{N,xI}$-modules of finite growth.*

Corollary. *The $spc_{N,xI}$-modules $\mathbb{C}^N[\partial]_\alpha$ $\alpha \in \mathbb{C}$, exhaust all finite irreducible $spc_{N,xI}$-modules.*

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