An asymmetric isochronous pendulum

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In this work, we propose a novel asymmetric isochronous pendulum, where half of the trajectory corresponds to that of the (non-isochronous) simple pendulum, while the remaining part corresponds to a specific trajectory. The pendulum's complete swing is isochronous—i.e., its period is not dependent on the oscillation amplitude. This new design is inspired by the symmetric isochronous Huygens pendulum, in which the trajectory is modified by cycloidal guides. In our case, only one guide is needed, for which analytical expressions are given for the first time. The objective of the paper is to add a new pedagogical tool in the undergraduate-level physics courses for understanding the isochronism concept, specifically the time delay for large amplitudes in the simple pendulum must be compensated by accelerating it along the new trajectory. We also introduce a novel numerical method to find the trajectory, based on the convergent superposition of straight trajectories (inclined-plane type) to approximate the curve. The procedure is not only accurate, but it is also appropriate for introducing the concepts of differential calculus and inclined-plane mechanics.
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I. INTRODUCTION

The simple pendulum (SP) turns out to be one of the most studied systems in courses of elementary mechanics and experimental physics. The equations that describe its dynamics are simple and easy to integrate under the approximation of small oscillations (i.e., oscillations of small amplitude around the equilibrium position[1]). In this regime, the period (i.e., the time required for a complete cycle) is independent of the amplitude, a well-known property called *isochronism*. However, as the oscillation amplitude is increased, the small-oscillation approach is no longer valid and isochronism lost. In fact, what happens is that the period increases with the oscillation amplitude. This phenomenon had been first investigated by Marin Mersenne (1588–1648). Later, in 1673, Christiaan Huygens modified the SP in order to obtain perfect isochronism.[2]

Initially, Huygens discovered that when a particle is forced to slide on a specific trajectory, called a *cycloid* or tautochrone curve, the obtained oscillation is isochronous (see Fig. 1). He incorporated two guides for the pendulum string (one on each side), to shorten its length forcing the particle to follow the tautochrone. These guides correspond to the evolute of the pendulum trajectory.[3]

The interest of Huygens for its isochronous oscillator was due to a technological question, namely, the construction of the pendulum clock. Since then, the problem has been a subject of interest to both theoretical and experimental physicists. From a theoretical point of view, there is work that focuses on finding the necessary and sufficient mathematical conditions that result in isochronous orbits in nonlinear mechanical models. Antón and Brun[4] and Terra *et al.*[5] analyzed the isochronous oscillations of a particle in an arbitrary potential as an inverse problem—i.e., given the period, how to find the potentials that produce it. In addition, Mohazzabi[6] studied the path that a particle must follow in the vertical plane to produce an oscillatory isochronous movement, concluding that there exist an infinite number of asymmetric potentials that satisfy such a condition.

In regard to experimental work, there exist several studies related to the period measurement of pendular systems. Aggarwal *et al.*[7] used a light-sensitive resistor and a laser to perform reliable period measurements and used them to analyze the SP undergoing large-amplitude oscillations. Gil and Di Gregorio[8,9] used a photo-gate to measure the period of several pendula, among which were the Huygens pendulum (HP) and the interrupted pendulum. Onorato *et al.*[10] used standard video tracking software[11] to analyze the two-dimensional path of a HP.

The main goal of this work is to design, construct, and measure an asymmetric isochronous pendulum (AIP) composed of two halves. The first half (from now on, the left side) corresponds to a SP, while the second half (the right side) is the exact path to make the whole trajectory isochronous. First, we present the analytical expression for the arc length as a function of the vertical coordinate y, and present the differential equation and solutions for the right-side curve and evolute. Then, we present a numerical method to alternatively solve the problem. It consists in a discretization of the height, and the consideration of a succession of inclined planes whose slopes are chosen in such a way that the time it takes the particle to travel along them compensates the increase in period associated with the left side (just as Mungan and Lipscombe did in Ref. 12). This method allows us to understand the nature of the problem, and simultaneously, to integrate the equations of motion in an intuitive way. Next, we perform several experimental trials by building the evolute of this trajectory curve on a wooden plate.

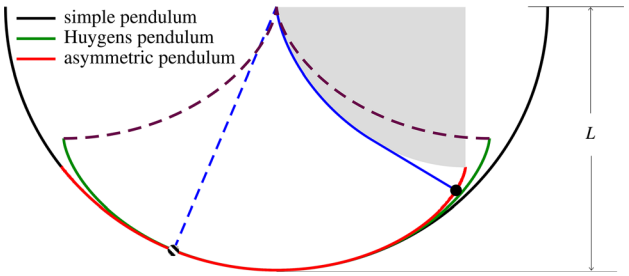

Fig. 1. Schematic plot describing the trajectory and the evolute curve corresponding to three different pendulums of length L. The lower line is the circular SP trajectory. Its complete evolute is the top central point from which the string hangs. Above it we get the HP, the two dashed lines, symmetrical located around the central axis represents its evolute. Above the HP on the right hand side and coinciding with the SP on the left side it is the AIP trajectory, whose evolute is composed by the same central point for the SP at the left side and by the lower boundary of the gray region to the right. Other lines which goes from the SP evolute point to the circular dots (bob) represent the string at different parts of the trajectory.

The total period and the half-periods corresponding to each side of the pendulum, as well as its trajectory, were determined by tracking the mass position with a standard video camera. A complete analysis of the pendulum dynamics is presented, including the $\dot{s} - s$ phase space and damping.

All time-dependent measurements were also contrasted with numerical integration of the equations of motion. In this work, we used the Runge-Kutta (RK) method,[13] a powerful technique to integrate numerically a set of ordinary differential equations.

We believe that this work covers several important topics to enrich the formation of physics and engineering students. First, we have replaced the usual isochronous Huygens pendulum by an asymmetric system, which compensates the loss of isochronism found in the simple pendulum. The second topic is related to the treatment of the mathematical analysis that defines the trajectory and evolute. The third topic corresponds to the inclined-planes algorithm, ruled by the isochronism condition. The last deals with the construction of the pendulum and the experimental determination of its dynamics using particle tracking from a video recording.

II. STATEMENT OF THE PROBLEM

Consider the problem of a SP, in the case where the circular movement starts with zero velocity at an initial position defined by the angle φ. If energy loss is neglected, the oscillation amplitude remains unchanged, and after a time $T(\varphi)$ the particle will return to its initial position and repeat the cycle indefinitely. The exact relation between the period of the simple pendulum and its oscillation amplitude is given in terms of an elliptic integral of the first kind[14]

$$T(\varphi) = T_0 \frac{2}{\pi} \int_0^{\pi/2} \frac{d\varphi'}{\sqrt{\sin^2(\varphi/2) - \sin^2(\varphi'/2)}}, \tag{1}$$

where φ is angular amplitude, $T_0 = 2\pi\sqrt{L/g}$, L is the string length, and g is the gravitational acceleration. If φ remains bounded between $(-\pi, \pi)$, the integral given in Eq. (1) can be written as

$$T(\varphi) = T_0(1 + f(\varphi)), \tag{2}$$

where[15]

$$\begin{aligned} f(\varphi) &= \sum_{n=1}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2} \right)^2 \sin^{2n}\left(\frac{\varphi}{2}\right) \\ &= \frac{\sin^2(\varphi/2)}{4} + \frac{9\sin^4(\varphi/2)}{64} + \frac{25\sin^6(\varphi/2)}{256} + \cdots \\ &= \frac{\varphi^2}{16} + \frac{11}{3072}\varphi^4 + \frac{173}{737280}\varphi^6 + \cdots . \end{aligned} \tag{3}$$

The series given in Eq. (3) converges rapidly—e.g., with six terms the relative error of $T(\pi/3)$ is smaller than 0.002, while for small angles, $\sin^2(\varphi/2) \ll 1$ and the SP becomes isochronous with period T_0. A simple formula approximating the series in Eq. (3) was given in Ref. 16, and was used to estimate the large-amplitude corrections to the period.

As mentioned in Sec. I, there exist an infinite number of isochronous pendula whose period are T_0 for any value of the initial amplitude. The HP (shown in Fig. 1) is the only case with a symmetric trajectory.[1] This trajectory is an inverted cycloid, whose parametric equation is

$$x(L, \varphi) = \frac{L}{4}(\sin(\varphi) + \varphi), \tag{4}$$

$$y(L, \varphi) = \frac{L}{4}(\cos(\varphi) - 1), \tag{5}$$

where L is the length of the pendulum and $-\pi \le \varphi \le \pi$. This implies that the oscillations of a bob that slides on this curve have the same period, regardless of the starting point of the oscillation. In addition, we can affirm that each side of the movement contributes to the period with half of its value, and that this trajectory is unique.

Aside from the HP, the rest of the isochronous pendula are all asymmetric. A particularly obvious family of them corresponds to the set of Huygens-type pendula constructed asymmetrically with cycloids of different longitudes L_R and L_L, smoothly matched at the minimal energy point. Since each half corresponds to a tautochrone with time of descent equal to $\pi\sqrt{L_R/g}$ and $\pi\sqrt{L_L/g}$, respectively, the composed system would be isochronous with period $\pi(\sqrt{L_R/g} + \sqrt{L_L/g})$. We can force the proposed system to have period $2\pi\sqrt{L/g}$ for any value of L_L between 0 and L by choosing: $\sqrt{L_R} = 2\sqrt{L} - \sqrt{L_L}$. Each half of the system would have a guide defined by the evolute curve of the corresponding cycloid (also a cycloid) characterized by the lengths L_L and L_R.

Many other isochronous pendula exist, composed of non-isochronous halves, where the opposite trajectories must be balanced for all amplitudes. In fact, suppose a pendulum characterized by two different trajectories $s_L(y)$ and $s_R(y)$ corresponding to the left and right sides, respectively, where s is the arc length and y is the vertical coordinate. The isochronism condition reads[6]

$$s_R(y) = s_L(y) + 2c\sqrt{y}, \tag{6}$$

where c is an arbitrary integration constant that can be determined by defining the oscillation period. This means that for almost any physical $s_L(y)$ trajectory we can find an isochronic potential by defining $s_R(y)$ from Eq. (6). Let us consider, for example, the case where $s_L(y)$ corresponds to the trajectory of a well-known movement: the circular path of a simple pendulum. Since the simple pendulum is

non-isochronous, the left part of the trajectory would contribute to the period with half of the time given in Eq. (2)

$$T_L(\varphi) = \frac{T_0}{2}(1 + f(\varphi)), \tag{7}$$

and the complementary part of trajectory should be also non-isochronous, but with a travel time which decreases with the oscillation amplitude in such a way to have a complete period of T_0

$$T_R(\varphi) = T_0 - T_L(\varphi). \tag{8}$$

The question arises as to whether this kind of motion is even possible. The answer is yes, as will be demonstrated through basic physical considerations in Sec. III.

Now let us continue with the exact analytical solution $s_R(y)$ for the special case where $s_L(y)$ matches the SP curve. The advantage of this system over other possible systems is that it is analytically integrable, and it is the most-studied pendular system. In addition, this system is relatively easy to build, since there is no need to construct a guide for the string corresponding to the SP part (the circle's evolute is a single point).

The left-side arc length of the SP is given by $s_L(y) = -L\arccos(1 - (y/L))$, where the minus sign is needed because we associate $s = 0$ to the minimal potential energy point, having negative values to the left and positive values to the right. $s_R(y)$ (≥ 0) is obtained from Eq. (6), where the constant c must be defined as $c = \sqrt{2L}$ in order to get a system with period T_0

$$s_R(y) = \sqrt{8Ly} - L\arccos(1 - (y/L)). \tag{9}$$

It is worth noting that, according to the result given in Eq. (9), s_R is independent of g, which means that the system would be isochronous on any planet. Since the equation for arc length can be written as

$$\left(\frac{dx}{dy}\right)^2 + 1 = \left(\frac{ds_R}{dy}\right)^2, \tag{10}$$

the differential equation for the trajectory of the bob $y(x)$ on the right side of the pendulum becomes

$$\left(\frac{dx}{dy}\right)^2 = \left(\frac{L}{\sqrt{y(2L-y)}} - \sqrt{2L/y}\right)^2 - 1. \tag{11}$$

From Eq. (11), we can immediately see that there is a limiting amplitude, which occurs when $dx/dy = 0$. Its exact value corresponds to an irrational number, whose numerical value (to 10 digits) is $y_{max}/L = 0.39153462858$, or $\varphi \simeq 52.52^o$, where φ is defined with respect to the left side.

Exact integration of Eq. (11) is not immediate, but a good approximation can be obtained through a Taylor-series expansion of the square root of the right-hand side. Accurate solutions can also be found by direct numerical integration. Once the trajectory is found, its evolute is described by[17]

$$\begin{aligned} x_e &= x(y) + \frac{x'(y)^2 + 1}{x''(y)}, \\ y_e &= y - x'(y)\frac{x'(y)^2}{x''(y)}, \end{aligned} \tag{12}$$

while the evolute of the left side (for a circular trajectory) corresponds to a single point.

A description of the trajectory for this pendulum is represented with a red line in Fig. 1, where it is compared with the circular (black) trajectory of the SP. In Fig. 1, the shaded region represents the guide, defined by the evolute curve given in Eq. (12).

We can see from Fig. 1 that for the cycloid the curve and its gradient (and therefore the potential energy and the restorative force) are above the circular trajectory, in such a way that the oscillation period is lowered respect to the SP, except at $x = 0$, where both are equal. The right side of the asymmetric pendulum (red curve) is even above the cycloid. This is because the restorative force must be higher in order to compensate for the increased period on the left side. That is the reason why there is a limiting amplitude.

We will describe in Sec. III an alternative way to find the trajectory on the right-hand side of the asymmetric pendulum and its evolute curve—i.e., to find solutions to Eqs. (11) and (12). The proposed methodology is a physical approximation to the differential calculus, based on the construction of the curve as a superposition of inclined planes, whose arrangement is defined by the isochronism condition, following a procedure similar to that described in Ref. 12.

III. THE INCLINED-PLANES ALGORITHM

Suppose we do not know the analytical expressions given in Eqs. (9) and (12), so we want to approximate the right-side curve with a superposition of inclined planes. According to the trajectory-discretization shown in Fig. 2, we could impose isochronism at each amplitude by considering the dynamics in each segment defined by the coordinates (x_{i+1}, y_{i+1}) and (x_i, y_i), where $i = 0, 1, 2, \ldots, N$. The value of y_i is chosen to be $i \times \Delta y$ with Δy having some fixed (small) value, while the x_i have to be found, except for $x_0 = 0$.

Before we continue, we should note that the trajectory is not differentiable at the points (x_i, y_i) corresponding to the intersection of two inclined planes. We can convert the sharp point of the curve into a smooth region by joining the segments with an arc of a circle of radius R, as shown in Fig. 3. If we take the limit $R \to 0$, the time needed to traverse this part of the trajectory also goes to zero. The role of this arc is to change the direction of the bob velocity in an infinitesimally small time interval. The energetic quantities used in

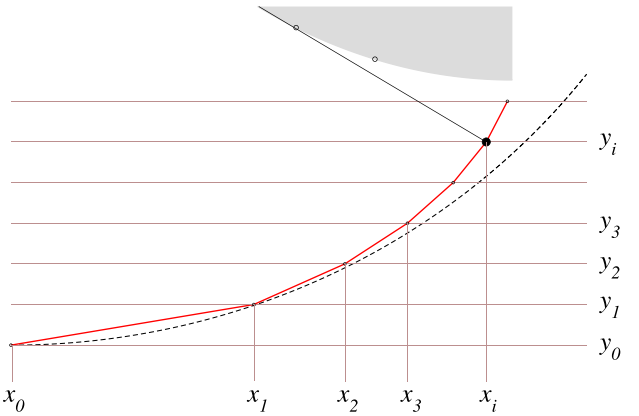

Fig. 2. Discretization procedure and interpretation of the unknown curve as a combination of inclined planes.

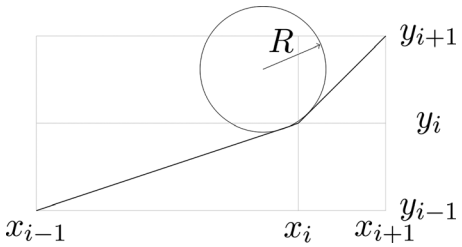

Fig. 3. The plane trajectories are connected through a circular arc of radius $R \to 0$.

this algorithm are not perturbed by the above mentioned consideration.

Now we can proceed with the calculation of the grid (x_i, y_i), where $i = 1, 2, \ldots, N$. If we let t_i denote the time it takes the particle to go from (x_i, y_i), characterized by the amplitude $\varphi_i = \arccos(1 - (y_i/L))$, to (x_0, y_0) starting from rest, the isochronism hypothesis reads

$$t_i = T_R(\varphi_i)/4. \tag{13}$$

Starting with the first inclined plane, we see that t_1 is the time it takes the particle to move along the first inclined plane, starting with zero velocity. Integration of the (Fig. 4) acceleration through the time t_1 gives

$$v(t_1) = \int_0^{t_1} g \sin\alpha_1 dt = t_1 g \sin\alpha_1, \tag{14}$$

where $\alpha_1 = \arctan(\Delta y/x_1 - x_0)$, while $v(t_1)$ must also satisfy energy conservation

$$v(t_1) = \sqrt{2g\Delta y}. \tag{15}$$

From Eqs. (14) and (15), it follows that

$$x_1 = \sqrt{\Delta y}\sqrt{\frac{t_1^2 g}{2} - \Delta y}. \tag{16}$$

We see from Eq. (16) that the time must always be greater than the free-fall time from an altitude Δy to have a well defined x_1. We also see from Eqs. (8) and (16) that the dependence on g cancels out, which is in accordance with the absence of g in Eq. (9).

The time $t_2 = T_R(\varphi_2)/4$ can be divided in two parts: $t_2 = t_2^{(1)} + t_2^{(2)}$, where $t_2^{(1)}$ corresponds to the time elapsed in the first (already known) plane, now starting with nonzero velocity, which is obtained from energy conservation, and $t_2^{(2)}$ is the time elapsed along the second (unknown) inclined plane. Integration of the velocity along each inclined plane and rearranging gives

$$t_2^{(1)} = \frac{\sqrt{2gy_2} - \sqrt{2g\Delta y}}{g\sin\alpha_1}, \tag{17}$$

and

$$t_2^{(2)} = \frac{\sqrt{2\Delta y}}{\sqrt{g}\sin\alpha_2}, \tag{18}$$

from which we can obtain x_2

$$x_2 = x_1 + \sqrt{\Delta y}\sqrt{\left(t_2^{(2)}\right)^2 \frac{g}{2} - \Delta y}, \tag{19}$$

where $t_2^{(2)} = \frac{T_R(\varphi_2)}{4} - t_2^{(1)}$.

More generally, we can divide the time of descent in two parts, $t_i = t_i^{(1)} + t_i^{(2)}$, where $t_i^{(1)}$ denotes the time elapsed along the known set of inclined planes and $t_i^{(2)}$ denotes the time along the last (unknown) inclined plane. Then

$$t_i^{(1)} = \sqrt{2\Delta y}\sum_{j=1}^{i-1}\frac{\sqrt{i-j+1} - \sqrt{i-j}}{\sqrt{g}\sin\alpha_j}, \tag{20}$$

$$t_i^{(2)} = \frac{T_R(\varphi_i)}{4} - t_i^{(1)}, \tag{21}$$

and

$$x_i = x_{i-1} + \sqrt{\Delta y}\sqrt{\left(t_i^{(2)}\right)^2 \frac{g}{2} - \Delta y}. \tag{22}$$

Equations (13)–(22) allow us to define an algorithm to find all the positions (x_i, y_i), which will stop when $t_i^{(2)} < \sqrt{2\Delta y/g}$. The physical meaning of $t_i^{(2)} = \sqrt{2\Delta y/g}$ is that $x_i = x_{i-1}$—i.e., the last plane is vertical. This is the limiting case of a restorative force, which cannot be increased above mg. At this point, the pendulum reaches a limiting amplitude y_{max}.

Note that this procedure allows us to find a discrete set of amplitudes for which the period is constant and equal to $2\pi\sqrt{L/g}$. If the initial amplitude does not exactly match one of our discrete values, the period will not be exactly $2\pi\sqrt{L/g}$, although it would correspond to a very close value. The limit of taking an infinite number of inclined planes $N \to \infty$ with $\Delta y = y_{max}/N \to 0$ converges to the well-defined curve given in Sec. II. For example, using $N = 10\,000$ we cannot visually distinguish the numerical solution from the exact curve, and obtain $y_{max}/L = 0.3916$.

The evolute curve must continuously interrupt the pendulum string displacement in order to force the bob to follow the desired trajectory. For an analytical parametric curve, the evolute is given in Eq. (12). For the approximation procedure described above, one can look at the intersection of two normal lines corresponding to two adjacent planes, as an approximation to the center of curvature. The first point corresponds to the center of curvature of the SP. The second point to the intersection of the $x = 0$ line with the normal to the first inclined plane, and so on. The collection of such points is a good approximation to the evolute curve.

The inclined-planes algorithm can also be applied to other problems. For example, we applied it to one member of the

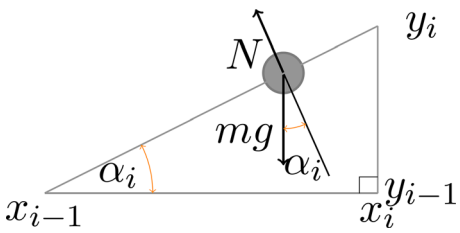

Fig. 4. ith inclined plane corresponding to the uniform discretization of the y coordinate.

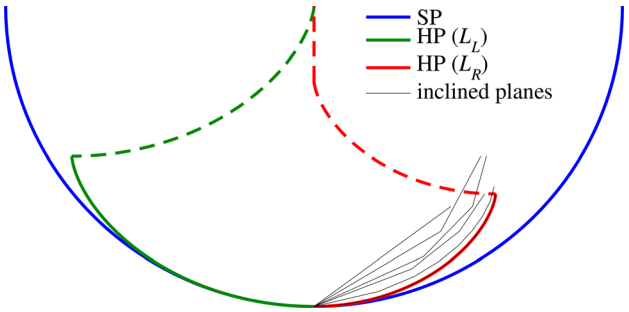

Fig. 5. Isochronous asymmetric Huygens pendulum, composed of two Huygens-type trajectories corresponding to wire longitudes $L_L = 16(7-4\sqrt{3})L$ (left) and $L_R = 12(7-4\sqrt{3})L$ (right). The dashed lines are the evolutes curves and the straight lines the successive inclined planes which gives synchronism for different discretization orders.

Huygens asymmetric pendula family: the one in which $L_R = 3L_L/4$. If we fix the period to be $2\pi\sqrt{L/g}$, the characteristic lengths are $L_L = 16(7-4\sqrt{3})L$ and $L_R = 12(7-4\sqrt{3})L$. In this case, we suppose that only the left side is known, while the right has to be approximated with the algorithm. Figure 5 shows the two Huygens halves corresponding to wires of length L (left) and $3L/4$ (right), together with the set of planes which give isochronism for different discretization orders of the height.

It seems to be natural to extend the exposed methodology by using polynomials instead of planes to match each pair of points. Then one would need an additional condition to determine the extra coefficients, for which one could use the continuity of the derivatives at each matching point. For example, using a second degree polynomial $y = a_i x^2 + b_i x + c_i$, we would start by imposing zero derivative at the point (0, 0) (i.e., $b_1 = c_1 = 0$). Isochronism would then determine the a_1 coefficient and x_1. By matching the second polynomial and its derivative to the first at (x_1, y_1), it would be possible to determine two of the three parameters, while the third one would by fixed by isochronism, etc. We think the implementation of such a methodology would be a good exercise and repeatability test for experimental or computational physics courses.

IV. CONSTRUCTION OF THE PENDULUM

The theoretical and numerical solutions found in Secs. II and III were tested through a series of experiments. In order to perform these trials, the asymmetric pendulum shown in Fig. 1 was built. The structure of the right side of the device was made from a 12 mm thick wooden panel, cut according to the exact shape of our evolute curve for $L = 1$ m. The bob was a 30 mm diameter metallic ball of 77 g hanging from the top left side of the plate (i.e., the origin, and evolute, of the circular left-side trajectory) via a 1 m nylon string. It is worth noting that this chord should be as thin as possible to reduce frictional forces, and also have a high elastic constant to avoid spurious changes of its length in different parts of the oscillation cycle. The string guide in the wooden piece was covered with plastic tape to create a smooth contact surface. The whole guide structure was painted in white and mounted on a wall of the same color, while the bob was painted in red as shown in Fig. 6. The objective of the latter was to create high visual contrast between the pendulum bob and the rest of the frame, and thus, increase the precision of the position tracking algorithm. The experiments were carried out in a sealed room free of air currents.

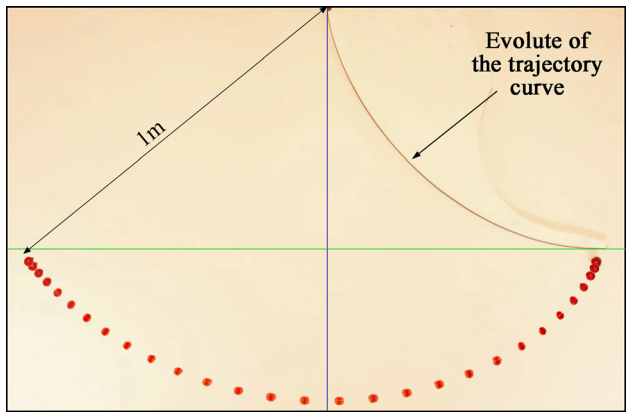

Fig. 6. Experimental setup. The asymmetric pendulum with the evolute guide made from the wood at the right side of the picture. The wall and pendulum bob were painted with contrasting colors to ease the tracking process.

V. MEASUREMENT AND MODELING OF THE PENDULUM DYNAMICS

The experiments consisted essentially of determining the bob position as a function of time. To perform this tracking process, we used a standard video camera, *Olimpus XZ-10*, to film the pendulum swing with a frame rate ranging from 30 fps (frames per second) to 240 fps (i.e., 33.3 ms between frames with a resolution of 1920×1682 pixels, to 4.2 ms between frames with a resolution of 640×480 pixels). The camera and the illumination lamps were placed about 3 m away from the wall and oriented normal to the plane of the pendulum. The films were analyzed with a license-free particle–tracking software *Tracker*,[11] which allowed us to compute the main characteristics of the pendulum dynamics on each side of the cycle (e.g., trajectory, velocity, acceleration, period, and amplitude). The tracking was performed by filtering the image to preserve just the red pixels in order to prevent a potential displacement of the observed position, produced by the shadow cast by the spherical mass. Two series of measurements were made: for the first, we recorded just a few cycles (around 8) of the swing using the highest temporal resolution available (i.e., a rate of 240 fps), releasing the bob from different amplitudes in order to observe the movement in detail; for the second, we took long-duration films (∼30 min) at 30 fps, releasing the bob from its maximum amplitude. The information acquired in this second trial was used to evaluate the amplitude decay rate and then estimate the magnitude of the frictional forces acting on the system.

It is important to stress that these kinds of measurements can be performed with a wide variety of devices (e.g., cellphones, tablets, webcams, etc.) and the videos can be processed with other license-free tracking software available on the web like ImageJ[18] or *OpenPTV*,[19] which makes this an ideal experiment to be included in any undergraduate or graduate science laboratory. In order to ease the implementation of this work in the classroom, the numerical code is available for public use and modification at https://osf.io/4r95p/, containing the inclined-planes algorithm and the RK simulation. We also included some sample videos of the

swinging pendulum taken with different frame rates, along with the results obtained from the tracking procedure.

The theoretical background of our measurements was given by the fourth-order RK method (one of the many algorithms of the RK family of methods) applied to the following system of equations:

$$\dot{s} = \frac{p}{m}, \tag{23}$$

$$\dot{p} = -g\frac{\partial y(s)}{\partial s} - (\alpha/m)p - (\beta/m)p^2, \tag{24}$$

where $y(s)$ is defined by the inverse of the analytic expressions s_L and s_R given in Sec. II, and α and β are two damping parameters which are introduced to simulate the energy loss of the system. The value of g used for this model is $g = 9.79994\ \mathrm{m/s^2}$, obtained from the web page in Ref. 20 for San Carlos de Bariloche, Argentina.

VI. RESULTS AND DISCUSSION

The first property analyzed in the experiments was the dependence of the period on the amplitude of the oscillation in order to check for isochronism. One of the long-duration measurements of position $s(t)$ as a function of time is shown in Fig. 7. There, the frictional forces between the mass, the wire, and the air, and also between the wire and guide, cause energy to be dissipated, which in turn causes the amplitude to decay over time. The experimental results shown indicate that the energy loss rate is small enough to have several periods of oscillations with almost the same amplitude. In Fig. 7, there is almost 2200 oscillation cycles over 1400 s, so the dynamics of the period cannot be appreciated from individual oscillations. However, this data allowed us to characterize the amplitude-damping of the pendulum.

The theoretical description of the system was obtained from Eqs. (23) and (24). In this case, the damping coefficients α and β were obtained by minimizing the difference between the experimental and theoretical envelopes. We found a minimum at $\alpha = 0.0012$ g/s and $\beta = 0.0380\ \mathrm{m^{-1}}$ for $m = 77$ g.

The pendulum trajectory s was numerically interpolated to precisely determine the instant when the particle passed through the origin $x = 0$, and to compute the moment where $s = 0$. On the other hand, interpolation was not needed to measure the angular amplitude at each extreme of the cycle, since the velocity of the bob reaches a null value at the extremes of the swing. Results for half of the total period of oscillation for each side of the pendulum are shown in Fig. 8 as a function of the amplitude. The lines correspond to the exact values given by T_L (black) and T_R (red), while the dots are the experimental results. As expected, the left side behaved as an increasing function of the amplitude, while the right side showed an oppositely decreasing behavior, in such a way that the whole system was isochronous with period 2.006 s (half of its value is shown in blue, in order to improve the visualization).

We can observe from Fig. 8 that the experimental results are larger than the theoretical predictions, especially for the right side of the movement and large amplitudes. We suppose that this is due to the damping forces acting on the system. Actually, we have observed that the numerical simulations introduce a retarded phase shift in the oscillation with respect to the free movement when using the damping coefficients that fit the experimental envelope curve (Fig. 7). For a more in-depth analysis, one must take into account that there exist two different kinds of damping forces, the one between the wire and the guide, which acts only on the right side of the movement, and damping between the bob and wire with the air, which acts during the whole cycle.

From the high-velocity measurements (performed with 240 fps) we obtained very smooth plots of the s coordinate as a function of time, as shown in Fig. 9. In this figure, we present just a few oscillation cycles due to the camera limitation of 20 s for the slow-motion recording mode. Excellent agreement between the experimental results and RK simulations was found, which confirms the reliability of the description given in Eqs. (23) and (24) integrated with the RK method.

To better understand the system dynamics, we used the high-velocity measurements to determine $\dot{s}$, and then plotted $\dot{s}$ as a function of s in Fig. 10. These phase space

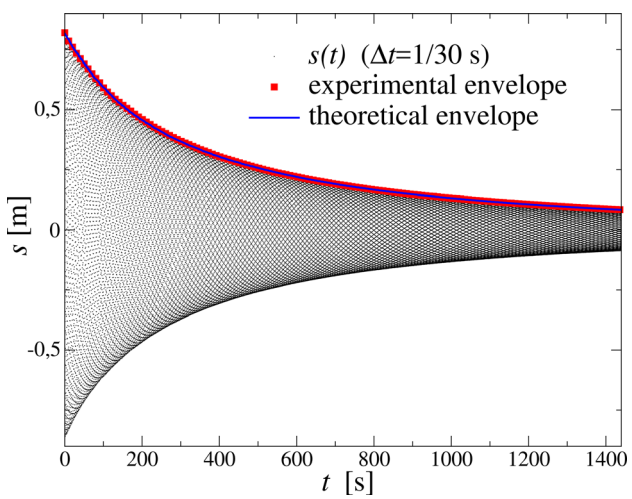

Fig. 7. Temporal evolution of the pendulum trajectory s after its mass is released from the right side maximum (given by extreme of the evolute). The small dots represent the experimental data obtained for a time period of 30 min. In squares we show the experimental envelope curve, while the straight line represents its theoretical counterpart given by equations (23) and (24) (see text).

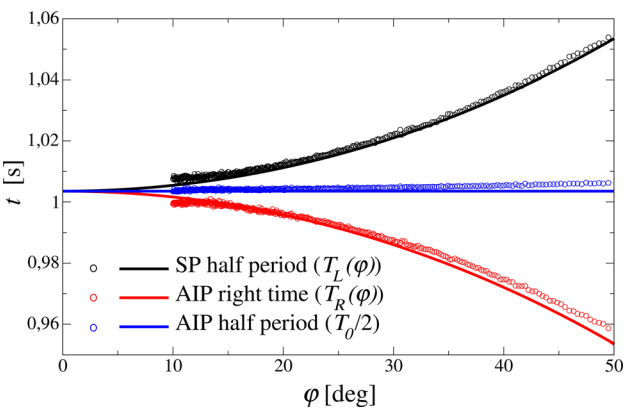

Fig. 8. Time elapsed by the particle in each side of the AIP and in a whole cycle, as a function of the amplitude. Upper circles: $T_L(\varphi)$ measurements. Lower circles: $T_R(\varphi)$ measurements. Middle circles: $(T_L(\varphi) + T_R(\varphi))/2$ measurements. The experimental data is compared in lines with expressions given by equations (7) and (8).

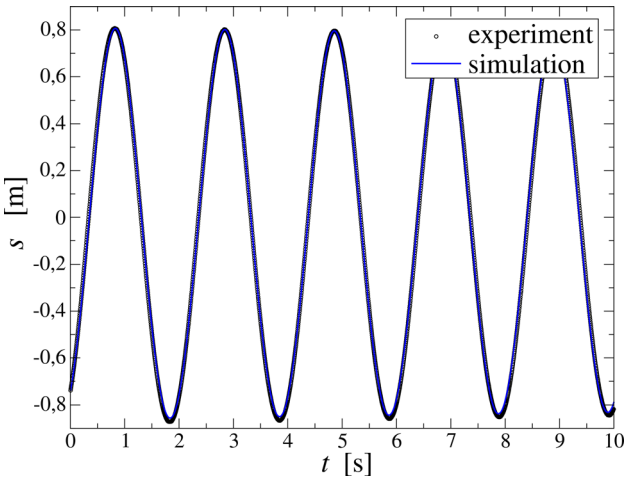

Fig. 9. Comparison between the trajectory obtained from high-speed video recordings of the pendulum swing and its simulated counterpart, for the case of an angular amplitude $\varphi = 46.1°$.

measurements were repeated for seven different amplitudes, and compared with simulations finding excellent agreement.

Although it is not easy to notice, there is an asymmetry with respect to the line $s = 0$ in the $\dot{s}$–s phase space. The latter can be seen by comparison with RK simulations of the simple pendulum (dark green) and Huygens pendulum (blue) for the maximum-energy curve. In Fig. 10, the SP result lies on top of the theoretical curve for the $s < 0$ region, while it reaches larger s values for $s > 0$. The HP curve has a smaller minimal value of s, while its maximum value is between the simple and the asymmetric pendulum. In the maximum velocity region, all systems behave in the same way, matching the SP curve. For lower energies, the curves become closer to each other and the differences are more difficult to discern, indicating how the pendula behave identically in the small-amplitude limit.

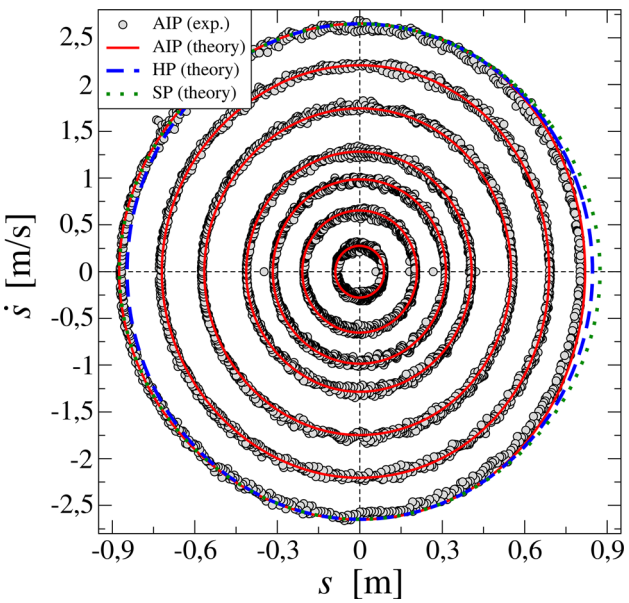

Fig. 10. Phase space trajectories of the AIP for different amplitudes corresponding to $\varphi = 47.3°$, $39.6°$, $31.7°$, $22.8°$, $17.4°$, $11.3°$, $7.9°$, and $3.7°$. The experimental data (dots) is compared with numerical simulations performed for the AIP (straight), the HP (dashed) and the SP (dotted). The right side of the figure shows the loss of symmetry in s for the AIP case.

VII. SUMMARY AND CONCLUSION

In this work we constructed, measured, and modelled an asymmetric isochronous pendulum. Although there are an infinite number of possibilities for defining such an oscillatory system, we chose the one which was half composed by the well-known SP, and the other half obtained from an isochronism hypothesis, compensating the change in period of the SP half outside the small-amplitude oscillation regime.

A solution for the evolute and potential curves was obtained using two different approaches: (i) an analytical approach, based on the isochronism hypothesis given in Eq. (6), and (ii) a numerical approach, based on simple physical considerations. The numerical strategy could be introduced first as a thought experiment, asking if there exists a single inclined plane that could compensate the loss of isochronism of the SP side for a given initial amplitude. We then solved the problem by employing an algorithm to construct a curve from a set of inclined planes, reaching the exact solution when the number of inclined planes tends to infinity. The proposed numerical method combines the physics of inclined planes with basic notions of differential calculus. The evolute of the curve was also obtained from a discrete approximation of differentiable functions. These methods are easy to implement even with spreadsheet software, making them ideal for beginning-level courses. Furthermore, the extension of these numerical methods to higher-order polynomial segments could be an appropriate project for advanced graduate students. In addition, other systems can be considered by replacing the SP half with some other curve, such as a parabola, or with an interrupted SP device.

We found that the proposed pendulum has a limiting angle ($\varphi_0 \simeq 52.52°$) from which the right side can no longer complement the time to maintain isochronism. This angle is independent of L, m, and g.

The obtained evolute guide was constructed and dynamic measurements were performed using a very simple experimental set up, based on a video camera and license-free tracking software (i.e., *Tracker*). Using this set up, whose complexity is adequate for an undergraduate experimental course, we were able to completely determine the pendulum dynamics and measure its isochronism. A series of high-speed video results were presented for several amplitudes, which allowed us to determine the $\dot{s}$–s phase space dynamics. The damping constants α and β were also characterized by comparison to Runge-Kutta simulations.

In summary, the asymmetric isochronous pendulum represents an excellent teaching tool for introductory laboratory or mechanics physics courses, as well as computational courses.

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Chladni Plate with Resonance Tube

This device, made by Marloye et Cie. of Paris in the middle years of the 19th century, is in the collection of historical physics apparatus at Dartmouth College. Areas of maxima and minima can be seen on the surface of a Chladni plate by strewing sand onto the surface of the plate and seeing where it collects. However, the maxima can also be verified acoustically by placing the vertical resonance tube over an area on which no sand has collected, and varying the length of the tube until the loudest sound can be heard. Marloye was the acoustical apparatus maker who placed tuning forks on half or quarter wave resonating boxes. (Picture and Notes by Thomas B. Greenslade, Jr., Kenyon College)