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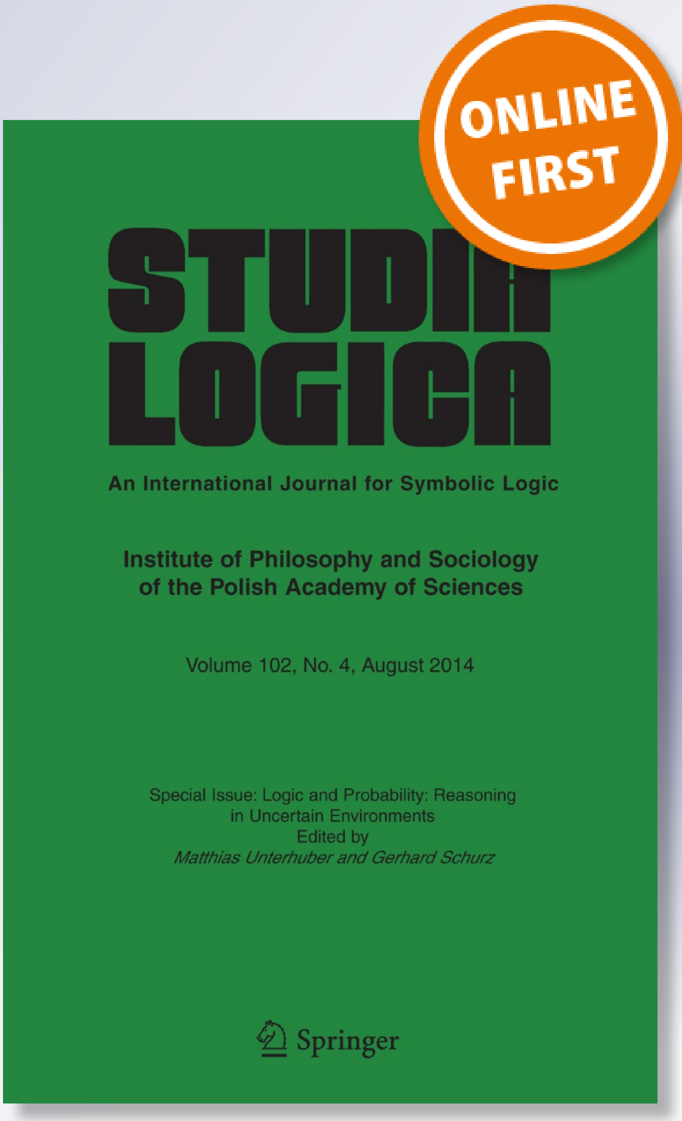

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Properties of Saturation in Monotonic Neighbourhood Models and Some Applications

Abstract. In this paper we shall discuss properties of saturation in monotonic neighbourhood models and study some applications, like a characterization of compact and modally saturated monotonic models and a characterization of the maximal Hennessy-Milner classes. We shall also show that our notion of modal saturation for monotonic models naturally extends the notion of modal saturation for Kripke models.

Keywords: Monotonic modal logic, Monotonic neighbourhood frames and models, m-saturated models, Kripke m-saturated models, Maximal Hennessy-Milner classes of monotonic models.

Mathematics Subject Classification: 06D05, 06D16, 03G10.

1. Introduction

Monotonic neighbourhood semantics is a generalization of Kripke semantics, and it is the standard tool for reasoning about monotonic modal logics. A *neighbourhood model*, is a structure $\mathcal{M} = \langle X, R, V \rangle$ where $R \subseteq X \times \mathcal{P}(X)$, and V is a valuation defined on X. The elements of $R(x)$ are called the neighbourhoods of x. A neighbourhood model $\mathcal{M}$ is *monotonic* if the set $R(x)$ is closed under supersets for each $x \in X$. Neighbourhood semantics is used for the classical modal logics that are strictly weaker than the normal modal logic **K**. For example, the neighbourhood semantics is used in [4] to study monotonic modal logics related to the Von Wright's logic of place. Also in [15] and [14] is used a neighbourhood semantic to study fused modal logics. On the other hand, in [17] a neighbourhood semantic is used to prove completeness for a monotonic modal logic called the modal Logic of Deductive Closure. The principal reference on neighbourhood semantics are Segerberg's monograph [18], Chellas's textbook [6], and Hansen's Master's thesis [10].

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The notion of bisimulation between Kripke models plays a very important role in the model theory of normal modal logics. Bisimulation is intended to characterize states in Kripke models with "the same behaviour". It is well-known that modal formulas about Kripke models (frames with valuations) can be understood as first-order formulas. However, not every first-order formula (in the relevant language) is equivalent to a modal formula. The question is which first-order formulas are equivalent to modal formulas. The answer is the ones invariant under bisimulation, i.e., normal modal logic is the fragment of first-order logic invariant under bisimulation [2]. This is a remarkable result given by Van Benthem (see also Theorem 18 in [8]).

Given M a class of Kripke models, it is said that it is a *Hennessy-Milner class* if for every $\mathcal{M}, \mathcal{N} \in \mathsf{M}$, the relation of modal equivalence $\approx$ is a bisimulation between $\mathcal{M}$ and $\mathcal{N}$. An interesting problem is to identify classes of models where the relation of modal equivalence $\approx$ be a bisimulation. A first example is the class of all image-finite Kripke models [2]. Other important example, that afore mentioned, is the class of *modally saturated* models, or m-saturated models. The notion of m-saturation is a kind of compactness property. A construction on models that returns m-saturated models is the *ultrafilter extension* of a model (see [2] or Theorem 2.10 in [13]). Since each point in a Kripke model is modally equivalent to the corresponding principal ultrafilter in the ultrafilter extension, and using the fact the ultrafilter extension is m-saturated, we have that two points in a model are modally equivalent if and only if their associated principal ultrafilters in the ultrafilter extension are bisimilar (see Section 2.5 of [2]). Others results and applications of the notion of bisimulation can be found in [2,8,9], and [13].

For monotonic neighbourhood models the notion of bisimulation was studied in detail in [10] (see also [11]), and independently in [1]. For arbitrary neighbourhood models different kinds of bisimulations have been studied in [12].

In [10], Hansen introduces a notion of modal saturation in monotonic models and proves that over the class of modally saturated monotonic models, modal equivalence implies bisimilarity. But Hansen's definition of modal saturation has, among others, the disadvantage that the ultrafilter extension of a monotonic model fails to meet one of the conditions of modal saturation given by Hansen. In [3] a different notion of modal saturation was defined (see Definition 5 of [3]). This new notion enables us to extended several theorems from Kripke semantics to neighbourhood semantics. For example, now it is possible to prove that the ultrafilter extension of a monotonic model is modally saturated (Theorem 18 in [3]). This notion also enables a characterization of the class compact and modally saturated models (see Theorem 23), and allows us to show that the class of all modally saturated models is a maximal Hennessy-Milner class (Theorem 29).

On the other hand, in [5] it was introduced the class of *normal monotonic frames*, or monotonic neighbourhood frames. As it is shown in Lemma 11 of [5], there is a bijective correspondence between normal monotonic frames and Kripke frames. For Kripke models, it is easy to check that we have a similar situation. For each normal monotonic model $\mathcal{M}$, there exists a Kripke model $\mathcal{M}^{\mathrm{krp}}$. Conversely, for each Kripke model $\mathcal{K}$ there exists a normal monotonic model $\mathcal{M}^{\mathrm{n}}$, and these correspondences are bijective. Thus, we get that the normal monotonic models are interdefinable with Kripke models. In Theorem 17, we will prove that our notion of monotonic modal saturation extends, in a natural way, the usual notion of m-saturation in Kripke models. We note that the notions of monotonic modal saturation and neighbourhood defined in [10] and [12], respectively, are not entirely satisfactory (see Remarks 4.8 and 4.9 in [12]). These observations could be interpreted as arguments to support that, the definition of modal saturation for monotonic neighbourhood models given in [3], is the right one.

The approach taken in [3] is mainly topological. For each monotonic model $\mathcal{M} = \langle X, R, V \rangle$ the family $D_V = \{V(\varphi) \mid \varphi \in Fm\}$ is a Boolean algebra closed under the monotonic operator $\Diamond_R(V(\varphi)) = \big\{x \in X \mid \exists Y \in R(x)$ such that $Y \subseteq V(\varphi)\big\}$. So, $\langle D_V, \Diamond_R \rangle$ is a monotonic modal algebra. We can also consider the topological space $\langle X, \mathcal{T}_{D_V} \rangle$ where the topology $\mathcal{T}_{D_V}$ is generated by taking D_V as the basis of open sets. On the other hand, we can define the *lower topology* $\mathcal{L}_{D_V}$ on $\mathcal{K}_R = \{Y \subseteq X \mid \exists x \in X \ (Y \in R(x))\}$ by taking as sub-basis of $\mathcal{L}_{D_V}$ the collection of all sets of the form $L_{V(\varphi)} = \{Y \in \mathcal{K}_R \mid Y \cap V(\varphi) \neq \emptyset\}$. The topological space $\langle \mathcal{K}_R, \mathcal{L}_{D_V} \rangle$ is called the *lower hyperspace* of $\langle X, \mathcal{T}_{D_V} \rangle$ *relative to* $\mathcal{K}_R$ (see [16]). In [3] the notions of image-compact, point-compact, point-closed and modally saturated monotonic models are defined using these spaces. As it is shown in [3], this approach seems more suitable to extend the classical notions of modal saturation for monotonic neighbourhood models. Following this line of research, the main objective of this paper is to discuss and generalize some results of the model theory of normal modal logic to monotonic modal logics. Particularly, we are interested in investigating some properties of saturation in monotonic neighbourhood models, together with some important applications, like a characterization of the class of compact and modally saturated models, and a characterization of the maximal Hennessy-Milner classes.

The paper is organized in the following fashion. In Section 2, we will recall the principal results on the relational and algebraic semantics for monotonic modal logic. In Section 3, we will review some special classes of monotonic models, like image-compact, point-closed, and modally saturated monotonic models. We will use the notion of modal saturation defined in [3]. We shall prove that the canonical model $\mathcal{M}_\Lambda$ of a monotonic logic Λ is compact and modally saturated. We will prove that the class of all modally saturated models is a Hennessy-Milner class. A similar statement was proved in [10] but a different notion of saturation is used. Finally, we will prove that a normal monotonic model $\mathcal{M}$ is modally saturated iff the Kripke model $\mathcal{M}^{\mathrm{krp}}$ is m-saturated.

In [3] it was proved that the concepts of compact and point-compact models are preserved by surjective bounded morphisms between monotonic models. In Section 4 we will extend these results proving that the notions of compact and point-compact are preserved by total bisimulations. Also, we prove that if $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a surjective bounded morphism and $\mathcal{M}_2$ is image-compact, then $\mathcal{M}_1$ is also image-compact. These results are needed in the next sections. In Section 5, we shall give a characterization of the class **CMSAT** of all compact and modally saturated models. A class of Kripke models **M** has the Hennessy-Milner property if modal equivalence between models in **M** implies (and hence is equivalent to) bisimulation equivalence. For instance, the class of all finite Kripke models has the Hennessy-Milner property. A more general result shows that the class of modally saturated Kripke models has the Hennessy–Milner property (see [8] and [13]). In Section 6, we will prove that the class **MSAT** of all modally saturated monotonic models is a Hennessy-Milner class not properly included in any Hennessy-Milner class, i.e. **MSAT** is a maximal Hennessy-Milner class.

2. Preliminaries

2.1. Spaces

Given a set X, we denote by $\mathcal{P}(X)$ the powerset of X, and for a subset Y of X, we write Y^c for the complement $X - Y$ of Y in X. Let us recall that a topological *basis* is a collection $D \subseteq \mathcal{P}(X)$ of subsets of a set X such that (1) $\emptyset \in D$, (2) $\bigcup D = X$, and (3) for all $U, V \in D$ and $x \in U \cap V$, there exists $W \in D$ such that $x \in W$ and $W \subseteq U \cap V$. A topological basis D generates a topology on X that we will denote by $\mathcal{T}_D$. In all this paper a *space* will be a topological space $\langle X, \mathcal{T}_D \rangle$, where the topological basis D is a subalgebra of the Boolean algebra of $\mathcal{P}(X)$. In this case the elements of D are clopen (closed and open) subsets of X, because D is a Boolean algebra, but an arbitrary clopen set does not need to be an element of D. Given a space $\langle X, \mathcal{T}_D \rangle$ and $Y \subseteq X$, we will use the notation $\mathrm{cl}(Y)$ to express the *closure* of Y. The set of all closed subsets (compact subsets) of $\langle X, \mathcal{T}_D \rangle$ will be denoted by $\mathcal{C}(X)$ ($\mathcal{K}(X)$). We note that $\mathcal{C}(X)$ and $\mathcal{K}(X)$ are posets under the inclusion relation.

Let A be a Boolean algebra. The lattice of filters of A will be denoted by $\mathrm{Fi}(A)$. The set of all prime filters or *ultrafilters* of A is denoted by $\mathrm{Ul}(A)$. To each Boolean algebra A we can associate a Stone space $\langle \mathrm{Ul}(A), \mathcal{T}_A \rangle$ whose points are the elements of $\mathrm{Ul}(A)$ with the topology $\mathcal{T}_A = \mathcal{T}_{\beta[A]}$ determined by the basis $\beta[A] = \{\beta(a) \mid a \in A\}$, where $\beta(a) = \{x \in \mathrm{Ul}(A) \mid a \in x\}$.

Some topological properties of a space $\langle X, \mathcal{T}_D \rangle$ can be characterized in terms of the map $\varepsilon_D : X \to \mathrm{Ul}(D)$ defined by $\varepsilon_D(x) = \{U \in D \mid x \in U\}$. For instance, $\langle X, \mathcal{T}_D \rangle$ is Hausdorff iff ε_D is injective, and $\langle X, \mathcal{T}_D \rangle$ is compact iff ε_D is surjective. A space $\langle X, \mathcal{T}_D \rangle$ is called a *Stone space* if it is *compact, Hausdorff,* and *zero dimensional* (i.e. the sets which are both open and closed form a basis for the open sets).

If $\langle X, \mathcal{T}_D \rangle$ is a Stone space, then the map ε_D is an homeomorphism between $\langle X, \mathcal{T}_D \rangle$ and $\langle \mathrm{Ul}(D), \mathcal{T}_{\beta[D]} \rangle$. If A is a Boolean algebra, then $A \cong \beta[A]$, by means of the map β. Moreover, it is known that the map $F \longmapsto \hat{F} = \{x \in \mathrm{Ul}(A) \mid F \subseteq x\}$ establishes a bijective correspondence between the lattice of all filters of A and the lattice $\mathcal{C}(\mathrm{Ul}(A))$ of all closed subsets of $\langle \mathrm{Ul}(A), \mathcal{T}_A \rangle$.

Definition 1. Let $\langle X, \mathcal{T}_D \rangle$ be a space. Let $\mathcal{K} \subseteq \mathcal{P}(X)$. The *lower topology* $\mathcal{L}_D$ on $\mathcal{K}$ is the topology defined on $\mathcal{K}$ taking as sub-basis the collection of all sets of the form

$$L_U = \{Y \in \mathcal{K} \mid Y \cap U \neq \emptyset\},$$

for $U \in D$. The pair $\mathcal{K} = \langle \mathcal{K}, \mathcal{L}_D \rangle$ is called the lower *hyperspace* of $\langle X, \mathcal{T}_D \rangle$ relative to $\mathcal{K}$.

Let $D_U = \{Y \in \mathcal{K} \mid Y \subseteq U\}$, for $U \in D$. We note that $(L_{U^c})^c = D_U$. Recall that if $\langle X, \mathcal{T}_D \rangle$ is a Stone space, then $\langle \mathcal{C}(X), \mathcal{L}_D \rangle$ is a Stone space (see [16] for the details).

The following technical result is needed in the next section. For a set $Y \subseteq X$, let $\varepsilon_D[Y] = \{\varepsilon_D(x) \mid x \in Y\}$.

Lemma 2. *Let $\langle X, \mathcal{T}_D \rangle$ be a space. Then*

(1) *For all $Y, Z \in \mathcal{P}(X)$, if $\varepsilon_D[Z] \subseteq \varepsilon_D[Y]$, then $Z \subseteq \mathrm{cl}(Y)$.*

(2) *If $Y \in \mathcal{K}(X)$, and $Z \subseteq \mathrm{cl}(Y)$, then $\varepsilon_D[Z] \subseteq \varepsilon_D[Y]$.*

Proof. (1) Let $Y, Z \in \mathcal{P}(X)$. Let $x \in Z$ and suppose that $x \notin \mathrm{cl}(Y)$. Then there exists $U \in D$ such that $x \notin U$ and $Y \subseteq U$. Then $U^c \in \varepsilon_D(x) \in \varepsilon_D[Z] \subseteq \varepsilon_D[Y]$. So there exists $y \in Y$ such that $\varepsilon_D(x) = \varepsilon_D(y)$. It follows $y \notin U$, which is a contradiction.

(2) Let $Z \in \mathcal{P}(X)$ and $Y \in \mathcal{K}(X)$. Suppose that $\varepsilon_D[Z] \nsubseteq \varepsilon_D[Y]$. Then there exists $z \in Z$ such that $\varepsilon_D(z) \neq \varepsilon_D(y_i)$, for all $y_i \in Y$. So, for each $y_i \in Y$ there exists $U_i \in D$ such that $z \notin U_i$ and $y_i \in U_i$. As $Y \subseteq \bigcup \{U_i \in D \mid z \notin U_i \text{ and } y_i \in U_i\}$ and Y is compact, there exists $U_1, \ldots, U_n$ such that $Y \subseteq U_1 \cup \ldots \cup U_n = U$ and $z \notin U$. Then, $Z \nsubseteq \mathrm{cl}(Y)$. ■

2.2. Monotonic Modal Logic and Monotonic Frames

Let us consider a propositional language $\mathcal{L}$ defined by using a denumerable set of propositional variables Var, the connectives $\vee$ and $\wedge$, the negation $\neg$, the modal connective $\Diamond$, and the propositional constant $\top$. We shall denote by $\Box$ the operator defined as $\Box p = \neg\Diamond\neg p$, for $p \in Var$. The set of formulas as well as the formula algebra are denoted by Fm.

A *monotonic modal logic* is a set of formulas Λ in the propositional language $\mathcal{L}$ which contains the Classical Propositional Calculus and is closed under the following inference rules:

R1. If $\varphi, \varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$ (Modus Ponens).

R2. If $\varphi \to \psi \in \Lambda$, then $\Box\varphi \to \Box\psi \in \Lambda$.

The smallest *monotonic modal logic* will be denoted by **MON**.

The algebraic semantics for monotonic modal logics is given by the class of Boolean algebras with a monotonic operator [10]. Recall that a *monotonic algebra* is a pair $A = \langle A, \Diamond \rangle$, where A is a Boolean algebra, and $\Diamond : A \to A$ is a monotonic function, i.e. if $a \leq b$ then $\Diamond a \leq \Diamond b$, for all $a, b \in A$.

Definition 3. *A monotonic neighbourhood frame,* or monotonic frame, is a structure $\mathcal{F} = \langle X, R \rangle$ where $R \subseteq X \times \mathcal{P}(X)$, and $R(x) = \{Z \in \mathcal{P}(X) \mid (x, Z) \in R\}$ is an increasing subset of $\mathcal{P}(X)$, for each $x \in X$, i.e. is closed under supersets for each $x \in X$.

Every monotonic frame $\mathcal{F}$ gives rise to a monotonic algebra of sets in the following way.

Definition 4. The *monotonic algebra,* or *complex algebra,* of a monotonic frame $\mathcal{F} = \langle X, R \rangle$ is the pair

$$A(\mathcal{F}) = \langle \mathcal{P}(X), \Diamond_R \rangle,$$

where the monotonic map $\Diamond_R : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by:

$$\Diamond_R(U) = \{x \in X \mid \exists Y \in R(x)\ (Y \subseteq U)\} = \{x \in X \mid R(x) \cap D_U \neq \emptyset\},$$

for each $U \in \mathcal{P}(X)$, where $(L_{U^c})^c = D_U$.

Let $\mathcal{F}$ be a monotonic frame. The dual map $\Box_R : \mathcal{P}(X) \to \mathcal{P}(X)$ of $\Diamond_R$ is defined by:

$$\Box_R(U) = \{x \in X \mid \forall Y \in R(x)(Y \cap U \neq \emptyset)\} = \{x \in X \mid R(x) \subseteq L_U\}, \tag{2.1}$$

for each $U \in \mathcal{P}(X)$.

Definition 5. The *ultrafilter frame* of a monotonic algebra A is a pair

$$\mathcal{F}(A) = \langle \mathrm{Ul}(A), R_\Diamond \rangle,$$

where the relation $R_\Diamond \subseteq \mathrm{Ul}(A) \times \mathcal{P}(\mathrm{Ul}(A))$ is defined by:

$$(x, Y) \in R_\Diamond \Leftrightarrow \exists F \in \mathrm{Fi}(A)\ (\hat{F} \subseteq Y \text{ and } F \subseteq \Diamond^{-1}(x)). \tag{2.2}$$

with $\hat{F} = \{y \in \mathrm{Ul}(A) \mid F \subseteq y\}$.

Let A be a monotonic algebra. We note that for any $F \in \mathrm{Fi}(A)$, and for each $x \in \mathrm{Ul}(A)$,

$$(x, \hat{F}) \in R_\Diamond \text{ iff } F \subseteq \Diamond^{-1}(x).$$

We note also that $\Diamond_{R_\Diamond}(\beta(a)) = \beta(\Diamond(a))$, for all $a \in A$. Thus the map $\beta : A \to \mathcal{P}(\mathrm{Ul}(A))$ is a monomorphism of monotonic algebras (see [3,5] or [10]).

A *valuation* V based on a monotonic frame $\mathcal{F}$ is homomorphism from Fm into $A(\mathcal{F})$. A *monotonic model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a monotonic frame and V is a valuation defined on it. We note that $V(\Diamond\varphi) = \Diamond_R V(\varphi)$, for any $\varphi \in Fm$. The notions of truth at a point, validity in a model and validity in a frame for formulas are defined as usual.

If $\mathcal{M} = \langle X, R, V \rangle$ is a monotonic model, then $D_V = \{V(\varphi) \mid \varphi \in Fm\}$ is a Boolean algebra closed under the monotonic operator $\Diamond_R$. So, we can consider the topological space $\langle X, \mathcal{T}_{D_V} \rangle$ where the topology $\mathcal{T}_{D_V}$ is generated by taking D_V as basis of open sets.

Let Λ be a monotonic modal logic over a language $\mathcal{L}$ with a fixed set of variables Var. A *theory* of Λ, or a Λ-*theory*, is a set of formulas T such that for any formulas φ and ψ, if $\varphi \in T$, $\varphi \to \psi \in T$, then $\psi \in T$. A Λ-theory T is consistent if $T \neq Fm$. A consistent theory T of Λ is *maximal* if for any formula φ, we get $\varphi \in T$ or $\neg\varphi \in T$.

Let us define the *canonical frame* of a monotonic modal logic Λ as the structure $\mathcal{F}_\Lambda = \langle X_\Lambda, R_\Lambda \rangle$ where X_Λ is the set of all maximal theories of Λ, and R_Λ is a subset of $X_\Lambda \times \mathcal{P}(X_\Lambda)$ defined by:

$$(P, Y) \in R_\Lambda \text{ iff there exists a } \Lambda\text{-theory } T \; (\hat{T} \subseteq Y \text{ and } T \subseteq \Diamond^{-1}(P)), \tag{2.3}$$

where $\hat{T} = \{Q \in X_\Lambda \mid T \subseteq Q\}$ (see [10]).

The *canonical model* of Λ is the structure

$$\mathcal{M}_\Lambda = \langle \mathcal{F}_\Lambda, V_\Lambda \rangle,$$

where $V_\Lambda(p) = \{P \in X_\Lambda \mid p \in P\}$. It is simple to verify that the valuation V_Λ can be extended to Fm, i.e. $V_\Lambda(\varphi) = \{P \in X_\Lambda \mid \varphi \in P\}$, for all formula φ. We note that $\langle X_\Lambda, \mathcal{T}_{D_\Lambda} \rangle$ is a topological space where $D_\Lambda = \{V_\Lambda(\varphi) : \varphi \in Fm\}$.

It is well known that if L_Λ is the *Lindenbaum-Tarski* algebra of Λ over the set Var, then the canonical frame $\mathcal{F}_\Lambda$ is isomorphic to the frame of L_Λ (for more details see [10]).

Let Λ be a monotonic modal logic and let $\mathcal{M} = \langle X, R, V \rangle$ be a monotonic model. Since $\langle X, \mathcal{T}_{D_V} \rangle$ is a space, we can consider the map $\varepsilon_{D_V} : X \to \mathrm{Ul}(D_V)$ defined by $\varepsilon_{D_V}(x) = \{V(\varphi) \mid x \in V(\varphi)\}$. By simplicity we write ε_V instead of ε_{D_V}.

In addition to the function ε_V, we shall also consider the function

$$F^{\mathcal{M}} : X \to X_\Lambda,$$

defined by

$$F^{\mathcal{M}}(x) = \{\varphi \in Fm \mid x \in V(\varphi)\}.$$

We note that $\varphi \in F^{\mathcal{M}}(x)$ iff $V(\varphi) \in \varepsilon_V(x)$, for any $\varphi \in Fm$.

2.3. Monotonic Bisimulations

Consider two monotonic models $\mathcal{M}_1$ and $\mathcal{M}_2$ and elements $x \in X_1$ and $y \in X_2$. We say that x and y are *modally equivalent*, in symbols $x \approx y$, if $F^{\mathcal{M}_1}(x) = F^{\mathcal{M}_2}(y)$. For to define the notion of bisimulation we shall use the following notation. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ two models. Let B be a relation between X_1 and X_2. We define a relation $\preccurlyeq_B \subseteq \mathcal{P}(X_1) \times \mathcal{P}(X_2)$ by:

$$Y \preccurlyeq_B Z \text{ iff } \forall y \in Y \exists z \in Z \text{ such that } (y, z) \in B.$$

If $B = \approx$ we will write $\preccurlyeq$ instead of $\preccurlyeq_\approx$. In this case the relation $\preccurlyeq$ can be defined in terms of the maps $F^{\mathcal{M}_1} : X_1 \to X_\Lambda$ and $F^{\mathcal{M}_2} : X_2 \to X_\Lambda$ as follows:

$$Y \preccurlyeq Z \text{ iff } F^{\mathcal{M}_1}[Y] \subseteq F^{\mathcal{M}_2}[Z].$$

DEFINITION 6. [11] Let $\mathcal{M}_1$ and $\mathcal{M}_2$ two monotonic models. A relation $B \subseteq X_1 \times X_2$ is a *bisimulation* if whenever $(a, b) \in B$ the following conditions hold:

B0. $a \in V_1(p)$ iff $b \in V_2(p)$, for any $p \in Var$,

B1. $\forall Y \in R_1(a)\ \exists Z \in R_2(b)$ such that and $Z \preccurlyeq_{B^{-1}} Y$,

B2. $\forall Z \in R_2(b)\ \exists Y \in R_1(a)$ such that $Y \preccurlyeq_B Z$.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models and let $x \in X_1$ and let $y \in X_2$. We say that x and y are *bisimilar* if there exists a bisimulation B between $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $(x, y) \in B$.

One of the reasons why bisimilarity is such an important notion in modal logic is that the truth of modal formulas is invariant under bisimilarity as is shown in Lemma 7, while for some important classes of models the converse inclusion holds as well (see Theorem 12). Given M a class of monotonic models, it is said that it is a *Hennessy-Milner class* if for every $\mathcal{M}, \mathcal{N} \in \mathsf{M}$, the relation $\approx$ is a bisimulation between $\mathcal{M}$ and $\mathcal{N}$. We shall say that $\mathcal{M}$ has the *Hennessy-Milner Property (HMP)* if it is logically bisimilar to itself, i.e. if $\approx$ is a bisimulation on $\mathcal{M}$. A bisimulation B is *total* if the domain of B is X_1 and the image is X_2. The proof of the following result is given in [10].

LEMMA 7. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models. For all states x in $\mathcal{M}_1$ and y in $\mathcal{M}_2$, if x and y are bisimilar, then $x \approx y$.*

A *bounded morphism* f between the monotonic frames $\mathcal{F}_1 = \langle X_1, R_1 \rangle$ to $\mathcal{F}_2 = \langle X_2, R_2 \rangle$ can be defined as a function $f : X_1 \to X_2$ whose graph $\{(x, f(x)) \mid x \in X_1\}$ is a bisimulation. This is equivalent to the following definition:

BM1. $\forall x \in X \forall Y \subseteq X\ ((x, Y) \in R_1$, then $(f(x), f[Y]) \in R_2)$,

BM2. If $(f(x), Z) \in R_2$, then there exists $Y \subseteq X$ such that $(x, Y) \in R_1$ and $f[Y] \subseteq Z$.

If f is surjective, then it is called a bounded epimorphism. A *bounded morphism* from the model $\mathcal{M}_1 = \langle X_1, R_1, V_1 \rangle$ to the model $\mathcal{M}_2 = \langle X_2, R_2, V_2 \rangle$ is a bounded morphism between the frames $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $V_1(p) = f^{-1}(V_2(p))$, for any $p \in Var$.

We say that the model $\mathcal{M}_1$ is a *generated submodel* of the model $\mathcal{M}_2$, in symbols $\mathcal{M}_1 \subseteq_{gs} \mathcal{M}_2$, if $X_1 \subseteq X_2$ and the inclusion function $X_1 \hookrightarrow X_2$ is a bounded morphism from $\mathcal{M}_1$ to $\mathcal{M}_2$.

3. Modally Saturated Monotonic Models

Let us recall that a Kripke model $\mathcal{M}$ is called *modally saturated*, or *m-saturated*, if for every state $x \in X$, and for every set Γ of formulas which is finitely satisfiable in the set of successors of x, is itself satisfiable in the set of successors of x. Then a result, originally due to A. Visser, states that the class of m-saturated models is a Hennessy-Milner class (see [13], or [2]). Now we attempt to generalize the notion of saturation along the lines of what has been accomplished in the classical normal modal setting.

A notion of modal saturation was introduced by Pauly (unpublished, see [10] for the details) for monotonic models. In this section we use the notion of modal saturation defined in [3]. Every modally saturated model according to Hansen's definition is modally saturated in our sense, but the converse is not true.

Let $\mathcal{M}$ be a monotonic model. Then we can consider the *hyperspace* $\langle \mathcal{K}_R, \mathcal{L}_{D_V} \rangle$ of $\langle X, \mathcal{T}_{D_V} \rangle$ relative to $\mathcal{K}_R = \{Y \subseteq X \mid \exists x \in X\ ((x, Y) \in R)\}$ (see definition 1).

Definition 8. Let $\mathcal{M} = \langle X, R, V \rangle$ be a monotonic model. We shall say that:

(1) $\mathcal{M}$ is *compact* if the topological space $\langle X, \mathcal{T}_{D_V} \rangle$ is compact.

(2) $\mathcal{M}$ is *image-compact* if for all $x \in X$ and for all $Y \in R(x)$, there exists a compact subset Z of $\langle X, \mathcal{T}_{D_V} \rangle$ such that $Z \subseteq Y$ and $Z \in R(x)$.

(3) $\mathcal{M}$ is *point-compact* if $R(x)$ is a compact subset in the topological space $\mathcal{K}_R = \langle \mathcal{K}_R, \mathcal{L}_{D_V} \rangle$, for each $x \in X$.

(4) $\mathcal{M}$ is *modally saturated* if it is image-compact and point-compact.

Now, we will recall the following result shown in [3], Lemma 13.

Lemma 9. *Let $\mathcal{M}$ be a model. Then the following conditions are equivalent:*

(1) *The topological space $\langle X, \mathcal{T}_{D_V} \rangle$ is compact.*

(2) *The map $\varepsilon_V : X \to \mathrm{Ul}(D_V)$ is onto.*

Now we prove that the canonical model $\mathcal{M}_\Lambda$ of a monotonic logic Λ is compact and modally saturated. This result is needed in Section 6.

Theorem 10. *Let Λ be a monotonic modal logic. The canonical model $\mathcal{M}_\Lambda = \langle \mathcal{F}_\Lambda, V_\Lambda \rangle$ is compact and modally saturated.*

Proof. We prove that the space $\left\langle X_\Lambda, \mathcal{T}_{D_{V_\Lambda}} \right\rangle$ is compact. From Lemma 9 it is enough to prove that the map $\varepsilon_{V_\Lambda} : X_\Lambda \to \mathrm{Ul}\left(D_{V_\Lambda}\right)$ given by $\varepsilon_{V_\Lambda}(T) =$

$\{V_\Lambda(\phi) \mid T \in V_\Lambda(\phi)\}$ is surjective. Let $P \in \mathrm{Ul}\,(D_{V_\Lambda})$. Consider the set $T = \{\phi \in Fm \mid V_\Lambda(\phi) \in P\}$. As P is an ultrafilter of D_{V_Λ}, we get that $T \in X_\Lambda$. Now, it is easy to see that $\varepsilon_{V_\Lambda}(T) = P$. Thus ε_{V_Λ} is surjective.

Now we prove that $\left\langle X_\Lambda, \mathcal{T}_{D_{V_\Lambda}} \right\rangle$ is point-compact. Let $P \in X_\Lambda$ and let Y be a subset of X_Λ such that $(P, Y) \in R_\Lambda$.Then there exists a theory T such that $\hat{T} \subseteq Y$ and $T \subseteq \Diamond^{-1}(P)$, where we recall that $\hat{T} = \{Q \in X_\Lambda \mid T \subseteq Q\}$. Assume that

$$R_\Lambda(P) \subseteq \bigcup \left\{L_{V_\Lambda(\alpha)} : \alpha \in \Gamma \subseteq Fm\right\}. \tag{3.1}$$

Suppose that $R_\Lambda(P) \nsubseteq \bigcup \left\{L_{V_\Lambda(\alpha)} : \alpha \in \Gamma_j\right\}$, for any finite subset Γ_j of Γ. Consider the theory F generated by the set $\{\neg\alpha : \alpha \in \varphi\}$. we prove that $F \subseteq \Diamond^{-1}(P)$. Let $\varphi \in F$. Then there exists a finite subset $\{\alpha_1, \cdots ., \alpha_n\}$ of Γ such that $(\neg\alpha_1 \wedge \cdots \wedge \neg\alpha_n) \to \varphi \in \Lambda \subset F$. Thus, $V_\Lambda(\neg\alpha_1 \wedge \cdots \wedge \neg\alpha_n) \subseteq V_\Lambda(\varphi)$. By hypothesis,

$$R_\Lambda(P) \nsubseteq L_{V_\Lambda(\neg\alpha_1)} \cup \cdots \cup L_{V_\Lambda(\neg\alpha_n)} = L_{V_\Lambda(\neg\alpha_1 \vee \cdots \vee \neg\alpha_n)}.$$

So, there exists $Z \in R_\Lambda(P)$ such that $Z \cap V_\Lambda(\neg\alpha_1 \vee \cdots \vee \neg\alpha_n) = \emptyset$, i.e., $Z \subseteq V_\Lambda(\neg\alpha_1 \wedge \cdots \wedge \neg\alpha_n)$. Then, $Z \subseteq V_\Lambda(\varphi)$, and as $Z \in R_\Lambda(P)$, we get that $\Diamond\varphi \in P$. Therefore $F \subseteq \Diamond^{-1}(P)$. Then $\hat{F} \in R_\Lambda(P)$. By (3.1), there exists $\alpha_0 \in \Gamma$ such that $\hat{F} \cap V_\Lambda(\alpha_0) \neq \emptyset$. So, there is $Q \in X_\Lambda$ such that $F \subseteq Q$ and $\alpha_0 \in Q$. But as $V_\Lambda(\neg\alpha_0) \subseteq \hat{F}$, we have that $\alpha_0 \in Q$, which is impossible because Q is a maximal theory. Thus, there exists a finite subset Γ_j of Γ such that $R_\Lambda(P) \subseteq \bigcup \left\{L_{V_\Lambda(\alpha)} : \alpha \in \Gamma_j\right\}$.

By definition of the relation R_Λ we have that $\mathcal{M}_\Lambda$ is image-compact. Thus, $\mathcal{M}_\Lambda$ is modally saturated. ■

The proof of the following Theorem 12 is essentially the same as is given in [10, Theorem 4.31] for locally core-finite monotonic models using Hansen's definition of modal saturation. Here we include a proof using our new notion of models modally saturated. First we needed the following auxiliary result.

Lemma 11. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models. Let $Z \in \mathcal{P}(X_2)$ and $K \in \mathcal{P}(X_1)$. If K is a compact subset of $\left\langle X_1, \mathcal{T}_{D_{V_1}} \right\rangle$, then*

$$Z \preccurlyeq K \text{ iff } \forall\varphi \in Fm\ (K \subseteq V_1(\varphi), \text{ implies } Z \subseteq V_2(\varphi)).$$

Proof. Suppose that $Z \not\preccurlyeq K$. Then there exists an element $z \in Z$ such that $F^{\mathcal{M}_2}(z) \neq F^{\mathcal{M}_1}(k_i)$, for all $k_i \in K$. So, for each $k_i \in K$ there exists a formula φ_i such that $z \in V_2(\varphi_i)$ and $k_i \in V_1(\neg\varphi_i)$. Then $K \subseteq \bigcup\{V_1(\neg\varphi_i) \mid k_i \in K\}$. By compactness there exists a finite subset $\{\neg\varphi_1, \ldots, \neg\varphi_n\}$ such that

$$K \subseteq V_1(\neg\varphi_1) \cup \cdots \cup V_1(\neg\varphi_n) = V_1(\neg\varphi_1 \vee \cdots \vee \neg\varphi_n) = V_1(\neg\varphi).$$

Then $Z \cap V_2(\varphi) \neq \emptyset$ and $K \cap V_1(\varphi) = \emptyset$. The other direction is easy and left to the reader. ■

THEOREM 12. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two modally saturated models. Then the relation $\approx$ is a bisimulation.*

PROOF. We prove the condition **B1**. Let $a \in X_1$ and $b \in X_2$. Assume that $a \approx b$. Let $(a, Y) \in R_1$. As $\mathcal{M}_1$ is image-compact, there exists a compact subset K of $\left\langle X_1, \mathcal{T}_{D_{V_1}} \right\rangle$ such that $K \subseteq Y$ and $(a, K) \in R_1$.

Now we suppose that $Z_i \not\preccurlyeq K$, for any $Z_i \in R_2(b)$. Then by Lemma 11, for each $Z_i \in R_2(b)$ there exists a formula φ_i such that $Z_i \cap V_2(\varphi_i) \neq \emptyset$ and $K \cap V_1(\varphi_i) = \emptyset$. Then

$$R_2(b) \subseteq \bigcup \left\{ L_{V_2(\varphi_i)} \mid Z_i \in R_2(b) \right\}.$$

Since $\mathcal{M}_2$ is point-compact, there exists a finite set $\{\varphi_1, \ldots, \varphi_n\}$ such that

$$R_2(b) \subseteq L_{V_2(\varphi_1)} \cup \cdots \cup L_{V_2(\varphi_n)} = L_{V_2(\varphi_1 \vee \cdots \vee \varphi_n)} = L_{V_2(\varphi)},$$

i.e. for every $Z_i \in R_2(b)$, we have that $Z_i \cap V_2(\varphi) \neq \emptyset$. So, $b \in V_2(\Box\varphi) = V_2(\neg\Diamond\neg\varphi)$. As $a \approx b$, we have $a \in V_1(\Box\varphi)$. But as $(a, K) \in R_1$ and $K \cap (V_1(\varphi_1) \cup \cdots \cup V_1(\varphi_n)) = K \cap V_1(\varphi) = \emptyset$, we get $a \in V_1(\neg\Box\varphi)$, which is a contradiction. Thus, there exists $Z \in R_2(b)$ such that $Z \preccurlyeq K$. Since $K \subseteq Y$, we get $Z \preccurlyeq Y$.

The proof of condition **B2** is similar and left to the reader. ■

COROLLARY 13. *The class of all modally saturated models is a Hennessy-Milner class.*

3.1. Normal Monotonic Saturated Models

We recall that there exists a bijective correspondence between normal monotonic models and Kripke models, and we see that the notions of modally saturated in normal monotonic models and Kripke models are equivalent. This is another argument to say that the definition of modal saturation for monotonic neighbourhood models given in [3], is the right one.

DEFINITION 14. A monotonic frame $\langle X, R \rangle$ is called *normal* if for any $x \in X$ and for any $Y \in R(x)$ there exists $y \in Y$, such that $\{y\} \in R(x)$.

Given a Kripke model $\mathcal{K} = \langle X, S, V \rangle$, we obtain a normal monotonic model

$$\mathcal{K}^{\mathrm{n}} = \langle X, R_S, V \rangle$$

where the relation $R_S \subseteq X \times \mathcal{P}(X)$ is defined as

$$(x, Y) \in R_S \Leftrightarrow S\,(x) \cap Y \neq \emptyset.$$

It is easy to prove that

$$\Diamond_{R_S}(V(\varphi)) = \{x \in X \mid S\,(x) \cap V(\varphi) \neq \emptyset\}\,.$$

Conversely, given a normal monotonic model $\mathcal{M} = \langle X, R, V\rangle$, we define the Kripke model

$$\mathcal{M}^{\mathrm{krp}} = \langle X, S_R, V\rangle$$

by taking

$$S_R(x) = \{y \in X : (x, \{y\}) \in R\}\,.$$

It is easy to prove that $V(\Diamond\varphi) = \big\{x \in X \mid R(x) \cap L_{V(\varphi)}\big\} = \big\{x \in X \mid S_R(x) \cap V(\varphi) \neq \emptyset\big\}$. These transformations are mutually inverse of each other (see Lemma 11 in [5]). Thus, the semantic definition (2.1) of the diamond operator $\Diamond_R$ is an adequate extension of the usual interpretation of the diamond operator in Kripke frames. Similar considerations can be given for the box operator $\Box_R$.

We recall that a Kripke model $\mathcal{K} = \langle X, S, V\rangle$ is *Kripke modally saturated,* if for all $x \in X$ and all set Ψ of formulas the following condition is verified:

KMS: If Ψ is finitely satisfiable in $S(x)$, then Ψ is satisfiable in $S(x)$.

It is well known that over the class of modally saturated Kripke models, modal equivalence implies Kripke bisimilarity (see e.g. [2]).

Remark 15. We note that a Kripke model $\mathcal{K} = \langle X, S, V\rangle$ is Kripke modally saturated iff $S(x) = \{y \in X : (x, y) \in S\}$ is a compact subset in the topological space $\langle X, \mathcal{T}_{D_V}\rangle$, for each $x \in X$.

Lemma 16. *Every normal monotonic model is image-compact.*

Proof. Let $\mathcal{M} = \langle X, R, V\rangle$ be a normal monotonic model. Let $Y \in R(x)$. As $\mathcal{M}$ is normal, there exists $y \in Y$ such that $\{y\} \in R(x)$. As $\{y\} \subseteq Y$ and $\{y\}$ is a compact subset of $\langle X, \mathcal{T}_{D_V}\rangle$, we have that $\mathcal{M}$ is image-compact. ■

Theorem 17. *Let $\mathcal{M} = \langle X, R, V\rangle$ be a normal monotonic model. Then $\mathcal{M}$ is modally saturated iff the Kripke model $\mathcal{M}^{\mathrm{krp}} = \langle X, S_R, V\rangle$ is Kripke modally saturated.*

Proof. We prove that for any set of formulas Ψ,

$$R(x) \subseteq \bigcup \big\{L_{V(\varphi)} : \varphi \in \Psi\big\} \text{ iff } S_R(x) \subseteq \bigcup \{V(\varphi) : \varphi \in \Psi\}\,.$$

Assume that $R(x) \subseteq \bigcup \left\{ L_{V(\varphi)} : \varphi \in \Psi \right\}$. Let $y \in S_R(x)$. Then $\{y\} \in R(x)$. Thus there exists $\varphi \in \Psi$ such that $\{y\} \cap V(\varphi) \neq \emptyset$, i.e., $y \in V(\varphi)$. Assume that $S_R(x) \subseteq \bigcup \{V(\varphi) : \varphi \in \Psi\}$, and let $Y \in R(x)$. As $\mathcal{M}$ is normal, there exists $y \in Y$ such that $\{y\} \in R(x)$. So, $y \in S_R(x)$, and consequently we get that $y \in V(\varphi)$, for some $\varphi \in \Psi$. Thus, $R(x) \subseteq \bigcup \left\{ L_{V(\varphi)} : \varphi \in \Psi \right\}$.

Now, it is easy to see that $R(x)$ is a compact subset in the topological space $\mathcal{K}_R = \langle \mathcal{K}_R, \mathcal{L}_{D_V} \rangle$ iff $S_R(x)$ is a compact subset in the topological space $\langle X, \mathcal{T}_{D_V} \rangle$, for each $x \in X$. Therefore, $\mathcal{M}$ is modally saturated iff the Kripke model $\mathcal{M}^{\mathrm{krp}} = \langle X, S_R, V \rangle$ is Kripke modally saturated. ■

4. Preservation Properties of Models

We will now prove certain properties that are preserved by total bisimulations and bounded morphisms of monotonic models.

PROPOSITION 18. *Let B be a total bisimulation between the models $\mathcal{M}_1$ and $\mathcal{M}_2$. Then,*

(1) *$\mathcal{M}_1$ is compact iff $\mathcal{M}_2$ is compact.*

(2) *$\mathcal{M}_1$ is point-compact iff $\mathcal{M}_2$ is point-compact.*

PROOF. We prove only the implications from left to right. The other implication follow by the first ones because the inverse of a total bisimulation is a total bisimulation.

(1) Assume that $\mathcal{M}_1$ is compact. By Lemma 9 it is suffices to prove that the map $\varepsilon_{V_2} : X_2 \to \mathrm{Ul}(D_{V_2})$ is onto. Let $P \in \mathrm{Ul}(D_{V_2})$. Consider the subset $Q = \{V_1(\varphi) : V_2(\varphi) \in P\}$. Since P is an ultrafilter of D_{V_2}, it is easy to see that Q is an ultrafilter of D_{V_1}. Since $\mathcal{M}_1$ is compact, there exists $x \in X_1$ such that $\varepsilon_{V_1}(x) = Q$. As B is total there exists $y \in X_2$ such that $(x, y) \in B$. Then, $V_2(\varphi) \in \varepsilon_{V_2}(y)$ iff $y \in V_2(\varphi)$ iff $x \in V_1(\varphi)$ iff $V_1(\varphi) \in \varepsilon_{V_1}(x) = Q$ iff $V_2(\varphi) \in P$. Thus, $\varepsilon_{V_2}(y) = P$ and we can conclude that $\mathcal{M}_2$ is compact.

(2) Let $b \in X_2$ and let $\Gamma \subseteq Fm$. We prove that $R_2(b)$ is a compact subset of the hyperspace $\langle \mathcal{K}_{R_2}, \mathcal{T}_{D_{V_2}} \rangle$. Suppose that for any finite subset Γ_j of Γ we get

$$R_2(b) \cap \bigcap \left\{ L^c_{V_2(\varphi)} \mid \varphi \in \Gamma_j \right\} \neq \emptyset.$$

We prove that $R_2(b) \cap \bigcap \left\{ L^c_{V_2(\varphi)} \mid \varphi \in \Gamma \right\} \neq \emptyset$. Since B is total there exists $a \in X_1$ such that $(a, b) \in B$. As a and b are modally equivalent we have that

$$R_1(a) \cap \bigcap \left\{ L^c_{V_1(\varphi)} \mid \varphi \in \Gamma_j \right\} \neq \emptyset,$$

for any finite subset Γ_j of Γ. Then there exists $Y \in R_1(a)$ and $Y \cap V_1(\varphi) = \emptyset$, for all $\varphi \in \Gamma$. As B is a bisimulation, there exists $Z \subseteq X_2$ such that $(b, Z) \in R_2$ and $Z \preccurlyeq_{B^{-1}} Y$. We note that if there exists $\varphi \in \Gamma$ such that $Z \cap V_2(\varphi) \neq \emptyset$, as $Z \preccurlyeq_{B^{-1}} Y$, $Y \cap V_1(\varphi) \neq \emptyset$, which is impossible. Thus, $Z \in R_2(b) \cap \bigcap \left\{ L^c_{V_2(\varphi)} \mid \varphi \in \Gamma \right\}$, and consequently $\mathcal{M}_2$ is point-compact. ■

Proposition 19. *Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be a bounded morphism between the monotonic models $\mathcal{M}_1$ and $\mathcal{M}_2$. If f is surjective and $\mathcal{M}_2$ is image-compact, then $\mathcal{M}_1$ is image-compact.*

Proof. Let $a \in X_1$ and let $Y \in R_1(a)$. Then $f[Y] \in R_2(f(a))$. As $\mathcal{M}_2$ is image-compact, then there exists a compact subset H of X_2 such that $H \subseteq f[Y]$ and $H \in R_2(f(a))$. As f is a bounded morphism, there exists $Z \subseteq X_1$ such that $(a, Z) \in R_1$ and $f[Z] \subseteq H$. So, $Z \subseteq f^{-1}[H]$, and as $\mathcal{M}_1$ is monotonic, $(a, f^{-1}[H]) \in R_1$. We prove that $f^{-1}[H]$ is compact. Let $W \subseteq Fm$ such that $f^{-1}[H] \subseteq \bigcup \{V_1(\varphi) : \varphi \in W\}$. As f is a bounded morphism, $V_1(\varphi) = f^{-1}[V_2(\varphi)]$, for each $\varphi \in W$. We prove that

$$H \subseteq \bigcup \{V_2(\varphi) : \varphi \in W\}.$$

Let $y \in H$. As f is surjective, there exists $x \in X_1$ such that $f(x) = y$. So, $x \in f^{-1}[H]$, and consequently there exists $\varphi \in W$ such that $x \in V_1(\varphi) = f^{-1}[V_2(\varphi)]$, i.e., $f(x) = y \in \bigcup \{V_2(\varphi) : \varphi \in W\}$. As H is compact, there exist $\varphi_1, \ldots, \varphi_n \in W$ such that $H \subseteq V_2(\varphi_1) \cup \ldots \cup V_2(\varphi_2)$. So, $f^{-1}[H] \subseteq V_1(\varphi_1) \cup \ldots \cup V_1(\varphi_2)$, and thus $f^{-1}[H]$ is compact. So, $f^{-1}[H] \subseteq Y$, and we get that $\mathcal{M}_1$ is image-compact. ■

Proposition 20. *Let $\mathcal{M}_1$ be a generated submodel of $\mathcal{M}_2$. If $\mathcal{M}_2$ is image-compact, then $\mathcal{M}_1$ is image-compact.*

Proof. Assume that $\mathcal{M}_2$ is image-compact. Let $x \in X_1$ and $(x, Y) \in R_1$. Then, $(x, Y) \in R_2$. As $\mathcal{M}_2$ is image-compact, there exists a compact subset H of $\langle X_2, \mathcal{T}_{D_{V_2}} \rangle$ such that $H \subseteq Y$ and $(x, H) \in R_2$. As $H \subseteq Y \subseteq X_1$, it is easy to see that H is also a compact subset of $\langle X_1, \mathcal{T}_{D_{V_1}} \rangle$. So, $\mathcal{M}_1$ is image-compact. ■

We define the following operations on a class of monotonic models M:

$\mathbf{S}(\mathsf{M}) = \{\mathcal{M} \mid \exists \mathcal{M}' \in \mathsf{M}$ such that $\mathcal{M} \subseteq_{gs} \mathcal{M}'\}$,
$\mathbf{B}(\mathsf{M}) = \{\mathcal{M} \mid \exists \mathcal{M}' \in \mathsf{M}$ such that $\mathcal{M}$ and $\mathcal{M}'$ are bisimilar$\}$.
$\mathbf{H}_\mathbf{p}^{-1}(\mathsf{M}) = \{\mathcal{M} \mid \exists \mathcal{M}' \in \mathsf{M}$ and a surjective bounded morphism $f : \mathcal{M} \to \mathcal{M}'\}$

We note that $\mathbf{H}_{\mathbf{p}}^{-1}(\mathsf{M}) \subseteq \mathbf{B}(\mathsf{M})$, because every surjective bounded morphism is also a total bisimulation.

Let **MSAT** and **CMSAT** be the class of all modally saturated and the class of compact and modally saturated, respectively.

COROLLARY 21. $\mathbf{H}_{\mathbf{p}}^{-1}\mathbf{S}(\mathbf{MSAT}) \subseteq \mathbf{MSAT}$ *and* $\mathbf{H}_{\mathbf{p}}^{-1}\mathbf{S}(\mathbf{CMSAT}) \subseteq \mathbf{CMSAT}$.

PROOF. It follows by Proposition 19 and Proposition 20. ■

5. Characterization of CMSAT

In this section we give a characterization of the class **CMSAT** in terms of the operators $\mathbf{H}_{\mathbf{p}}^{-1}$ and $\mathbf{S}$.

The following result was proved in [3] and we will used in Theorem 23.

PROPOSITION 22. *Let $\mathcal{M}$ be a compact model. Then $\mathcal{M}$ is point-compact iff for all $x \in X$, for all $Y \in \mathcal{P}(X)$, if $\bigcap \{\varepsilon_V(y) \mid y \in Y\} \subseteq \Diamond_R^{-1}(\varepsilon_V(x))$, then there exists a subset $Z \subseteq X$ such that $Z \in R(x)$ and $Z \subseteq \mathrm{cl}(Y)$,*

where $\mathrm{cl}(Y)$ is the closure of Y in the space $\langle X, \mathcal{T}_{D_V}\rangle$.

THEOREM 23. *Let $\mathcal{M}_\Lambda$ be the canonical model of* **MON**. *Then,* $\mathbf{H}_{\mathbf{p}}^{-1}\mathbf{S}(\mathcal{M}_\Lambda) = \mathbf{CMSAT}$.

PROOF. By Theorem 10 we get that $\mathcal{M}_\Lambda = \langle X_\Lambda, R_\Lambda, V_\Lambda\rangle \in \mathbf{CMSAT}$. From Corollary 21 we have that $\mathbf{H}_{\mathbf{p}}^{-1}\mathbf{S}(\mathcal{M}_\Lambda) \subseteq \mathbf{CMSAT}$.

Let $\mathcal{N} = \langle X, R, V\rangle \in \mathbf{CMSAT}$. Let us consider the map $f : X \to X_\Lambda$ defined by

$$f(x) = \{\varphi \in Fm \mid x \in V(\varphi)\}.$$

It is readily checked that $f(x)$ is a maximal theory. We note that f is a continuous function between the spaces $\langle X, \mathcal{T}_{D_V}\rangle$ and $\left\langle X_\Lambda, \mathcal{T}_{D_{V_\Lambda}}\right\rangle$, because

$$f^{-1}(V_\Lambda(\varphi)) = V(\varphi),$$

for each formula φ. Thus, if C is a closed subset of $\left\langle X_\Lambda, \mathcal{T}_{D_{V_\Lambda}}\right\rangle$, then $f^{-1}[C]$ is a closed subset of $\langle X, \mathcal{T}_{D_V}\rangle$, and therefore it is compact, because $\langle X, \mathcal{T}_{D_V}\rangle$ is compact.

We prove that f is a surjective bounded morphism between $\mathcal{N}$ and $\mathcal{M}_\Lambda$. Let $P \in X_\Lambda$. Let us consider the set

$$\Gamma = \{V(\varphi) \mid \varphi \in P\}.$$

It is easy to see that Γ is an ultrafilter of $D_V = \{V(\varphi) \mid \varphi \in Fm\}$. As $\mathcal{N}$ is compact, we have by Lemma 9 that the map $\varepsilon_V : X \to X_\Lambda$ is surjective. So, $\varepsilon_V(x) = \Gamma$, for some $x \in X$. Then,

$$\begin{aligned} V(\varphi) \in f(x) \quad & \text{iff } x \in V(\varphi) \text{ iff } V(\varphi) \in \varepsilon_V(x) \\ & \text{iff } V(\varphi) \in \Gamma \text{ iff } \varphi \in P. \end{aligned}$$

Thus, $f(x) = P$, and consequently f is surjective.

Now we prove that f satisfies the conditions **BM1** and **BM2** given on page 7.

Let $x \in X$ and let $H \subseteq Y$. Assume that $(x, H) \in R$. We prove that $(f(x), f[H]) \in R_\Lambda$. We analyzed two cases.

(a) Suppose that H is a compact subset of X. As f is a continuous function, $f[H]$ is a compact subset of $\left\langle X_\Lambda, \mathcal{T}_{D_{V_\Lambda}} \right\rangle$. So,

$$\begin{aligned} \varphi \in \bigcap \{f(h) \mid h \in H\} & \Leftrightarrow \forall h \in H \ (h \in V(\varphi)) \\ & \Leftrightarrow H \subseteq V(\varphi). \end{aligned}$$

As $(x, H) \in R$, we get that $x \in V(\Diamond\varphi)$. Thus, $\varphi \in \Diamond^{-1}(f(y))$. This completes the proof that $(f(x), f(H)) \in R_\Lambda$.

(b) If H is not a compact subset of X, then as $\mathcal{N}$ is image-compact, there exists a compact subset H' of $\langle X, \mathcal{T}_{D_V} \rangle$ such that $H' \subseteq H$ and $(x, H') \in R$. As in the point (a), $f[H']$ is a compact subset of $\left\langle X_\Lambda, \mathcal{T}_{D_{V_\Lambda}} \right\rangle$ such that $(f(y), f[H']) \in R_\Lambda$. As $f[H'] \subseteq f[H]$, we deduce that $(f(x), f[H]) \in R_\Lambda$.

Let $x \in X$ and $Y \subseteq X_\Lambda$ such that $(x, Y) \in R_\Lambda$. Then there exists a Λ-theory F such that $\hat{F} \subseteq Y$ and $F \subseteq \Diamond^{-1}(f(x))$. Since $\hat{F}$ is a closed subset of $\left\langle X_\Lambda, \mathcal{T}_{D_{V_\Lambda}} \right\rangle$ and f is continuous, $f^{-1}(\hat{F}) = Z$ is a closed subset

and thus it is a compact subset of $\langle X, \mathcal{T}_{D_V} \rangle$. We prove that

$$\bigcap \{\varepsilon_V(z) \mid z \in Z\} \subseteq \Diamond_R^{-1}(\varepsilon_V(x)).$$

Take a formula φ such that $V(\varphi) \in \bigcap \{\varepsilon_V(z) \mid z \in Z\}$. Then $\varphi \in f(z)$, for all $z \in Z$. So, $\varphi \in F \subseteq \Diamond^{-1}(f(x))$. It follows $x \in V(\Diamond\varphi)$). As $\mathcal{N}$ is point-compact and compact, from Proposition 22 there exists a subset H of X such that $(x, H) \in R$ and $H \subseteq \mathrm{cl}(Z) = Z$. So, $f[H] \subseteq f[Z] \subseteq \hat{F} \subseteq Y$. Therefore, $(x, H) \in R$, and $f[H] \subseteq Y$. Therefore f is a surjective bounded morphism. ■

6. Maximal Hennessy-Milner Classes

In this section we will prove that the class of all modally saturated models **MSAT** is a Hennessy-Milner class not properly included in any Hennessy-Milner class, i.e. **MSAT** is a maximal Hennessy-Milner class. The proof follows a similar argument to the proof of the fact that the class of m-saturated Kripke models is a Hennessy-Milner maximal class, as is expounded in [13].

Let $\langle X_\Lambda, R_\Lambda, V_\Lambda \rangle$ be the canonical model of the monotonic logic **MON**. This model is also called the Henkin model of **MON**.

Definition 24. A *Henkin-like* model is a structure $\mathcal{M} = \langle X_\Lambda, R^{\mathcal{M}}, V_\Lambda \rangle$ with universe and valuation the same as in the canonical model of **MON** and a relation $R^{\mathcal{M}} \subseteq R_\Lambda$ such that for every formula ϕ and every $P \in X_\Lambda$

$$\mathcal{M} \models_P \phi \quad \text{iff} \quad \phi \in P.$$

We note that a Henkin-like model is compact, because the space $\left\langle X_\Lambda, \mathcal{T}_{D_{V_\Lambda}} \right\rangle$ is compact. Now we will prove that, each Henkin-like model produces a maximal Hennessy-Milner class. This result will be used to prove that any Hennessy-Milner class is contained in a maximal Hennessy-Milner class (see Theorem 28). The analogue of this result for the normal modal logic case is due to A. Visser; see [13] for its proof.

Lemma 25. *If $\mathcal{M}$ is a Henkin-like model, then* $\mathbf{BS}(\mathcal{M})$ *is a maximal Hennessy-Milner class.*

Proof. Let $\mathcal{M}$ be a Henkin-like model. In $\mathcal{M}$, $P \approx P'$ iff $P = P'$. So, $\approx$ is a bisimulation in $\mathcal{M}$, and thus $\mathcal{M}$ has the Hennessy-Milner property. First of all we see that the class $\mathbf{BS}(\mathcal{M})$ is a Hennessy-Milner class. In $\mathbf{S}(\mathcal{M})$ the relation $\approx$ between two models is the identity, because this holds in $\mathcal{M}$, and therefore is a bisimulation. Hence $\mathbf{S}(\mathcal{M})$ is a Hennessy-Milner class. Moreover, as the composition of total bisimulations is again a total bisimulation, we get if M is a Hennessy-Milner class, then $\mathbf{B}(\mathsf{M})$ is too. We conclude that $\mathbf{BS}(\mathcal{M})$ is a Hennessy-Milner class as desired.

Let us see that $\mathbf{BS}(\mathcal{M})$ is maximal among the Hennessy-Milner classes. Assume that M is a Hennessy-Milner class that includes $\mathbf{BS}(\mathcal{M})$. We prove that $\mathsf{M} \subseteq \mathbf{BS}(\mathcal{M})$. Let $\mathcal{N} = \langle X, R, V \rangle \in \mathsf{M}$. We prove that there exists a generated submodel $\mathcal{N}'$ of $\mathcal{M}$ and a surjective bounded morphism $f : \mathcal{N} \to \mathcal{N}'$.

Consider the model

$$\mathcal{N}' = \langle \{F^{\mathcal{N}}(x) \mid x \in X\}, R', V' \rangle,$$

where R' is defined by

$$(F^{\mathcal{N}}(x), F^{\mathcal{N}}[Y]) \in R' \quad \text{iff} \quad \bigcap\{F^{\mathcal{N}}(y) \mid y \in Y \subseteq X\} \subseteq \Diamond^{-1}(F^{\mathcal{N}}(x)),$$

and V' is defined by

$$V'(p) = \{F^{\mathcal{N}}(x) \mid p \in F^{\mathcal{N}}(x)\},$$

for each propositional variable p. Clearly

$$\{F^{\mathcal{N}}(x) \mid x \in X\} \subseteq X_{\Lambda},$$

because for each $x \in X$ the set $F^{\mathcal{N}}(x)$ is a maximal and consistent theory. It is clear that if $(x, Y) \in R$, then $(F^{\mathcal{N}}(x), F^{\mathcal{N}}[Y]) \in R'$.

Let us see that $\mathcal{N}' \in \mathbf{S}(\mathcal{M})$. Also it is clear that $V(p) = (F^{\mathcal{N}})^{-1}(V'(p))$, for each propositional variable p.

Assume that $x \in X$, $T \subseteq X_{\Lambda}$ and $(F^{\mathcal{N}}(x), T) \in R^{\mathcal{M}}$. Since, M is a Hennessy-Milner class and $\mathcal{M}, \mathcal{N} \in \mathsf{M}$,we have that $\approx$ is a bisimulation between $\mathcal{N}$ and $\mathcal{M}$. Therefore, there is $Z \in R(x)$ such that $F^{\mathcal{N}}[Z] \subseteq T$. Thus $F^{\mathcal{N}}$ is a surjective bounded morphism, and thus $\mathcal{N}'$ is a generated submodel of $\mathcal{M}$.

Since $\mathcal{N} \in \mathsf{M}$, $\mathcal{N}' \in \mathbf{H}_{\mathbf{p}}^{-1}\mathbf{S}(\mathcal{M}) \subseteq \mathbf{BS}(\mathcal{M}) \subseteq \mathsf{M}$ and M is a Hennessy-Milner class, we obtain that the relation $x \approx F^{\mathcal{N}}(x)$ is an bisimulation between $\mathcal{N}$ and $\mathcal{N}'$. Therefore, $\mathcal{N} \in \mathbf{BS}(\mathcal{M})$. ■

Let M be any class of models. Our next task is to define a Henkin-like model associated with M. We will define a relation $R_{\mathsf{M}} \subseteq X_{\Lambda} \times \mathcal{P}(X_{\Lambda})$ such that the map $F^{\mathcal{N}} : \mathcal{N} \to \langle X_{\Lambda}, R_{\mathsf{M}}\rangle$ be a bounded morphism for all $\mathcal{N} \in \mathsf{M}$.

We denote by $\mathcal{M}_{\mathsf{M}}$ the monotonic model whose accessibility relation R_{M} is defined by:

$$(P, Y) \in R_{\mathsf{M}} \quad \text{iff} \quad \begin{array}{l} (1)\ (P, Y) \in R_{\Lambda} \text{ and } P \neq F^{\mathcal{N}}(x) \text{ for all } \mathcal{N} \in \mathsf{M}, \text{ or} \\ (2) \text{ there exists a model } \mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}, \\ \text{an element } x \in X, \text{ and a subset } Z \subseteq X \text{ such that} \\ P = F^{\mathcal{N}}(x),\ (x, Z) \in R, \text{ and } F^{\mathcal{N}}[Z] \subseteq Y. \end{array}$$

Lemma 26. *$\mathcal{M}_{\mathsf{M}} = \langle X_{\Lambda}, R_{\mathsf{M}}, V_{\Lambda}\rangle$ is a Henkin-like model.*

Proof. We need to prove that

$$P \in V_{\Lambda}(\varphi) \text{ iff } \varphi \in P,$$

for any formula φ, and for all $P \in X_{\Lambda}$. The proof is by induction on the complexity of the formulas. We prove only the case of formulas $\Diamond\varphi$.

$\Rightarrow$) Assume that $P \in V_\Lambda(\Diamond\varphi) = \{Q \in X_\Lambda \mid \exists Y \in R_{\mathsf{M}}(Q)\,(Y \subseteq V_\Lambda(\varphi))\}$. Then there exists $Y \in R_{\mathsf{M}}(P)$ such that $Y \subseteq V_\Lambda(\varphi)$. By induction hypothesis, $\varphi \in Q$, for all $Q \in Y$. By definition of R_{M} we have two cases:

(a) If $(P, Y) \in R_\Lambda$ and $P \neq F^{\mathcal{N}}(x)$ for all $\mathcal{N} \in \mathsf{M}$. As $Y \subseteq V_\Lambda(\varphi)$, we have by induction hypothesis that $\varphi \in \bigcap\{Q \in X_\Lambda \mid Q \in Y\} = F_Y$, and since $F_Y \subseteq \Diamond^{-1}(P)$, we get $\Diamond\varphi \in P$.

(b) Suppose that there exists a model $\mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}$, there exists an element $x \in X$, and there exist a subset $Z \subseteq X$ such that

$$P = F^{\mathcal{N}}(x), (x, Z) \in R \text{ and } F^{\mathcal{N}}[Z] \subseteq Y.$$

Since $F^{\mathcal{N}}[Z] \subseteq Y \subseteq V_\Lambda(\varphi)$, we have $F^{\mathcal{N}}(z) \in V_\Lambda(\varphi)$, for all $z \in Z$, i.e., $z \in V(\varphi)$. So, $Z \subseteq V(\varphi)$. Since $(x, Z) \in R$, $x \in V(\Diamond\varphi)$. Thus, $\Diamond\varphi \in F^{\mathcal{M}}(x) = P$.

$\Leftarrow$) Suppose that $\Diamond\varphi \in P$. We have two cases:

(a) If $(P, Y) \in R_\Lambda$ and $P \neq F^{\mathcal{N}}(x)$ for all $\mathcal{N} \in \mathsf{M}$. Take the set $Y_\varphi = \{Q \in X_\Lambda \mid \varphi \in Q\}$. Then it is easy to see that $F_{Y_\varphi} \subseteq \Diamond^{-1}(P)$, i.e., $(P, Y_\varphi) \in R_\Lambda$.

(b) Suppose that there exists a model $\mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}$, and there exists an element $x \in X$ such that $P = F^{\mathcal{N}}(x)$. As $\Diamond\varphi \in P = F^{\mathcal{N}}(x)$, there exists $Z \subseteq X$ such that $(x, Z) \in R$ and $Z \subseteq V(\varphi)$. Thus, $F^{\mathcal{N}}[Z] \subseteq V_\Lambda(\varphi)$. If we take $Y = F^{\mathcal{N}}[Z]$, then $(P, Y) \in R_{\mathsf{M}}$. ∎

The next result is needed in the proof of Theorem 28.

Lemma 27. *Let* M *be a Hennessy-Milner class of models. Then for all models* $\mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}$, *the map* $F^{\mathcal{N}} : X \to \mathcal{M}_{\mathsf{M}}$ *is a bounded morphism.*

Proof. Let $x \in X$ and $Z \subseteq X$ such that $(x, Z) \in R$. We need to prove that $(F^{\mathcal{N}}(x), F^{\mathcal{N}}[Z]) \in R_{\mathsf{M}}$, but this is immediate by the definition of relation R_{M}.

Let $Y \subseteq X_\Lambda$. Assume that $(F^{\mathcal{N}}(x), Y) \in R_{\mathsf{M}}$. By the definition of R_{M}, we have a model $\mathcal{N}' = \langle X', R', V'\rangle \in \mathsf{M}$, an element $x' \in X'$ and $Z \in R'(x)$ such that $F^{\mathcal{N}}(x) = F^{\mathcal{N}'}(x')$, and $F^{\mathcal{N}'}[Z] \subseteq Y$. So, $x \approx x'$. Since M is a Hennessy-Milner class, $\approx$ is a bisimulation between $\mathcal{N}$ and $\mathcal{N}'$. So, given $(x', Z) \in R'$, there exists $Z \in R(x)$ such that $F^{\mathcal{N}}[Z] \subseteq F^{\mathcal{N}'}[Z'] \subseteq Y$. Thus $F^{\mathcal{N}}$ is a bounded morphism. ∎

The next Theorem and its proof are a monotonic version of an analogous results for arbitrary Hennessy-Milner classes of Kripke models due to A. Visser (see [13] for a proof), which states that every Hennessy-Milner class can be extended to a maximal one.

Theorem 28. *Let* M *be a Hennessy-Milner class. Then* $\mathsf{M} \subseteq \mathbf{BS}(\mathcal{M}_{\mathsf{M}})$.

Proof. Let $\mathcal{M} = \langle X, R, V\rangle \in \mathsf{M}$. By Lemma 27 $F^{\mathcal{M}} : \mathcal{M} \to \mathcal{M}_{\mathsf{M}}$ is a bounded morphism. So, the model $F^{\mathcal{M}}[\mathcal{M}] = \mathcal{M}'$ is a generated submodel of $\mathcal{M}_{\mathsf{M}}$, and the graph of $F^{\mathcal{M}}$ is total bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. Hence, $\mathcal{M} \in \mathbf{BS}(\mathcal{M}_{\mathsf{M}})$. ■

Corollary 29. *A Hennessy-Milner class* M *is maximal iff there exists a Henkin-like model* $\mathcal{M}$ *such that* $\mathsf{M} = \mathbf{BS}(\mathcal{M})$.

Proof. Let M be a maximal Hennessy-Milner class. Consider the Henkin-like model $\mathcal{M}_{\mathsf{M}}$. From Theorem 28, $\mathsf{M} \subseteq \mathbf{BS}(\mathcal{M}_{\mathsf{M}})$, and as M is maximal, we have that $\mathsf{M} = \mathbf{BS}(\mathcal{M}_{\mathsf{M}})$. The converse is immediate. ■

7. Conclusions

In this paper we study a notion of modal saturation that enables us to extended several theorems from Kripke semantics to neighbourhood semantics. The main idea is to replace the notion of finiteness involved in the Kripke version of modal saturation with the compactness in the hyperspace associated together with the new notion of image-compactness. We prove that some of these properties are preserved by total bisimulations and bounded morphisms of monotonic models, and we prove also that the generated submodels of image-compact models are image-compact. We show that the class of compact and modally saturated is generated by the canonical model of the minimal monotonic modal logic **MON** by means of the operators $\mathbf{H_p}^{-1}$ and $\mathbf{S}$. Finally, we prove that the class of modally saturated models **MSAT** is a maximal Hennessy-Milner class.

Every modally saturated model, according to Hansen's definition, is modally saturated in our sense, but the converse is not true. On the other hand, in [12, Definition 4.4], other notion of m-saturation for neighbourhood models, called *neighbourhood modal saturation,* is defined. However, it is not clear whether monotonic modal saturation and neighbourhood modal saturation coincides in all monotonic models (see Remark 4.8 in [12, Definition 4.4]). Neither is it clear whether Definition 8 and neighbourhood modal saturation coincides in all monotonic models. Therefore, it is interesting to make a comprehensive comparative analysis of all these notions of saturation. On the other hand, we prove that the notion of monotonic modal saturation defined in [3] extended the usual notion of m-saturation in Kripke models. This fact is important because the notions of neighbourhood modal saturation defined in [12, Definition 4.4], and the notion monotonic and the Hansen's definition of monotonic saturation defined in [10] are not accurate generalizations of the Kripke modal saturation.

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