ON THE ROOTS OF THE GENERALIZED ROGERS-RAMANUJAN FUNCTION

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ABSTRACT. We give simple proofs of the fact that for certain parameters the roots of the generalized Rogers-Ramanujan function are irrational numbers and, for example, that at least one of the following two numbers is irrational: $\left\{\sum_{n=1}^{\infty} \frac{F_n}{m^n \prod_{i=0}^{n-1} \phi(k+i)}, \sum_{n=1}^{\infty} \frac{F_n}{m^n \prod_{i=0}^{n-1} \phi(k+i+1)}\right\}$ where $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0, F_1 = 1$ (the Fibonacci sequence), m is a natural number $> \frac{1+\sqrt{5}}{2}$ and $\phi(k)$ is *any* function taking positive integer values such that $\limsup_{k\to\infty} \phi(k) = \infty$.

1. Introduction and results.

The irrationality of π was first proved by J. H. Lambert in 1761 using the continued fraction for the function $\tan x$. Nowadays proofs avoid the use of continued fractions and use a variant of Hermite's ideas; a proof of this type was given by I. Niven [3]. M. Laczkovich's proof of the irrationality of π presented in [5] is particularly simple and contains ideas from J. Popken's paper [6].

The aim of this note is to give short proofs of two irrationality theorems, both inspired by M. Laczkovich's proof. In fact, we use ideas that are of elementary nature. One may say that the crux of Laczkovich's proof is based on the existence of a one-parameter family satisfying a certain recursion. To prove our theorems we follow the same path using three one-parameter families, namely (1), (3) and (4).

Our first and most important result is a general theorem which implies that, for certain parameters, the roots of the generalized Rogers-Ramanujan function are irrational numbers.

We write $Q := a_1 a_2^2 \cdots a_r^r$ for short and define $f_k = f_k(x, a_1, \ldots, a_r)$ by

$$f_k := \sum_{n=0}^{\infty} \frac{1}{a_1^{nk+\frac{n(n-1)}{2}} \cdots a_r^{rnk+\frac{rn^2}{2}+n(1-\frac{3r}{2})}} \frac{x^n}{\prod_{i=1}^{n}(1-Q^i)}, \tag{1}$$

where if $n = 0$ it is understood that $\prod_{i=1}^{n}(1-Q^i) = 1$.

The following theorem holds.

Theorem 1. *Assume $a_i \in \mathbb{Z}$ and $|a_1 \cdots a_r| \geq 2$. If $x \neq 0$ is a rational number and $k = 0, 1, 2, \ldots$ then $f_k \neq 0$ and $\frac{f_{k+1}}{f_k}$ is irrational.*

2010 *Mathematics Subject Classification.* 11J72, 11P84.

Key words and phrases. Irrationality. Rogers-Ramanujan functions.

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Corollary. *Assume that* $1/q = \pm 2, \pm 3, \dots$ *and* $k = 0, 1, 2, \dots$. *If a real number* x_0 *satisfies*

$$1 + \sum_{n=1}^{\infty} \frac{x_0^n q^{n^2+kn}}{(1-q)\cdots(1-q^n)} = 0, \tag{2}$$

then x_0 *is irrational.*

The function appearing in (2) is the generalized Rogers-Ramanujan function (see [1], [2] and [4]). Observe that (2) can be written as $f_k(-x_0, 1/q) = 0$ and therefore the conclusion follows from the theorem.

Note that

$$f_0(-1, 1/q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

and

$$f_1(-1, 1/q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})},$$

are the Rogers-Ramanujan functions, see [2] or [4] pp. 78.

To state the next theorem we need to define the functions h_k and g_k.

Definition. (i) If A, B are real numbers, let F_n be defined recursively by

$$F_n = A(F_{n-1} - BF_{n-2}),$$

for $2 \leq n$ with initial values $F_0 = 0, F_1 = 1$.

Let $\phi(k)$ be a function taking non-zero real values and whose domain is $\mathbb{N} \bigcup \{0\}$. If $k = 0, 1, 2, \dots$ we set

$$h_k := h_k(x) = \sum_{n=1}^{\infty} \frac{F_n x^n}{\prod_{i=0}^{n-1} \phi(k+i)}. \tag{3}$$

(ii) Let $\eta(k), \phi(k)$ be two functions taking non-zero real values and whose domain is $\mathbb{N} \bigcup \{0\}$. If $k = 0, 1, 2, \dots$ we set

(4)

$$g_k := g_k(x) = \sum_{n=1}^{\infty} x^n \frac{\{\phi(k) + \phi(k)\phi(k+1) + \cdots + \phi(k)\phi(k+1)\cdots\phi(k+n-1)\}}{\prod_{i=0}^{n-1} \eta(k+i)}.$$

By looking at the coefficient of x^n of h_k one observes that *formally* the following recursion holds

$$\frac{B}{\phi(k+1)} x^2 h_{k+2} = x h_{k+1} - \frac{\phi(k)}{A} h_k + \frac{x}{A}. \tag{5}$$

Similarly, g_k *formally* satifies

$$\frac{\eta(k)}{\phi(k)} g_k - \left\{1 + \frac{1}{\phi(k+1)}\right\} x g_{k+1} + \frac{1}{\eta(k+1)} x^2 g_{k+2} = x. \tag{6}$$

The following theorem holds.

Theorem 2. *(i) In the definition of h_k let A, B, x be rational non-zero numbers such that $\frac{1}{xB}$, $\frac{1}{x^2AB}$ are integers. Assume also that $\phi(k)$ takes positive integer values, $0 < x$, $0 \le F_n$ for all n and at least one of the two following conditions holds: (i1) $\lim_{k\to\infty}\phi(k) = \infty$ (i2) $\limsup_{k\to\infty}\phi(k) = \infty$ and there exists $x_0 > x$ such that $\sum_1^\infty F_n x_0^n$ converges.*

Then for any $k = 1, 2, \ldots$ at least one of $\{h_k(x), h_{k+1}(x)\}$ is an irrational number.

(ii) In the definition of g_k assume that $\eta(k) = P_kQ_k$, $\phi(k) = \frac{P_{k-1}}{R_k}$ where P_k, Q_k, R_k are positive integers for all k and $\lim_{i\to\infty} Q_i = \infty$. Furthermore assume that $\sup_k \frac{P_{k-1}}{R_kP_k} \le M$ for some $1 \le M$.

Then at least one of $\{g_k(1/m), g_{k+1}(1/m)\}$ is irrational for any $m = 1, 2, 3, \ldots; k = 0, 1, 2, \ldots$.

Remarks. a) As we shall see, g_k is entire function and in case that condition $(i1)$ holds the same is true for h_k.

b) The result of the abstract follows taking $A = 1, B = -1, x = 1/m$ in $(i2)$ observing that $\sum_1^\infty F_n x_0^n$ converges if $\frac{1+\sqrt{5}}{2} < 1/x_0$.

c) The following example follows from (ii): let P_k, R_k be two sequences of positive integers such that P_{k-1}/P_k is bounded (this is satisfied, for example, if P_k is non-decreasing) and $Q_0 = 2, Q_1 = 3, Q_2 = 5, Q_3 = 7, \ldots$ (that is, Q_k is the $k+1$ prime). Then at least one of $\{g_0(1), g_1(1)\}$ is irrational.

2. Proof of Theorem 1.

Claim 1: If f_k is defined by (1) then the following recursion holds

$$f_{k+1} - f_k = x f_{k+2} \frac{1}{a_1^{k+1} \cdots a_r^{rk+1}}. \tag{7}$$

In fact, the coefficient of x^n in the expression $f_{k+1} - f_k$ is

$$\frac{1}{Q^{nk} a_1^{\frac{n(n-1)}{2}} \cdots a_r^{\frac{rn^2}{2} + n(1 - \frac{r3}{2})} \prod_{i=1}^n (1 - Q^i)} \left(\frac{1}{Q^n} - 1 \right) =$$

$$\frac{1}{Q^{nk+n} a_1^{\frac{n(n-1)}{2}} \cdots a_r^{\frac{rn^2}{2} + n(1 - \frac{r3}{2})} \prod_{i=1}^{n-1} (1 - Q^i)},$$

which is the coefficient of x^n of $x f_{k+2} \frac{1}{a_1^{k+1} \cdots a_r^{rk+1}} = x f_{k+2} \frac{1}{Q^k a_1 \cdots a_r}$.

Claim 2: One has that $f_k \to 1$ if $k \to \infty$. In fact, by hypothesis $a_i \in \mathbb{Z}$ and $|a_1 \cdots a_r| \ge 2$. For simplicity assume that $2 \le |a_1|$, the other cases are similar. Then,

$$|f_k - 1| \le \frac{|x|}{2^k} + \cdots + \frac{|x|^n}{2^{nk + \frac{n(n-1)}{2}}} + \cdots = O\left(\frac{1}{2^k} \right),$$

and the claim follows.

Claim 3: Let $C \ne 0$ be a natural number such that $\frac{C}{x}$ is a non-zero integer. Recall that $|Q| \ge 2$. Take a fixed natural number $i \ge 1$ such that $|\frac{C}{a_1^i \cdots a_r^{ri}} = \frac{C}{Q^i}| < 1$ and set

$$G_n := f_{k+n} \frac{C^n}{Q^{in}}.$$

We have $G_n \to 0$ if $n \to \infty$ and $G_n \neq 0$ if n is large enough; these facts follow from Claim 2.

Claim 4: As $C, C/x, a_i$ are all integers, the recursion (7) can be written in terms of G_n more simply as

$$G_{n+2} = S_n Q^{n-i} G_{n+1} + T_n Q^{n-2i} G_n,$$

if $0 \leq n$, where S_n and T_n are integers: in fact, using (7) one obtains

$$G_{n+2} = f_{k+n+2} \frac{C^{n+2}}{Q^{i(n+2)}} = Q^{k+n} \frac{a_1 \cdots a_r}{x} (f_{k+n+1} - f_{k+n}) \frac{C^{n+2}}{Q^{i(n+2)}} =$$

$$\left\{ Q^k a_1 \cdots a_r \frac{C}{x} \right\} Q^{n-i} G_{n+1} + \left\{ -Q^k a_1 \cdots a_r \frac{C^2}{x} \right\} Q^{n-2i} G_n.$$

Now the proof of the theorem goes as follows. Assume that the conclusion of the theorem is false: then one may write $f_k = Ay$ and $f_{k+1} = By$ for some real non-zero number y and integers A, B. Notice that we allow A, B to be zero.

This gives $G_0 = Ay$ and $G_1 = CBy/Q^i$. The last recursion yields that G_{2i}, G_{2i+1} are integer multiples of y/Q^{j_0} for some $j_0 \geq 0$. But for $n \geq 2i$ the above recursion has integer coeficients and therefore G_n is an integer multiple of y/Q^{j_0}. This is in contradiction with Claim 3.

3. Proof of Theorem 2.

(i) **Claim 1:** One has that $h_k > 0$ for all k. In case that condition ($i1$) holds then $h_k \to 0$ if $k \to \infty$. This follows from the fact that there exists some fixed $\alpha > 0$ such that $0 \leq F_n x^n \leq \alpha^n$ for all n and that $\phi(k) \to \infty$ as $k \to \infty$. This also yields that h_k is an entire function.

If condition ($i2$) holds then $h_{k_i} \to 0$ for some subsequence $k_i \to \infty$.

Claim 2: Set

$$G_n := \frac{h_{k+n}}{\phi(k+n-1)}.$$

In any case one has from Claim 1 that $G_{n_j} \to 0$ for some subsequence $n_j \to \infty$ and $G_n \neq 0$ for all n.

Claim 3: The recursion (5) can be written in terms of G_n more simply as

$$G_{n+2} = \frac{\phi(k+n)}{xB} G_{n+1} - \frac{\phi(k+n)\phi(k+n-1)}{x^2 AB} G_n + \frac{1}{xAB},$$

if $0 \leq n$.

Assume that the conclusion of the theorem is false, that is, both h_k and h_{k+1} ($1 \leq k$) are rational numbers. Then G_n and G_{n+1} are both rational numbers, say, they are integer multiples of $1/D$ with $D \in \mathbb{N}$. Recall that $1/xB$ and $1/x^2AB$ are integers and $1/xAB$ is a rational number, say, with denominator $K \in \mathbb{N}$. Then the last recursion yields that G_n is an integer multiple of $1/KD$ for all n.

This is in contradiction with Claim 2.

(ii) **Claim 1:** We show that g_k, which is a series of positive terms, is an entire function. It is enough to prove that the series (4) converges for any $0 < x$.

Next observe that if $1 \leq i \leq n$ then

$$0 < \frac{\prod_{j=0}^{i-1} \phi(k+j)}{\prod_{j=0}^{n-1} \eta(k+j)} = \left(\prod_{j=0}^{i-1} \frac{P_{k-1+j}}{R_{k+j} P_{k+j} Q_{k+j}} \right) \frac{1}{\prod_{j=i}^{n-1} \eta(k+j)} \leq \frac{M^n}{\prod_{j=0}^{n-1} Q_{k+j}}.$$

Therefore, if $0 < x$ then $0 < g_k(x) \leq \frac{Mx}{Q_k} + \cdots + \frac{n(Mx)^n}{\prod_{j=0}^{n-1} Q_{k+j}} + \cdots$, where this last function converges because $\lim_{i\to\infty} Q_i = \infty$. Thus g_k is an entire function.

In other words: if $0 < x$ then $0 \neq g_k(x) \to 0$ when $k \to \infty$.

Claim 2: Set

$$G_n := \frac{g_{k+n}}{\eta(k+n-1)} = \frac{g_{k+n}}{P_{k+n-1}Q_{k+n-1}}.$$

From Claim 1 one gets that $G_n \to 0$ if $n \to \infty$ and $G_n \neq 0$ for all n.

Claim 3: Dividing by x^2 and putting $k+n$ instead of k, the recursion (6) can be written in terms of G_n as

$$\frac{\eta(k+n)\eta(k+n-1)}{x^2\phi(k+n)} G_n - \left\{1 + \frac{1}{\phi(k+n+1)}\right\} \frac{\eta(k+n)}{x} G_{n+1} + G_{n+2} = \frac{1}{x},$$

or putting $x = 1/m$ and using the hypothesis the last can be written as

$$m^2 P_{k+n}Q_{k+n}Q_{k+n-1}R_{k+n}G_n - m(P_{k+n}Q_{k+n} + Q_{k+n}R_{k+n+1})G_{n+1} + G_{n+2} = m,$$

which is a recursion of the form

$$G_{n+2} = G_{n+1}A_n + G_nB_n + C_n,$$

if $0 \leq n$, where A_n, B_n, C_n are integers.

To prove the theorem, assume that both $g_k(1/m)$ and $g_{k+1}(1/m)$ are rational numbers. Then G_0 and G_1 are both rational numbers, say, they are integer multiples of $1/D$ with $D \in \mathbb{N}$. Then the last recursion yields that G_n is an integer multiple of $1/D$ for all n.

This is in contradiction with Claim 2.

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