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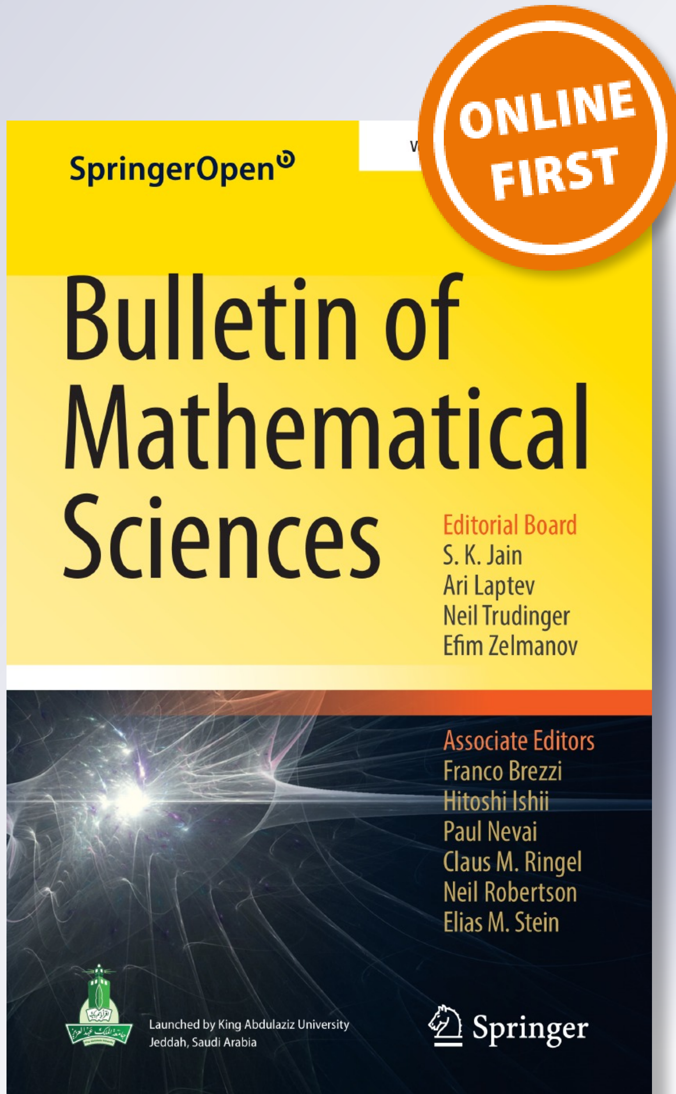

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Conjugacy theorems for loop reductive group schemes and Lie algebras

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Abstract The conjugacy of split Cartan subalgebras in the finite-dimensional simple case (Chevalley) and in the symmetrizable Kac–Moody case (Peterson–Kac) are fundamental results of the theory of Lie algebras. Among the Kac–Moody Lie algebras the affine algebras stand out. This paper deals with the problem of conjugacy for a class of algebras—extended affine Lie algebras—that are in a precise sense higher nullity analogues of the affine algebras. Unlike the methods used by Peterson–Kac, our approach is entirely cohomological and geometric. It is deeply rooted on the theory of reductive group schemes developed by Demazure and Grothendieck, and on the work of Bruhat–Tits on buildings. The main ingredient of our conjugacy proof is the classification of loop torsors over Laurent polynomial rings, a result of its own interest.

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1 Introduction

Let **g** be a split simple finite-dimensional Lie algebra over a field k of characteristic 0. From the work of Cartan and Killing one knows that **g** is determined by its root system. The problem, of course, is that a priori the type of the root system may depend on the choice of a split Cartan subalgebra. One of the most elegant ways of establishing that this does not happen, hence that the type of the root system is an invariant of **g**, is the conjugacy theorem of split Cartan subalgebras due to Chevalley: all split Cartan subalgebras of **g** are conjugate under the adjoint action of $\mathbf{G}(k)$ where **G** is the split simply connected group corresponding to **g**.

Variations of this theme are to be found on the seminal work of Peterson and Kac on conjugacy of "Cartan subalgebras" for symmetrizable Kac–Moody Lie algebras [35]. Except for the toroidal case, nothing is known about conjugacy for extended affine Lie algebras (EALAs for short); a fascinating class of algebras which can be thought of as higher nullity analogues of the affine algebras.

The aim of this paper is two-fold. First, to show the existence and conjugacy of what we call Borel–Mostow subalgebras; an important class of "Cartan subalgebras" of multiloop algebras (Borel–Mostow subalgebras are rather special. A general conjugacy result fails, as we show in Sect. 9). As an application of conjugacy we show that the root system attached to a Lie torus is an invariant (see Theorem 13.2). Second, it turns out that to solve the conjugacy problem we are, out of necessity, faced with the classification problem of loop reductive group schemes over a Laurent polynomials ring $R_n = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. Our second main result provides us with a local-global principle for classification of loop torsors over R_n, a result that we believe is of its own interest. The case $n = 1$ was done in our paper [13] and here we consider the general case. For details and precise statements we refer to Sects. 11 and 14.

The philosophy that we follow is motivated by two assumptions:

(1) The affine Kac–Moody and extended affine Lie algebras are among the most relevant infinite-dimensional Lie algebras today.
(2) Since the affine and extended affine algebras are closely related to finite-dimensional simple Lie algebras, a proof of conjugacy ought to exist that is faithful to the spirit of finite-dimensional Lie theory.

That this much is true for toroidal Lie algebras (which correspond to the "untwisted case" in this paper) has been shown in [33]. The present work is much more ambitious. Not only it tackles the twisted case, but it does so in arbitrary nullity.

Some of the algebras covered by our result are related to extended affine Lie algebras, but our work depicts a more global point of view. For every k-algebra R which is a normal ring it builds a bridge between ad-k-diagonalizable subalgebras of twisted forms of semisimple Lie algebras over R (viewed as infinite-dimensional Lie algebras over the base field k), and split tori of the corresponding reductive group schemes over

R. Using this natural one-to-one correspondence, shown in Theorem 7.1, we are able to attach a cohomological obstacle to conjugacy which eventually leads to the proof of our main conjugacy result in Theorem 12.1. The main ingredient of the proof of conjugacy is the classification of loop reductive torsors over Laurent polynomial rings given by Theorem 14.1.

2 Generalities on multiloop algebras and forms

2.1 Notation and conventions

Throughout this work, with the exception of the Appendix, k will denote a field of characteristic 0 and $\overline{k}$ an algebraic closure of k. For integers $n \geq 0$ and $m > 0$ we set

$$R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \quad K_n = k(t_1, \dots, t_n), \quad F_n = k((t_1)) \cdots ((t_n)),$$

and

$$R_{n,m} = k[t_1^{\pm \frac{1}{m}}, \dots, t_n^{\pm \frac{1}{m}}], \; K_{n,m} = k(t_1^{\frac{1}{m}}, \dots, t_n^{\frac{1}{m}}), \; F_{n,m} = k((t_1^{\frac{1}{m}})) \cdots ((t_n^{\frac{1}{m}})).$$

The category of commutative associate unital algebras over k will be denoted by k-alg. If $\mathfrak{X}$ is a scheme over $\mathrm{Spec}(k)$, by an $\mathfrak{X}$-*group* we will understand a group scheme over $\mathfrak{X}$. When $\mathfrak{X} = \mathrm{Spec}(R)$ for some object R of k-alg, we use the expression R-*group*. If R is an object in k-alg we will denote the corresponding multiplicative and additive groups by $\mathbf{G}_{m,R}$ and $\mathbf{G}_{a,R}$.

We will use bold roman characters, e.g. $\mathbf{G}$, $\mathbf{g}$ to denote k-groups and their Lie algebras. The notation $\mathfrak{G}$ and $\mathfrak{g}$ will be reserved for R-groups (which are usually not obtained from a k-group by base change) and their Lie algebras.

2.2 Forms

Let $\mathbf{g}$ be a finite-dimensional split semisimple Lie algebra over k. Recall that a Lie algebra $\mathcal{L}$ over R is called a *form* of $\mathbf{g} \otimes_k R$ (or simply a form of $\mathbf{g}$) if there exists a faithfully flat and finitely presented R–algebra $\widetilde{R}$ such that

$$\mathcal{L} \otimes_R \widetilde{R} \simeq (\mathbf{g} \otimes_k R) \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R}, \tag{2.1}$$

where all the above are isomorphisms of Lie algebras over $\widetilde{R}$. The set of isomorphism classes of such forms is measured by the pointed set

$$H^1_{fppf}(\mathrm{Spec}(R), \mathbf{Aut}(\mathbf{g})_{\mathrm{R}})$$

where $\mathbf{Aut}(\mathbf{g})_{\mathrm{R}}$ is the R-group obtained by base change from the k–linear algebraic group $\mathbf{Aut}(\mathbf{g})$. We have a split exact sequence of k-groups

$$1 \longrightarrow \mathbf{G}_{ad} \longrightarrow \mathbf{Aut}(\mathbf{g}) \longrightarrow \mathbf{Out}(\mathbf{g}) \longrightarrow 1 \tag{2.2}$$

where $\mathbf{G}_{ad}$ is the adjoint group corresponding to $\mathbf{g}$ and $\mathbf{Out}(\mathbf{g})$ is the constant k-group corresponding to the finite (abstract) group of symmetries of the Coxeter–Dynkin diagram of $\mathbf{g}$.

By base change we obtain an analogous sequence over R. In what follows we will denote $H^1_{fppf}(\mathrm{Spec}(R), \mathbf{Aut}(\mathbf{g})_{\mathrm{R}})$ simply by $H^1_{fppf}(R, \mathbf{Aut}(\mathbf{g}))$ when no confusion is possible. Similarly for the Zariski and étale topologies, as well as for k-groups other than $\mathbf{Aut}(\mathbf{g})$.

Remark 2.1 Since $\mathbf{Aut}(\mathbf{g})$ is smooth and affine over $\mathrm{Spec}(R)$

$$H^1_{\acute{e}t}(R, \mathbf{Aut}(\mathbf{g})) \simeq H^1_{fppf}(R, \mathbf{Aut}(\mathbf{g})).$$

Remark 2.2 Let $R = R_n$ be as in Sect. 2.1. By the Isotriviality Theorem of [18] the trivializing algebra $\widetilde{R}$ in (2.1) may be taken to be of the form

$$\widetilde{R} := R_{n,m} \otimes_k \tilde{k} = \tilde{k}[t_1^{\pm\frac{1}{m}}, \ldots, t_n^{\pm\frac{1}{m}}]$$

for some m and some Galois extension $\tilde{k}$ of k containing all m-th roots of unity of $\overline{k}$. The extension $\widetilde{R}/R$ is Galois.

2.3 Multiloop algebras

Assume now that k is algebraically closed. We fix a compatible set of primitive m–th roots of unity ξ_m, namely such that $\xi_{me}^e = \xi_m$ for all $e > 0$. Let $R = R_n$ and $\widetilde{R} = R_{n,m}$. Then $\widetilde{R}/R$ is Galois. Via our choice of roots of unity, we can identify $\mathrm{Gal}(\widetilde{R}/R)$ with $(\mathbb{Z}/m\mathbb{Z})^n$ as follows: For each $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ the corresponding element $\overline{\mathbf{e}} = (\overline{e}_1, \cdots, \overline{e}_n) \in \mathrm{Gal}(\widetilde{R}/R)$ acts on $\widetilde{R}$ via ${}^{\overline{\mathbf{e}}}(t_i^{\frac{1}{m}}) = \xi_m^{e_i} t_i^{\frac{1}{m}}$.

The primary example of forms $\mathcal{L}$ of $\mathbf{g} \otimes_k R$ which are trivialized by a Galois extension $\widetilde{R}/R$ as above are the multiloop algebras based on $\mathbf{g}$. These are defined as follows. Consider an n–tuple $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of commuting elements of $\mathrm{Aut}_k(\mathbf{g})$ satisfying $\sigma_i^m = 1$. For each n–tuple $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ we consider the simultaneous eigenspace

$$\mathbf{g}_{i_1 \ldots i_n} = \{x \in \mathbf{g} : \sigma_j(x) = \xi_m^{i_j} x \ \text{ for all } 1 \le j \le n\}.$$

Then $\mathbf{g} = \sum \mathbf{g}_{i_1 \ldots i_n}$, and $\mathbf{g} = \bigoplus \mathbf{g}_{i_1 \ldots i_n}$ if we restrict the sum to those n–tuples $(i_1, \ldots, i_n)$ for which $0 \le i_j < m_j$, where m_j is the order of σ_j.

The *multiloop algebra based on* $\mathbf{g}$ *corresponding to* $\boldsymbol{\sigma}$, commonly denoted by $L(\mathbf{g}, \boldsymbol{\sigma})$, is defined by

$$L(\mathbf{g}, \boldsymbol{\sigma}) = \bigoplus_{(i_1, \ldots, i_n) \in \mathbb{Z}^n} \mathbf{g}_{i_1 \ldots i_n} \otimes t_1^{\frac{i_1}{m}} \ldots t_n^{\frac{i_n}{m}} \subset \mathbf{g} \otimes_k \widetilde{R} \subset \mathbf{g} \otimes_k \overline{R}_\infty$$

where $\overline{R}_\infty = \varinjlim \overline{k}[t_1^{\pm\frac{1}{m}}, \ldots, t_n^{\pm\frac{1}{m}}]$.[1] Note that $L(\mathbf{g}, \boldsymbol{\sigma})$, which does not depend on the choice of common period m, is not only a k–algebra (in general infinite-dimensional),

[1] The ring $\overline{R}_\infty$ is a useful artifice that allows us to see *all* multiloop algebras based on a given $\mathbf{g}$ as subalgebras of one Lie algebra.

but also naturally an R–algebra. A rather simple calculation shows that

$$L(\mathbf{g}, \boldsymbol{\sigma}) \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R} \simeq (\mathbf{g} \otimes_k R) \otimes_R \widetilde{R}.$$

Thus $L(\mathbf{g}, \boldsymbol{\sigma})$ corresponds to a torsor over $\mathrm{Spec}(R)$ under $\mathbf{Aut}(\mathbf{g})$.

It is worth to point out that the cohomological information is always about the twisted forms viewed as algebras over R (and *not* k). In practice, as the affine Kac–Moody case illustrates, one is interested in understanding these algebras as objects over k (and *not* R). We find in Theorem 7.1 a bridge between these two very different and contrasting kinds of mathematical worlds.

3 Preliminaries I: Reductive group schemes

3.1 Some terminology

Let $\mathfrak{X}$ be a k-scheme. A *reductive* $\mathfrak{X}$-group is to be understood in the sense of [41]. In particular, a reductive k-group is a reductive *connected* algebraic group defined over k in the sense of Borel. We recall now two fundamental notions about reductive $\mathfrak{X}$-groups.

Definition 3.1 Let $\mathfrak{G}$ be a reductive $\mathfrak{X}$-group. We say that $\mathfrak{G}$ is reducible if $\mathfrak{G}$ admits a proper parabolic subgroup $\mathfrak{P}$ which has a Levi subgroup, and irreducible otherwise.

Definition 3.2 We say that $\mathfrak{G}$ is isotropic if $\mathfrak{G}$ admits a subgroup isomorphic to $\mathbf{G}_{m,\mathfrak{X}}$. Otherwise we say that $\mathfrak{G}$ is anisotropic.

We denote by $\mathbf{Par}(\mathfrak{G})$ the $\mathfrak{X}$–scheme of parabolic subgroup of $\mathfrak{G}$. This scheme is smooth and projective over $\mathfrak{X}$ [41, XXVI, 3.5]. Since by definition $\mathfrak{G}$ is a parabolic subgroup of $\mathfrak{G}$, when $\mathfrak{X}$ is connected, to say that $\mathfrak{G}$ admits a proper parabolic subgroup is to say that $\mathbf{Par}(\mathfrak{G})(\mathfrak{X}) \neq \{\mathfrak{G}\}$.

Remark 3.3 If $\mathfrak{X}$ is connected, to each parabolic subgroup $\mathfrak{P}$ of $\mathfrak{G}$ corresponds a "type" $\mathbf{t} = \mathbf{t}(\mathfrak{P})$ which is a subset of the corresponding Coxeter-Dynkin diagram. Given a type $\mathbf{t}$, the scheme $\mathbf{Par}_{\mathbf{t}}(\mathfrak{G})$ of parabolic subgroups of $\mathfrak{G}$ of type $\mathbf{t}$ is also smooth and projective over $\mathfrak{X}$ (*ibid.* cor.3.6).

Let $\mathfrak{H}$ denote a reductive $\mathfrak{X}$-group. If $\mathfrak{T}$ is a subgroup of $\mathfrak{H}$ the expression "$\mathfrak{T}$ is a maximal torus of $\mathfrak{H}$" has a precise meaning ([41, XII, Définition 1.3]). A maximal torus may or may not be split. If it is, we say that $\mathfrak{T}$ is a *split maximal torus*. This is in contrast with the concept of *maximal split torus* which we also need. This is a closed subgroup of $\mathfrak{H}$ which is a split torus and which is not properly included in any other split torus of $\mathfrak{H}$. Note that split maximal tori (even maximal tori) need not exist, while maximal split tori always do exist if $\mathfrak{X}$ is noetherian.

If $\mathfrak{S} < \mathfrak{H}$ are $\mathfrak{X}$-groups and $\mathfrak{s} \subset \mathfrak{g}$ are their respective Lie algebras we will denote by $Z_{\mathfrak{H}}(\mathfrak{S})$ [resp. $Z_{\mathfrak{g}}(\mathfrak{s})$] the centralizer of $\mathfrak{S}$ in $\mathfrak{H}$ [resp. of $\mathfrak{s}$ in $\mathfrak{g}$]. If $\mathfrak{S} \subset \mathfrak{H}$ is a split torus then $Z_{\mathfrak{H}}(\mathfrak{S})$ is a closed reductive subgroup (see [41, XIX, 2.2]). Also, if $\mathfrak{X}$ is connected and $\mathfrak{T}$ a torus of $\mathfrak{H}$ then $\mathfrak{T}$ contains a unique maximal split subtorus $\mathfrak{T}_d$ (see [41, XXVI, 6.5, 6.6]).

We now recall and establish for future reference some basic useful facts.

Lemma 3.4 *Let $\mathfrak{H}$ be a reductive $\mathfrak{X}$-group and $\mathfrak{S} \subset \mathfrak{H}$ a split torus. Then there exists a parabolic subgroup $\mathfrak{P} \subset \mathfrak{H}$ such that $Z_{\mathfrak{H}}(\mathfrak{S})$ is a Levi subgroup of $\mathfrak{P}$.*

Proof See [41, XXVI, cor. 6.2]. □

Lemma 3.5 *Let $\mathfrak{S}$ be a split torus of $\mathfrak{H}$, and let $\mathfrak{T}$ be the radical of the reductive group $\mathfrak{C} = Z_{\mathfrak{H}}(\mathfrak{S})$.[2] If $\mathfrak{X}$ is connected then $Z_{\mathfrak{H}}(\mathfrak{T}_d) = \mathfrak{C}$.*

Proof Since $\mathfrak{T}$ is the centre of $\mathfrak{C}$ we have $\mathfrak{C} \subset Z_{\mathfrak{H}}(\mathfrak{T})$. Also, the inclusions $\mathfrak{S} \subset \mathfrak{T}_d \subset \mathfrak{T}$ yield

$$Z_{\mathfrak{H}}(\mathfrak{T}) \subset Z_{\mathfrak{H}}(\mathfrak{T}_d) \subset Z_{\mathfrak{H}}(\mathfrak{S}) = \mathfrak{C},$$

whence the result. □

Proposition 3.6 *Let $\mathfrak{H}$ be a reductive group scheme over $\mathfrak{X}$. Assume $\mathfrak{X}$ is connected. Let $\mathfrak{S}$ be a split subtorus of $\mathfrak{H}$ and let $\mathfrak{P}$ be a parabolic subgroup of $\mathfrak{H}$ containing $Z_{\mathfrak{H}}(\mathfrak{S})$ as Levi subgroup. Then following are equivalent:*

(1) *The reductive group scheme $Z_{\mathfrak{H}}(\mathfrak{S})$ has no proper parabolic subgroups.*
(2) *$\mathfrak{P}$ is a minimal parabolic subgroup of $\mathfrak{H}$.*
If $\mathfrak{S}$ is the maximal split subtorus of the radical of $Z_{\mathfrak{H}}(\mathfrak{S})$ these two conditions are equivalent to
(3) *The reductive group scheme $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$ is anisotropic.*

Proof According to [41, XXVI.1.20], there is a bijective correspondence

$$\{\text{ parabolics } \mathfrak{Q} \text{ of } \mathfrak{H} \text{ included in } \mathfrak{P}\} \quad < -- > \quad \left\{\text{ parabolics } \mathfrak{M} \text{ of } Z_{\mathfrak{H}}(\mathfrak{S})\right\}$$

Thus the left handside consists of one element if and only if so does the right handside. □

Proposition 3.7 *Let $\mathfrak{G}$ be a reductive group scheme over a connected base scheme $\mathfrak{X}$, $\mathfrak{S}$ a split subtorus of $\mathfrak{G}$, and let $\mathfrak{g}$ and $\mathfrak{s}$ denote their respective Lie algebras. Then*

(1) $\mathrm{Lie}(Z_{\mathfrak{G}}(\mathfrak{s})) = Z_{\mathfrak{g}}(\mathfrak{s})$.
(2) *$Z_{\mathfrak{G}}(\mathfrak{S})$ is a Levi subgroup of $\mathfrak{G}$ and $Z_{\mathfrak{G}}(\mathfrak{S}) = Z_{\mathfrak{G}}(\mathfrak{s})$.*

Proof (1) This is a particular case of [41, II théo. 5.3.1(i)].
(2) That $Z_{\mathfrak{G}}(\mathfrak{S})$ is a Levi subgroups of $\mathfrak{G}$ follows from Lemma 3.4. To establish the equality $Z_{\mathfrak{G}}(\mathfrak{S}) = Z_{\mathfrak{G}}(\mathfrak{s})$ we reason in steps.
   (a) Assume $\mathfrak{X} = \mathrm{Spec}(k)$ *and* $\mathfrak{G}$ *simply connected*: Then this is a result of Steinberg. See [44, 3.3 and 3.8] and [44, 0.2].

[2] Recall that the radical of a reductive $\mathfrak{X}$-group is the unique maximal torus of its centre [41, XXII, 4.3.6].

(b) Assume $\mathfrak{X} = \mathrm{Spec}(k)$ *and* $\mathfrak{G}$ *reductive*: Embed $\mathfrak{G}$ into $\mathbf{SL}_n$ for a suitable n. Then

$$Z_{\mathfrak{G}}(\mathfrak{S}) = \mathfrak{G} \cap Z_{\mathbf{SL_n}}(\mathfrak{S}) \text{ and } Z_{\mathfrak{G}}(\mathfrak{s}) = \mathfrak{G} \cap Z_{\mathbf{SL_n}}(\mathfrak{s})$$

and we are reduced to the previous case.

(c) In general, we proceed by étale descent. This reduces the problem to the case $\mathfrak{S} \subset \mathfrak{T} \subset \mathfrak{G}$ where $\mathfrak{G}$ is a Chevalley group and $\mathfrak{T}$ its standard split maximal torus. This sequence is obtained by base change to $\mathfrak{X}$ from a similar sequence over k by [41, VII cor. 1.6]. Over k our equality holds. Since both centralizers commute with base change the equality follows.

□

4 Loop torsors and loop reductive group schemes

Throughout this section $\mathfrak{X}$ will denote a connected and noetherian scheme over k and $\mathbf{G}$ a k-group which is locally of finite presentation.[3]

4.1 The algebraic fundamental group

If $\mathfrak{X}$ is a k–scheme and if a of is a geometric point of $\mathfrak{X}$ i.e. a morphism $a : \mathrm{Spec}(\Omega) \to \mathfrak{X}$ where Ω is an algebraically closed field, we denote the algebraic fundamental group of $\mathfrak{X}$ at a by $\pi_1(\mathfrak{X}, a)$ (see [40] for details).

Suppose now that our $\mathfrak{X}$ is a geometrically connected k–scheme. We will denote $\mathfrak{X} \times_k \overline{k}$ by $\overline{\mathfrak{X}}$. Fix a geometric point $\overline{a} : \mathrm{Spec}(\overline{k}) \to \overline{\mathfrak{X}}$. Let a (resp. b) be the geometric point of $\mathfrak{X}$ (resp. $\mathrm{Spec}(k)$) given by the composite maps $a : \mathrm{Spec}(\overline{k}) \overset{\overline{a}}{\to} \overline{\mathfrak{X}} \to \mathfrak{X}$ (resp. $b : \mathrm{Spec}(\overline{k}) \overset{\overline{a}}{\to} \mathfrak{X} \to \mathrm{Spec}(k)$). Then by [40, théo. IX.6.1] $\pi_1(\mathrm{Spec}(k), b) \simeq \mathrm{Gal}(k) := \mathrm{Gal}(\overline{k}/k)$ and the sequence

$$1 \to \pi_1(\overline{\mathfrak{X}}, \overline{a}) \to \pi_1(\mathfrak{X}, a) \to \mathrm{Gal}(k) \to 1 \tag{4.1}$$

is exact.

4.2 The algebraic fundamental group of R_n

We refer the reader to [18,19] for details. The simply connected cover $\mathfrak{X}^{sc}$ of $\mathfrak{X} = \mathrm{Spec}\,(R_n)$ is $\mathrm{Spec}(\overline{R}_{n,\infty})$ where

$$\overline{R}_{n,\infty} = \varinjlim \overline{R}_{n,m}$$

[3] The case most relevant to our work is that of the group of automorphism of a reductive k-group.

with $\overline{R}_{n,m} = \overline{k}[t_1^{\pm\frac{1}{m}}, \ldots, t_n^{\pm\frac{1}{m}}]$. The "evaluation at 1" provides a geometric point that we denote by a. The algebraic fundamental group is best described as

$$\pi_1(\mathfrak{X}, a) = \widehat{\mathbb{Z}}(1)^n \rtimes \text{Gal}\,(k). \tag{4.2}$$

where $\widehat{\mathbb{Z}}(1)$ denotes the abstract group $\varprojlim_m \boldsymbol{\mu}_m(\overline{k})$ equipped with the natural action of the absolute Galois group $\text{Gal}(k)$.

4.3 Loop torsors

Because of the universal nature of $\mathfrak{X}^{sc}$ we have a natural group homomorphism

$$\mathbf{G}(\overline{k}) \longrightarrow \mathbf{G}(\mathfrak{X}^{sc}). \tag{4.3}$$

The group $\pi_1(\mathfrak{X}, a)$ acts on $\overline{k}$, hence on $\mathbf{G}(\overline{k})$, via the group homomorphism $\pi_1(\mathfrak{X}, a) \to \text{Gal}\,(k)$ of (4.1). This action is continuous, and together with (4.3) yields a map

$$H^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})) \to H^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})),$$

where we remind the reader that these H^1 are defined in the "continuous" sense. On the other hand, by [19, prop. 2.3] and basic properties of torsors trivialized by Galois extensions we have a natural inclusion

$$H^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})) \subset H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G}).$$

By means of the foregoing observations we make the following.

Definition 4.1 A torsor $\mathfrak{E}$ over $\mathfrak{X}$ under $\mathbf{G}$ is called a loop torsor if its isomorphism class $[\mathfrak{E}]$ in $H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G})$ belongs to the image of the composite map

$$H^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})) \to H^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})) \subset H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G}).$$

We will denote by $H^1_{loop}(\mathfrak{X}, \mathbf{G})$ the subset of $H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G})$ consisting of classes of loop torsors. They are given by (continuous) cocycles in the image of the natural map $Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})) \to Z^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G})$, which we call *loop cocycles.*

This fundamental concept is used in the definition of loop reductive groups which we will recall momentarily. The following examples illustrate the immensely rich class of objects that fit within the language of loop torsors.

Examples 4.2 (a) If $\mathfrak{X} = \text{Spec}\,(k)$ then $H^1_{loop}(\mathfrak{X}, \mathbf{G})$ is nothing but the usual Galois cohomology of k with coefficients in $\mathbf{G}$.

(b) Assume that k is algebraically closed. Then the action of $\pi_1(\mathfrak{X}, a)$ on $\mathbf{G}(\overline{k})$ is trivial, so that

$$H^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})) = \text{Hom}(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k}))/\,\text{Int}\,\mathbf{G}(\overline{k})$$

where the group Int $\mathbf{G}(\overline{k})$ of inner automorphisms of $\mathbf{G}(\overline{k})$ acts naturally on the right on $\mathrm{Hom}(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k}))$. Two particular cases are important:

(b1) $\mathbf{G}$ abelian: In this case $H^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k}))$ is just the group of continuous homomorphisms from $\pi_1(\mathfrak{X}, a)$ to $\mathbf{G}(\overline{k})$.

(b2) $\pi_1(\mathfrak{X}, a) = \widehat{\mathbb{Z}}(1)^n$: In this case $H^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k}))$ is the set of conjugacy classes of n–tuples $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of commuting elements of finite order of $\mathbf{G}(\overline{k})$.[4]

This last example is exactly the setup of multiloop algebras, and the motivation for the "loop torsor" terminology.

4.4 Geometric and arithmetic part of a loop cocycle

By means of the decompositions (4.1) and (4.2) we can think of loop cocycles as being comprised of a geometric and an arithmetic part, as we now explain.

Let $\eta \in Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k}))$. The restriction $\eta_{|\mathrm{Gal}(k)}$ is called the *arithmetic part* of η and it is denoted by η^{ar}. It is easily seen that η^{ar} is in fact a cocycle in $Z^1(\mathrm{Gal}(k), \mathbf{G}(\overline{k}))$. If η is fixed in our discussion, we will at times denote the cocycle η^{ar} by the more traditional notation z. In particular, for $s \in \mathrm{Gal}(k)$ we write z_s instead of η^{ar}_s.

Next we consider the restriction of η to $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ that we denote by η^{geo} and called the *geometric part* of η. We thus have a map

$$\begin{array}{rccc} \Theta \ : & Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})) & \longrightarrow & Z^1(\mathrm{Gal}(k), \mathbf{G}(\overline{k})) \times \mathrm{Hom}(\pi_1(\overline{\mathfrak{X}}, \overline{a}), \mathbf{G}(\overline{k})) \\ & \eta & \mapsto & (\ \eta^{ar}\ ,\ \eta^{geo}) \end{array}$$

The group $\mathrm{Gal}(k)$ acts on $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ by conjugation. On $\mathbf{G}(\overline{k})$, the Galois group $\mathrm{Gal}(k)$ acts on two different ways. There is the natural action arising from the action of $\mathrm{Gal}(k)$ on $\overline{k}$, and there is also the twisted action given by the cocycle $\eta^{ar} = z$. Following standard practice to view the abstract group $\mathbf{G}(\overline{k})$ as a $\mathrm{Gal}(k)$–module with the twisted action by z we write ${}_z\mathbf{G}(\overline{k})$.

Lemma 4.3 *The map Θ described above yields a bijection between $Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k}))$ and couples (z, η^{geo}) with $z \in Z^1(\mathrm{Gal}(k), \mathbf{G}(\overline{k}))$ and $\eta^{geo} \in \mathrm{Hom}_{\mathrm{Gal}(k)}(\pi_1(\overline{\mathfrak{X}}, \overline{a}), {}_z\mathbf{G}(\overline{k}))$.*

Proof See [19, lemma 3.7]. □

Remark 4.4 Assume that $\mathfrak{X} = \mathrm{Spec}(R_n)$. It is easy to verify that η^{geo} arises from a unique k-group homomorphism

$${}_\infty\boldsymbol{\mu} = (\varprojlim \boldsymbol{\mu}_m)^n \to {}_z\mathbf{G}$$

We finish this section by recalling some basic properties of the twisting bijection (or torsion map) $\tau_z : H^1(\mathfrak{X}, {}_z\mathbf{G}) \to H^1(\mathfrak{X}, \mathbf{G})$. Take a cocycle $\eta \in Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k}))$

[4] That the elements are of finite order follows from the continuity assumption.

and consider its corresponding pair $\Theta(\eta) = (z, \eta^{geo})$. We can apply the same construction to the twisted k-group ${}_z\mathbf{G}$. This would lead to a map Θ_z that will attach to a cocycle $\eta' \in Z^1(\pi_1(\mathfrak{X}, a), {}_z\mathbf{G}(\bar{k}))$ a pair $(z', \eta'^{\mathrm{geo}})$ along the lines explained above.

Lemma 4.5 *Let $\eta \in Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\bar{k}))$. With the above notation, the inverse of the twisting map* [42]

$$\tau_z^{-1} : Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\bar{k})) \xrightarrow{\sim} Z^1(\pi_1(\mathfrak{X}, a), {}_z\mathbf{G}(\bar{k}))$$

satisfies $\Theta_z \circ \tau_z^{-1}(\eta) = (1, \eta^{\mathrm{geo}})$. □

Remark 4.6 The notion of loop torsor behaves well under twisting by a Galois cocycle $z \in Z^1(\mathrm{Gal}(k), \mathbf{G}(\bar{k}))$. Indeed the torsion map $\tau_z^{-1} : H^1_{ét}(\mathfrak{X}, \mathbf{G}) \to H^1_{et}(\mathfrak{X}, {}_z\mathbf{G})$ maps loop classes to loop classes.

4.5 Loop reductive groups

Let $\mathfrak{H}$ be a reductive group scheme over $\mathfrak{X}$. Since $\mathfrak{X}$ is connected, for all $x \in \mathfrak{X}$ the geometric fibers $\mathfrak{H}_{\overline{x}}$ are reductive group schemes of the same "type" [41, XXII, 2.3]. By Demazure's theorem there exists a unique split reductive group $\mathbf{H}$ over k such that $\mathfrak{H}$ is a twisted form (in the étale topology of $\mathfrak{X}$) of $\mathfrak{H}_0 = \mathbf{H} \times_k \mathfrak{X}$. We will call $\mathbf{H}$ the *Chevalley k–form of* $\mathfrak{H}$. The $\mathfrak{X}$-group $\mathfrak{H}$ corresponds to a torsor $\mathfrak{E}$ over $\mathfrak{X}$ under the group scheme $\mathbf{Aut}(\mathfrak{H}_0)$, namely $\mathfrak{E} = \mathbf{Isom}_{gr}(\mathfrak{H}_0, \mathfrak{H})$. We recall that $\mathbf{Aut}(\mathfrak{H}_0)$ is representable by a smooth and separated group scheme over $\mathfrak{X}$ by [41, XXII, 2.3]. It is well-known that $\mathfrak{H}$ is then the contracted product $\mathfrak{E} \wedge^{\mathbf{Aut}(\mathfrak{H}_0)} \mathfrak{H}_0$ (see [15] III §4 n°3 for details).

We now recall one of the central concepts needed for our work.

Definition 4.7 We say that a group scheme $\mathfrak{H}$ over $\mathfrak{X}$ is loop reductive if it is reductive and if $\mathfrak{E}$ is a loop torsor.

5 Preliminaries II: Reductive group schemes over a normal noetherian base

We begin with a useful variation of Lemma 3.5 under some extra assumptions on our connected base k–scheme $\mathfrak{X}$.

Lemma 5.1 *Assume that $\mathfrak{X}$ is normal noetherian and integral. Let $\mathfrak{H}$ be a reductive $\mathfrak{X}$-group. Then there exists an étale cover $(\mathfrak{U}_i)_{i=1,..,l} \to \mathfrak{X}$ such that:*

(i) *$\mathfrak{H} \times_{\mathfrak{X}} \mathfrak{U}_i$ is a split reductive $\mathfrak{U}_i$-group scheme,*
(ii) *$\mathfrak{U}_i = \mathrm{Spec}(R_i)$ with R_i a normal noetherian domain.*
(iii) *If $\mathfrak{H}$ is a torus and $\mathfrak{X} = \mathrm{Spec}(R)$ there exists a Galois extension $\widetilde{R}/R$ that splits $\mathfrak{H}$.*

Proof Since $\mathfrak{X}$ is normal noetherian, $\mathfrak{H}$ is a locally isotrivial group scheme [41, XXIV.4.1.6]. We can thus cover $\mathfrak{X}$ by affine Zariski open subsets $\mathfrak{X}_1, \ldots, \mathfrak{X}_l$ where $\mathfrak{X}_i = \mathrm{Spec}(A_i)$ and such that there exists a finite étale cover $\mathfrak{V}_i \to \mathfrak{X}_i$ for $i = 1, .., l$

which splits $\mathfrak{H}_{\mathfrak{X}_i}$. For each i, choose a connected component $\mathfrak{U}_i$ of $\mathfrak{V}_i$. According to the classification of étale maps over $\mathfrak{X}$ (see [22, 18.10.12]) we know that $\mathfrak{U}_i$ is a finite étale cover of $\mathfrak{X}_i$ and that $\mathfrak{U}_i = \mathrm{Spec}(R_i)$ where R_i is a normal domain. Since R_i is finite over the noetherian ring A_i, it is noetherian as well.

(iii) By [41, X, théo. 5.16] there exists a finite étale extension of $\mathfrak{X}$ that splits $\mathfrak{H}$. The result now follows by considering a connected component of this extension and basic properties of the algebraic fundamental group (see [45, 5.3.9]). □

Remark 5.2 If $\mathfrak{X}$ is local, one single $\mathfrak{U}_i$ suffices.

Proposition 5.3 *Let $\mathfrak{X}$ be normal and noetherian. Let $\mathfrak{H}$ be a reductive $\mathfrak{X}$-group, $\mathfrak{P} \subset \mathfrak{H}$ be a parabolic subgroup and $\mathfrak{L} \subset \mathfrak{P}$ a Levi subgroup.*[5] *Let $\mathfrak{T}$ be the radical of $\mathfrak{L}$ and $\mathfrak{T}_d$ its maximal split subtorus. Then $Z_{\mathfrak{H}}(\mathfrak{T}_d) = \mathfrak{L}$.*

Proof Since $\mathfrak{T}$ is the centre of $\mathfrak{L}$ we have $\mathfrak{L} \subset Z_{\mathfrak{H}}(\mathfrak{T})$. The inclusion $\mathfrak{T}_d \subset \mathfrak{T}$ yields $Z_{\mathfrak{H}}(\mathfrak{T}) \subset Z_{\mathfrak{H}}(\mathfrak{T}_d)$. Thus we have $\mathfrak{L} \subset Z_{\mathfrak{H}}(\mathfrak{T}_d)$. By the Lemma below and by [41, XXVI, prop. 6.8] the above inclusion is an equality locally in the Zariski topology, hence globally. □

Lemma 5.4 *Assume $\mathfrak{X} = \mathrm{Spec}(R)$ is affine and as in the Proposition. Let $x \in \mathfrak{X}$ and consider the localized ring R_x. Then $(\mathfrak{T}_d)_{R_x}$ is the maximal split subtorus of $\mathfrak{T}_{R_x}$. In particular, if K denotes the quotient field of R then $\mathfrak{T}_d \times_R K$ is the maximal split subtorus of $\mathfrak{T} \times_R K$.*

Proof It suffices to show that $(\mathfrak{T}_d)_K$ is the maximal split subtorus of $\mathfrak{T}_K$. Recall that $\mathfrak{T}$ is determined by its lattice of characters $X(\mathfrak{T})$ equipped with an action of $\mathrm{Gal}\,(\widetilde{R}/R)$, and that $\mathfrak{T}_d$ corresponds to the maximal sublattice in $X(\mathfrak{T})$ stable (elementwise) with respect to $\mathrm{Gal}\,(\widetilde{R}/R)$. Similar considerations apply to $\mathfrak{T}_K$. It remains to note that $\mathfrak{T}_K$ and $\mathfrak{T}$ have the same lattices of characters and that $\mathrm{Gal}\,(\widetilde{R}/R) \simeq \mathrm{Gal}\,(\widetilde{K}/K)$ by [9, Ch5 §2.2 theo.2]). □

Proposition 5.5 *Let $\mathfrak{G}$ be a reductive group over a normal ring*[6] *R. If $\mathfrak{G}$ contains a proper parabolic subgroup $\mathfrak{P}$ then it contains a split non-central subtorus $\mathbf{G}_{m,R}$.*

Proof We may assume that $\mathfrak{G}$ is semisimple. Since the base is affine, $\mathfrak{P}$ contains a Levi subgroup $\mathfrak{L}$. Let $\mathfrak{T}$ be the radical of $\mathfrak{L}$ and $\mathfrak{T}_d$ its maximal split subtorus. By Proposition 5.3, $Z_{\mathfrak{H}}(\mathfrak{T}_d) = \mathfrak{L}$. Hence $\mathfrak{T}_d \neq 1$. □

Corollary 5.6 *For a reductive group scheme $\mathfrak{G}$ over a normal ring R to contain a proper parabolic subgroup it is necessary and sufficient that it contains a non-central split subtorus.*

6 AD and MAD subalgebras

Let R be an object in k-alg and $\mathfrak{G}$ be an R-group, i.e a group scheme over R. Recall (see [15] II §4.1) that to $\mathfrak{G}$ we can attach an R–functor on Lie algebras $\mathfrak{Lie}(\mathfrak{G})$ which

5 The existence of $\mathfrak{L}$ is automatic if the base scheme is affine by [41, XXVI.2.3].

6 All of our normal rings are hereon assumed to be integral and noetherian.

associates to an object S of R-alg the kernel of the natural map $\mathfrak{G}(S[\epsilon]) \to \mathfrak{G}(S)$ where $S[\epsilon]$ is the algebra of dual numbers over S. Let $\mathrm{Lie}(\mathfrak{G}) = \mathfrak{Lie}(\mathfrak{G})(R)$. This is an R–Lie algebra that will be denoted by $\mathfrak{g}$ in what follows.

Remark 6.1 If $\mathfrak{G}$ is smooth, the additive group of $\mathrm{Lie}(\mathfrak{G})$ represents $\mathfrak{Lie}(\mathfrak{G})$, that is $\mathfrak{Lie}(\mathfrak{G})(S) = \mathrm{Lie}(\mathfrak{G}) \otimes_R S$ as S–Lie algebras (this equality is strictly speaking a functorial family of canonical isomorphisms).

If S is in R-alg, $g \in \mathfrak{G}(S)$ and $x \in \mathfrak{Lie}(\mathfrak{G})(S)$, then $gxg^{-1} \in \mathfrak{Lie}(\mathfrak{G})(S)$. This last product is computed in the group $\mathfrak{G}(S[\epsilon])$ where g is viewed as an element of $\mathfrak{G}(S[\epsilon])$ by functoriality. The above defines an action of $\mathfrak{G}$ on $\mathfrak{Lie}(\mathfrak{G})(S)$, called the adjoint action and denoted by $g \mapsto \mathrm{Ad}(g)$. This action in fact induces an R-group homomorphism

$$\mathrm{Ad} : \mathfrak{G} \to \mathfrak{Aut}(\mathfrak{Lie}(\mathfrak{G}))$$

whose kernel is the centre of $\mathfrak{G}$.

Given a k-subspace V of $\mathfrak{g}$ consider the R-group functor $Z_{\mathfrak{G}}(V)$ defined by

$$Z_{\mathfrak{G}}(V) : S \to \{g \in \mathfrak{G}(S) : \mathrm{Ad}(g)(v_S) = v_S \text{ for every } v \in V\} \tag{6.1}$$

for all S in R-alg, where v_S denotes the image of v in $\mathfrak{g} \otimes_R S$.

We will denote by RV the R-span of V inside $\mathfrak{g}$, i.e. RV is the R-submodule of $\mathfrak{g}$ generated by V.

Remark 6.2 Note that $Z_{\mathfrak{G}}(V) = Z_{\mathfrak{G}}(RV)$. This follows from the fact that the adjoint action of $\mathfrak{G}$ on $\mathfrak{g}$ is "linear" (in a functorial way).

We now introduce some of the central concepts of this work.

A subalgebra $\mathfrak{m}$ of the k-Lie algebra $\mathfrak{g}$ is called an AD *subalgebra* if the adjoint action of each element $x \in \mathfrak{m}$ on $\mathfrak{g}$ is k-diagonalizable, i.e. $\mathfrak{g}$ admits a k-basis consisting of eigenvectors of $\mathrm{ad}_{\mathfrak{g}}(x)$. A maximal AD subalgebra of $\mathfrak{g}$, namely one which is not properly included in any other AD subalgebra of $\mathfrak{g}$ is called a MAD subalgebra of $\mathfrak{g}$.[7]

Example 6.3 Let $\mathbf{G}$ be a semisimple Chevalley k-group and $\mathbf{T}$ its standard maximal split torus. Let $\mathbf{h}$ be the Lie algebra of $\mathbf{T}$; it is a split Cartan subalgebra of $\mathbf{g}$. For all R we have $\mathfrak{g} := \mathrm{Lie}(\mathbf{G}_R) = \mathbf{g} \otimes_k R$. Assume that R is *connected*. Then $\mathfrak{m} = \mathbf{h} \otimes 1$ is a MAD subalgebra of $\mathfrak{g}$ by [33, cor. to theo.1(i)]. We have $Z_{\mathbf{G}_R}(\mathfrak{m}) = \mathbf{T}_R$.

Note that $\mathfrak{m}$ is not its own normalizer. Indeed $N_{\mathfrak{g}}(\mathfrak{m}) = Z_{\mathfrak{g}}(\mathfrak{m}) = \mathbf{h} \otimes_k R$. Thus $\mathbf{h} \otimes 1$ is not a Cartan subalgebra of $\mathfrak{g}$ in the usual sense. However, in infinite-dimensional Lie theory—for example, in the case of Kac–Moody Lie algebras—these types of subalgebras do play the role that the split Cartan subalgebras play in the classical theory. This is our motivation for studying conjugacy questions related to MAD subalgebras.

[7] It is not difficult to see that any such $\mathfrak{m}$ is necessarily abelian, so AD can be thought as shorthand for abelian k-diagonalizable or ad k-diagonalizable.

Remark 6.4 Let $\mathfrak{s}$ be an abelian Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two subalgebras of $\mathfrak{s}$ which are AD subalgebras of $\mathfrak{g}$. Because $\mathfrak{s}$ is abelian their sum $\mathfrak{m}_1 + \mathfrak{m}_2$ is also an AD subalgebra of $\mathfrak{g}$. By considering the sum of all such subalgebras we see that $\mathfrak{s}$ contains a *unique* maximal subalgebra $\mathfrak{m}(\mathfrak{s})$ which is an AD subalgebra of $\mathfrak{g}$. Of course this AD subalgebra need not be a MAD subalgebra of $\mathfrak{g}$.

We will encounter this situation when $\mathfrak{s}$ is the Lie algebra of a torus $\mathfrak{S}$ inside a reductive group scheme $\mathfrak{G}$. In this case we denote $\mathfrak{m}(\mathfrak{s})$ by $\mathfrak{m}(\mathfrak{S})$.

Remark 6.5 Let $\mathfrak{m}$ be an AD subalgebra of $\mathfrak{g}$. Then for any extension S/R in k-*alg* the image $\mathfrak{m} \otimes 1$ of $\mathfrak{m}$ in $\mathfrak{g} \otimes_R S$ is an AD subalgebra of $\mathfrak{g} \otimes_R S$. Indeed if $x \in \mathfrak{m}$ and $v \in \mathfrak{g}$ are such that $[x, v] = \lambda v$ for some $\lambda \in k$, then $[x \otimes 1, v \otimes s] = v \otimes \lambda s = \lambda(v \otimes s)$ for all $s \in S$. Thus $\mathfrak{g} \otimes_R S$ is spanned as a k–space by eigenvectors of $\mathrm{ad}_{\mathfrak{g}\otimes_R S}(x \otimes 1)$. Note that if the map $\mathfrak{g} \to \mathfrak{g} \otimes_R S$ is injective, for example if S/R is faithfully flat, then we can identify $\mathfrak{m}$ with $\mathfrak{m} \otimes 1$ and view $\mathfrak{m}$ as an AD subalgebra of $\mathfrak{g} \otimes_R S$.

The main thrust of this work is to investigate the question of conjugacy of MAD subalgebras of $\mathfrak{g}$ when $\mathfrak{g}$ is a twisted form of $\mathbf{g} \otimes_k R_n$. The result we aim for is in the spirit of Chevalley's work, as explained in the Introduction. In the "untwisted case" the result is as expected.

Theorem 6.6 *Let* $\mathbf{g}$ *be a split finite-dimensional semisimple Lie algebra over* k *and* $\mathbf{G}$ *the corresponding simply connected Chevalley group. Then all MAD subalgebras of* $\mathbf{g} \otimes_k R_n$ *are conjugate to* $\mathbf{h} \otimes 1$ *under* $\mathbf{G}(R_n)$.

This is a particular case of Theorem 1 of [33] by taking Cor 2.3 of [18] into consideration. The proof is cohomological in nature, which is also the approach that we will pursue here. As we shall see, the general twisted case holds many surprises in place.

We finish by stating and proving a simple result for future use.

Lemma 6.7 *Let* $\mathbf{G}$ *be a semisimple algebraic group over a field* L *of characteristic* 0*. Let* $\mathbf{T} \subset \mathbf{G}$ *be a torus and* $\mathbf{T}_d$ *be the (unique) maximal split subtorus of* $\mathbf{T}$*. Set* $\mathbf{g} = \mathrm{Lie}\,(\mathbf{G})$*,* $\mathbf{t} = \mathrm{Lie}\,(\mathbf{T})$ *and* $\mathbf{t}_d = \mathrm{Lie}\,(\mathbf{T}_d)$*. Then*

- (i) *The adjoint action of* $\mathbf{T}_d$ *on* $\mathbf{g}$ *is* L*-diagonalizable. In particular,* $\mathbf{t}_d$ *is an* AD *subalgebra of* $\mathbf{g}$.
- (ii) $\mathbf{t}_d$ *is the largest subalgebra of* $\mathbf{t}$ *satisfying the condition given in* (i).

Proof Part (i) is clear. As for (ii) we may assume that $\mathbf{G}$ is semisimple adjoint. Let $\mathbf{T}_a$ be the largest anisotropic subtorus of $\mathbf{T}$. The product morphism $\mathbf{T}_d \times \mathbf{T}_a \to \mathbf{T}$ is a central isogeny, hence $\mathbf{t} = \mathbf{t}_d \oplus \mathbf{t}_a$ where $\mathbf{t}_a = \mathrm{Lie}\,(\mathbf{T}_a)$. We must show that $\mathbf{t}_a$ does not contain any nonzero element whose adjoint action on $\mathbf{g}$ is L-diagonalizable. Let h be such an element. Fix a basis $\{ v_1, \ldots, v_n \}$ of $\mathbf{g}$ and scalars $\lambda_i \in L$ such that

$$[\, h, v_i \,] = \lambda_i v_i \quad \forall\, 1 \le i \le n.$$

By means of this basis we identify $\mathbf{GL}(\mathbf{g})$ with $\mathbf{GL}_{n,L}$. Consider the adjoint representation diagrams

$$\mathbf{T} \hookrightarrow \mathbf{G} \xrightarrow{\mathrm{Ad}} \mathbf{GL}(\mathbf{g}) \simeq \mathbf{GL}_{n,L}$$

and

$$\mathfrak{t} \hookrightarrow \mathbf{g} \xrightarrow{\text{ad}} \mathfrak{gl}(\mathbf{g}) \simeq \mathfrak{gl}_{n,L}.$$

Since $\mathbf{G}$ is of adjoint type Ad is injective, so that we can identify $\mathbf{T}$ with a subtorus, say $\widetilde{\mathbf{T}}$, of $\mathbf{GL}_{n,L}$. Similarly for $\mathbf{T}_d$ and $\mathbf{T}_a$. Since $\mathbf{T} \simeq \widetilde{\mathbf{T}}$ we see that $\widetilde{\mathbf{T}}_d$ and $\widetilde{\mathbf{T}}_a$ are the maximal split and anisotropic parts of $\widetilde{\mathbf{T}}$.

Let $\mathbf{D}_n$ be the diagonal subgroup of $\mathbf{GL}_{n,L}$. By construction we see that

$$\text{ad}_{\mathfrak{g}}(h) \in \text{Lie}\,(\mathbf{D}_n) \cap \text{Lie}\,(\widetilde{\mathbf{T}}_a) = \text{Lie}\,(\mathbf{D}_n \cap \widetilde{\mathbf{T}}_a),$$

this last by [24, theo. 12.5] since $\text{char}(k) = 0$. Thus $\mathbf{D}_n \cap \widetilde{\mathbf{T}}_a$ has dimension > 0. But then the connected component of the identity of $\mathbf{D}_n \cap \widetilde{\mathbf{T}}_a$ is a non-trivial split torus which contradicts the fact that $\widetilde{\mathbf{T}}_a$ is anisotropic. □

7 The correspondence between MAD subalgebras and maximal split tori

Throughout this section R will denote an object of k-*alg* such that $\mathfrak{X} = \text{Spec}(R)$ is normal integral and noetherian and K its fraction field. The purpose of this section is to establish the following fundamental correspondence.

Theorem 7.1 *Let $\mathfrak{G}$ be a semisimple simply connected R-group and $\mathfrak{g}$ its Lie algebra.*

(1) *Let $\mathfrak{m}$ be a* MAD *subalgebra of $\mathfrak{g}$. Then $Z_{\mathfrak{G}}(\mathfrak{m})$ is a reductive R-group and its radical contains a unique maximal split torus $\mathfrak{S}(\mathfrak{m})$ of $\mathfrak{G}$.*
(2) *Let $\mathfrak{S}$ is a maximal split torus of $\mathfrak{G}$, and let $\mathfrak{m}(\mathfrak{S})$ be the unique maximal subalgebra of Lie algebra* Lie $(\mathfrak{S})$ *which is an* AD *subalgebra of $\mathfrak{g}$ (see Remark 6.4). Then $\mathfrak{m}(\mathfrak{S})$ is a* MAD *subalgebra of $\mathfrak{g}$.*
(3) *The process $\mathfrak{m} \to \mathfrak{S}(\mathfrak{m})$ and $\mathfrak{S} \to \mathfrak{m}(\mathfrak{S})$ described above gives a bijection between the set of* MAD *subalgebras of $\mathfrak{g}$ and the set of maximal split tori of $\mathfrak{G}$.*
(4) *If $\mathfrak{m}$ and $\mathfrak{m}'$ are two* MAD *subalgebras of $\mathfrak{g}$, then for $\mathfrak{m}$ and $\mathfrak{m}'$ to be conjugate under the adjoint action of $\mathfrak{G}(R)$ it is necessary and sufficient that the maximal split tori $\mathfrak{S}(\mathfrak{m})$ and $\mathfrak{S}(\mathfrak{m}')$ be conjugate under the adjoint action of $\mathfrak{G}(R)$ on $\mathfrak{g}$.*

Remark 7.2 Since $\mathfrak{S}$ is split we have $\text{Lie}\,(\mathfrak{S}) = X(\mathfrak{S})^{\text{o}} \otimes_{\mathbb{Z}} R$ where $X(\mathfrak{S})^{\text{o}}$ is the cocharacter group of $\mathfrak{S}$. As we shall see in the proof of Lemma 7.5 $\mathfrak{m}(\mathfrak{S}) = X(\mathfrak{S})^{\text{o}} \otimes_{\mathbb{Z}} k$.

The proof of the Theorem will be given at the end of this section after a long list of preparatory results. What is remarkable about this correspondence is that MAD subalgebras exist over k but not over R while, in general, the exact opposite is true for split tori of $\mathfrak{G}$. It is this correspondence that allows us to use the methods from [41] to the study of conjugacy questions.

We begin with some general observations and fixing some notation that will be used throughout the proofs of this section. Since $\mathfrak{X}$ is connected all geometric fibers of $\mathfrak{G}$ are of the same type. Let $\mathbf{G}$ be the corresponding Chevalley group over k and $\mathbf{g}$ its Lie algebra.

Lemma 7.3 *Let $\mathfrak{m}$ be an AD subalgebra of $\mathfrak{g}$. Then*

(1) $\dim_k(\mathfrak{m}) \leq \text{rank}\,(\mathbf{g})$. *In particular any AD subalgebra of $\mathfrak{g}$ is included inside a MAD subalgebra of $\mathfrak{g}$.*
(2) *The natural map $\mathfrak{m} \otimes_k R \to R\,\mathfrak{m}$ is an R–module isomorphism. In particular $R\,\mathfrak{m}$ is a free R–module of rank $= \dim_k(\mathfrak{m})$.*
(3) *Let $\{v_1, \dots, v_m\}$ be a k-basis of $\mathfrak{m}$. For every $x \in \mathfrak{X}$ the elements $v_i \otimes 1 \in \mathfrak{g} \otimes_R R_x$ are R_x-linearly independent. Similarly if we replace R_x by K or any field extension of K.*

Proof The three assertions are of local nature, so we can assume that R is local. We will establish the Lemma by first reducing the problem to the split case. According to Remark 5.2 there exists a finite étale extension $\widetilde{R}/R$ such that $\widetilde{R}$ is integral and normal and $\mathfrak{G} \times_R \widetilde{R} \simeq \mathbf{G}_{\widetilde{R}}$. Note that the canonical map $\mathfrak{g} \to \mathfrak{g} \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R}$ is injective and that if $\{v_1, \dots, v_m\}$ are k-linearly independent elements of $\mathfrak{m}$ which are R-linearly dependent, then the image of the elements $\{v_1, \dots, v_m\}$ on $\text{Lie}(\mathfrak{G} \times_R \widetilde{R}) \simeq \mathbf{g} \otimes_k \widetilde{R}$ are k-linearly independent and are $\widetilde{R}$-linearly dependent.

Let $\widetilde{K}$ be the field of fractions of $\widetilde{R}$. By Remark 6.5 the image of $\mathfrak{m}$ under the injection $\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R}$ is an AD subalgebra of $\mathbf{g} \otimes_k \widetilde{R}$. By [33, theo.1.(i)] the dimension of $\mathfrak{m}$ is at most the rank of $\mathbf{g}$. This establishes (1).

As for (2) and (3), the crucial point—as explained in [33, prop. 4]—lies in the fact that the image $\widetilde{\mathfrak{m}}$ of $\mathfrak{m}$ under the injection $\mathfrak{g} \hookrightarrow \mathbf{g} \otimes_k \widetilde{K}$ sits inside a split Cartan subalgebra $\mathcal{H}$ of the split semisimple $\widetilde{K}$-algebra $\mathbf{g} \otimes_k \widetilde{K}$. Consider the basis $\{\check{\omega}_1, \dots, \check{\omega}_\ell\}$ of $\mathcal{H}$ consisting of the fundamental coweights for a base $\alpha_1, \dots, \alpha_\ell$ of the root system of $(\mathbf{g} \otimes_k \widetilde{K}, \mathcal{H})$. Let $1 \leq n \leq m$ be such that $\{\widetilde{v}_1, \dots \widetilde{v}_n\}$ is a maximal set of $\widetilde{K}$-linearly independent elements of $\mathbf{g}(\widetilde{K})$. To establish (2) and (3) it will suffice to show that $n = m$.

Assume on the contrary that $n < m$. Write $\widetilde{v}_i = \sum c_{ji} \check{\omega}_j$ with $c_{1i}, \dots c_{\ell i}$ in $\widetilde{K}$. The fact that the eigenvalues of $\text{ad}_{\mathbf{g}(\widetilde{K})}(\widetilde{v}_i)$ belong to k show that the c_{ji} necessarily belong to k. Indeed $\widetilde{v}_i$ acts on $\mathbf{g}(\widetilde{K})_{\alpha_j}$ as multiplication by the scalar c_{ji}.

Let $v = v_{n+1}$. Write $\widetilde{v} = \sum_{i=1}^n a_i \widetilde{v}_i$ with $a_1, \dots, a_n$ in $\widetilde{K}$. Let $c_{jn+1} = \lambda_j$. Then $\langle \alpha_j, \widetilde{v} \rangle = \lambda_j$ and

$$\widetilde{v} = \sum_j \left(\sum_i a_i c_{ji} \right) \check{\omega}_j = \sum_j \lambda_j \check{\omega}_j.$$

Therefore for all $1 \leq j \leq \ell$ we have $\sum_i a_i c_{ji} = \lambda_j$.

Write $\widetilde{K} = k \oplus W$ as a k–space and use this decomposition to write $a_i = d_i + w_i$. Then $\sum_i d_i c_{ji} = \lambda_j$. A straightforward calculation shows that $\langle \alpha_j, \widetilde{v} - \sum_i d_i \widetilde{v}_i \rangle = 0$ for all j. This forces

$$v_{n+1} = v = \sum_i d_i v_i$$

which contradicts the linear independence of the $v_i' s$ over k. ◻

Remark 7.4 Let $\mathfrak{S} < \mathfrak{G}$ be a split torus. Then there exist characters $\lambda_i : \mathfrak{S} \to \mathbf{G}_{m,R}$ for $1 \leq i \leq l$ such that $\mathfrak{g} = \oplus_{i=1}^l \mathfrak{g}_{\lambda_i}$ where

$$\mathfrak{g}_{\lambda_i} = \{v \in \mathfrak{g} : \mathrm{Ad}(g)v = \lambda_i(g)v \; \forall g \in \mathfrak{S}(R)\}.$$

At the Lie algebra level the situation is as follows. Let $\mathfrak{s} = \mathrm{Lie}\,(\mathfrak{S}) \subset \mathfrak{g}$. Then $\mathfrak{s} \subset \mathfrak{S}(R[\varepsilon])$. We avail ourselves of the useful convention that if $s \in \mathfrak{s}$ then to view s as an element of $\mathfrak{S}(R[\varepsilon])$ we write $e^{s\varepsilon}$. There exist unique R–linear functionals $d\lambda_i : \mathfrak{s} \to R$ such that

$$\lambda_i(e^{s\varepsilon}) = 1 + d\lambda_i(s)\varepsilon \in R[\varepsilon]^{\times} = \mathbf{G}_{m,R}(R[\varepsilon]).$$

Then for $s \in \mathfrak{s}$ and $v \in \mathfrak{g}_{\lambda_i}$ we have the following equality in $\mathfrak{g}$

$$[s, v] = d\lambda_i(s)v. \tag{7.1}$$

Lemma 7.5 *Consider the restriction* $\mathrm{Ad}_{\mathfrak{S}} : \mathfrak{S} \to \mathbf{Gl}\,(\mathfrak{g})$ *of the adjoint representation of* $\mathfrak{G}$ *to* $\mathfrak{S}$*. There exists a finite number of characters* $\lambda_1, \ldots, \lambda_l$ *of* $\mathfrak{S}$ *such that* $\mathfrak{g} = \bigoplus_{i=1}^{l} \mathfrak{g}_{\lambda_i}$*. The* λ_i *are unique and*

$$\mathfrak{m}(\mathfrak{S}) = \{s \in \mathrm{Lie}\,(\mathfrak{S}) \subset \mathfrak{S}\,(R[\varepsilon]) \;:\; d\lambda_i(s) \in k\}.$$

Furthermore

$$\dim_k(\mathfrak{m}(\mathfrak{S})) = \mathrm{rank}(\mathfrak{S}) = \mathrm{rank}_{R-mod}(R\mathfrak{m}(\mathfrak{S}))$$

and $\mathrm{Lie}(\mathfrak{S}) = R\mathfrak{m}(\mathfrak{S})$.

Proof We appeal to the explanation given in Remark 7.4. Let

$$\mathfrak{n} = \{s \in \mathfrak{s} \;:\; d\lambda_i(s) \in k \; \forall i\}.$$

Then (7.1) shows not only that $\mathfrak{n} \subset \mathfrak{s}$ is an AD subalgebra of $\mathfrak{g}$, but in fact that $\mathfrak{m}(\mathfrak{S}) \subset \mathfrak{n}$. By maximality we have $\mathfrak{m}(\mathfrak{S}) = \mathfrak{n}$ as desired.

We now establish the last assertions. Let n be the rank of $\mathfrak{S}$, so $\mathfrak{S} \simeq \mathbf{G}_{m,R}^n$ and the character lattice $X(\mathfrak{S})$ of $\mathfrak{S}$ is generated by the projections $\pi_i : \mathbf{G}_{m,R}^n \to \mathbf{G}_{m,R}$. Since the kernel of the adjoint representation of $\mathfrak{G}$ is finite the sublattice of $X(\mathfrak{S})$ generated by $\lambda_1, \ldots, \lambda_\ell$ has finite index; in particular every character π of $\mathfrak{S}$ can be written as a linear combination $\pi = a_1\lambda_1 + \cdots + a_\ell\lambda_\ell$ with rational coefficients $a_1, \ldots, a_\ell$ and hence $d\pi = a_1 d\lambda_1 + \cdots + a_n d\lambda_\ell$. Similarly π can be written as $\pi = a_1\pi_1 + \cdots + a_n\pi_n$ with $a_1, \ldots, a_n \in \mathbb{Z}$ and we then have $d\pi = a_1 d\pi_1 + \cdots + a_n d\pi_n$. It follows that

$$\begin{aligned}\mathfrak{m}(\mathfrak{S}) &= \{s \in \mathfrak{s} \;:\; d\lambda_i(s) \in k \; \forall i\} \\ &= \{s \in \mathfrak{s} \;:\; d\pi(s) \in k \; \forall \pi \in X(\mathfrak{S})\} \\ &= \{s \in \mathfrak{s} \;:\; d\pi_i(s) \in k \; \forall i\}.\end{aligned}$$

The identification $\mathfrak{S} \simeq \mathbf{G}_{m,R}^n$ induces the identification $\mathfrak{s} \simeq \mathbf{G}_{a,R}^n$. The above equalities yield

$$\mathfrak{m}(\mathfrak{S}) \simeq \{(s_1, \ldots, s_n) \;:\; s_i \in k \; \forall i\},$$

hence the last assertions follow immediately. □

Proposition 7.6 *Let* $\mathfrak{m}$ *be an* AD *subalgebra of* $\mathfrak{g}$. *Then the submodule* $R\mathfrak{m}$ *is a direct summand of* $\mathfrak{g}$.

Proof Let $M = \mathfrak{g}/R\mathfrak{m}$. Assume for a moment that M is a projective R–module. Then the exact sequence

$$0 \longrightarrow R\mathfrak{m} \longrightarrow \mathfrak{g} \longrightarrow M \longrightarrow 0$$

is split and the Proposition follows.

Thus it remains to show that M is a projective R–module or, equivalently, that for every prime ideal x of R the localized R_x–module M_x is free. Since localization is a left exact functor, and by Lemma 7.3 we have $(R\mathfrak{m})_x = R_x\mathfrak{m}$ the sequence

$$0 \longrightarrow R_x\mathfrak{m} \longrightarrow \mathfrak{g}_{R_x} \longrightarrow M_x \longrightarrow 0$$

is exact. By Lemma 7.3(3), the elements

$$v_1 \otimes 1, \dots, v_m \otimes 1 \in R_x\mathfrak{m} \subset \mathfrak{g} \otimes_R R_x = \mathfrak{g}_x$$

and the module $\mathfrak{g}_x$ satisfy the variation of Nakayama's lemma stated in [28, cor. 1.8]. Hence $R_x\mathfrak{m}$ is a direct summand of $\mathfrak{g}$ and this implies that M_x is free. □

Proposition 7.7 *Let* $\mathfrak{m}$ *be an AD subalgebra of* $\mathfrak{g}$. *Then* $Z_{\mathfrak{G}}(\mathfrak{m})$ *is an affine* R*-group whose geometric fibres are (connected) reductive groups.*

Proof By Proposition 7.6 $R\mathfrak{m}$ is a direct summand of $\mathfrak{g}$. It follows from [15, II prop.1.4] that $Z_{\mathfrak{G}}(R\mathfrak{m}) = Z_{\mathfrak{G}}(\mathfrak{m})$ is a closed subgroup of $\mathfrak{G}$. In particular, $Z_{\mathfrak{G}}(\mathfrak{m})$ is an affine scheme which is of finite type over $\mathrm{Spec}(R)$.

Let $x \in \mathrm{Spec}(R)$ be a point and let $k(\overline{x})$ be an algebraic closure of $k(x)$. Since the functor $Z_{\mathfrak{G}}(\mathfrak{m}) = Z_{\mathfrak{G}}(R\mathfrak{m})$ commutes with base change, to verify the nature of its geometric fibers $Z_{\mathfrak{G}}(\mathfrak{m})(\overline{x})$ we may look at

$$Z_{\mathfrak{G}}(R\mathfrak{m}) \otimes_R k(\overline{x}) = Z_{\mathfrak{G}(\overline{x})}(k(\overline{x})\,\mathfrak{m}(\overline{x}))$$

where $\mathfrak{G}(\overline{x}) = \mathfrak{G} \otimes_R k(\overline{x})$ and $\mathfrak{m}(\overline{x})$ is the image of $\mathfrak{m}$ under $\mathfrak{g} \to \mathfrak{g} \otimes_R k(\overline{x})$. Thus we may assume without loss of generality that the ground ring is a field. By results of Steinberg ([44, 3.3 and 3.8] and [44, 0.2]) we conclude that $Z_{\mathfrak{G}}(\mathfrak{m})(\overline{x})$ is connected and reductive. □

7.1 Flatness of $Z_{\mathfrak{G}}(\mathfrak{m})$

Fix a split Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$. With respect to the adjoint representation ad : $\mathbf{g} \to \mathrm{End}_k(\mathbf{g})$ we have the weight space decomposition

$$\mathbf{g} = \oplus_{\alpha \in \Sigma}\, \mathbf{g}_\alpha$$

where $\alpha : \mathbf{h} \to k$ is a linear function such that the corresponding eigenspace $\mathbf{g}_\alpha$ is non-zero. The kernel of the adjoint representation of $\mathbf{g}$ is trivial, $\dim \mathbf{g}_\alpha = 1$ if $\alpha \neq 0$ and $\mathbf{g}_0 = \mathbf{h}$.

Lemma 7.8 *Let* $\mathbf{a} \subset \mathbf{h}$ *be a subalgebra. Then:*

(1) *The centralizer* $Z_{\mathbf{g}}(\mathbf{a})$ *is a reductive Lie algebra whose centre is contained in* $\mathbf{h}$.
(2) *If* $a \in \mathbf{a}$ *is in generic position then* $Z_{\mathbf{g}}(\mathbf{a}) = Z_{\mathbf{g}}(a)$.

Proof (1) The centralizer of $\mathbf{a}$ is generated by $\mathbf{h}$ and those $\mathbf{g}_\alpha$ for which $\alpha(x) = 0$ for every $x \in \mathbf{a}$. It is a well-known fact that this algebra is reductive.
(2) The inclusion $\subset$ is obvious. Conversely, the centralizer of a is generated by $\mathbf{h}$ and those $\mathbf{g}_\alpha$ for which $\alpha(a) = 0$. Since a is generic all such roots α also satisfy $\alpha(x) = 0$ for all $x \in \mathbf{a}$.

□

Lemma 7.9 *Let* $a_\alpha \in k,\ \alpha \in \Sigma$. *Then there exists at most one element* $h \in \mathbf{h}$ *such that* $\alpha(h) = a_\alpha$.

Proof Since the kernel of the adjoint representation of $\mathbf{g}$ is trivial the result follows.
□

Lemma 7.10 *Let S be an object of k-alg. Let* $v \in \mathbf{h} \otimes_k S$ *be an ad k-diagonalizable element of* $\mathbf{g} \otimes_k S$. *If S is an integral domain then* $v \in \mathbf{h}$.

Proof Let F be a field of quotients of S and view v as an element of $\mathbf{g} \otimes_k F$. The eigenvalues a_α of v with respect to the adjoint representation are $a_\alpha = \alpha(v)$. By assumption they all belong to k. Thus the nonhomogeneous linear system $\alpha(x) = a_\alpha$, $\alpha \in \Sigma$, has a solution over F, namely v. Since the coefficients of this system of equations are in k it also has a solution over k [see the proof of Lemma 7.3(2)]. By Lemma 7.9 such a solution is unique, hence $v \in \mathbf{h}$. □

Fix an arbitrary element $h \in \mathbf{h}$. Recall that $\mathbf{G}$ acts on $\mathbf{g}$ by conjugation and it is known that the orbit $\mathcal{O}_h = \mathbf{G} \cdot h$ is a Zariski closed subset of $\mathbf{g}$ (because h is semisimple). Let $\mathbf{L} \subset \mathbf{G}$ be the isotropy subgroup of h in $\mathbf{G}$. As we saw above $\mathbf{L}$ is a reductive subgroup and we have an exact sequence

$$1 \longrightarrow \mathbf{L} \longrightarrow \mathbf{G} \xrightarrow{\phi} \mathbf{G}/\mathbf{L} \longrightarrow 1.$$

The algebraic k–varieties $\mathcal{O}_h$ and $\mathbf{G}/\mathbf{L}$ have the distinguished points h and the coset $e = 1 \cdot \mathbf{L}$ respectively. The group $\mathbf{G}$ acts on both $\mathcal{O}_h$ and $\mathbf{G}/\mathbf{L}$ in a natural way and there exists a natural $\mathbf{G}$-equivariant isomorphism $\lambda : \mathcal{O}_h \simeq \mathbf{G}/\mathbf{L}$ which takes h into e (see [5, III §9] for details). Hence if R is an object in k-alg and $x \in \mathcal{O}_h(R)$, then x and h are conjugate by an element in $\mathbf{G}(R)$ if and only if $\lambda(x) \in \mathbf{G}(R) \cdot e$.

We now return to our simply connected semisimple R-group $\mathfrak{G}$ and its Lie algebra $\mathfrak{g}$.

Lemma 7.11 *Let* $\mathfrak{m}$ *be an AD subalgebra of* $\mathfrak{g}$. *The affine scheme* $Z_{\mathfrak{G}}(\mathfrak{m})$ *is flat over* $\mathrm{Spec}(R)$.

Proof That $Z_{\mathfrak{G}}(\mathfrak{m})$ is an affine scheme over R has already been established. Since flatness is a local property it will suffice to establish the result after we replace R by its localization at each element of $\mathfrak{X}$. Lemma 5.1 provides a finite étale connected cover $\widetilde{R}/R$ which splits $\mathfrak{G}$. By replacing R by $\widetilde{R}$ we reduce the problem to the split case. Summarizing, without loss of generality we may assume that $\mathfrak{G} = \mathbf{G} \times_k R$, $\mathfrak{g} = \mathbf{g} \otimes_k R := \mathbf{g}_R$ and R is a local domain.

As observed in Lemma 7.3 $\mathfrak{m}$ is contained in a split Cartan subalgebra $\mathcal{H}$ of $\mathbf{g} \otimes_k K := \mathbf{g}_K$. Fix a generic vector $v \in \mathfrak{m} \subset \mathbf{g}_K$. Let $\{a_\alpha, \alpha \in \Sigma\}$ be the family of all eigenvalues of v with respect to the adjoint representation of $\mathbf{g}_K$. Since $\mathfrak{m}$ is an AD subalgebra of $\mathbf{g}_R$, we have $a_\alpha \in k$ for every $\alpha \in \Sigma$. □

Sublemma 7.12 *There exists a unique vector $h \in \mathbf{h}$ whose eigenvalues with respect to the adjoint representation are $\{a_\alpha, \alpha \in \Sigma\}$. Moreover if v and h are viewed as elements of $\mathbf{g}_K$, then they are conjugate under $\mathbf{G}(K)$.*

Proof Uniqueness follows from Lemma 7.9. As for existence, we note that $\mathcal{H}$ and $\mathbf{h}_K$ are conjugate over K, hence $\mathbf{h}_K$ clearly contains an element with the prescribed property. By Lemma 7.10 this element is contained in $\mathbf{h}$. The conjugacy assertion follows from the construction of h. □

We now come back to the $\mathbf{G}$-orbit $\mathcal{O}_h$ of h. We remind the reader that this is a closed subvariety of $\mathbf{g}$.

Sublemma 7.13 $v \in \mathcal{O}_h(R)$.

Proof The element $v \in \mathbf{g}_R$ can be viewed as a morphism

$$\phi_v : \mathrm{Spec}\,(R) \to \mathbf{g}.$$

The image of the generic point $\mathrm{Spec}(K) \to \mathrm{Spec}(R) \to \mathbf{g}$ is contained in $\mathcal{O}_h$ for v and h are conjugate over K. Since $\mathcal{O}_h$ is a closed subvariety of $\mathbf{g}$ and since $\mathrm{Spec}(R)$ is irreducible it follows that ϕ_v factors through the embedding $\mathcal{O}_h \hookrightarrow \mathfrak{g}$. □

To finish the proof of Lemma 7.11 we first consider the particular case when $\mathfrak{m}$ is contained in $\mathbf{h}$. Then $Z_{\mathfrak{G}}(\mathfrak{m})$ is obtained from the variety $Z_{\mathbf{G}}(\mathfrak{m})$ by the base change R/k so that flatness is clear.

In the general case, let $h \in \mathbf{h}$ be the element provided by Sublemma 7.12. By Sublemma 7.13 we have $v \in \mathcal{O}_h(R) = (\mathbf{G}/\mathbf{L})(R)$. Denote by R^{sh} the strict henselisation of the local ring R, that is the simply connected cover of R attached to a separable closure K_s of K (see [37, §X.2]). Since the map $p : \mathbf{G} \to \mathbf{G}/\mathbf{L}$ is smooth and surjective, Hensel's lemma [29, §4] shows that $\mathbf{G}(R^{sh}) \to (\mathbf{G}/\mathbf{L})(R^{sh})$ is surjective. But R^{sh} is the inductive limit of the finite (connected) Galois covers of R, so there exists one such cover R' and a point $g' \in \mathbf{G}(R')$ such that $v = g'.h$. Up to replacing R by R' (which is a noetherian normal domain) we may assume that $v = h$.

We now recall that $Z_{\mathbf{g}_R}(h) = Z_{\mathbf{g}_R}(\mathfrak{m})$ since $h = v \in \mathfrak{m}$ is a generic vector. Since the center of $Z_{\mathbf{g}_R}(h)$ is contained in $\mathbf{h}_R$ and since $\mathfrak{m}$ is contained in the center of its centralizer we have $\mathfrak{m} \subset \mathbf{h}_R$. Applying Lemma 7.10 then shows that $\mathfrak{m} \subset \mathbf{h}$. Thus we have reduced the general case to the previous one. □

Proposition 7.14 *If $\mathfrak{m}$ is an AD subalgebra of $\mathfrak{g}$ then $Z_{\mathfrak{G}}(\mathfrak{m})$ is a reductive R-group.*

Proof Since $Z_{\mathfrak{G}}(\mathfrak{m})$ is flat and also finitely presented over R the differential criteria for smoothness shows that $Z_{\mathfrak{G}}(\mathfrak{m})$ is in fact smooth over R because of Proposition 7.7. Furthermore, geometric fibers of $Z_{\mathfrak{G}}(\mathfrak{m})$ are (connected) reductive groups in the usual sense (this last again by Proposition 7.7). By definition $Z_{\mathfrak{G}}(\mathfrak{m})$ is a reductive R-group. □

Proof of Theorem 7.1 (1) Let $\mathfrak{m}$ be a MAD subalgebra of $\mathfrak{g}$, and let $\mathfrak{S}$ denote the maximal split torus of the radical $\mathfrak{T}$ of the reductive R-group $Z_{\mathfrak{G}}(\mathfrak{m})$. By Remark 6.4 the Lie algebra of $\mathfrak{S}$ contains a unique maximal subalgebra $\mathfrak{m}(\mathfrak{S})$ which is an AD–subalgebra of $\mathfrak{g}$. By definition $\mathfrak{S} < \mathfrak{H} = Z_{\mathfrak{G}}(R\mathfrak{m})$. Denote Lie$(\mathfrak{S})$ by $\mathfrak{s}$. Since $\mathfrak{s} \subset \mathfrak{S}(R[\varepsilon])$ it follows that in $\mathfrak{g}$ we have $[\mathfrak{s}, R\mathfrak{m}] = 0$. In particular since $\mathfrak{m}(\mathfrak{S}) \subset \mathfrak{s}$ we have $[\,\mathfrak{m}(\mathfrak{S}), \mathfrak{m}\,] = 0$. But then by Remark 6.4 $\mathfrak{m} + \mathfrak{m}(\mathfrak{S})$ is an AD subalgebra of $\mathfrak{g}$. Since $\mathfrak{m}$ is a MAD subalgebra we necessarily have $\mathfrak{m}(\mathfrak{S}) \subset \mathfrak{m}$ and now we are going to show that $\mathfrak{m}(\mathfrak{S}) = \mathfrak{m}$.

Recall that K denotes the quotient field of R. By Lemma 7.5 we have $\dim(\mathfrak{m}(\mathfrak{S})) = \operatorname{rank}(\mathfrak{S})$, so that to establish that $\mathfrak{m}(\mathfrak{S}) = \mathfrak{m}$ it will suffice to show that $\operatorname{rank}(\mathfrak{S}) \geq \dim_k(\mathfrak{m})$, or equivalently that $\dim_K(\mathfrak{S}_K) \geq \dim_k(\mathfrak{m})$.

We have $\mathfrak{H}_K = Z_{\mathfrak{G}_K}(R\mathfrak{m}) = Z_{\mathfrak{G}_K}(K\mathfrak{m})$, as can be seen from the fact that the computation of the centralizer commutes with base change. Since $\mathfrak{S}$ is the maximal split torus of $\mathfrak{T}$ then $\mathfrak{S}_K$ is the maximal split torus of $\mathfrak{T}_K = \operatorname{rad}(\mathfrak{H}_K)$ by Lemma 5.4. We also have

$$\operatorname{Lie}(\mathfrak{H}_K) = \operatorname{Lie}(Z_{\mathfrak{G}_K}(R\mathfrak{m})) = \operatorname{Lie}(Z_{\mathfrak{G}_K}(K\mathfrak{m})) = Z_{\mathfrak{g}_K}(K\mathfrak{m}).$$

Since $K\mathfrak{m}$ is in the centre of $Z_{\mathfrak{g}_K}(K\mathfrak{m}) = \operatorname{Lie}(\mathfrak{H}_K)$ and the centre of $\operatorname{Lie}(\mathfrak{H}_K)$ coincides with $\operatorname{Lie}(\mathfrak{T}_K)$ we conclude that $K\mathfrak{m} \subset \operatorname{Lie}(\mathfrak{T}_K)$. On the other hand $K\mathfrak{m}$ is an AD subalgebra of $\mathfrak{g}_K$, so that by Lemma 6.7 $K\mathfrak{m} \subset \operatorname{Lie}(\mathfrak{S}_K)$. This shows that $\dim_K(K\mathfrak{m}) \leq \dim_K(\mathfrak{S}_K)$. But by Lemma 7.3(3) we have $\dim_k(\mathfrak{m}) = \dim_K(K\mathfrak{m})$. This completes the proof that $\mathfrak{m}(\mathfrak{S}) = \mathfrak{m}$.

Now it is easy to finish the proof that $\mathfrak{S}$ is a maximal split torus in $\mathfrak{G}$. If $\mathfrak{S}$ is contained in a split torus $\mathfrak{S}'$ of larger rank then $\mathfrak{m}(\mathfrak{S}) \subset \mathfrak{m}(\mathfrak{S}')$ is a proper subalgebra which contradicts to the fact that $\mathfrak{m} = \mathfrak{m}(\mathfrak{S})$ is a MAD subalgebra.

(2) Let $\mathfrak{S}$ be a maximal split torus of $\mathfrak{G}$, and let $\mathfrak{s} = \operatorname{Lie}(\mathfrak{S})$ be its Lie algebra. By Remark 6.4 $\mathfrak{s}$ contains a unique maximal subalgebra $\mathfrak{m}(\mathfrak{S}) = \mathfrak{m}$ which is an AD–subalgebra of $\mathfrak{g}$. We have by Lemma 7.5 that $R\mathfrak{m} = \operatorname{Lie}(\mathfrak{S})$. Thus, appealing to Proposition 3.7 and Lemma 7.3(1) we obtain

$$Z_{\mathfrak{G}}(\mathfrak{m}) = Z_{\mathfrak{G}}(R\mathfrak{m}) = Z_{\mathfrak{G}}(\mathfrak{s}) = Z_{\mathfrak{G}}(\mathfrak{S}).$$

We claim that $\mathfrak{m}$ is maximal. Assume otherwise. Then by Lemma 7.3(1) $\mathfrak{m}$ is properly included in a MAD subalgebra $\mathfrak{m}'$ of $\mathfrak{g}$. We have

$$\mathfrak{H}' := Z_{\mathfrak{G}}(R\,\mathfrak{m}') \subset \mathfrak{H} := Z_{\mathfrak{G}}(R\mathfrak{m}) = Z_{\mathfrak{G}}(\mathfrak{S}).$$

By Proposition 7.14 $\mathfrak{H}'$ and $\mathfrak{H}$ are reductive R-groups. Let $\mathfrak{T}'$ and $\mathfrak{T}$ be their radicals and let $\mathfrak{T}'_d, \mathfrak{T}_d$ be their maximal split tori. We have $\mathfrak{S} \subset \mathfrak{T} \subset \mathfrak{T}'$ and hence $\mathfrak{S} \subset \mathfrak{T}_d \subset \mathfrak{T}'_d$. But $\mathfrak{S}$ is a maximal split torus in $\mathfrak{G}$. Therefore $\mathfrak{S} = \mathfrak{T}'_d = \mathfrak{T}_d$ and this implies $\mathfrak{m} = \mathfrak{m}(\mathfrak{S}) = \mathfrak{m}(\mathfrak{T}_d) = \mathfrak{m}(\mathfrak{T}'_d)$. Recall that in part (1) we showed that $\mathfrak{m}(\mathfrak{T}'_d) = \mathfrak{m}'$ and thus $\mathfrak{m} = \mathfrak{m}'$ – a contradiction.

(3) If $\mathfrak{m}$ is a MAD subalgebra of $\mathfrak{g}$, the corresponding maximal split torus $\mathfrak{S}(\mathfrak{m})$ is the maximal split torus of the radical of $\mathfrak{H} = Z_{\mathfrak{G}}(R\mathfrak{m})$. The proof of (1) shows that the MAD subalgebra corresponding to $\mathfrak{S}(\mathfrak{m})$ is $\mathfrak{m}$.

Conversely, if $\mathfrak{S}$ is a maximal split torus of $\mathfrak{G}$ then the maximal split torus corresponding to $\mathfrak{m}(\mathfrak{S})$ is the maximal split torus of the radical of the reductive group $Z_{\mathfrak{G}}(R\mathfrak{m}(\mathfrak{S})) = Z_{\mathfrak{G}}(\mathfrak{s}) = Z_{\mathfrak{G}}(\mathfrak{S})$ as explained in the proof of (1). Clearly $\mathfrak{S}$ is inside the radical of $Z_{\mathfrak{G}}(\mathfrak{S})$. Since $\mathfrak{S}$ is maximal split in $\mathfrak{G}$ it is maximal split in the radical of $Z_{\mathfrak{G}}(\mathfrak{S})$. Thus $\mathfrak{S} = \mathfrak{S}'$.

(4) Follows from the construction and functoriality in the definition of the adjoint action at the Lie algebra and group level. □

8 A sufficient condition for conjugacy

In this section R denotes a normal noetherian domain and K its field of quotients. Let $\mathfrak{G}$ be a reductive group scheme over R. We say that a maximal split torus $\mathfrak{S}$ of $\mathfrak{G}$ is *generically maximal split* if $\mathfrak{S}_K$ is a maximal split torus of $\mathfrak{G}_K$.

Proposition 8.1 *Let $\mathfrak{S}$ be a generically maximal split torus of $\mathfrak{G}$. If*

$$H^1_{Zar}(R, Z_{\mathfrak{G}}(\mathfrak{S})) = 1 \tag{8.1}$$

then all generically maximal split tori of $\mathfrak{G}$ are conjugate under $\mathfrak{G}(R)$.

We begin with two preliminary results.

Lemma 8.2 *Let $\mathfrak{W}$ be a finite étale R-group with R normal. Let K be the field of quotients of R. Then*

(1) *The canonical map*

$$\chi : H^1_{\acute{e}t}(R, \mathfrak{W}) \longrightarrow H^1(K, \mathfrak{W}_K)$$

is injective.

(2) $H^1_{Zar}(R, \mathfrak{W}) = 1$.

Proof (1) Because of the assumptions on $\mathfrak{W}$ we can compute $H^1_{\acute{e}t}(R, \mathfrak{W})$ as the limit of $H^1_{\acute{e}t}(S/R, \mathfrak{W})$ with S a connected finite Galois extension of R.

Let $\Gamma = \mathrm{Gal}(S/R)$. It is well-known that $\mathfrak{W}$ corresponds to a finite group W together with an action of the algebraic fundamental group of R, and that $H^1_{\acute{e}t}(S/R, \mathfrak{W}) = H^1(\Gamma, \mathfrak{W}(S))$ (see [40, XI §5]). If L denotes the field of quotients of S then L/K is also Galois with Galois group naturally isomorphic to Γ as explained in [9, Ch.5 §2.2 theo. 2]. Our map χ is obtained by the base change K/R. By the above considerations the problem reduces to the study of the map

$$\chi : H^1(\Gamma, \mathfrak{W}(S)) \longrightarrow H^1(\Gamma, \mathfrak{W}(S \otimes_R K))$$

when passing to the limit over S. Since R is normal by [22, 18.10.8 and 18.10.9] we have $S \otimes_R K = L$. If S is sufficiently large, $\mathfrak{W}(S) = W = \mathfrak{W}(L)$. The compatibility of the two Galois actions gives the desired injectivity.

(2) It is clear that $H^1_{Zar}(R, \mathfrak{W})$ is in the kernel of χ. □

Lemma 8.3 *Let $\mathfrak{S}$ and $\mathfrak{S}'$ be generically maximal split tori of $\mathfrak{G}$. Then the transporter $\tau_{\mathfrak{S},\mathfrak{S}'} = \mathbf{Trans}_{\mathfrak{G}}(\mathfrak{S}, \mathfrak{S}')$ is a (Zariski) locally trivial $N_{\mathfrak{G}}(\mathfrak{S})$-torsor over R.*

Proof By [41, XI, 6.11 (a)], $\tau_{\mathfrak{S},\mathfrak{S}'}$ is a closed subscheme of $\mathfrak{G}$. It is clearly a right (formal) torsor under the affine R-group $N_{\mathfrak{G}}(\mathfrak{S})$. Since $\mathfrak{S}_{R_\mathfrak{p}}$ and $\mathfrak{S}'_{R_\mathfrak{p}}$ are maximal split tori of $\mathfrak{G}_{R_\mathfrak{p}}$ they are conjugate under $\mathfrak{G}(R_\mathfrak{p})$ by [41, XXVI, 6.16]. Thus $\tau_{\mathfrak{S},\mathfrak{S}'}$ is an $N_{\mathfrak{G}}(\mathfrak{S})$-torsor which is locally trivial (i.e. there exists a Zariski open cover $\mathfrak{X} = \cup \mathfrak{X}_i$ such that $\tau_{\mathfrak{S},\mathfrak{S}'}(\mathfrak{X}_i) \neq \emptyset$). □

Proof of Proposition 8.1 Let $\mathfrak{S}'$ be a generically maximal split torus of $\mathfrak{G}$. The transporter $\tau_{\mathfrak{S},\mathfrak{S}'}$ yields according to Lemma 8.3 an element $\alpha \in H^1_{Zar}(R, N_{\mathfrak{G}}(\mathfrak{S}))$. Our aim is to show that α is trivial.

Consider the exact sequence (on $\mathfrak{X}_{\acute{e}t}$) of R-groups

$$1 \longrightarrow Z_{\mathfrak{G}}(\mathfrak{S}) \longrightarrow N_{\mathfrak{G}}(\mathfrak{S}) \longrightarrow \mathfrak{W} \longrightarrow 1$$

with $\mathfrak{W} = N_{\mathfrak{G}}(\mathfrak{S})/Z_{\mathfrak{G}}(\mathfrak{S})$. Then $\mathfrak{W}$ is a finite étale group over R (see [41, XI, 5.9]). By Lemma 8.2(2) the image of α in $H^1_{\acute{e}t}(R, \mathfrak{W})$, which we know lies in $H^1_{Zar}(R, \mathfrak{W})$, is trivial. Thus we may assume $\alpha \in H^1_{\acute{e}t}(R, Z_{\mathfrak{G}}(\mathfrak{S}))$. To finish the proof we need to show that

$$\alpha \in \mathrm{Im}\,[H^1_{Zar}(R, Z_{\mathfrak{G}}(\mathfrak{S})) \longrightarrow H^1_{\acute{e}t}(R, Z_{\mathfrak{G}}(\mathfrak{S}))].$$

For this it suffices to show that the image $\alpha_\mathfrak{p}$ of α in

$$H^1_{\acute{e}t}(R_\mathfrak{p}, Z_{\mathfrak{G}}(\mathfrak{S}) \times_R R_\mathfrak{p}) = H^1_{\acute{e}t}(R_\mathfrak{p}, Z_{\mathfrak{G}_{R_\mathfrak{p}}}(\mathfrak{S}_{R_\mathfrak{p}}))$$

is trivial for all $\mathfrak{p} \in \mathfrak{X}$.

Since $\mathfrak{S}$ is generically maximal split, $\mathfrak{S}_{R_\mathfrak{p}}$ is a maximal split torus of $\mathfrak{G}_{R_\mathfrak{p}}$. Similarly for $\mathfrak{S}'_{R_\mathfrak{p}}$. Now by [41, XXVI prop. 6.16] $\mathfrak{S}_{R_\mathfrak{p}}$ and $\mathfrak{S}'_{R_\mathfrak{p}}$ are conjugate under $\mathfrak{G}_{R_\mathfrak{p}}(R_\mathfrak{p}) = \mathfrak{G}(R_\mathfrak{p})$, Thus the image of α under the composition of the natural maps

$$H^1_{\acute{e}t}(R, Z_{\mathfrak{G}}(\mathfrak{S})) \to H^1_{\acute{e}t}(R, N_{\mathfrak{G}}(\mathfrak{S})) \to H^1_{\acute{e}t}(R_\mathfrak{p}, N_{\mathfrak{G}_{R_\mathfrak{p}}}(\mathfrak{S}_{R_\mathfrak{p}})) \to H^1_{\acute{e}t}(R_\mathfrak{p}, \mathfrak{G}_{R_\mathfrak{p}})$$

is trivial. Let $\mathfrak{P}$ be a parabolic subgroup of $\mathfrak{G}_{R_\mathfrak{p}}$ containing $Z_{\mathfrak{G}_{R_\mathfrak{p}}}(\mathfrak{S}_{R_\mathfrak{p}})$ as a Levi subgroup (see Lemma 3.4). Then (see the proof of [41, XXVI cor. 5.10]) we have

$$H^1_{\acute{e}t}(R_\mathfrak{p}, Z_{\mathfrak{G}_{R_\mathfrak{p}}}(\mathfrak{S}_{R_\mathfrak{p}})) \simeq H^1_{\acute{e}t}(R_\mathfrak{p}, \mathfrak{P}) \hookrightarrow H^1_{\acute{e}t}(R_\mathfrak{p}, \mathfrak{G}_{R_\mathfrak{p}}).$$

It now follows that $\alpha_\mathfrak{p}$ is trivial. □

8.1 A counter-example to conjugacy for multiloop algebras

Let $\mathfrak{G}$ and $\mathfrak{g}$ be as in Theorem 7.1. We know that the conjugacy of two MAD subalgebras in $\mathfrak{g}$ is equivalent to the conjugacy of the corresponding maximal split tori. The following example shows that in general maximal split tori are not necessarily conjugate.

Let D be the quaternion algebra over $R = R_2 = k[t_1^{\pm 1}, t_2^{\pm 1}]$ with generators T_1, T_2 and relations $T_1^2 = t_1,\ T_2^2 = t_2$ and $T_2 T_1 = -T_1 T_2$ and let $A = M_2(D)$. We may view A as the D-endomorphism algebra of the free right rank 2 module $V = D \oplus D$ over D. Let $\mathfrak{G} = \mathbf{SL}\,(1, A)$. This is a simple simply connected R-group of absolute type $\mathbf{SL}_{4,R}$. It contains a split torus $\mathfrak{S}$ whose R–points are matrices of the form

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

where $x \in R^{\times}$. It is well-known that this is a maximal split torus of $\mathfrak{G}$.

Consider now the D-linear map $f : V = D \oplus D \to D$ given by

$$(u, v) \to (1 + T_1)u - (1 + T_2)v.$$

Let $\mathcal{L}$ be its kernel. It is shown in [17] that f splits and that $\mathcal{L}$ is a projective D–module of rank 1 which is not free. Since f is split, we have another decomposition $V \simeq \mathcal{L} \oplus D$. Let $\mathfrak{S}'$ be the split torus of $\mathfrak{G}$ whose R–points consist of linear transformations acting on the first summand $\mathcal{L}$ by multiplication $x \in R^{\times}$ and on the second summand by x^{-1}. As before, $\mathfrak{S}'$ is also a maximal split torus of $\mathfrak{G}$.

We claim that $\mathfrak{S}$ and $\mathfrak{S}'$ are not conjugate under $\mathfrak{G}(R)$. To see this we note that given $\mathfrak{S}$ we can restore the two summands in the decomposition $V = D \oplus D$ as eigenspaces of elements $\mathfrak{S}(R)$. Similarly, we can uniquely restore the two summands in the decomposition $V = \mathcal{L} \oplus D$ out of $\mathfrak{S}'$. Assuming now that $\mathfrak{S}$ and $\mathfrak{S}'$ are conjugate by an element in $\mathfrak{G}(R)$ we obtained immediately that the D–submodule $\mathcal{L}$ in V is isomorphic to one of the components of $V = D \oplus D$, in particular $\mathcal{L}$ is free – a contradiction.

9 The nullity one case

In this section we look in detail at the case $R = k[t^{\pm 1}]$ where k is assumed to be *algebraically closed*. It is known that twisted forms of $\mathbf{g} \otimes_k R$ are nothing but the derived algebras of the affine Kac–Moody Lie algebras modulo their centres [34]. We maintain all of our previous notation, except for the fact that now we specify that $n = 1$.

Lemma 9.1 *Every maximal split torus of* $\mathfrak{G}$ *is generically maximal split.*

Proof Let $\mathfrak{S}$ be a maximal split torus of our simply connected R-group $\mathfrak{G}$. We must show that $\mathfrak{S}_K$ is a maximal split torus of $\mathfrak{G}_K$. We consider the reductive R-group $\mathfrak{H} = Z_{\mathfrak{G}}(\mathfrak{S})$, its derived (semisimple) group $\mathcal{D}\,(\mathfrak{H})$ which we denote by $\mathfrak{H}'$, and the radical rad $(\mathfrak{H})$ of $\mathfrak{H}$. Recall that rad $(\mathfrak{H})$ is a central torus of $\mathfrak{H}$ and that we have an exact sequence of R-groups

$$1 \longrightarrow \boldsymbol{\mu} \longrightarrow \mathrm{rad}\,(\mathfrak{H}) \times_R \mathfrak{H}' \xrightarrow{m} \mathfrak{H} \longrightarrow 1$$

where m is the multiplication and $\boldsymbol{\mu}$ is a finite group of multiplicative type.

Since $\mathfrak{S}$ is central in $\mathfrak{H}$ it lies inside rad $(\mathfrak{H})$, hence it is the maximal split torus of rad $(\mathfrak{H})$. Recall that by Lemma 5.4, $\mathfrak{S}_K$ is still the maximal split torus of rad $(\mathfrak{H})_K$. If $\mathfrak{S}_K$ is not a maximal split torus of $\mathfrak{G}_K$, there exists a split torus $\mathfrak{S}'$ of $\mathfrak{H}_K$ such that $\mathfrak{S}'$ is not a subgroup of rad $(\mathfrak{H}_K)$. Thus if we set $(\mathfrak{S}' \cap \mathfrak{H}'_K)^\circ = \mathfrak{T}$ then $\mathfrak{T}$ is a non-trivial split torus of $\mathfrak{H}'_K$. Then $Z_{\mathfrak{H}'_K}(\mathfrak{T})$ is a Levi subgroup of a proper parabolic subgroup $\mathfrak{P}$ of $\mathfrak{H}'_K$.

Let $\mathbf{t} = \text{type}\,(\mathfrak{P})$ be the type of $\mathfrak{P}$. Let $\mathbf{Par_t}(\mathfrak{H}')$ be the R-scheme of parabolic subgroups of $\mathfrak{H}'$ of type $\mathbf{t}$. Then $\mathbf{Par_t}(\mathfrak{H}')(K) \neq \emptyset$. Since $\mathbf{Par_t}(\mathfrak{H}')$ is proper and R is regular of dimension 1, it follows that $\mathbf{Par_t}(\mathfrak{H}')(R) \neq \emptyset$. Let $\mathfrak{P}'$ be a parabolic subgroup $\mathfrak{H}'$ of type $\mathbf{t}$. It is a proper subgroup, so that by Proposition 5.5 $\mathfrak{P}'$ contains a copy of $\mathbf{G}_{m,R}$. But then $m : \mathfrak{S} \times \mathbf{G}_{m,R} \to \mathfrak{H}$ yields a split torus of $\mathfrak{H}$ that properly contains $\mathfrak{S}$ (since the multiplication map has finite kernel), which contradicts the maximality of $\mathfrak{S}$. □

Theorem 9.2 *In nullity one all MAD subalgebras of* $\mathfrak{g}$ *are conjugate under the adjoint action of* $\mathfrak{G}(R)$.

Proof In view of the last Lemma and Proposition 8.1 it will suffice to show that if $\mathfrak{S}$ is a maximal split torus of $\mathfrak{G}$, then $H^1_{Zar}(R, Z_{\mathfrak{G}}(\mathfrak{S})) = 1$. Since $Z_{\mathfrak{G}}(\mathfrak{S})$ is a reductive R-group one in fact has a much stronger result, namely that $H^1_{\acute{e}t}(R, Z_{\mathfrak{G}}(\mathfrak{S})) = 1$ (see [34, theo. 3.1]). □

Remark 9.3 Let G be the "simply connected" Kac–Moody (abstract) group corresponding to $\mathfrak{g}$ (see [PK], and also [27] and [MP] for details). We have the adjoint representation $\text{Ad} : G \to \text{Aut}_{k-\text{Lie}}(\mathfrak{g})$. The celebrated Peterson–Kac conjugacy theorem [35] for symmetrizable Kac–Moody (applied to the affine case) asserts that all MAD subalgebras of $\mathfrak{g}$ are conjugate under the adjoint action of the group Ad (G) on $\mathfrak{g}$, while our result gives conjugacy under the image of $\mathfrak{G}(R)$, where the image is that of the adjoint representation $\text{Ad} : \mathfrak{G} \to \text{Aut}\,(\mathfrak{g})$ evaluated at R. In the untwisted case it is known that the two groups induce the same group of automorphisms of $\mathfrak{g}$ (see for example [27]). The twisted case appears to remain unstudied.

10 A density property for points of loop groups

In this section $\mathfrak{X} = \text{Spec}(R_n)$. For a description of $\pi_1(\mathfrak{X}, a)$ see Sect. 4.2.

Let $\mathbf{G}$ be a linear algebraic k-group. Let $\eta \in Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\bar{k}))$ be a loop cocycle and recall the decomposition $\eta = (\eta^{geo}, z)$ into geometric and arithmetic parts described in Lemma 4.3. Recall that we may view η^{geo} as a k-group homomorphism ${}_\infty\boldsymbol{\mu} \to {}_z\mathbf{G}$. We denote below by $({}_z\mathbf{G})^{\eta^{geo}}$ the centralizer in ${}_z\mathbf{G}$ of the group homomorphism η^{geo}. Thus defined $({}_z\mathbf{G})^{\eta^{geo}}$ is a k–subgroup of ${}_z\mathbf{G}$

Remark 10.1 By continuity there exists m and a Galois extension $\tilde{k}$ of k such that η factors through

$$\eta : \widetilde{\Gamma}_{n,m} \to \mathbf{G}(\tilde{k})$$

where

$$\widetilde{\Gamma}_{n,m} := \text{Gal}(R_{n,m} \otimes_k \tilde{k}/R_n) = \boldsymbol{\mu}_m^n(\tilde{k}) \rtimes \text{Gal}(\tilde{k}/k)$$

where $m > 0$ and $\widetilde{k}/k$ is a finite Galois extension containing all m–roots of unity in $\overline{k}$. By means of this interpretation η can be viewed as a Galois cocycle in $Z^1(\widetilde{\Gamma}_{n,m}, \mathbf{G}(R_{n,m} \otimes_k \tilde{k}))$. We call this procedure "reasoning at the finite level".

We say that an abstract group M is *pro-solvable* if it admits a filtration

$$\cdots \subset M_{n+1} \subset M_n \subset \cdots \subset M_0 = M$$

by normal subgroups such that $\cap M_n = 1$ and M_n/M_{n+1} is abelian for all $n \geq 0$. If there exists a filtration such that M_n/M_{n+1} are k-vector spaces, we say that M is *pro-solvable in k-vector spaces*.

Theorem 10.2 *Let* $\mathbf{G}$ *be a linear algebraic k-group such that* $\mathbf{G}^\circ$ *is reductive. Let* $\eta \in Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k}))$ *be a loop cocycle such that the twisted* R_n*-group* $\mathfrak{H} = {}_\eta(\mathbf{G}_{R_n})$ *is anisotropic. There exists a family of pro-solvable groups in k-vector spaces* $(J_i)_{i=1,..,n}$ *such that*

$$\mathfrak{H}(F_n) \simeq J_n \rtimes J_{n-1} \rtimes \ldots \rtimes J_1 \rtimes ({}_z\mathbf{G})^{\eta^{geo}}(k) \simeq (J_n \rtimes J_{n-1} \rtimes \ldots \rtimes J_1) \cdot \mathfrak{H}(R_n).$$

Proof Twisting by z we may assume that z is trivial. It is convenient to work at a finite level, namely with a cocycle $\eta : \widetilde{\Gamma}_{n,m} \to \mathbf{G}(\tilde{k})$ as in Remark 10.1.

We proceed by induction on $n \geq 0$; the case $n = 0$ being obvious. We reason by means of a building argument and we view $\widetilde{F}_{n,m}$ and its subfield $F_n = (\widetilde{F}_{n,m})^{\widetilde{\Gamma}_{n,m}}$ as local complete fields with the residue fields $\widetilde{F}_{n-1,m}$ and F_{n-1} respectively. Let $\mathcal{B}_n = \mathcal{B}(\mathbf{G}_{\widetilde{F}_{n,m}})$ be the (enlarged) Bruhat–Tits building of the $\widetilde{F}_{n,m}$-group $\mathbf{G}_{\widetilde{F}_{n,m}}$ [10–12,46, §2.1]. Recall that $\mathcal{B}_n$ is equipped with a natural action of $\mathbf{G}(\widetilde{F}_{n,m}) \rtimes \widetilde{\Gamma}_{n,m}$. Since $\mathfrak{H}$ is anisotropic the algebraic F_n-group $\mathfrak{H}_{F_n}$ is also anisotropic by [19, cor. 7.4.3]. It is shown in [19, theo. 7.9] that the building of $\mathfrak{H}_{F_n}$ inside $\mathcal{B}_n$ consists of a single point ϕ whose stabilizer is $\mathbf{G}(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]])$. Since $\mathfrak{H}(F_n)$ stabilizes ϕ it follows that

$$\mathfrak{H}(F_n) = \{g \in \mathbf{G}(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) \mid \eta(\sigma)\,\sigma(g) = g \ \forall \sigma \in \widetilde{\Gamma}_{n,m}\}. \tag{10.1}$$

We next decompose $\boldsymbol{\mu}_m^n = \boldsymbol{\mu}_m^{n-1} \times \boldsymbol{\mu}_m$. The second component is a finite k-group of multiplicative type acting on $\mathbf{G}$ via η^{geo}. We let $\mathbf{G}_{n-1}$ denote the k–subgroup of $\mathbf{G}$ which is the centralizer of this action [15, II 1.3.7]. The connected component of $\mathbf{G}_{n-1}$ is reductive according to [38]. Since the action of $\boldsymbol{\mu}_m^{n-1}$ on $\mathbf{G}$ given by η^{geo} commutes with that of $\boldsymbol{\mu}_m$ the k-group morphism $\eta^{geo} : \boldsymbol{\mu}_m^n \to \mathbf{G}$ factors through $\mathbf{G}_{n-1}$.

Denote by η_{n-1}^{geo} the restriction of η^{geo} to the k–subgroup $\boldsymbol{\mu}_m^{n-1}$ of $\boldsymbol{\mu}_m^n$. Set $\tilde{\Gamma}_{n-1,m} := \boldsymbol{\mu}_m^{n-1}(\tilde{k}) \rtimes \mathrm{Gal}(\tilde{k}/k)$ and consider the loop cocycle

$$\eta_{n-1} : \tilde{\Gamma}_{n-1,m} \to \mathbf{G}_{n-1}(\tilde{k})$$

attached to $(1, \eta_{n-1}^{geo})$. We define

$$\mathfrak{H}_{n-1,R_{n-1}} = {}_{\eta_{n-1}}(\mathbf{G}_{n-1,R_{n-1}}).$$

The crucial point for the induction argument is the fact that the twisted F_{n-1}-group $_{\eta_{n-1}}\mathbf{G}_{n-1}$ is anisotropic. This is established just as in [19, theo. 7.9]. We look now at the specialization map

$$sp_n : \mathfrak{H}(F_n) \hookrightarrow \mathbf{G}(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) \to \mathbf{G}(\widetilde{F}_{n-1,m}).$$

Let P be the parahoric subgroup of $\mathfrak{H}^\circ(F_n)$ attached to the point ϕ. Since the building of $\mathfrak{H}_{F_n}$ consists of the single point we have $P = \mathfrak{H}^\circ(F_n)$. Recall that the notation P^* stands for the "pro-unipotent radical" of P as defined in Sect. 14 of the Appendix. □

Claim 10.3 We have $P^* = \ker(sp_n)$ and the image of sp_n is $\mathfrak{H}_{n-1}(F_{n-1})$.

Because $\mathbf{G}$ is a k-group it is clear that the kernel of the specialization map $\mathbf{G}(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) \to \mathbf{G}(\widetilde{F}_{n-1,m})$ is contained in $\mathbf{G}^\circ(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]])$. Since $(\mathfrak{H}/\mathfrak{H}^\circ)(F_n)$ injects into $(\mathfrak{H}/\mathfrak{H}^\circ)(\widetilde{F}_{n,m}) = (\mathbf{G}/\mathbf{G}^\circ)(\widetilde{F}_{n,m})$, the kernel of the specialization map sp_n is contained in $\mathfrak{H}^\circ(F_n)$. The parahoric subgroup of $\mathbf{G}^\circ(\widetilde{F}_{n,m})$ attached to the point ϕ is $Q = \mathbf{G}^\circ(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]])$ and we have

$$Q^* = \ker(Q \to \mathbf{G}^\circ(\widetilde{F}_{n-1,m}))$$

by the very definition of Q^*. Hence $\ker(sp_n) = P \cap Q^* = P^*$ by Corollary 15.6 applied to the point ϕ.

The group $\mathfrak{H}_{n-1}(F_{n-1})$ is a subgroup of $\mathfrak{H}_n(F_n)$ which maps identically to itself by sp_n, so we have to verify that the specialization h_{n-1} of an element $h \in \mathfrak{H}(F_n)$ belongs to $\mathfrak{H}_{n-1}(F_{n-1})$. Specializing (10.1) at $t_n = 0$, we get

$$\eta(\gamma)\ {}^{\gamma}h_{n-1} = h_{n-1} \quad \forall \gamma \in \widetilde{\Gamma}_{n,m}. \tag{10.2}$$

We now apply the relation (10.2) to the generator τ_n of the Galois group $\mathrm{Gal}(\widetilde{F}_{n,m}/\widetilde{F}_{n-1,m}((t_n)))$; it yields

$$\eta(\tau_n)\, h_{n-1} = h_{n-1}, \tag{10.3}$$

where $\eta(\tau_n) \in \mathbf{G}(\widetilde{k})$, so that $h_{n-1} \in \mathbf{G}_{n-1}(\tilde{F}_{n-1,m})$. Furthermore, the equality (10.2) restricted to $\tilde{\Gamma}_{n-1,m}$ shows that $h_{n-1} \in \mathfrak{H}_{n-1}(F_{n-1})$. This establishes the Claim.

We can now finish the induction process. The group $\mathfrak{H}_{n-1}(F_{n-1})$ is a subgroup of $\mathfrak{H}(F_n)$, so

$$\mathfrak{H}(F_n) = J_n \rtimes \mathfrak{H}_{n-1}(F_{n-1})$$

where $J_n := \ker(sp_n)$ is the "pro-unipotent radical" and hence it is pro-solvable in k–spaces. By using the induction hypothesis, we have

$$\mathfrak{H}_{n-1}(F_{n-1}) = (J_{n-1} \rtimes \cdots \rtimes J_1) \rtimes \mathbf{G}_{n-1}^{\eta_{n-1}^{geo}}(k).$$

Since $\mathbf{G}_{n-1}^{\eta_{n-1}^{geo}} = \mathbf{G}^{\eta^{geo}}$, we conclude that

$$\mathfrak{H}(F_n) = (J_n \rtimes \cdots \rtimes J_1) \rtimes \mathbf{G}^{\eta^{geo}}(k)$$

as desired.

We have $\mathbf{G}^{\eta^{geo}}(k) \subset \mathfrak{H}(R_n)$, so we get the second equality as well.

11 Acyclicity, I

Let $\mathfrak{H}$ be a loop reductive group scheme. We will denote by $H^1_{toral}(R_n, \mathfrak{H})$ (resp. $H^1_{toral}(R_n, \mathfrak{H})_{irr}$) the subset of $H^1(R_n, \mathfrak{H})$ consisting of isomorphism classes of $\mathfrak{H}$-torsors $\mathfrak{E}$ such that the twisted R_n-group ${}_{\mathfrak{E}}\mathfrak{H}$ admits a maximal torus (resp. admits a maximal torus and is irreducible).

Theorem 11.1 *Let $\mathfrak{H}$ be a loop reductive group scheme. Then the natural map*

$$H^1_{toral}(R_n, \mathfrak{H})_{irr} \to H^1(F_n, \mathfrak{H})$$

is injective.

Proof By twisting, it is enough to show that for an irreducible loop reductive group $\mathfrak{H}$ the canonical map $H^1_{toral}(R_n, \mathfrak{H}) \to H^1(F_n, \mathfrak{H})$ has trivial kernel. Indeed reductive R_n-group schemes admitting a maximal torus are precisely the loop reductive groups [19, theo. 6.1]. We now reason by successive cases.

Case 1 $\mathfrak{H}$ is adjoint and anisotropic. We may view $\mathfrak{H}$ as a twisted form of a Chevalley group scheme $\mathbf{H}_{R_n}$ by a loop cocycle $\eta : \pi_1(R_n) \to \mathrm{Aut}(\mathbf{H})(\overline{k})$. We have the following commutative diagram of torsion bijections

$$\begin{array}{ccc} H^1_{toral}(R_n, \mathbf{Aut}(\mathfrak{H})) & \longrightarrow & H^1(F_n, \mathbf{Aut}(\mathfrak{H})) \\ {\scriptstyle\tau_\eta}\downarrow\simeq & & {\scriptstyle\tau_\eta}\downarrow\simeq \\ H^1_{toral}(R_n, \mathbf{Aut}(\mathbf{H})) & \longrightarrow & H^1(F_n, \mathbf{Aut}(\mathbf{H})). \end{array}$$

The vertical maps are bijective by [20, III 2.5.4] and Remark 4.6, while the bottom map is bijective by [19, theo. 8.1]. We thus have a bijection

$$\psi : H^1_{toral}(R_n, \mathbf{Aut}(\mathfrak{H})) \xrightarrow{\sim} H^1(F_n, \mathbf{Aut}(\mathfrak{H})).$$

The exact sequence $1 \to \mathfrak{H} \to \mathbf{Aut}(\mathfrak{H}) \to \mathbf{Out}(\mathfrak{H}) \to 1$ gives rise to the commutative diagram of exact sequences of pointed sets

$$\begin{array}{ccccccc} \mathbf{Aut}(\mathfrak{H})(R_n) & \xrightarrow{\delta} & \mathbf{Out}(\mathfrak{H})(R_n) & \xrightarrow{\varphi} & H^1_{\acute{e}t}(R_n, \mathfrak{H}) & \longrightarrow & H^1_{\acute{e}t}(R_n, \mathbf{Aut}(\mathfrak{H})) \\ \downarrow & & || & & \downarrow & & \downarrow\psi \\ \mathbf{Aut}(\mathfrak{H})(F_n) & \xrightarrow{\gamma} & \mathbf{Out}(\mathfrak{H})(F_n) & \longrightarrow & H^1(F_n, \mathfrak{H}) & \longrightarrow & H^1(F_n, \mathbf{Aut}(\mathfrak{H})). \end{array}$$

Let $v \in H^1_{\acute{e}t}(R_n, \mathfrak{H})$ be a toral class mapping to $1 \in H^1(F_n, \mathfrak{H})$. Since ψ is bijective there exists $u \in \mathbf{Out}(\mathfrak{H})(R_n)$ such that $v = \varphi(u)$ and $u \in \mathrm{Im}\,\gamma$. Since $\mathbf{Out}(\mathfrak{H})(R_n)$ is a

finite group, the Density Theorem 10.2 shows that $\mathbf{Aut}(\mathfrak{H})(R_n)$ and $\mathbf{Aut}(\mathfrak{H})(F_n)$ have the same image in $\mathbf{Out}(\mathfrak{H})(F_n)$. So $u \in \operatorname{Im}\delta$, which implies that $\gamma = 1 \in H^1_{ét}(R_n, \mathfrak{H})$.
Case 2 $\mathfrak{H}$ *is irreducible.* Set $\mathfrak{Z} = Z(\mathfrak{H})$; it is an R_n-group of multiplicative type and we have an exact sequence of R_n-group schemes

$$1 \to \mathfrak{Z} \xrightarrow{i} \mathfrak{H} \to \mathfrak{H}_{ad} \to 1.$$

Here the adjoint group $\mathfrak{H}_{ad}$ is anisotropic since $\mathfrak{H}$ is irreducible. This exact sequence gives rise to the diagram

$$\begin{array}{ccccccc} \mathfrak{H}_{ad}(R_n) & \xrightarrow{\varphi_{R_n}} & H^1_{ét}(R_n,\mathfrak{Z}) & \xrightarrow{i_*} & H^1_{ét}(R_n,\mathfrak{H}) & \longrightarrow & H^1_{ét}(R_n,\mathfrak{H}_{ad}) \\ \downarrow & & \downarrow\simeq & & \downarrow & & \downarrow \\ \mathfrak{H}_{ad}(F_n) & \xrightarrow{\varphi_{F_n}} & H^1(F_n,\mathfrak{Z}) & \longrightarrow & H^1(F_n,\mathfrak{H}) & \longrightarrow & H^1(F_n,\mathfrak{H}_{ad}). \end{array}$$

Note that the second vertical map is bijective by [18, prop. 3.4.(3)] since $\mathfrak{Z}$ is of finite type ([41, XII, §3]).

Let $v \in H^1_{ét}(R_n, \mathfrak{H})$ be a toral class mapping to $1 \in H^1(F_n, \mathfrak{H})$. Taking into account the adjoint anisotropic case, a diagram chase provides an element $u \in H^1_{ét}(R_n, \mathfrak{Z})$ such that $v = i_*(u)$ and u belongs to the image of the characteristic map φ_{F_n}. Since $H^1_{ét}(R_n, \mathfrak{Z})$ is an abelian torsion group, the Density Theorem 10.2 shows that $\mathfrak{H}_{ad}(F_n)$ and $\mathfrak{H}_{ad}(R_n)$ have the same image in $H^1_{ét}(R_n, \mathfrak{Z})$. So u belongs to the image of φ_{R_n}, and this implies that $v = i_*(u) = 1 \in H^1_{ét}(R_n, \mathfrak{H})$ as desired. □

12 Conjugacy of certain parabolic subgroup schemes and maximal split tori

Theorem 12.1 *Let* $\mathfrak{H}$ *be a loop reductive group scheme over* R_n*. There exists a unique* $\mathfrak{H}(R_n)$*–conjugacy class of*

(a) *Couples* $(\mathfrak{L}, \mathfrak{P})$ *where* $\mathfrak{P}$ *is a minimal parabolic* R_n*–subgroup scheme of* $\mathfrak{H}$ *and* $\mathfrak{L}$ *is a Levi subgroup of* $\mathfrak{P}$ *such that* $\mathfrak{L}$ *is a loop reductive group scheme.*
(b) *Maximal split subtori* $\mathfrak{S}$ *of* $\mathfrak{H}$ *such that* $Z_{\mathfrak{H}}(\mathfrak{S})$ *is a loop reductive group scheme.*

Remark 12.2 The counter-example in Sect. 8.1 shows that the assumption that $\mathfrak{L}$ and $Z_{\mathfrak{H}}(\mathfrak{S})$ be loop reductive group schemes is not superflous.

Proof (i) *Reduction to the semisimple simply connected case.* Let $\mathfrak{H}^{sc}$ be the simply connected covering of the derived group scheme of $\mathfrak{H}$, and let $\mathfrak{E}$ be the radical torus of $\mathfrak{H}$. There is a canonical central isogeny [23, §1.2]

$$1 \to \boldsymbol{\mu} \to \mathfrak{H}^{sc} \times \mathfrak{E} \xrightarrow{f} \mathfrak{H} \to 1.$$

Let $(\mathfrak{L}, \mathfrak{P})$ be a pair where $\mathfrak{P}$ is a parabolic subgroup of $\mathfrak{H}$ containing a Levi subgroup $\mathfrak{L}$. Then

$$f^{-1}(\mathfrak{P}) = \mathfrak{P}^{sc} \times \mathfrak{E},\ f^{-1}(\mathfrak{L}) = \mathfrak{L}^{sc} \times \mathfrak{E}$$

where $\mathfrak{P}^{sc}$ is a minimal parabolic subgroup of the R_n-group $\mathfrak{H}^{sc}$ and $\mathfrak{L}^{sc}$ is a Levi subgroup of $\mathfrak{P}^{sc}$. Conversely, from a couple $(\mathfrak{M}, \mathfrak{Q})$ for $\mathfrak{H}^{sc}$, we can define a couple $((\mathfrak{M}\times\mathfrak{E})/\boldsymbol{\mu}, (\mathfrak{Q}\times\mathfrak{E})/\boldsymbol{\mu})$ for $\mathfrak{H}$. By [19, cor. 6.3], loop group schemes are exactly those carrying a maximal torus. Since the last property is insensitive to central extensions [41, XII.4.7], the correspondence described above exchanges loop objects $\mathfrak{L}$ with loop objects $\mathfrak{L}^{sc}$. Also it exchanges minimal parabolics of $\mathfrak{H}$ with minimal parabolics of $\mathfrak{H}^{sc}$. Thus without loss of generality we may assume that $\mathfrak{H}$ is simply connected.

(ii) *Existence (a).* Let $\mathbf{H}$ be the Chevalley k–form of $\mathfrak{H}$ and let $\eta : \pi_1(R_n) \to \mathbf{Aut}(\mathbf{H})(\overline{k})$ be a loop cocycle such that $\mathfrak{H} = {}_\eta(\mathbf{H}_{R_n})$. Let $(\mathbf{T}, \mathbf{B})$ be a Killing couple of $\mathbf{H}$ and $\Pi \subset \Delta(\mathbf{H}, \mathbf{T})$ be the base of the root system associated to $(\mathbf{T}, \mathbf{B})$. We denote by $\mathbf{H}_{ad}$ the adjoint group of $\mathbf{H}$ and by $(\mathbf{T}_{ad}, \mathbf{B}_{ad})$ the corresponding Killing couple. We have $\mathbf{Aut}(\mathbf{H}) = \mathrm{Aut}(\mathbf{H}_{ad})$. For each $I \subset \Pi$, we have the standard parabolic subgroup $\mathbf{P}_I$ of $\mathbf{H}$ and its Levi subgroup $\mathbf{L}_I$, as well as $\mathbf{P}_{I,ad}$ and $\mathbf{L}_{I,ad}$ for $\mathbf{H}_{ad}$.

Let $I \subset \Pi$ be the subset of circled vertices in the Witt–Tits diagram of $\mathfrak{H}_{F_n}$. The version of the "Witt-Tits decomposition" given in [19, cor. 8.4] applied to $\mathrm{Aut}(\mathbf{H}_{ad})$ shows that

$$[\eta] \in \mathrm{Im}(H^1_{loop}(R_n, \mathbf{Aut}(\mathbf{H}_{ad}, \mathbf{P}_{I,ad}, \mathbf{L}_{I,ad}))_{irr} \to H^1_{loop}(R_n, \mathbf{Aut}(\mathbf{H}_{ad}))).$$

Thus we may assume that η has values in

$$\mathbf{Aut}(\mathbf{H}, \mathbf{P}_I, \mathbf{L}_I)(\overline{k}) = \mathbf{Aut}(\mathbf{H}_{ad}, \mathbf{P}_{I,ad}, \mathbf{L}_{I,ad})(\overline{k}).$$

The twisted R_n-group schemes $\mathfrak{P} = {}_\eta(\mathbf{P}_I)$ and $\mathfrak{L} = {}_\eta(\mathbf{L}_I)$ are as desired for $\mathfrak{P}_{F_n}$ is a minimal F_n–parabolic subgroup of $\mathfrak{H}_{F_n}$ by the definition of the Witt-Tits index.

(iii) *Existence (b).* Consider the pair $(\mathfrak{L}, \mathfrak{P})$ constructed in (ii) and let $\mathfrak{S}$ be the maximal split subtorus of the radical $\mathfrak{T}$ of $\mathfrak{L}$. By Proposition 5.3 we have $Z_{\mathfrak{H}}(\mathfrak{S}) = \mathfrak{L}$ so that $Z_{\mathfrak{H}}(\mathfrak{S})$ is a loop reductive group. To show that $\mathfrak{S}$ is a maximal split torus of $\mathfrak{H}$ it suffices to establish that so is $\mathfrak{S}_{F_n}$.

Assume that $\mathfrak{S}_{F_n} \subset \mathfrak{S}'$ is a proper inclusion where $\mathfrak{S}'$ is a split torus in $\mathfrak{H}_{F_n}$. By construction $\mathfrak{P}_{F_n}$ is a minimal parabolic subgroup over F_n. Hence $\mathfrak{L}_{F_n} = C_{\mathfrak{H}_{F_n}}(\mathfrak{S}_{F_n}) = C_{\mathfrak{H}_{F_n}}(\mathfrak{S}')$. This implies that $\mathfrak{S}'$ is contained in the radical $\mathfrak{T}_{F_n}$ of $\mathfrak{L}_{F_n}$. But by Lemma 5.4, $\mathfrak{S}$ is still maximal split in $\mathfrak{T}$ over K_n and hence over F_n because $\mathfrak{T}$ is split over a Galois extension $\widetilde{R}_{n,m}/R_n$ for some integer m – a contradiction.

(iv) *Conjugacy (a).* Let $(\mathfrak{L}, \mathfrak{P})$ be the couple constructed in (ii). Consider the R_n–scheme $\mathfrak{Y} = \mathfrak{H}/\mathfrak{P}$ of parabolic subgroups of type $\mathbf{t}(\mathfrak{P})$. The exact sequence $1 \to \mathfrak{P} \to \mathfrak{H} \xrightarrow{f} \mathfrak{Y} \to 1$ induces exact sequences of pointed sets

$$\begin{array}{ccccccc} \mathfrak{H}(R_n) & \xrightarrow{\psi} & \mathfrak{Y}(R_n) & \xrightarrow{\varphi} & H^1_{\acute{e}t}(R_n, \mathfrak{P}) & \longrightarrow & H^1_{\acute{e}t}(R_n, \mathfrak{H}) \\ & & & & \uparrow \simeq & & \\ & & & & H^1_{\acute{e}t}(R_n, \mathfrak{L}) & & \end{array}$$

(note that the natural mapping $H^1_{\acute{e}t}(R_n, \mathfrak{L}) \to H^1_{\acute{e}t}(R_n, \mathfrak{P})$ is a bijection by [41, XXVI, 3.2]) and by base change

$$\begin{array}{ccccccc} \mathfrak{H}(F_n) & \xrightarrow{\psi_{F_n}} & \mathfrak{Y}(F_n) & \xrightarrow{\varphi_{F_n}} & H^1(F_n, \mathfrak{P}) & \longrightarrow & H^1(F_n, \mathfrak{H}) \\ & & & & \uparrow \simeq & & \\ & & & & H^1(F_n, \mathfrak{L}) & & \end{array}$$

Let $(\mathfrak{M}, \mathfrak{Q})$ be another couple satisfying the conditions of Theorem 12.1. By [19], $\mathfrak{Q}_{F_n} \subset \mathfrak{H}_{F_n}$ is still a minimal parabolic subgroup; in particular $\mathfrak{Q}$ has the same type $\mathbf{t}(\mathfrak{P})$ and hence it corresponds to a point $y \in \mathfrak{Y}(R_n)$. □

Claim 12.3 $\varphi(y) \in H^1_{toral}(R_n, \mathfrak{L}) \simeq H^1_{toral}(R_n, \mathfrak{P})$.

Indeed, $\varphi(y)$ is the class of the $\mathfrak{P}$-torsor $\mathfrak{E} := f^{-1}(y)$. We can assume without loss of generality that $\mathfrak{E}$ is obtained from an $\mathfrak{L}$-torsor $\mathfrak{F}$. Then $\mathfrak{Q}$ is isomorphic[8] to the twist ${}_{\mathfrak{F}}\mathfrak{P}$, and ${}_{\mathfrak{F}}\mathfrak{L}$ is a Levi subgroup of the R_n-group ${}_{\mathfrak{F}}\mathfrak{P}$. Since Levi subgroups of ${}_{\mathfrak{F}}\mathfrak{P}$ are conjugate under $R_u({}_{\mathfrak{F}}\mathfrak{P})(R_n)$ [41, XXVI, 1.8], it follows that ${}_{\mathfrak{F}}\mathfrak{L}$ is R_n–isomorphic to $\mathfrak{M}$. The group scheme ${}_{\mathfrak{F}}\mathfrak{L}$ carries then a maximal torus and the claim is proved.

On the other hand, since $\mathfrak{P}_{F_n}$ and $\mathfrak{Q}_{F_n}$ are minimal parabolic subgroups of $\mathfrak{H}_{F_n}$ they are conjugate under $\mathfrak{H}(F_n)$. Then y viewed as an element of $\mathfrak{Y}(F_n)$ is in the image of ψ_{F_n}, hence $\varphi_{F_n}(y) = 1$. It follows that $\varphi(y)$ belongs to the kernel of

$$H^1_{toral}(R_n, \mathfrak{L})_{irr} \to H^1(F_n, \mathfrak{L})$$

which is trivial by Theorem 11.1. Thus $y \in \operatorname{Im} \psi$, i.e. $\mathfrak{P}$ and $\mathfrak{Q}$ are $\mathfrak{H}(R_n)$–conjugate and so are the couples $(\mathfrak{L}, \mathfrak{P})$ and $(\mathfrak{M}, \mathfrak{Q})$.

(v) *Conjugacy (b).* We still denote by $(\mathfrak{L}, \mathfrak{P})$ the couple constructed in (ii). Let $\mathfrak{S}'$ be a maximal split subtorus of $\mathfrak{H}$ such that its centralizer $\mathfrak{L}' = Z_{\mathfrak{H}}(\mathfrak{S}')$ is a loop reductive group scheme. By Lemma 3.4, $Z_{\mathfrak{H}}(\mathfrak{S}')$ is a Levi subgroup of a parabolic subgroup of $\mathfrak{P}'$ of $\mathfrak{H}$. By Proposition 3.6 (c), $\mathfrak{P}'$ is a minimal parabolic subgroup of $\mathfrak{H}$. By (iv), the couple $(\mathfrak{L}', \mathfrak{P}')$ is conjugate under $\mathfrak{H}(R_n)$ to $(\mathfrak{L}, \mathfrak{P})$. We may thus assume that $\mathfrak{L} = \mathfrak{L}'$, i.e. $Z_{\mathfrak{H}}(\mathfrak{S}) = Z_{\mathfrak{H}}(\mathfrak{S}')$. It follows $\mathfrak{S}'$ is a central split subtorus of $\mathfrak{L}$, hence $\mathfrak{S}' \subset \mathfrak{S}$. But $\mathfrak{S}'$ is a maximal split subtorus of $\mathfrak{H}$, so we conclude that $\mathfrak{S} = \mathfrak{S}'$ as desired. □

13 Applications to infinite-dimensional Lie theory

Throughout this section we assume that k is algebraically closed of characteristic zero, $\mathbf{G}$ is a simple simply connected Chevalley group over k, and $\mathbf{g}$ its Lie algebra. We fix integers $n \geq 0$, $m > 0$ and an n–tuple $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of commuting elements of $\operatorname{Aut}_k(\mathbf{g})$ satisfying $\sigma_i^m = 1$. Let $R = R_n$ and $\widetilde{R} = R_{n,m}$. Recall that $\widetilde{R}/R$ is Galois and that we can identify $\operatorname{Gal}(\widetilde{R}/R)$ with $(\mathbb{Z}/m\mathbb{Z})^n$ via our choice of compatible primitive roots of unity.

Recall also from the Introduction the multiloop algebra based on $\mathbf{g}$ corresponding to $\boldsymbol{\sigma}$, is

[8] Surprisingly enough, this compatibility is not in Giraud's book. A proof can be found in [14, lemme 4.2.33].

$$L(\mathbf{g}, \boldsymbol{\sigma}) = \bigoplus_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} \mathbf{g}_{i_1\ldots i_n} \otimes t_1^{\frac{i_1}{m}} \ldots t_n^{\frac{i_n}{m}} \subset \mathbf{g} \otimes_k \widetilde{R}$$

It is a twisted form of the R–Lie algebra $\mathbf{g} \otimes_k R$ which is split by $\widetilde{R}$. The $\widetilde{R}/R$ form $L(\mathbf{g}, \boldsymbol{\sigma})$ is given by a natural loop cocycle

$$\eta = \eta(\boldsymbol{\sigma}) \in Z^1(\Gamma, \mathbf{Aut}(\mathbf{g})(k)) \subset Z^1(\Gamma, \mathbf{Aut}(\mathbf{g})(\widetilde{R})).$$

Since $\mathbf{Aut}(\mathbf{g}) \simeq \mathbf{Aut}(\mathbf{G})$ we can also consider by means of η the twisted R-group $\mathfrak{G} = {}_\eta\mathbf{G}_R$. As before we denote the Lie algebra of $\mathfrak{G}$ by $\mathfrak{g}$. Clearly, $\mathfrak{g} \simeq L(\mathbf{g}, \boldsymbol{\sigma})$.

13.1 Borel–Mostow MAD subalgebras

By a Theorem of Borel and Mostow [6] there exists a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$ that is stable under the action of $\boldsymbol{\sigma}$ (by which we mean that each of the σ_i stabilizes $\mathbf{h}$). By restricting $\boldsymbol{\sigma}$ to $\mathbf{h}$ we can consider the loop algebra based on $\mathbf{h}$ with respect to $\boldsymbol{\sigma}$,

$$L(\mathbf{h}, \boldsymbol{\sigma}) = \bigoplus_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} \mathbf{h}_{i_1\ldots i_n} \otimes t_1^{\frac{i_1}{m}} \ldots t_n^{\frac{i_n}{m}} \subset \mathbf{h} \otimes_k \widetilde{R}$$

Let $\mathbf{T}$ be the maximal torus of $\mathbf{G}$ corresponding to $\mathbf{h}$. Denote by $\mathbf{T}^{\boldsymbol{\sigma}}$ (resp. $\mathbf{h}^{\boldsymbol{\sigma}}$) the fixed point subgroup of $\mathbf{T}$ (resp. subalgebra of $\mathbf{h}$) under $\boldsymbol{\sigma}$, i.e the elements of $\mathbf{T}$ (resp. $\mathbf{h}$) that are fixed by each of the σ_i. Since the torus $\mathbf{T}$ is also $\boldsymbol{\sigma}$–stable, just as above, we can consider its twisted form $\mathfrak{T} = {}_\eta\mathbf{T}_R$ and the corresponding Lie algebra $\mathfrak{h} = {}_\eta\mathbf{h}_R$. The same formalism already mentioned yields that $\mathfrak{h} \simeq L(\mathbf{h}, \boldsymbol{\sigma})$.

Let $\mathfrak{T}_d$ be the maximal split torus of $\mathfrak{T}$. It is easy to see that

$$\mathfrak{T}_d \simeq \mathbf{T}_R^{\boldsymbol{\sigma}} = {}_\eta(\mathbf{T}_R^{\boldsymbol{\sigma}}) \subset \mathfrak{G} = {}_\eta\mathbf{G}_R.$$

According to Remark 6.4 its Lie algebra $\mathfrak{t}_d$ contains a unique maximal subalgebra $\mathfrak{m}$ which is an AD subalgebra of $\mathfrak{g}$. The description of this algebra is quite simple:

$$\mathfrak{m} = \mathbf{h}^{\boldsymbol{\sigma}} \otimes_k 1 = \mathbf{h}_{0,\ldots,0} \otimes_k 1 \subset L(\mathbf{g}, \boldsymbol{\sigma}) \simeq \mathfrak{g}.$$

By Theorem 7.1 $\mathfrak{m}$ is a MAD subalgebra if and only if $\mathfrak{T}_d$ is a maximal split torus of $\mathfrak{G}$, in which case $\mathfrak{m} = \mathfrak{m}(\mathfrak{T}_d)$. We will call MAD subalgebras of a multiloop algebra which are of this form *Borel Mostow* MAD subalgebras of $\mathfrak{g}$.

Clearly, $Z_{\mathfrak{g}}(\mathfrak{m})$ is precisely the multiloop algebra $L(Z_{\mathbf{g}}(\mathbf{h}^{\boldsymbol{\sigma}}), \boldsymbol{\sigma})$. Note that by Proposition 3.7, $Z_{\mathbf{g}}(\mathbf{h}^{\boldsymbol{\sigma}})$ is the Lie algebra of the reductive k-group $\mathbf{H} := Z_{\mathbf{G}}(\mathbf{h}^{\boldsymbol{\sigma}}) = Z_{\mathbf{G}}(\mathbf{T}^{\boldsymbol{\sigma}})$ and hence by twisting we conclude that $Z_{\mathfrak{g}}(\mathfrak{m})$ is the Lie algebra of $Z_{\mathfrak{G}}(\mathfrak{T}_d) = Z_{\mathfrak{G}}(\mathbf{T}_R^{\boldsymbol{\sigma}}) \simeq {}_\eta\mathbf{H}_R$.

Proposition 13.1 (1) *$Z_{\mathfrak{G}}(\mathfrak{T}_d)$ is a loop reductive group.*

(2) $\mathfrak{m}$ *is a MAD subalgebra if and only if the dimension of* $\mathbf{h}_{0,\ldots,0}$ *is maximal among the Cartan subalgebras of* $\mathbf{g}$ *normalised by* $\boldsymbol{\sigma}$. *In particular, Borel–Mostow MAD subalgebras exist.*

Proof (1) We have explained above that $Z_{\mathfrak{G}}(\mathfrak{T}_d) \simeq {}_\eta\mathbf{H}_R$. This last group is loop reductive by definition since η is a loop cocycle.

(2) It follows from Theorem 12.1 that all maximal split tori in $\mathfrak{G}$ whose centralizers are loop reductive groups and corresponding MAD subalgebras are conjugate, hence have the same dimension, say r, equal to the rank of $\mathfrak{G}$ over F_n. Since $\mathfrak{m}$ is an AD subalgebra whose centralizer is a multiloop algebra Theorem 12.1 applied to $Z_{\mathfrak{G}}(\mathfrak{T}_d)$ shows that $\dim_k \mathfrak{m} = \dim_k(\mathbf{h}_{0,\ldots,0}) \leq r$ and hence $\mathfrak{m}$ is a MAD subalgebra if and only if $\dim_k(\mathbf{h}_{0,\ldots,0}) = r$. It is then enough to show that there exists a Borel–Mostow AD subalgebra of rank r, that is we need to find a Cartan subalgebra $\mathbf{h}'$ of $\mathbf{g}$ normalized by $\boldsymbol{\sigma}$ such that $\dim_k(\mathbf{h}'_{0,\ldots,0}) = r$.

If $r = 0$ there is nothing to prove. Assume that $r > 0$. Denote by I the type of minimal parabolic subgroups of $\mathfrak{G}$ over F_n. Fix a Cartan subalgebra $\mathbf{h}_0 \subset \mathbf{g}$, the corresponding maximal torus $\mathbf{T}_0 \subset \mathbf{G}$ and a basis of the root system $\Sigma(\mathbf{T}_0, \mathbf{G})$. In the course of the proof of Theorem 12.1 we showed that up to conjugacy by an element of $\mathbf{G}(k)$, we can assume that $\boldsymbol{\sigma}$ normalizes the standard parabolic group $\mathbf{P}_I$ and also the standard Levi subgroup $\mathbf{L}_I$.

Let $\mathbf{S} \subset \mathbf{T}_0$ be the torus consisting of the fixed point subgroup of the radical of $\mathbf{L}_I$ under $\boldsymbol{\sigma}$. Then $\mathbf{S}_R \hookrightarrow {}_\eta(\mathbf{L}_R) \subset {}_\eta\mathbf{G}_R$ is the maximal split torus in the radical of ${}_\eta(\mathbf{L}_I)_R$. Since the twist ${}_\eta(\mathbf{P}_I)_R \otimes_R F_n$ is a minimal parabolic subgroup of $\mathfrak{G}$ over F_n and ${}_\eta(\mathbf{L})_I \otimes_R F_n$ is its Levi subgroup it follows that $\mathbf{S}_R \otimes_R F_n$ is a maximal split torus of $\mathfrak{G}$ over F_n; in particular $\dim_k(\mathbf{S}) = r$.

Let $\mathfrak{s} \subset \mathbf{h}_0$ be the Lie algebra of $\mathbf{S}$. We have $\dim_k(\mathfrak{s}) = \dim_k(\mathbf{S}) = r$ and by our construction $\boldsymbol{\sigma}$ acts trivially on $\mathfrak{s}$. The reductive subalgebra $Z_{\mathbf{g}}(\mathfrak{s})$ is stable under $\boldsymbol{\sigma}$, so the application of Borel–Mostow's theorem provides a Cartan subalgebra $\mathbf{h}'$ of $Z_{\mathbf{g}}(\mathfrak{s})$ stable under $\boldsymbol{\sigma}$. Its fixed subalgebra has dimension $\leq r$ and contains $\mathfrak{s}$, hence it coincides with $\mathfrak{s}$. □

According to our Conjugacy Theorem all Borel–Mostow MAD subalgebras of a multiloop algebra are conjugate under $\mathfrak{G}(R_n)$. There is a very important class of multiloop algebras, the so-called Lie tori, where Borel–Mostow MAD subalgebras play a crucial role. We now turn our attention to them.[9]

Theorem 13.2 *Let* $\mathcal{L}$ *be a centreless Lie torus which is finitely generated over its centroid. The* (*relative*) *type* Δ *is an invariant of* $\mathcal{L}$.

Proof After sorting through the several relevant definitions, the Theorem follows from our conjugacy of Borel–Mostow MAD subalgebras in view of the realization of the Lie tori in question as multiloop algebras as established in [3]. □

The spirit of this result should be interpreted as the analogue that on $\mathbf{g}$ we cannot choose two different Cartan subalgebras that will lead to root systems of different

[9] Lie tori were introduced by Yoshii [47,48] and further studied by Neher in [31,32]. The terminology is consistent with that of tori in the theory of non-associative algebras, e.g. Jordan tori. But in the presence of algebraic groups, where tori are well defined objects, the terminology is a bit unfortunate.

type. More generally, it is the analogue of the fact that the relative type of a finite-dimensional simple Lie algebra (in characteristic 0) or of a simple algebraic group is an invariant of the algebra or group in question.

The relevance of centreless Lie tori is that they sit at the "bottom" of every Extended Affine Lie Algebra (see [2,31,32]). A good example is provided by the affine Kac–Moody Lie algebras. They are of the form (see [26])

$$\mathcal{E} = \mathcal{L} \oplus kc \oplus kd$$

where $\mathcal{L}$ is a loop algebra of the form $L(\mathbf{g}, \pi)$ for some (unique) $\mathbf{g}$ and some (unique up to conjugacy) diagram automorphism π of $\mathbf{g}$. The element c is central and d is a degree derivation for a natural grading of $\mathcal{L}$. If $\mathbf{h}$ is the standard Chevalley split Cartan subalgebra of $\mathbf{g}$, then $\mathcal{H} = \mathbf{h}^{\pi} + kc + kd$ plays the role of the Cartan subalgebra for $\mathcal{E}$.

Remark 13.3 The invariance of the relative type was established in [1] by using strictly methods from EALA theory. Allison also showed that under the assumption that conjugacy (as established in this paper) holds, any isotopy between Lie tori necessarily preserves the external root data information. This is a very important result for the theory of EALAs for, together with conjugacy, it yields a very precise description of the group of automorphisms of Lie tori.

14 Acyclicity, II

Theorem 14.1 *Let $\mathfrak{H}$ be a loop reductive group scheme over R_n. Then the natural map*

$$H^1_{toral}(R_n, \mathfrak{H}) \to H^1(F_n, \mathfrak{H}).$$

is bijective.

Remark 14.2 The theorem generalizes (in characteristic 0) our main result in [13]. Indeed, in that paper we showed that if $n = 1$ and $\mathbf{G}$ is a reductive group over an arbitrary field k of good characteristic then $H^1_{\acute{e}t}(R_1, \mathbf{G}) \to H^1(F_1, \mathbf{G})$ is bijective and that every $\mathbf{G}$-torsor is toral. The Theorem also generalizes the Acyclicity result of [19], which is used in the present proof and covers the case when $\mathfrak{H}$ is "constant".

The proof of the theorem is based on the following statement which generalizes the Density Theorem 10.2 to the case of arbitrary loop reductive group schemes, not necessary anisotropic.

Theorem 14.3 *Let $\mathbf{H}$ be a linear algebraic k-group whose connect component of the identity is reductive. Let $\eta : \pi_1(R_n) \to \mathbf{H}(\overline{k})$ be a loop cocycle and consider the loop reductive R_n-groups $\mathfrak{H} = {}_{\eta}\mathbf{H}_{R_n}$ and $\mathfrak{H}^{\circ} = {}_{\eta}\mathbf{H}^{\circ}{}_{R_n}$. Let $(\mathfrak{P}, \mathfrak{L})$ be a couple given by Theorem 12.1 for $\mathfrak{H}^{\circ}$. Then there exists a normal subgroup J of $\mathfrak{L}(F_n)$ which is a quotient of a group admitting a composition serie whose quotients are pro-solvable groups in k–vector spaces such that*

$$\mathfrak{H}(F_n) = \langle\, \mathfrak{H}(R_n),\ J,\ \mathfrak{H}(F_n)^{+} \rangle$$

where $\mathfrak{H}(F_n)^+$ stands for the normal subgroup of $\mathfrak{H}(F_n)$ generated by one parameter additive F_n–subgroups.

Remark 14.4 If $\mathfrak{H}$ is semisimple simply connected, isotropic and F_n-simple we know that $\mathfrak{H}(F_n)/\mathfrak{H}(F_n)^+ \cong \mathfrak{H}(F_n)/R$ [16, 7.2], where R is an R-equivalence, so that the group $\mathfrak{H}(F_n)/\mathfrak{H}(F_n)^+$ has finite exponent (*ibid*, 7.6). In this case, the decomposition reads $\mathfrak{H}(F_n) = \langle\, \mathfrak{H}(R_n),\ \mathfrak{H}(F_n)^+\rangle$.

Proof Case 1 $\mathfrak{H}$ is a torus $\mathfrak{T}$. We leave it to the reader to reason by induction on n to establish the case of a split torus $\mathbf{T} = \mathbf{G}_m^n$ (the case $n = 1$ follows from the identity $F_1^\times = R_1^\times \cdot \ker(k[[t_1]]^\times \to k^\times)$. Since all finite connected étale coverings of R_n are also Laurent polynomial rings over field extensions of k [19, lemma 2.8] and the statement is stable under products, the theorem also holds for induced tori.

Let $\mathfrak{T}$ be an arbitrary torus. Since $\mathfrak{T}$ is isotrivial, it is a quotient of an induced torus $\mathfrak{E}$. We have then an exact sequence

$$1 \to \mathfrak{S} \xrightarrow{i} \mathfrak{E} \xrightarrow{f} \mathfrak{T} \to 1$$

of multiplicative R_n-group schemes. It gives rise to a commutative diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & \mathfrak{S}(R_n) & \xrightarrow{i_{R_n}} & \mathfrak{E}(R_n) & \xrightarrow{f_{R_n}} & \mathfrak{T}(R_n) & \xrightarrow{\varphi_{R_n}} & H^1_{\acute{e}t}(R_n, \mathfrak{S}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \simeq\downarrow & & \\
1 & \longrightarrow & \mathfrak{S}(F_n) & \xrightarrow{i_{F_n}} & \mathfrak{E}(F_n) & \xrightarrow{f_{F_n}} & \mathfrak{T}(F_n) & \xrightarrow{\varphi_{F_n}} & H^1(F_n, \mathfrak{S}) & \longrightarrow & 1
\end{array}$$

with exact rows. Note that the right vertical map is an isomorphism by [18, prop. 3.4] and that surjectivity on the right horizontal maps is due to the fact $H^1_{\acute{e}t}(R_n, \mathfrak{E}) = H^1(F_n, \mathfrak{E}) = 1$. By diagram chasing we see that

$$\mathfrak{T}(R_n)/f_{F_n}(\mathfrak{E}(R_n)) \xrightarrow{\sim} \mathfrak{T}(F_n)/f_{R_n}(\mathfrak{E}(F_n)).$$

Therefore the case of the induced torus $\mathfrak{E}$ provides a suitable group J such that $\mathfrak{T}(F_n) = \mathfrak{T}(R_n) \cdot f_{F_n}(J)$.

Case 2 $\mathfrak{H} = \mathfrak{L}$ is irreducible. Let $\mathfrak{C}$ be the radical torus of $\mathfrak{L}$. We have an exact sequence [41, XXI, 6.2.4]

$$1 \longrightarrow \boldsymbol{\mu} \xrightarrow{\ i\ } \mathcal{D}\mathfrak{L} \times_{R_n} \mathfrak{C} \xrightarrow{\ f\ } \mathfrak{L} \longrightarrow 1.$$

Here f is a natural multiplication map and $\boldsymbol{\mu}$ is its kernel. It gives rise to a commutative diagram of exact sequences of pointed sets

$$\begin{array}{ccccccc}
(\mathcal{D}\mathfrak{L} \times \mathfrak{C})(R_n) & \xrightarrow{f_{R_n}} & \mathfrak{L}(R_n) & \xrightarrow{\varphi_{R_n}} & H^1_{\acute{e}t}(R_n, \boldsymbol{\mu}) & \xrightarrow{i_{*,R_n}} & H^1_{loop}(R_n, \mathcal{D}\mathfrak{L} \times \mathfrak{C}) \\
\downarrow & & \downarrow & & \simeq\downarrow & & \downarrow \\
(\mathcal{D}\mathfrak{L} \times \mathfrak{C})(F_n) & \xrightarrow{f_{F_n}} & \mathfrak{L}(F_n) & \xrightarrow{\varphi_{F_n}} & H^1(F_n, \boldsymbol{\mu}) & \xrightarrow{i_{*,F_n}} & H^1(F_n, \mathcal{D}\mathfrak{L} \times \mathfrak{C}).
\end{array}$$

Note that the image of the map $H^1_{ét}(R_n, \boldsymbol{\mu}) \to H^1_{ét}(R_n, \mathcal{D}\mathfrak{L})$ is contained in $H^1_{toral}(R_n, \mathcal{D}\mathfrak{L})$. So taking into consideration Theorem 11.1 (applied to the irreducible loop reductive group scheme $\mathcal{D}\mathfrak{L}$ and chasing the above diagram we see that

$$\mathfrak{L}(R_n)/f_{R_n}((\mathcal{D}\mathfrak{L})(R_n) \times \mathfrak{C}(R_n)) \xrightarrow{\sim} \mathfrak{L}(F_n)/f_{F_n}((\mathcal{D}\mathfrak{L})(F_n) \times \mathfrak{C}(F_n)).$$

The case of $\mathcal{D}\mathfrak{L}$ done in Proposition 10.2 together with the case of the torus $\mathfrak{C}$ provide a suitable normal group J such that $\mathfrak{L}(F_n) = \mathfrak{L}(R_n) \cdot J$.

Case 3 $\mathfrak{H} = \mathfrak{H}^\circ$. Since $\mathfrak{H}$ is loop reductive by assumption it suffices to observe that $\mathfrak{H}(F_n)$ is generated by $\mathfrak{L}(F_n)$ and $\mathfrak{H}^+(F_n)$ [7, 6.11].

Case 4 For the general case it remains to show that for an arbitrary element $g \in \mathfrak{H}(F_n)$ the coset $g\mathfrak{H}^\circ(F_n)$ contains at least one R_n–point of $\mathfrak{H}$.

Let $\mathfrak{S}$ be the maximal split torus of the radical of $\mathfrak{L}$. The torus $g\mathfrak{S}_{F_n}g^{-1} \subset \mathfrak{H}^\circ_{F_n}$ is maximal split, hence $g\mathfrak{S}_{F_n}g^{-1} = g_1\mathfrak{S}_{F_n}g_1^{-1}$ for some $g_1 \in \mathfrak{H}^\circ(F_n)$. Thus replacing g by $g_1^{-1}g$ if necessary, we may assume that $g\mathfrak{S}_{F_n}g^{-1} = \mathfrak{S}_{F_n}$. Then we also have $g(\mathfrak{L}_{F_n})g^{-1} = \mathfrak{L}_{F_n}$, so that $g \in N_{\mathfrak{H}}(\mathfrak{L})(F_n)$.

The torus $\mathfrak{S}$ is clearly normal in $N_{\mathfrak{H}}(\mathfrak{L})$. Hence we have an exact sequence

$$1 \longrightarrow \mathfrak{S} \longrightarrow N_{\mathfrak{H}}(\mathfrak{L}) \longrightarrow \mathfrak{H}' := N_{\mathfrak{H}}(\mathfrak{L})/\mathfrak{S} \longrightarrow 1.$$

Note that since $H^1_{ét}(R_n, \mathfrak{S}) = 1$, the natural maps $N_{\mathfrak{H}}(\mathfrak{L})(R_n) \to \mathfrak{H}'(R_n)$ and $N_{\mathfrak{H}}(\mathfrak{L})(F_n) \to \mathfrak{H}'(F_n)$ are surjective. Furthermore, $\mathfrak{H}'$ satisfies all conditions of Theorem 10.2, so that the required fact follows immediately from that theorem applied to $\mathfrak{H}'$ and from the surjectivity of the above maps. □

We can proceed to the proof of Theorem 14.1.

Proof Injectivity: By twisting, it is enough to show that the natural map $H^1_{toral}(R_n, \mathfrak{H}) \to H^1(F_n, \mathfrak{H})$ has trivial kernel.

We first assume that $\mathfrak{H}$ is adjoint. We may view $\mathfrak{H}$ as the twisted form of a Chevalley group scheme $\mathbf{H}_{R_n}$ by a loop cocycle $\eta : \pi_1(R_n) \to \mathrm{Aut}(\mathbf{H}(\overline{k}))$. The same reasoning given in Case 1 of the proof of Theorem 11.1 shows that we have a natural bijection

$$H^1_{toral}(R_n, \mathbf{Aut}(\mathfrak{H})) \xrightarrow{\sim} H^1(F_n, \mathbf{Aut}(\mathfrak{H})). \tag{14.1}$$

The exact sequence

$$1 \to \mathfrak{H} \to \mathbf{Aut}(\mathfrak{H}) \to \mathbf{Out}(\mathfrak{H}) \to 1$$

gives rise to a commutative diagram of exact sequence of pointed sets

$$\begin{array}{ccccccc}
\mathbf{Aut}(\mathfrak{H})(R_n) & \xrightarrow{\gamma} & \mathbf{Out}(\mathfrak{H})(R_n) & \xrightarrow{\varphi} & H^1_{ét}(R_n, \mathfrak{H}) & \longrightarrow & H^1_{ét}(R_n, \mathbf{Aut}(\mathfrak{H})) \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\mathbf{Aut}(\mathfrak{H})(F_n) & \xrightarrow{\psi} & \mathbf{Out}(\mathfrak{H})(F_n) & \longrightarrow & H^1(F_n, \mathfrak{H}) & \longrightarrow & H^1(F_n, \mathbf{Aut}(\mathfrak{H})).
\end{array}$$

Let $v \in H^1_{ét}(R_n, \mathfrak{H})$ be a toral class mapping to $1 \in H^1(F_n, \mathfrak{H})$. In view of bijection (14.1) there exists $u \in \mathrm{Out}(\mathfrak{H})(R_n)$ such that $v = \varphi(u)$ and u belongs to the image of ψ. Since $\mathrm{Out}(\mathfrak{H})(R_n)$ is a finite group, the Density Theorem 14.3 shows that $\mathrm{Aut}(\mathfrak{H})(R_n)$ and $\mathrm{Aut}(\mathfrak{H})(F_n)$ have same image in $\mathrm{Out}(\mathfrak{H})(F_n)$. So u belongs to the image of γ, hence $v = 1 \in H^1_{ét}(R_n, \mathfrak{H})$.

Let now $\mathfrak{H}$ be an arbitrary reductive group. Set $\mathfrak{C} = Z(\mathfrak{H})$. This is an R_n-group of multiplicative type and we have an exact (central) sequence of R_n-group schemes

$$1 \to \mathfrak{C} \xrightarrow{i} \mathfrak{H} \to \mathfrak{H}_{ad} \to 1.$$

This exact sequence gives rise to the diagram of exact sequences of pointed sets

$$\begin{array}{ccccccccc}
\mathfrak{H}_{ad}(R_n) & \xrightarrow{\varphi_{R_n}} & H^1_{ét}(R_n, \mathfrak{C}) & \xrightarrow{i_*} & H^1_{ét}(R_n, \mathfrak{H}) & \longrightarrow & H^1_{ét}(R_n, \mathfrak{H}_{ad}) & \xrightarrow{\Delta} & H^2_{ét}(R_n, \mathfrak{C}) \\
\downarrow & & \downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \simeq \\
\mathfrak{H}_{ad}(F_n) & \xrightarrow{\varphi_{F_n}} & H^1(F_n, \mathfrak{C}) & \longrightarrow & H^1(F_n, \mathfrak{H}) & \longrightarrow & H^1(F_n, \mathfrak{H}_{ad}) & \xrightarrow{\Delta_{F_n}} & H^2_{ét}(F_n, \mathfrak{C}).
\end{array}$$

The isomorphisms $H^i_{ét}(R_n, \mathfrak{C}) \cong H^i(F_n, \mathfrak{C})$ comes from [18, prop. 3.4.(3)] for $i = 1, 2$.

Let $v \in H^1_{ét}(R_n, \mathfrak{H})$ be a toral class mapping to $1 \in H^1(F_n, \mathfrak{H})$. Taking into account the adjoint case, a diagram chase provides $u \in H^1_{ét}(R_n, \mathfrak{C})$ such that $v = i_*(u)$ and u_{F_n} belongs to the image of the characteristic map φ_{F_n}. Since $H^1_{ét}(R_n, \mathfrak{C})$ is an abelian torsion group, the Density Theorem 14.3 shows that $\mathfrak{H}_{ad}(F_n)$ and $\mathfrak{H}_{ad}(R_n)$ have the same images in $H^1_{ét}(R_n, \mathfrak{C})$. So u belongs to the image of φ_{R_n}. Hence $v = i_*(u) = 1 \in H^1_{ét}(R_n, \mathfrak{H})$.

Surjectivity Follows by a simple chasing in the diagrams above. □

Question. Assume that $\mathfrak{H}$ is loop semisimple simply connected, isotropic and F_n-simple. Let $\mathfrak{H}(R_n)^+ \subset \mathfrak{H}(R_n)$ be the (normal) subgroup generated by the $R_u(\mathfrak{P})(R_n)$ where $\mathfrak{P}$ runs over the set of parabolic subgroups of $\mathfrak{H}$ considered in Theorem 12.1. Is the map

$$\mathfrak{H}(R_n)/\mathfrak{H}(R_n)^+ \to \mathfrak{H}(F_n)/\mathfrak{H}(F_n)^+$$

an isomorphism?

Note that the map is surjective by Remark 14.4.

Appendix: Greenberg functors, Bruhat–Tits theory and pro-unipotent radicals

We are given a complete discrete valuation field K of valuation ring $O = O_K$ and of perfect residue field $k = O/\pi O$. Here $\pi \in O$ is a uniformizer. In the inequal characteristic case denote by e_0 the absolute ramification index of O, i.e. $p = u\pi^{e_0}$ for a unit $u \in O$ where $p = \mathrm{char}(k)$; in the equal characteristic case, put $e_0 = 1$. We denote by O^{sh} the strict henselization of O, or in other words, its maximal unramified extension.

Greenberg functor

We recall here basic facts, see the references [21], [30, §III.4], [8], [4].

Assume first that we are in the unequal characteristic case, that is K is of characteristic 0 and k is of characteristic $p > 0$.

For each k-algebra Λ and $r \geq 0$, we denote by $W_r(\Lambda)$ the group of Witt vectors of length r and by $W(\Lambda) = \varprojlim W_r(\Lambda)$ the ring of Witt vectors (see [43, §II.6]). There exists a unique ring homomorphism $W(k) \to O$ commuting with the projection on $k = W_0(k)$ (*ibid*, II.5).

Let $\mathfrak{S}$ be an affine $W(k)$-scheme. Recall that for each $r \geq 0$, the functor $k\text{-}alg \to Sets$ given by

$$\Lambda \to \mathfrak{S}(W_r(\Lambda))$$

is representable by an affine k–scheme $\mathrm{Green}_r(\mathfrak{S})$. The projective limit

$$\mathrm{Green}\,(\mathfrak{S}) := \varprojlim_r \mathrm{Green}_r(\mathfrak{S})$$

is a scheme which satisfies $\mathrm{Green}(\mathfrak{S})(\Lambda) = \mathfrak{S}(W(\Lambda))$. If $\mathfrak{X}$ is an affine O–scheme, we deal also with the relative versions of the Greenberg functor

$$\underline{G}_r(\mathfrak{X}) := \mathrm{Green}_r\left(\prod_{O/W(k)} \mathfrak{X}\right), \quad \underline{G}(X) := \mathrm{Green}\left(\prod_{O/W(k)} \mathfrak{X}\right).$$

We have $\underline{G}_r(\mathfrak{X})(k) = \mathfrak{X}(O/p^r O)$ and $\underline{G}(\mathfrak{X})(k) = \mathfrak{X}(O)$. We also have $\underline{G}(\mathrm{Spec}(O)) = \mathrm{Spec}(k)$; if $\mathfrak{X}$ is a O-group scheme, then $\underline{G}(\mathfrak{X})$ and the $\underline{G}_r(\mathfrak{X})$ carry a natural k-group structure [4, 4.1].

Lemma 15.1 *Let L/K be a finite extension, O_L the valuation ring of L and l/k the corresponding residue extension. Let $\mathfrak{Y}/O_L$ be an affine scheme. Let $\underline{H}/l$ be the relative Greenberg functor of $\mathfrak{Y}$ with respect to $W(l)$. Then we have natural isomorphisms of k-schemes (for all $r \geq 1$)*

$$\underline{G}_r\left(\prod_{O_L/O} \mathfrak{Y}\right) \simeq \prod_{l/k} \underline{H}_r(\mathfrak{Y}), \quad \underline{G}\left(\prod_{O_L/O} \mathfrak{Y}\right) \simeq \prod_{l/k} \underline{H}(\mathfrak{Y}).$$

In particular if $k = l$ then we have $\underline{G}_r\Big(\prod_{O_L/O} \mathfrak{Y}\Big) = \underline{H}_r(\mathfrak{Y})$ and $\underline{G}\Big(\prod_{O_L/O} \mathfrak{Y}\Big) \simeq \underline{H}(\mathfrak{Y})$.

Proof We have a commutative square

$$\begin{array}{ccc} O & \longrightarrow & O_L \\ \uparrow & & \uparrow \\ W(k) & \longrightarrow & W(l). \end{array}$$

So by the functorial properties of the Weil restriction, we have

$$\prod_{O/W(k)} \prod_{O_L/O} \mathfrak{Y} = \prod_{O_L/W(k)} \mathfrak{Y} = \prod_{W(l)/W(k)} \prod_{O_L/W(l)} \mathfrak{Y}. \tag{15.1}$$

Let Λ be a k–algebra. Using (15.1) and the definitions of the Greenberg functors, we have

$$\begin{aligned} \underline{G}_r\left(\prod_{O_L/O} \mathfrak{Y}\right)(\Lambda) &= \mathrm{Green}_r\left(\prod_{O_L/W(k)} \mathfrak{Y}\right)(\Lambda) \\ &= \left(\prod_{W(l)/W(k)} \prod_{O_L/W(l)} \mathfrak{Y}\right)(W_r(\Lambda)) \\ &= \left(\prod_{O_L/W(l)} \mathfrak{Y}\right)(W(l) \otimes_{W(k)} W_r(\Lambda)). \end{aligned}$$

Since $W_r(\Lambda)$ is a $W_r(k)$-module, we have

$$W(l) \otimes_{W(k)} W_r(\Lambda) = W_r(l) \otimes_{W_r(k)} W_r(\Lambda) = W_r(\Lambda \otimes_k l)$$

by [25, 1.5.7]. Hence

$$\underline{G}_r\left(\prod_{O_L/O} \mathfrak{Y}\right)(\Lambda) = \left(\prod_{O_L/W(l)} \mathfrak{Y}\right)(W_r(\Lambda \otimes_k l)) = R_{l/k}(\underline{H}_r)(\Lambda)$$

as desired. By passing to the limit, we get the second identity. □

Lemma 15.2 (1) *Let $\mathfrak{X}/O$ be an affine scheme of finite type such that $\mathfrak{X}_K = \emptyset$. Then* $\underline{G}(\mathfrak{X}) = \emptyset$.

(2) *Let $\mathfrak{N}/O$ be an affine group scheme of finite type such that $\mathfrak{N}_K = \mathrm{Spec}\,(K)$. Then* $\underline{G}(\mathfrak{N}) = \underline{G}(\mathrm{Spec}\,(O)) = \mathrm{Spec}(k)$.

Proof (1) We have $\mathfrak{X} = \mathrm{Spec}\,(A)$ where A is an $O/\pi^d O$–algebra of finite type for d large enough Put $r_0 = d\, e_0$. Then $p^{r_0} A = 0$. For a k–algebra Λ we have by definition

$$\underline{G}(\mathfrak{X})(\Lambda) = \mathrm{Hom}_O(A, W(\Lambda) \otimes_{W(k)} O).$$

But $W(\Lambda) \otimes_{W(k)} O$ is p–torsion free, so $\underline{G}(\mathfrak{X}) = \emptyset$.

(2) We have $N = \mathrm{Spec}(B)$ and we have the decomposition $B = O \oplus I$ where I is the kernel of the co-unit of the corresponding Hopf algebra. The O-module I is an ideal of B which is an $O/\pi^d O$-algebra of finite type. The same reasoning as above shows that

$$\begin{aligned}\underline{G}(\mathfrak{N})(\Lambda) &= \mathrm{Hom}_O(B, W(\Lambda) \otimes_{W(k)} O) \\ &= \mathrm{Hom}_O(O, W(\Lambda) \otimes_{W(k)} O) \\ &= \underline{G}(\mathrm{Spec}(O))(\Lambda).\end{aligned}$$

Thus $\underline{G}(\mathfrak{N}) = \underline{G}(\mathrm{Spec}(O))$ which is nothing but $\mathrm{Spec}(k)$ as explained above.

□

Secondly, assume that k and K have the same characteristic (0 or $p > 0$) and we still assume that k is perfect. Then k embeds in O (in a unique way, [22, 21.5.3]) and for an O-scheme $\mathfrak{X}$ the functors

$$\underline{G}(\mathfrak{X}) := \prod_{O|k} \mathfrak{X} \text{ and } \underline{G}_r(\mathfrak{X}) := \prod_{O/\pi^r O \ |k} (\mathfrak{X} \times_O O/\pi^r O)$$

play the desired role [8, §9.6] and allow us to write

$$\mathfrak{X}(O) = \varprojlim_r \mathfrak{X}(O/\pi^r O) \ = \ \varprojlim_r \underline{G}_r(\mathfrak{X})(k)$$

where the $\underline{G}_r(\mathfrak{X})$ are k-schemes (by Weil restriction considerations [8, §7.6]). The two lemmas are true as well.

Congruence filtration

Let $\mathbf{G}$ be a reductive K-group and denote by $\mathcal{B} = \mathcal{B}(\mathbf{G}, K)$ its (extended) Bruhat–Tits building. Let x be a point of $\mathcal{B}$ and denote by P_x the parahoric subgroup

$$P_x = \{g \in \mathbf{G}(K) \ | \ g(x) = x\}.$$

Denote by $\mathfrak{P}_x$ the canonical smooth group scheme over O defined by Bruhat–Tits [11, §5.1] with generic fiber $\mathbf{G}$ and such that $\mathfrak{P}_x(O) = P_x$ or, more precisely,

$$\mathfrak{P}_x(O^{sh}) = \{g \in \mathbf{G}(K^{sh}) \ | \ g(x) = x\}$$

where x is viewed as an element in $\mathcal{B}(\mathbf{G}, K^{sh})$ via the canonical mapping $\mathcal{B}(\mathbf{G}, K) \hookrightarrow \mathcal{B}(\mathbf{G}, K^{sh})$. Since $\mathfrak{P}_x$ is smooth we have

$$\mathfrak{P}_x(O) = \varprojlim_{n \geq 1} \mathfrak{P}_x(O/\pi^n O)$$

and the transition maps $\mathfrak{P}_x(O/\pi^{n+1} O) \to \mathfrak{P}_x(O/\pi^n O)$ are surjective with kernel $\mathrm{Lie}(\mathfrak{P}_x) \otimes_O k$ ([30, III.4.3])

The application of the relative Greenberg functor to the smooth affine group scheme $\mathfrak{P}_x$ defines a projective system of affine k-groups $\mathbf{P}_{x,n}$ $(n \geq 1)$ such that

$$\mathbf{P}_{x,n}(k) = \mathfrak{P}_x(O/\pi^{ne_0}O).$$

The $\mathbf{P}_{x,n}$ are smooth according to [4, Lemme 4.1.1]. The kernel $\mathbf{P}_{x,n+1/n}$ of the transition maps $\mathbf{P}_{x,n+1} \to \mathbf{P}_{x,n}$ are k-unipotent abelian groups which are successive extensions of the vector group of $\mathrm{Lie}(\mathfrak{P}_x) \otimes_O k$ (*ibid.* or [30, III.4.3]).

For each $n \geq 1$, we denote by $\mathbf{R}_{n,x} := R_u(\mathbf{P}_{x,n})$ the unipotent radical of $\mathbf{P}_{x,n}$; since k is perfect, it is defined over k and split [15, IV.2.3.9]. The quotient $\mathbf{M}_x$ of $\mathbf{P}_{x,n}$ by $\mathbf{R}_{x,n}$ is independent of n. It is nothing but the quotient of the special fiber of $\mathfrak{P}_x$ by its k-unipotent radical R_x. The k-group $\mathbf{M}_x^\circ$ is reductive according to [11, 4.6.12].

We consider the "maximal pro-unipotent normal subgroup"

$$P_x^* := \ker(\mathfrak{P}_x(O) \to \mathbf{M}_x(k))$$

which is of analytic nature. Denote by

$$\mathbf{P}_x/k := \varprojlim_{n\geq 1} \mathbf{P}_{x,n}$$

and by $\mathbf{P}_x^*/k = \ker(\mathbf{P}_x \to \mathbf{M}_x)$. By construction we have $P_x^* = \mathbf{P}_x^*(k)$.

Lemma 15.3 *For each $n \geq 1$, there is a short exact sequence of affine k-groups*

$$1 \to \ker(\mathbf{P}_x \to \mathbf{P}_{x,n}) \to \mathbf{P}_x^* \to \mathbf{R}_{x,n} \to 1.$$

Proof Apply the snake lemma to the commutative diagram of k-groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{P}_x^* & \longrightarrow & \mathbf{P}_x & \longrightarrow & \mathbf{M}_x & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbf{R}_{x,n} & \longrightarrow & \mathbf{P}_{x,n} & \longrightarrow & \mathbf{M}_x & \to & 1. \end{array}$$

□

Lemma 15.4 *The k-group $\mathbf{P}_x^*$ is the unique maximal split pro-unipotent closed normal subgroup of the pro-algebraic affine k-group $\mathbf{P}_x$.*

Proof Since

$$\ker(\mathbf{P}_x \to \mathbf{P}_{x,1}) = \varprojlim_n \ker(\mathbf{P}_{x,n} \to \mathbf{P}_{x,1})$$

is pro-unipotent, the above exact sequence shows that $\mathbf{P}_x^*$ is pro-unipotent. Let $\mathbf{U}_x$ be a pro-unipotent normal closed subgroup of $\mathbf{P}_x$. The image of $\mathbf{U}_x$ by the map $\mathbf{P}_x \to \mathbf{M}_x$ is a normal unipotent connected k-subgroup. Since $\mathbf{M}_x^\circ$ is reductive, its image is trivial. Therefore $\mathbf{U}_x \subset \mathbf{P}_x^*$ which completes the proof. □

Behaviour under a Galois extension

Just as does the whole theory, the construction of P_x^* has a very nice behaviour with respect to unramified extensions of K. The behaviour under a given tamely ramified finite Galois field extension L/K is subtle. Since such an extension is a tower of an unramified extension and a totally ramified one, we may concentrate on the case when L/K is totally (tamely) ramified. Then L/K is cyclic of degree e invertible in $k = \overline{K} = \overline{L}$. The Galois group $\Gamma = \mathrm{Gal}(L/K)$ acts on the building $\mathcal{B}(\mathbf{G}, L)$. The Bruhat–Tits–Rousseau theorem ([39, §5], see also [36]) states that the natural map

$$j : \mathcal{B}(\mathbf{G}, K) \to \mathcal{B}(\mathbf{G}, L)$$

induces a bijection $\mathcal{B}(\mathbf{G}, K) \xrightarrow{\sim} \mathcal{B}(\mathbf{G}, L)^\Gamma$. For $z \in \mathcal{B}(\mathbf{G}, L)$, we denote by Q_z the parahoric subgroup of $\mathbf{G}(L)$ and by $\mathfrak{Q}_z$ the canonical group scheme over O_L attached to the point z.

For $\sigma \in \Gamma$, we have $\sigma(Q_z) = Q_{\sigma(z)}$. Hence for the canonical group schemes over O_L attached to z and $\sigma(z)$, there is a natural cartesian square

$$\begin{array}{ccc} \mathfrak{Q}_{\sigma(z)} & \xrightarrow{f_{\sigma,z}} & \mathfrak{Q}_z \\ \downarrow & & \downarrow \\ \mathrm{Spec}(O_L) & \xrightarrow{(\sigma^{-1})^*} & \mathrm{Spec}(O_L). \end{array}$$

Put $y = j(x) \in \mathcal{B}(\mathbf{G}, L)^\Gamma$. We then have an O-action of Γ on the scheme $\mathfrak{Q}_y$. We note that

$$P_x = \mathbf{G}(K) \cap Q_y = \mathbf{G}(L)^\Gamma \cap Q_y = Q_y^\Gamma. \qquad (15.2)$$

As above we consider the groups $\mathbf{Q}_{y,n}$ and their projective limit $\mathbf{Q}_y$. Since k is the residue field of O_L, all $\mathbf{Q}_{y,n}$ and $\mathbf{Q}_y$ are k-groups. The action of Γ on $\mathfrak{Q}_y$ induces its action on $\mathbf{Q}_{y,n}$, hence on $\mathbf{M}_y$ where $\mathbf{M}_y$ stands for the reductive k-group attached to y, and on their projective limit $\mathbf{Q}_y$. By Lemma 15.4, $\mathbf{Q}_y^*$ is a characteristic k-subgroup of $\mathbf{Q}_y$, hence Γ also acts on the pro-algebraic k-group $\mathbf{Q}_y^*$. Our goal is to prove the following fact:

Proposition 15.5 *There is a natural closed embedding* $\mathbf{P}_x \to \mathbf{Q}_y$ *and we have*

$$\mathbf{P}_x^* = \mathbf{P}_x \cap \mathbf{Q}_y^*.$$

This gives rise to an isomorphism $\mathbf{M}_x \xrightarrow{\sim} \mathbf{M}_y^\Gamma$.

By taking k–points we get the following wished compatibility, namely.

Corollary 15.6 *We have*

$$P_x^* \xrightarrow{\sim} P_x \cap Q_y^*.$$

Consider the Weil restriction $\mathfrak{J}_x := \prod_{O_L/O}(\mathfrak{Q}_y)$ and recall it is a smooth O-scheme [49, §2.5]. Let $\mathfrak{N}$ be the kernel of the natural map $\mathfrak{P}_x \to \mathfrak{J}_x$, its generic fiber is trivial. As above, applying the Greenberg functors to the O-schemes $\mathfrak{J}_x$ and $\mathfrak{N}$ we get k-groups $\mathbf{J}_{x,n},\ \mathbf{J}_x$ and $\mathbf{N}_n,\ \mathbf{N}$.

Since the Greenberg functor is left exact, we get an exact sequence

$$1 \to \mathbf{N} \to \mathbf{P}_x \to \mathbf{J}_x.$$

Since $\mathfrak{N}_K = 1$, we have $\mathbf{N} = 1$ according to Lemma 15.2 (2). Hence we may view $\mathbf{P}_x$ as a closed subgroup of $\mathbf{J}_x$. But according to Lemma 15.1, $\mathbf{J}_{x,n}$ is nothing but $\mathbf{Q}_{y,n}$. This implies $\mathbf{J}_x$ is isomorphic in a natural way to $\mathbf{Q}_y$. Thus we have constructed a natural closed embedding $\mathbf{P}_x \to \mathbf{Q}_y$.

Define the k-subgroups $\mathbf{Q}_y^\Gamma := \varprojlim_n \mathbf{Q}_{y,n}^\Gamma$ and $(\mathbf{Q}_y^*)^\Gamma = \mathbf{Q}_y^\Gamma \cap \mathbf{Q}_y^*$ of $\mathbf{Q}_y$ and $\mathbf{Q}_y^*$ respectively.

Lemma 15.7 (1) *If k'/k is a finite extension of fields, the projective system $(\mathbf{Q}_{y,n}^\Gamma(k'))_{n\geq 1}$ has surjective transitions maps. Therefore the projective system of k-groups $(\mathbf{Q}_{y,n}^\Gamma)_{n\geq 1}$ has surjective transitions maps.*

(2) *If k'/k is a field finite extension, we have an exact sequence*

$$1 \to (\mathbf{Q}_y^*)^\Gamma(k') \to \mathbf{Q}_y^\Gamma(k') \to \mathbf{M}_y^\Gamma(k') \to 1;$$

hence the sequence of the pro-algebraic k-groups

$$1 \to (\mathbf{Q}_y^*)^\Gamma \to \mathbf{Q}_y^\Gamma \to \mathbf{M}_y^\Gamma \to 1$$

is also exact.

(3) *The algebraic k-group $\mathbf{M}_y^\Gamma$ is smooth and its connected component of the identity is reductive.*

Proof (1) Since Bruhat–Tits theory is insensitive to finite unramified extensions, we may assume without loss of generality that $k = k'$. Since $\mathbf{Q}_{y,n+1/n}$ is a k-split unipotent group, we have an exact sequence

$$1 \to \mathbf{Q}_{y,n+1/n}(k) \to \mathbf{Q}_{y,n+1}(k) \to \mathbf{Q}_{y,n}(k) \to 1.$$

It gives rise to the exact sequence of pointed sets

$$1 \to \mathbf{Q}_{y,n+1/n}(k)^\Gamma \to \mathbf{Q}_{y,n+1}(k)^\Gamma \to \mathbf{Q}_{y,n}(k)^\Gamma \to H^1(\Gamma, \mathbf{Q}_{y,n+1/n}(k)).$$

Since $\mathbf{Q}_{y,n+1/n}(k)$ admits a characteristic central composition serie in k-vector spaces and the order of Γ is invertible in k, the right hand side is trivial. A fortiori, the system $(\mathbf{Q}_{y,n}^\Gamma)$ of k-groups is surjective (because $\mathbf{Q}_{y,n}^\Gamma(k) = \mathbf{Q}_{y,n}(k)^\Gamma$).

(2) By part (1), the map $\mathbf{Q}_y^\Gamma(k) \to (\mathbf{Q}_{y,1})^\Gamma(k)$ is surjective. The same argument as in (1) shows that $(\mathbf{Q}_{y,1})^\Gamma(k) \to \mathbf{M}_y^\Gamma(k)$ is also surjective. By taking the composition of these maps we conclude the map $\mathbf{Q}_y^\Gamma(k) \to \mathbf{M}_y^\Gamma(k)$ is surjective whence the desired exactness of both sequences.

(3) The group Γ may be viewed as a finite abelian constant group scheme whose order is invertible in k. Hence Γ is also a (smooth) k-group of multiplicative type. Since $\mathbf{M}_y$ is affine and smooth, Grothendieck's theorem of smoothness of centralizers [41, XI, 5.3] shows that $\mathbf{M}_y^\Gamma$ is smooth. Its connected component of the identity is reductive by a result of Richardson [38, prop. 10.1.5].

□

We can now proceed to the proof of Proposition 15.5.

Proof We have to show that our closed embedding $\mathbf{P}_x \to \mathbf{Q}_y$ which we constructed above induces an isomorphism $\mathbf{P}_x^* \xrightarrow{\sim} \mathbf{P}_x \cap \mathbf{Q}_y^*$. Since $\mathbf{P}_x \cap \mathbf{Q}_y^*$ is a normal closed split pro-unipotent subgroup of $\mathbf{P}_x$ it is contained in $\mathbf{P}_x^*$. Hence it remains only to show that $\mathbf{P}_x^* \subset \mathbf{Q}_y^*$.

We now recall from (15.2) that $P_x = Q_y^\Gamma$ and $\mathbf{Q}_y^\Gamma(k) = \mathbf{Q}_y(k)^\Gamma = Q_y^\Gamma$. By Lemma 15.7, $\mathbf{Q}_y^\Gamma(k)$ projects onto $\mathbf{M}_y^\Gamma(k)$, so the composite map

$$P_x = \mathbf{P}_x(k) \to \mathbf{Q}_y^\Gamma(k) \to \mathbf{M}_y^\Gamma(k)$$

is surjective. Since this is true for all finite extensions of k, the homomorphism of k-algebraic groups $\mathbf{P}_x \to \mathbf{M}_y^\Gamma$ is surjective. But $(\mathbf{M}_y^\Gamma)^\circ$ is reductive, hence this map is trivial on the pro-unipotent radical $\mathbf{P}_x^*$. We get then a surjective map $\mathbf{M}_x \to \mathbf{M}_y^\Gamma$ and also a homomorphism $\mathbf{P}_x^* \to (\mathbf{Q}_y^*)^\Gamma \subset \mathbf{Q}_y^*$ as required. □

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