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Meromorphic and Harmonic Functions Inducing Continuous Maps from M_{H^∞} into the Riemann Sphere

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We study the class of maps from the open unit disk into the Riemann sphere or into $[-\infty, +\infty]$ that can be continuously extended to the maximal ideal space of H^∞. Several characterizations are given for these classes and the subclasses of meromorphic and harmonic functions in terms of cluster sets, spherical gradients, and Carleson measures. © 2001 Academic Press

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1. INTRODUCTION AND PRELIMINARIES

Let M_{H^∞} denote the maximal ideal space of the algebra H^∞ of bounded analytic functions on the open unit disk $\mathbb{D}$. The topology of M_{H^∞} is known to be rather complicated, but it also has several positive features. Via evaluations, the disk can be viewed as an open subset of M_{H^∞}, and thanks to the corona theorem of Carleson [7] we know that it is dense. There are two kind of points in M_{H^∞}. A point $x \in M_{H^\infty}$ is called nontrivial if its Gleason part consists of more than just itself. The part of such an x is a continuous one-to-one image of the disk (see [10]). The set $\mathcal{G}$ of nontrivial points is an open subset of M_{H^∞} that contains $\mathbb{D}$. Consequently the complement of $\mathcal{G}$, denoted here by Γ, is formed by the points $x \in M_{H^\infty}$ whose Gleason part is just $\{x\}$. They are called trivial points.

One obtains the Gelfand transform of $f \in H^\infty$ by looking at f in the bidual space of H^∞ and then restricting its domain to M_{H^∞}. Thus, the Gelfand transform provides a continuous extension of f to a map from M_{H^∞} into the complex plane $\mathbb{C}$. Using this together with exponentials and harmonic conjugates, one sees that bounded harmonic functions also can be extended continuously to M_{H^∞}.

The papers [2, 3, 5, 6, 17] study the possibility of extending by continuity different classes of maps from $\mathbb{D}$ into $\mathbb{C}$ or into the Riemann sphere

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$\hat{\mathbb{C}}$ to the whole space M_{H^∞} or just to $\mathcal{G}$. The purpose of this paper is to study this phenomenon for several classes of functions on $\mathbb{D}$. An early example of this research is due to Brown and Gauthier in [6], where they show that a meromorphic function f can be extended continuously to a map from $\mathcal{G}$ into $\hat{\mathbb{C}}$ if and only if f is uniformly continuous from the metric space $(\mathbb{D}, \rho)$ into the metric space $(\hat{\mathbb{C}}, \chi)$, where ρ and χ denote the pseudohyperbolic and the chordal metrics, respectively. From the general theory of meromorphic functions we know that the last condition is equivalent to the normality of f, which means that the family $\{f \circ \varphi : \varphi \in \mathrm{Aut}(\mathbb{D})\}$ is normal (see [15]). In our first result (Theorem 2.1) we will show that the theorem of Brown and Gauthier holds in a very general situation. Specifically, if f is a continuous function from the metric space $(\mathbb{D}, \rho)$ into a metric space (M, d) such that the closure of $f(\mathbb{D})$ in M is compact, then f can be extended continuously from $\mathcal{G}$ into M if and only if f is uniformly continuous.

If f is a meromorphic function on $\mathbb{D}$ and $x \in M_{H^\infty}$, the cluster set of f at x, denoted by $\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$, is the set of all $\lambda \in \hat{\mathbb{C}}$ that can be approached by $f(z)$ as z tends to x, with $z \in \mathbb{D}$. It is shown in [6] that a suitable Schwarz triangle function f is uniformly (ρ, χ) continuous but $\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x) = \hat{\mathbb{C}}$ for every $x \in \Gamma$. In particular, such f can be extended by continuity to $\mathcal{G}$, but it cannot be extended continuously to any set E, with $\mathbb{D} \subset E \subset M_{H^\infty}$, that contains a trivial point. The authors conjecture that the behavior of their example is indeed a general phenomenon. We will see that the conjecture is true, meaning that if f is meromorphic and $x \in M_{H^\infty}$, then $\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$ is a single point or the whole Riemann sphere. An immediate consequence is that if $f(\mathbb{D})$ is not dense in $\hat{\mathbb{C}}$ then f can be extended continuously to M_{H^∞}. Similar results will be shown to hold for harmonic functions.

The starting point of Section 3 is a couple of characterizations of $C(M_{H^\infty}, \mathbb{C})$, the space of continuous functions from M_{H^∞} into $\mathbb{C}$, obtained by Bishop in [5]. These characterizations involve gradients of C^∞ functions, uniform ρ-euclidean continuity and the notion of a Carleson measure. Our main goal here is to show that there are completely analogous characterizations of $C(M_{H^\infty}, \hat{\mathbb{C}})$ if we change the euclidean metric of $\mathbb{C}$ by the chordal metric of $\hat{\mathbb{C}}$, and the gradient operator by the *spherical gradient* operator.

The literature is full of examples where a discontinuous function on $\mathbb{D}$ satisfying some Carleson measure condition is regularized in such a way that the regularization also satisfies the same or similar conditions. Sometimes this is left to the reader and sometimes the regularization is effectively constructed (see, for instance [9, pp. 356–357]). In Section 3 we introduce and study an operator that achieves the same kind of regularization in a simpler way. This will allow us to avoid the language of distributions and deal with just good old-fashioned C^∞ functions. The operator is a technical aid that will be used for the rest of the paper.

Section 4 is the most extensive and difficult part of the paper. It is inspired by a remarkable theorem of Garnett [9, VIII, Theorem 6.1] stating that if u is the Poisson integral of a bounded (or BMO) function on $\partial\mathbb{D}$, then u can be uniformly approximated on $\mathbb{D}$ by C^∞ functions g such that $(1-|z|^2)\,|\nabla g(z)|$ is bounded and $|\nabla g(z)|\,dA(z)$ is a Carleson measure, where $dA(re^{it}) = (1/2\pi)\,r\,dr\,dt$. By Bishop's theorem, a bounded continuous function u satisfies the above conditions if and only if $u \in C(M_{H^\infty}, \mathbb{C})$, (i.e., u can be extended continuously to M_{H^∞}). Similarly, we show that if p is a complex-valued function such that $(1-|\xi|^2)\,|p(\xi)|\,dA(\xi)$ is a Carleson measure and $(1-|\xi|^2)^2\,|p(\xi)|$ is bounded, then the integral of $p(\xi)\,dA(\xi)$ against either of the kernels

$$\log\left|\frac{1-\bar{z}\xi}{z-\xi}\right|^2 \qquad \text{or} \qquad \frac{(1-|z|^2)(1-|\xi|^2)}{|1-\bar{z}\xi|^2} \qquad (z \in \mathbb{D})$$

belongs to $C(M_{H^\infty}, \mathbb{C})$. The fact that the variable ξ of the integral runs over the whole disk rather than its boundary produces several technical complications that did not appear (or were easier to deal with), in the case considered in [9, VIII, Theorem 6.1]. Consequently, the proof, although following the general lines of Garnett's argument, is considerably longer and more complicated. The reader interested in this kind of results should consult a paper of Dahlberg [8], where Garnett's result has being improved from the quantitative point of view and broadly generalized.

The typical situation where our theorem will be applied is when $p(\xi)$ is the Laplacian of a bounded C^2 function. Section 5 will grow from this type of application. The first predecessor of this section is a theorem of Sundberg (see [17]), where he proves that the Poisson integral u of a function in $\mathrm{BMO}(\partial\mathbb{D})$ ($u \in \mathrm{BMO}$ for short) can be continuously extended from M_{H^∞} into $\hat{\mathbb{C}}$. In particular, this holds for functions in BMOA, the analytic functions in BMO. It is also noticed by Sundberg that there are analytic functions in $C(M_{H^\infty}, \hat{\mathbb{C}})$ that do not belong to BMOA, because the first class is closed under squaring while the second is not. The proof uses truncations of the boundary values of u. Later, Axler and Shields found a very simple proof of Sundberg's theorem using the Fefferman–Stein decomposition of BMO (see [2, Theorem 2]). They also ask for a characterization of the harmonic or the analytic functions in $C(M_{H^\infty}, \hat{\mathbb{C}})$. It is fair to say that, although formally speaking our characterization of $C(M_{H^\infty}, \hat{\mathbb{C}})$ given in Theorem 3.4, together with harmonicity or analyticity, do characterize these classes, the conditions of Theorem 3.4 are usually not easy to check. So, it is reasonable to expect a better characterization when dealing with harmonic or analytic functions. Fortunately, a more precise answer can be given for harmonic or even meromorphic functions.

It is well known that an analytic function f is in BMOA if and only if $(1-|z|^2)\,|f'(z)|^2\,dA(z)$ is a Carleson measure, where f' denotes the derivative of f (see [9, VI]). In Theorem 5.5 we characterize the meromorphic functions in $C(M_{H^\infty}, \hat{\mathbb{C}})$ with two conditions resembling the above characterization of BMOA, where the derivative is replaced by the *spherical derivative*. Similarly, we give several characterizations of the real-valued harmonic functions that can be extended by continuity from M_{H^∞} into $[-\infty, \infty]$, the two-point compactification of $\mathbb{R}$. Finally, we give two sufficient conditions (along the same lines) for a harmonic function f to belong to $C(M_{H^\infty}, \hat{\mathbb{C}})$. The conditions are clearly weaker than $f \in BMO$, and we conjecture that they are also necessary.

We finish the paper with some examples illustrating negative behavior of the functions under study. In particular, we shall see that none of the classes of analytic functions in $C(\mathcal{G}, \hat{\mathbb{C}})$ or $C(M_{H^\infty}, \hat{\mathbb{C}})$ is closed under addition or multiplication.

1.1. *Preliminary Results*

We provide here most of the necessary background to read the paper. The maximal ideal space of H^∞ is

$$M_{H^\infty} = \{\varphi\colon H^\infty \to \mathbb{C}\colon \varphi \text{ is linear, multiplicative and } \varphi \neq 0\},$$

endowed with the weak $*$ topology. The topological space M_{H^∞} is a non-metrizable Hausdorff compactification of $\mathbb{D}$. Every function $f \in H^\infty$ extends continuously to M_{H^∞} via the Gelfand transform $\hat{f}(\varphi) = \varphi(f)$ $(\varphi \in M_{H^\infty})$. In the sequel we shall not write the hat for the Gelfand transform of f.

Every analytic automorphism of $\mathbb{D}$ has the form $\lambda\varphi_z$ for some $z \in \mathbb{D}$, where λ is a constant of modulus 1 and $\varphi_z(\omega) = (z-\omega)/(1-\bar{z}\omega)$. The pseudohyperbolic distance between $z, \omega \in \mathbb{D}$ is $\rho(z, \omega) = |\varphi_z(\omega)| = |\varphi_\omega(z)|$. The closed ρ-ball of center $z \in \mathbb{D}$ and radius $r<1$ will be denoted by $K(z, r) \overset{\text{def}}{=} \{\omega \in \mathbb{D} : \rho(z, \omega) \leqslant r\}$. We summarize several elementary formulas that will be used in the paper. For $z, \xi \in \mathbb{D}$,

$$1-|\varphi_z(\xi)|^2 = \frac{(1-|z|^2)(1-|\xi|^2)}{|1-\bar{z}\xi|^2}, \tag{1.1}$$

and

$$\begin{aligned} 1-|\varphi_z(\xi)|^2 &\leqslant \log\frac{1}{|\varphi_z(\xi)|^2} \\ &\leqslant \left(1+\log\frac{1}{a^2}\right)(1-|\varphi_z(\xi)|^2) \qquad \text{if} \quad \rho(z, \xi) > a. \end{aligned} \tag{1.2}$$

A stronger version of the triangle inequality holds for ρ. Let $z_0, z_1, z_2 \in \mathbb{D}$; then

$$\frac{\rho(z_0, z_2) - \rho(z_2, z_1)}{1 - \rho(z_0, z_2)\, \rho(z_2, z_1)} \leqslant \rho(z_0, z_1) \leqslant \frac{\rho(z_0, z_2) + \rho(z_2, z_1)}{1 + \rho(z_0, z_2)\, \rho(z_2, z_1)}. \tag{1.3}$$

For $x, y \in M_{H^\infty}$ the formula $\rho(x, y) = \sup\{|f(y)| : f \in H^\infty,\ f(x) = 0$ and $\|f\| \leqslant 1\}$ provides an extension of ρ to $M_{H^\infty} \times M_{H^\infty}$. The Gleason part of $x \in M_{H^\infty}$ is then defined as $P(x) = \{y \in M_{H^\infty} : \rho(x, y) < 1\}$. The set of Gleason parts is a pairwise disjoint covering of M_{H^∞}. There are only two possibilities, either $P(x) = \{x\}$ or $P(x)$ is an analytic disk. The later case means that there is a continuous one-to-one and onto map $L_x : \mathbb{D} \to P(x)$, such that $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. Reciprocally, any analytic disk is contained in a Gleason part, and any maximal analytic disk is a Gleason part. By Schwarz's lemma, the Gleason part of any $z \in \mathbb{D}$ is $\mathbb{D}$.

A Blaschke product b with zero sequence $\{z_n\}$ that satisfies $\delta(b) = \inf_k \prod_{n \neq k} \rho(z_n, z_k) > 0$ is called an interpolating Blaschke product, and $\{z_n\}$ is called an interpolating sequence. From the work of Hoffman [10] we know that if $x \in M_{H^\infty}$ then $P(x)$ is an analytic disk if and only if x belongs to the set

$$\mathcal{G} = \{y \in M_{H^\infty} : y \text{ is in the closure of some interpolating sequence}\}.$$

As said in the Introduction, the points of $\mathcal{G}$ are called nontrivial points, and the points of $\Gamma \overset{\text{def}}{=} M_{H^\infty} \setminus \mathcal{G}$ are called trivial points. Since an interpolating Blaschke product b only vanishes on the closure of its zero sequence, b is zero-free on the compact set Γ. More precisely, by [9, X, Lemma 1.4] there is a function $s(\alpha) > 0$, for $0 < \alpha < 1$, such that every Blaschke product b with $\delta(b) \geqslant \alpha$ satisfies $\inf_\Gamma |b| \geqslant s(\alpha)$. In addition, $s(\alpha) \to 1$ as $\alpha \to 1$. By a result of Mills (see [10, Theorem 3.2]) we also know that every interpolating Blaschke product b can be factorized as $b = b_1 b_2$, where $\delta(b_j) \geqslant \delta(b)^{1/2}$ for $j = 1, 2$.

A set of the form $Q = \{z \in \mathbb{D} : |z| \geqslant 1 - r,\ |\arg z - \theta_0| \leqslant \pi r\}$, where $0 < r \leqslant 1$ and $0 \leqslant \theta_0 \leqslant 2\pi$, is called a (circular) square; and the side length of Q is $\ell(Q) = r$. Observe that r is also the $d\theta/2\pi$ measure of the interval $[\theta_0 - \pi r, \theta_0 + \pi r]$, which can be identified with the "base" of Q. Let μ be a complex-valued measure on $\mathbb{D}$. It is said that μ is a Carleson measure if

$$\|\mu\|_* \overset{\text{def}}{=} \sup_Q \frac{|\mu|(Q)}{\ell(Q)} < \infty,$$

where Q runs over all the (circular) squares.

It is well known (see [9, p. 287]) that a sequence $\{z_n\}$ in $\mathbb{D}$ is interpolating if and only if $\inf_{n\neq k}\rho(z_n, z_k)>0$ and

$$\sum_{n\geqslant 1}(1-|z_n|^2)\,\delta_{z_n} \text{ is a Carleson measure,} \tag{1.4}$$

where δ_{z_n} is the probability measure with mass concentrated at z_n. A sequence that only satisfies (1.4) can be split into finitely many interpolating sequences (see the construction in [14, pp. 158–159]). The next two lemmas are well known. The first one is an easy consequence of Schwarz's lemma (see [9, p. 405] for a proof), while the second is a simple combination of the above results.

LEMMA 1.1. *Let* $0<r<1$ *and* b *be an interpolating Blaschke product with zero sequence* $\{z_n\}$. *Then* $\inf\{|b(z)|: z\notin\bigcup_{n\geqslant 1}K(z_n, r)\}>0$.

LEMMA 1.2. *Let* $0<r<1$ *and* $\{z_n\}$ *be a sequence in* $\mathbb{D}$ *that satisfies* (1.4). *Then the closure in* M_{H^∞} *of the set* $\bigcup_{n\geqslant 1}K(z_n, r)$ *is contained in* $\mathcal{G}$.

Proof. Factorize the Blaschke product b associated to the sequence $\{z_n\}$ as $b=b_1\cdots b_m$, where m is some positive integer and each b_j is an interpolating Blaschke product. If $\delta_0>0$ is so close to 1 that $s(\delta_0)>\sqrt{r}$, then factorize each b_j $(1\leqslant j\leqslant m)$ as $b_j=b_{j,1}\cdots b_{j,n_j}$, where $\delta(b_{j,k_j})>\delta_0$ for every $1\leqslant j\leqslant m$ and $1\leqslant k_j\leqslant n_j$. Therefore

$$\inf_{\Gamma}|b_{j,k_j}|\geqslant s(\delta_0)>\sqrt{r} \qquad \text{for all } j \text{ and } k_j,$$

while the closure in M_{H^∞} of $\bigcup_{n\geqslant 1}K(z_n, r)$ is contained in

$$\bigcup_{j=1}^{m}\bigcup_{k_j=1}^{n_j}\{x\in M_{H^\infty}: |b_{j,k_j}(x)|\leqslant r\}.$$

The last assertion holds because if z_n is a zero of b_{j,k_j} then $|b_{j,k_j}(z)|\leqslant|\varphi_{z_n}(z)|\leqslant r$ for every $z\in K(z_n, r)$. ∎

Consider the sphere $S_2\subset\mathbb{R}^3$ defined by the formula $x_1^2+x_2^2+(x_3-\frac{1}{2})^2=\frac{1}{4}$. The stereographic projection s from $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ onto S_2 is given by

$$z\mapsto\left(\frac{\operatorname{Re} z}{1+|z|^2}, \frac{\operatorname{Im} z}{1+|z|^2}, \frac{|z|^2}{1+|z|^2}\right)$$

and the inverse map is

$$(x_1, x_2, x_3)\mapsto\frac{x_1}{1-x_3}+i\,\frac{x_2}{1-x_3}.$$

It is implicit in these definitions that $s(\infty)=(0,0,1)$. The chordal distance between $z_1, z_2 \in \hat{\mathbb{C}}$ is defined as $\chi(z_1, z_2)=\|s(z_1)-s(z_2)\|_{\mathbb{R}^3}$. It can be shown that if $z_1, z_2 \in \mathbb{C}$ then

$$\chi(z_1, z_2)=\frac{|z_1-z_2|}{\sqrt{1+|z_1|^2}\,\sqrt{1+|z_2|^2}},$$

and if $z_2=\infty$ then

$$\chi(z_1, \infty)=\frac{1}{\sqrt{1+|z_1|^2}}.$$

2. THE CONDITION OF BROWN AND GAUTHIER

Let (M, d) be a metric space, $x \in M_{H^\infty}$ and $V \subset M_{H^\infty}$ be an open neighborhood of x. Suppose that $f\colon V \cap \mathbb{D} \to M$ is a continuous function such that the closure of $f(V \cap \mathbb{D})$ is a compact subset of M. In particular, $(\overline{f(V \cap \mathbb{D})}, d)$ is a *complete* metric space. The cluster set of f at x is

$$\begin{aligned}\mathrm{Cl}_M(f, x)=\{\lambda \in M\colon & \text{there is a net } (z_\alpha) \text{ in } V \cap \mathbb{D} \\ & \text{such that } z_\alpha \to x \text{ and } f(z_\alpha) \to \lambda\}.\end{aligned}$$

When $V=M_{H^\infty}$ we can define a multivalued function $F\colon M_{H^\infty} \to \mathscr{P}(M)$, where $\mathscr{P}(M)$ denotes the set of subsets of M, by the rule $F(x)=\mathrm{Cl}_M(f, x)$.

Let $U \subset M_{H^\infty}$ be an open set that contains the disk. By the density of $\mathbb{D}$ in M_{H^∞}, a simple diagonal argument implies that $f \in C(\mathbb{D}, M)$ admits a continuous extension from U into M if and only if $\mathrm{Cl}_M(f, x)$ is a single point for every $x \in U \backslash \mathbb{D}$. In that case, the extension is $F|_U$. In particular, this holds when $U=M_{H^\infty}$ or $\mathscr{G}$.

When $(M, d)=(\hat{\mathbb{C}}, \chi)$ and f is a meromorphic function on $\mathbb{D}$, the aforementioned result of Brown and Gauthier says that f can be continuously extended from $\mathscr{G}$ into $\hat{\mathbb{C}}$ if and only if f is uniformly continuous with respect to the metrics ρ and χ. As it turns out, their result holds in the general situation.

THEOREM 2.1. *Let (M, d) be a metric space and f be a continuous map from $\mathbb{D}$ into M such that $\overline{f(\mathbb{D})}$ is compact. Then f admits a continuous extension from $\mathscr{G}$ into M if and only if f is uniformly (ρ, d) continuous.*

Proof. Suppose that $f \in C(\mathscr{G}, M)$. If f is not uniformly (ρ, d) continuous there are two sequences $z_n, \omega_n \in \mathbb{D}$ such that $\rho(z_n, \omega_n) \to 0$ and $d(f(z_n), f(\omega_n)) \geqslant \delta > 0$ for every n. Since f is continuous on $\mathbb{D}$, by a compactness argument f is uniformly (ρ, d) continuous on any disk $|z| \leqslant r < 1$. Therefore the sequence $\{z_n\}$ cannot accumulate on $\mathbb{D}$. Consequently we can assume that $\{z_n\}$ is an interpolating sequence (otherwise we take a suitable subsequence of $\{z_n\}$). Let $x \in \overline{\{z_n\}} \backslash \{z_n\}$ and (z_α) be a subnet of $\{z_n\}$ that tends to x. Then $x \in \mathcal{G}$, and since every z_α is some $z_{n'}$, writing $\omega_\alpha = \omega_{n'}$ we have a subnet (ω_α) of the sequence $\{\omega_n\}$ such that

$$\rho(z_\alpha, \omega_\alpha) \to 0 \qquad \text{and} \qquad d(f(z_\alpha), f(\omega_\alpha)) \geqslant \delta \qquad \text{for all } \alpha. \tag{2.1}$$

By the lower semi-continuity of ρ [10, Theorem 6.2], $\omega_\alpha \to x$, and since f is continuous at x then $\lim f(\omega_\alpha) = f(x) = \lim f(z_\alpha)$, contradicting (2.1).

Now let us assume that f is uniformly (ρ, d) continuous on $\mathbb{D}$. Since the pseudohyperbolic and euclidean metrics are equivalent on compact subsets of $\mathbb{D}$, then $f \in C(\mathbb{D}, M)$. Let $x \in \mathcal{G} \backslash \mathbb{D}$ and $S \subset \mathbb{D}$ be an interpolating sequence such that $x \in \overline{S}$. For every positive n the compactness of $\overline{f(\mathbb{D})}$ implies that there is a finite partition P_n of $\overline{f(\mathbb{D})}$, such that the d-diameter of every $W \in P_n$ is bounded by $1/n$. We can also assume that each $W' \in P_{n+1}$ is contained in some $W \in P_n$. Therefore P_1 is a finite covering of $\overline{f(\mathbb{D})}$ whose elements are pairwise disjoint. For every $W \in P_1$ put

$$S(W) = \{z \in S : f(z) \in W\}.$$

Hence, $\{S(W) : W \in P_1\}$ is a finite partition of S. Since disjoint subsequences of an interpolating sequence have disjoint closures, then x belongs to the closure of one and only one $S(W)$. The corresponding $W \in P_1$ will be denoted by W_1. We can repeat this process with $S(W_1)$ instead of S and P_2 instead of P_1, and so forth. Doing so we obtain a chain of interpolating sequences $S \supset S(W_1) \supset S(W_2) \cdots$, and a chain of subsets of M, $W_1 \supset W_2 \supset \cdots$, such that

$$f(S(W_n)) \subset W_n, \quad x \in \overline{S(W_n)}, \quad \text{and} \quad \operatorname{diam}_d W_n < 1/n \quad \text{for all } n \geqslant 1.$$

Pick some $\omega_n \in f(S(W_n))$ for every n. Then $\omega_n, \omega_m \in W_p$ for $p = \min\{n, m\}$, implying that $d(\omega_n, \omega_m) \leqslant 1/p = \max\{1/n, 1/m\} \to 0$ as $n, m \to \infty$. So, $\{\omega_n\}$ is a Cauchy sequence in $f(\mathbb{D})$, and since $\overline{f(\mathbb{D})}$ is complete, then it converges to some point $\omega \in M$. Besides, since $\operatorname{diam}_d f(S(W_n)) \leqslant \operatorname{diam}_d W_n \leqslant 1/n$, it makes sense to define $f(x) = \omega$.

By previous comments, the theorem will follow if we show that $f(\xi_\alpha) \to \omega$ for every net (ξ_α) in $\mathbb{D}$ that tends to x. So, let (ξ_α) be a net in $\mathbb{D}$ tending to x, and $\varepsilon > 0$. By hypothesis there is $\eta > 0$ such that if $z_1, z_2 \in \mathbb{D}$,

$$\rho(z_1, z_2) \leqslant \eta \Rightarrow d(f(z_1), f(z_2)) < \varepsilon. \tag{2.2}$$

Fix some integer n_0 such that $n_0 > 1/\varepsilon$ and $d(\omega_{n_0}, \omega) < \varepsilon$, and write $S(W_{n_0}) = \{z_k\}$. Since $x \in \overline{\{z_k\}}$ and $\{z_k\}$ is interpolating, then Lemma 1.1 implies that the closure of $\bigcup_k K(z_k, \eta)$ in M_{H^∞} is a neighborhood of x. So, the fact that $\xi_\alpha \to x$ implies that there is a "tail" of the net (ξ_α) (call it (ξ_β)) that is contained in $\overline{\bigcup_k K(z_k, \eta)}$, and hence in $\bigcup_k K(z_k, \eta)$. That is, $\rho(\xi_\beta, S(W_{n_0})) \leqslant \eta$ for every β, which together with (2.2) gives $d(f(\xi_\beta), f(S(W_{n_0}))) \leqslant \varepsilon$. Thus

$$\begin{aligned} d(f(\xi_\beta), \omega) &\leqslant d(f(\xi_\beta), f(S(W_{n_0}))) + \operatorname{diam}_d f(S(W_{n_0})) + d(\omega_{n_0}, \omega) \\ &< \varepsilon + \frac{1}{n_0} + \varepsilon < 3\varepsilon, \end{aligned}$$

proving the theorem. ∎

In [6] the authors use elementary properties of a Schwarz triangle function to exhibit a meromorphic function f that can be continuously extended to $\mathscr{G}$, but such that $\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x) = \hat{\mathbb{C}}$ for every $x \in M_{H^\infty} \setminus \mathscr{G}$. According to the example, they conjecture that if f is meromorphic and $x \in M_{H^\infty}$, then $\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$ is a single point or the whole $\hat{\mathbb{C}}$. We prove this conjecture in the next subsection.

2.1. *Cluster Set Conditions*

The following result can be found in [16, Theorem 3.2.].

LEMMA 2.2. *Let $x \in M_{H^\infty}$ and $V \subset M_{H^\infty}$ be an open neighborhood of x. If $f \in H^\infty(V \cap \mathbb{D})$ then f can be extended by continuity to some open neighborhood of x.*

THEOREM 2.3. *Let $x \in M_{H^\infty} \setminus \mathbb{D}$, $V \subset M_{H^\infty}$ be an open neighborhood of x and f be a meromorphic function on $V \cap \mathbb{D}$. Then the following conditions are equivalent.*

(1) *f extends continuously to some neighborhood of x into $\hat{\mathbb{C}}$.*

(2) *$\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$ is a single point.*

(3) *$\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$ omits some point in $\hat{\mathbb{C}}$.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial. For $(3) \Rightarrow (1)$ let us assume first that $\infty \notin \mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$. This means that there is some open neighborhood U of x contained in V such that f is bounded on $U \cap \mathbb{D}$. Therefore f is an analytic bounded function on $U \cap \mathbb{D}$. By Lemma 2.2 f can be continuously extended from some open neighborhood of x into $\mathbb{C}$.

Now suppose that there is $a \in \mathbb{C}$ such that $a \notin \mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$. Then the function $g(z) = 1/(f(z) - a)$ is meromorphic on $V \cap \mathbb{D}$ and $\infty \notin \mathrm{Cl}_{\hat{\mathbb{C}}}(g, x)$. So, by the above argument there is a continuous (bounded) extension G of g to some open neighborhood of x. Since the map $\omega \mapsto (1/\omega)+a$ is continuous from $\hat{\mathbb{C}}$ into $\hat{\mathbb{C}}$ then $(1/G)+a$ is a continuous extension of f. ∎

Remark 2.4. The theorem has some strong consequences. It implies that a meromorphic function f can be extended continuously from M_{H^∞} into $\hat{\mathbb{C}}$ if and only if so can $|f|$ from M_{H^∞} into $[0, \infty]$ (with the topology induced by $\hat{\mathbb{C}}$). Also, f admits a continuous extension when $f(\mathbb{D})$ is not dense in $\hat{\mathbb{C}}$.

LEMMA 2.5. *Let $x \in M_{H^\infty} \setminus \mathbb{D}$. Then there is a fundamental system $\mathcal{U}$ of neighborhoods of x such that for each $U \in \mathcal{U}$, every connected component of $U \cap \mathbb{D}$ is simply connected.*

Proof. Since H^∞ is a separating algebra (see [16, Theorem 2.4]) there is a base of neighborhoods of x of the form $\{|f| < 1\}$ with $f \in H^\infty$. By the maximum modulus principle, each component of $\{|f| < 1\} \cap \mathbb{D}$ is simply connected. ∎

In the sequel $[-\infty, +\infty]$ denotes the two-point compactification of $\mathbb{R}$.

COROLLARY 2.6. *Let $x \in M_{H^\infty} \setminus \mathbb{D}$, $V \subset M_{H^\infty}$ be an open neighborhood of x and u be a real-valued harmonic function on $V \cap \mathbb{D}$. Then the following conditions are equivalent.*

(1) *u extends continuously to some neighborhood of x into $[-\infty, +\infty]$.*

(2) $\mathrm{Cl}_{[-\infty, +\infty]}(u, x)$ *is a single point.*

(3) $\mathrm{Cl}_{[-\infty, +\infty]}(u, x)$ *omits some pont in $[-\infty, +\infty]$.*

Again, the only nontrivial implication is $(3) \Rightarrow (1)$. By Lemma 2.5 we can assume that every component of $V \cap \mathbb{D}$ is simply connected. Therefore there is a harmonic function v on $V \cap \mathbb{D}$ such that $u + iv$ is analytic there (because this holds for every connected component of $V \cap \mathbb{D}$). Assuming (3) the cluster set of the function $f = e^{u+iv}$ at x omits 0, ∞, or some circle centered at the origin. In any case Theorem 2.3 implies that f can be continuously extended to some neighborhood of x as function into the Riemann sphere. Let F be such extension. Then $|F|$ is continuous as a map into $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$ with the topology induced by $\hat{\mathbb{C}}$. Since log: $\mathbb{R}_{\geqslant 0} \cup \{\infty\} \to [-\infty, +\infty]$ is a homeomorphism then $\log |F|$ is the desired extension of u. ∎

COROLLARY 2.7. *Let u be a real-valued harmonic function such that $u(\mathbb{D}) \neq \mathbb{R}$. Then u admits a continuous extension from M_{H^∞} into $[-\infty, +\infty]$.*

Proof. Suppose that $a \in \mathbb{R}$ is not in the image of u. Since $u(\mathbb{D})$ is connected then it is entirely to the right or to the left of a. Therefore $u(\mathbb{D})$ is not dense in $[-\infty, +\infty]$, and the result follows from the previous corollary. ∎

COROLLARY 2.8. *Let $x \in M_{H^\infty} \setminus \mathbb{D}$, $V \subset M_{H^\infty}$ be an open neighborhood of x and f be a harmonic function on $V \cap \mathbb{D}$. If $\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$ is bounded then f extends continuously to some neighborhood of x into $\mathbb{C}$. In particular, a harmonic function f on $\mathbb{D}$ extends continuously from M_{H^∞} into $\hat{\mathbb{C}}$ if and only if for every $x \in M_{H^\infty} \setminus \mathbb{D}$,*

$$\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x) \text{ is bounded or } \mathrm{Cl}_{\hat{\mathbb{C}}}(f, x) = \{\infty\}.$$

Proof. Write $f = u + iv$, where u and v are real-valued harmonic functions. If $x \in M_{H^\infty} \setminus \mathbb{D}$ is such that $\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$ is bounded then so are $\mathrm{Cl}_{[-\infty, +\infty]}(u, x)$ and $\mathrm{Cl}_{[-\infty, +\infty]}(v, x)$. Thus, by Corollary 2.6, u and v can be continuously extended from some open neighborhood of x into $[-\infty, +\infty]$. Therefore f can be continuously extended from this neighborhood into $\hat{\mathbb{C}}$. Since $\mathrm{Cl}_{\hat{\mathbb{C}}}(f, x)$ is bounded, we can shrink the neighborhood of x so that f never takes the value ∞ on that set.

The second assertion follows because the hypothesis and the fact just proved imply that all the cluster sets of f are singletons. ∎

3. THE CONDITIONS OF BISHOP

In [5] Bishop gives a useful characterization of the algebra $C(M_{H^\infty}, \mathbb{C})$. The proof makes use of a distance formula that he had previously found (see [4]) and of earlier work of Garnett.

By a Carleson contour we mean a rectifiable curve $\Lambda \subset \mathbb{C}$ such that the measure induced by the arc length of Λ is a Carleson measure. In the sequel, we write $dA(re^{i\theta}) = (1/2\pi)\, r\, dr\, d\theta$ for the area measure on $\mathbb{D}$ normalized so that the length of the circle $|z| = r$ is r. Also, if $M = \mathbb{C}$, $\hat{\mathbb{C}}$ or $[-\infty, +\infty]$ and $f \in C(\mathbb{D}, M)$, we abbreviate the phrase "f can be extended continuously from M_{H^∞} into M" by simply writing $f \in C(M_{H^\infty}, M)$.

THEOREM 3.1 (Bishop). *Let $f \in C(\mathbb{D}, \mathbb{C}) \cap L^\infty(\mathbb{D})$. The following conditions are equivalent.*

(1) $f \in C(M_{H^\infty}, \mathbb{C})$.

(2) *f is uniformly $(\rho, |\,|)$ continuous on $\mathbb{D}$, and for every $\varepsilon > 0$ there is a Carleson contour $\Lambda \subset \mathbb{D}$ and a function $h\colon \mathbb{D} \setminus \Lambda \to \mathbb{C}$, such that h is constant on every connected component $\mathcal{O}$ of $\mathbb{D} \setminus \Lambda$ and $|f(z) - h(z)| < \varepsilon$ if $z \in \mathcal{O}$.*

(3) *For every* $\varepsilon > 0$ *thee is* $g \in C^1(\mathbb{D}, \mathbb{C})$ *such that* $\|f - g\|_\infty < \varepsilon$, $(1 - |z|^2)\,|\nabla g(z)|$ *is bounded and* $|\nabla g(z)|\, dA(z)$ *is a Carleson measure.*

The main purpose of this section is to prove an analogue of the theorem, with $\mathbb{C}$ replaced by the Riemann sphere. This requires a technical step (next subsection) that will be useful later in the paper.

3.1. *The Regularization Operator*

As usual, when we say that a function satisfies some smoothness condition (C^∞, harmonic, etc) on a closed set, we mean that the condition is satisfied on an open neighborhood of it. For $0 < \eta < 1$ let g: $[0, 1] \to [0, 1]$ be a C^∞ function such that $g(x) = 1$ for $x \leqslant \eta^2/4$, $g(x) = 0$ for $x \geqslant \eta^2$, $|g'(x)| \leqslant c\eta^{-2}$ and $|g''(x)| \leqslant c\eta^{-4}$ for every x, where $c > 0$ is an absolute constant. If $z, \omega \in \mathbb{D}$, we define the regularization function

$$\wp(z, \omega) \overset{\text{def}}{=} g(|\varphi_z(\omega)|^2)(1 - |\omega|^2)^{-2} \Big/ \int_{\mathbb{D}} g(|\xi|^2)(1 - |\xi|^2)^{-2}\, dA(\xi). \tag{3.1}$$

The support set of $\wp(z, \omega)$ as a function of ω is contained in $K(z, \eta)$ for every $z \in \mathbb{D}$. Also, the conformal invariance of the measure $(1 - |\omega|^2)^{-2}\, dA(\omega)$ implies that $\int \wp(z, \omega)\, dA(\omega) = 1$ for every $z \in \mathbb{D}$. Therefore the operator $T_\eta h(z) = \int \wp(z, \omega)\, h(\omega)\, dA(\omega)$ is a contraction on the space $L^\infty(dA)$. Write $T = T_\eta$.

LEMMA 3.2. *Let* $h \in L^\infty(dA)$.

(a) *If* h *is harmonic on* $K(z, \eta)$ *then* $Th(z) = h(z)$.

(b) *If there is some* $\lambda_z \in \mathbb{C}$ *such that* ess sup $\{|h(\omega) - \lambda_z| : \omega \in K(z, \eta)\} \leqslant \alpha$ *then* $|Th(z) - \lambda_z| \leqslant \alpha$.

(c) *There is* $C_\eta > 0$ *depending only on* η *such that* $(1 - |z|^2)\,|\nabla(Th)(z)| \leqslant C_\eta \|h\|_\infty$ *for every* $z \in \mathbb{D}$.

(d) *There is* $C'_\eta > 0$ *depending only on* η *such that* $(1 - |z|^2)^2\,|\Delta(Th)(z)| \leqslant C'_\eta\,\|h\|_\infty$ *for every* $z \in \mathbb{D}$.

Proof. Suppose that $h \in L^\infty(dA)$ is harmonic on $K(z, \eta)$, where z is some point in $\mathbb{D}$. Therefore $h \circ \varphi_z$ is harmonic on $|\xi| \leqslant \eta$, and since $\wp(0, \xi)$ is a radial function supported on $|\xi| \leqslant \eta$, the change of variable $\omega = \varphi_z(\xi)$ yields

$$\begin{aligned} \int_{\mathbb{D}} \wp(z, \omega)\, h(\omega)\, dA(\omega) &= \int_{|\xi| \leqslant \eta} \wp(0, \xi)\, h(\varphi_z(\xi))\, dA(\xi) \\ &= (h \circ \varphi_z)(0) = h(z). \end{aligned} \tag{3.2}$$

If $h \in L^\infty(dA)$ is arbitrary, then $|Th(z)| \leqslant \text{ess sup}_{K(z,\eta)} |h|$. If h satisfies the hypothesis of (b), then the conclusion follows because T maps the constant function λ_z into itself. Since

$$\begin{aligned} |\nabla_z g(|\varphi_z(\omega)|^2)| &= 2\, \frac{1-|\omega|^2}{|1-\bar{\omega}z|^2}\, |\varphi_z(\omega)|\; |g'(|\varphi_z(\omega)|^2)| \\ &\leqslant 2\, \|g'\|_\infty \frac{1-|\omega|^2}{|1-\bar{\omega}z|^2} \end{aligned} \tag{3.3}$$

then

$$\begin{aligned} &\left|\nabla \int_{\mathbb{D}} \wp(z,\omega)\, h(\omega)\, dA(\omega)\right| \\ &\quad \leqslant \|h\|_\infty \int_{K(z,\eta)} |\nabla_z \wp(z,\omega)|\, dA(\omega) \\ &\quad \leqslant 2\, \|h\|_\infty\, \|g'\|_\infty \left[\int_{\mathbb{D}} g(|\xi|^2)\, \frac{dA(\xi)}{(1-|\xi|^2)^2}\right]^{-1} \frac{1}{(1-|z|)} \int_{|u|\leqslant\eta} \frac{dA(u)}{(1-|u|^2)} \\ &\quad \leqslant C_\eta\, \|h\|_\infty \frac{1}{(1-|z|^2)}, \end{aligned}$$

where the second inequality follows from the change of variable $u = \varphi_z(\omega)$. This proves (c). The proof of (d) is completely analogous, where instead of (3.3) we use

$$\begin{aligned} &(1-|z|^2)\, |\Delta_z g(|\varphi_z(\omega)|^2)| \\ &\quad = 4(1-|z|^2)\, \frac{(1-|\omega|^2)^2}{|1-\bar{\omega}z|^4}\, |\,|\varphi_z(\omega)|^2\, g''(|\varphi_z(\omega)|^2) + g'(|\varphi_z(\omega)|^2)| \\ &\quad \leqslant 4\, \frac{1-|\omega|^2}{|1-\bar{\omega}z|^2}\, (\|g''\|_\infty + \|g'\|_\infty). \end{aligned}$$

Here, the inequality comes from (1.1). ∎

If $Q \subset \mathbb{D}$ is a "square" and n is a positive integer, denote by nQ the square whose base on $\partial\mathbb{D}$ has the same center as Q, and such that $\ell(nQ) = \min\{n\ell(Q), 1\}$. The use of the minimum is just a way to say that $nQ = \mathbb{D}$ when $n\ell(Q) \geqslant 1$.

Suppose that $\Lambda \subset \mathbb{D}$ is a rectifiable curve whose arc length induces a Carleson measure λ, and that $h \in C^1(\mathbb{D}\backslash\Lambda)$ is a bounded function such that $|\nabla(h|_{\mathbb{D}\backslash\Lambda})(z)|\, dA(z)$ is a Carleson measure.

LEMMA 3.3. *Let* $0<\eta<1/2$. *If* h *is as above then* $|\nabla(Th)(z)|\,dA(z)$ *is a Carleson measure of constant bounded by* $C(\eta)(\|h\|_\infty\,\|\lambda\|_*+\|\,|\nabla(h|_{\mathbb{D}\setminus\Lambda})|\,dA\|_*)$, *where* $C(\eta)>0$ *only depends on* η. *If in addition* h *is constant on each connected component of* $\mathbb{D}\setminus\Lambda$ *then* $(1-|z|^2)\,|\Delta(Th)(z)|\,dA(z)$ *is a Carleson measure of constant bounded by* $C'(\eta)\,\|h\|_\infty\,\|\lambda\|_*$, *with* $C'(\eta)>0$ *depending only on* η.

Proof. Suppose first that $\rho(z,\Lambda)>\eta$. Then $\varphi_z(\omega)$ does not meet Λ when $|\omega|\leqslant\eta$. Thus, for every such fixed ω, the function (of z) $h(\varphi_z(\omega))$ is C^1 on some neighborhood of z, which enables us to take the gradient with respect to z,

$$\begin{aligned}|\nabla_z h(\varphi_z(\omega))| &\leqslant |(\nabla h)(\varphi_z(\omega))|\,|\nabla_z\varphi_z(\omega)|\\ &\leqslant c\,|(\nabla h)(\varphi_z(\omega))|\,|1-\bar{\omega}z|^{-1}.\end{aligned}$$

Hence, when $\rho(z,\Lambda)>\eta$,

$$\begin{aligned}&\left|\nabla\int_{\mathbb{D}}\wp(z,\omega)\,h(\omega)\,dA(\omega)\right|\\ &\leqslant\int_{|\xi|\leqslant\eta}\wp(0,\xi)\,|\nabla_z h(\varphi_z(\xi))|\,dA(\xi)\\ &\leqslant C_1(\eta)\int_{|\xi|\leqslant\eta}\frac{g(|\xi|^2)}{|1-\bar{\xi}z|}\,|(\nabla h)(\varphi_z(\xi))|\,\frac{dA(\xi)}{(1-|\xi|^2)^2}\\ &\leqslant\frac{C_1(\eta)}{1-\eta}\int_{K(z,\eta)}|\nabla h(\omega)|\,\frac{dA(\omega)}{(1-|\omega|^2)^2}.\end{aligned}\tag{3.4}$$

Let $Q\subset\mathbb{D}$ be a circular square and put $U=\{z\in\mathbb{D}:\rho(z,\Lambda)>\eta\}$. Since $\eta<1/2$ then

$$z\in Q\cap U\quad\text{and}\quad\omega\in K(z,\eta)\quad\Rightarrow\quad\omega\in 2Q\setminus\Lambda.$$

Hence, integrating (3.4) with respect to z on $Q\cap U$ and using the equality $\chi_{K(z,\eta)}(\omega)=\chi_{K(\omega,\eta)}(z)$, we get by Fubini's theorem

$$\begin{aligned}&\int_{Q\cap U}|\nabla Th(z)|\,dA(z)\\ &\leqslant\frac{C_1(\eta)}{1-\eta}\int_{Q\cap U}\left[\int_{\mathbb{D}}\chi_{K(z,\eta)}(\omega)\,|\nabla h(\omega)|\,\frac{dA(\omega)}{(1-|\omega|^2)^2}\right]dA(z)\\ &\leqslant\frac{C_1(\eta)}{1-\eta}\int_{2Q\setminus\Lambda}\left[\int_{K(\omega,\eta)}dA(z)\right]|\nabla h(\omega)|\,\frac{dA(\omega)}{(1-|\omega|^2)^2}\end{aligned}$$

$$= \frac{C_1(\eta)}{1-\eta} \int_{2Q\setminus\Lambda} \left[\frac{\eta^2(1-|\omega|^2)^2}{2(1-\eta^2\,|\omega|^2)^2} \right] |\nabla h(\omega)| \, \frac{dA(\omega)}{(1-|\omega|^2)^2}$$

$$\leqslant C_2(\eta) \int_{2Q\setminus\Lambda} |\nabla h(\omega)| \, dA(\omega).$$

On the other hand, if $z \notin U$ (that is, if $\rho(z, \Lambda) \leqslant \eta$) we can use (c) of Lemma 3.2, which together with simple geometrical considerations yields

$$\int_{Q\setminus U} |\nabla Th(z)| \, dA(z) \leqslant C_\eta \, \|h\|_\infty \int_{Q\setminus U} \frac{dA(z)}{1-|z|^2}$$

$$\leqslant C_3(\eta) \, \|h\|_\infty \int_{2Q} d\lambda, \tag{3.5}$$

proving our first assertion. Now suppose that in addition to the previous hypotheses h is constant on each connected component of $\mathbb{D}\setminus\Lambda$. If $z \in U$ then (a) of Lemma 3.2 tells us that Th is constant in a neighborhood of z. This means that the support set of ΔTh is contained in $\mathbb{D}\setminus U$. Hence, by (d) of Lemma 3.2 we obtain an inequality like (3.5) with $(1-|z|^2)\,|\Delta Th|$ in place of $|\nabla Th|$. ∎

3.2. *The Spherical Version of Bishop's Theorem*

Let $g\colon \mathbb{D} \to \hat{\mathbb{C}}$ and n be a positive integer. We will say that $g \in C^n(\mathbb{D}, \hat{\mathbb{C}})$ if g is continuous, and for every $z \in \mathbb{D}$ there is an open neighborhood $U \subset \mathbb{D}$ of z, such that at least one of the following conditions holds

(a) $g(U) \subset \mathbb{C}$ and $g \in C^n(U, \mathbb{C})$, or

(b) $(1/g)(U) \subset \mathbb{C}$ and $(1/g) \in C^n(U, \mathbb{C})$.

For $g \in C^1(\mathbb{D}, \hat{\mathbb{C}})$ define the spherical gradient as

$$g^{\#}(z) = \frac{1}{\sqrt{2}} \frac{|\nabla g(z)|}{1+|g(z)|^2}, \qquad z \in \mathbb{D}, \tag{3.6}$$

where the "modulus" of $|\nabla g(z)|$ is the euclidean norm of $\mathbb{C}^2$. In order to be formally correct we should say that $g^{\#}(z)$ is defined in (3.6) with $1/g$ instead of g at every point z where $g(z) = \infty$. The easy equality $g^{\#} = (1/g)^{\#}$ implies that $g^{\#} \in C(\mathbb{D}, \hat{\mathbb{C}})$. When g is a meromorphic function on $\mathbb{D}$, $g^{\#}$ is the usual *spherical derivative*, a name that comes from the calculation

$$\lim_{z' \to z} \frac{\chi(g(z), g(z'))}{|z-z'|} = \frac{|g'(z)|}{1+|g(z)|^2} = g^{\#}(z).$$

The factor $2^{-1/2}$ in (3.6) has being included to make the definition of the spherical gradient consistent with the well-established notion of spherical derivative. It is well known that a meromorphic function g on $\mathbb{D}$ is uniformly (ρ, χ) continuous if and only if $(1-|z|^2)\, g^{\#}(z)$ is bounded. That is, the theorem of Brown and Gauthier can be expressed in terms of the spherical derivative, by saying that a meromorphic function g can be continuously extended from $\mathscr{G}$ into $\hat{\mathbb{C}}$ if and only if $\sup_{z \in \mathbb{D}}(1-|z|^2)\, g^{\#}(z) < \infty$.

At this point it is natural to convince oneself that if we want a version of Theorem 3.1 with $(\mathbb{C}, |\ |)$ replaced by $(\hat{\mathbb{C}}, \chi)$, then the right candidate to replace $|\nabla g|$ in the theorem is $g^{\#}$. In fact, everything works out nicely, as we see next.

THEOREM 3.4. *Let f be a function from $\mathbb{D}$ into $\hat{\mathbb{C}}$. Then the following conditions are equivalent.*

(1) $f \in C(M_{H^\infty}, \hat{\mathbb{C}})$.

(2) *f is uniformly (ρ, χ) continuous on $\mathbb{D}$, and for every $\varepsilon > 0$ there is a Carleson contour $\Lambda \subset \mathbb{D}$ and a function h: $\mathbb{D} \backslash \Lambda \to \hat{\mathbb{C}}$, such that h is constant on every connected component $\mathcal{O}$ of $\mathbb{D} \backslash \Lambda$ and $\chi(f(z), h(z)) < \varepsilon$ if $z \in \mathcal{O}$.*

(3) *For every $\varepsilon > 0$ there is $g \in C^1(\mathbb{D}, \hat{\mathbb{C}})$ such that $(1-|z|^2)\, g^{\#}(z)$ is bounded, $g^{\#}(z)\, dA(z)$ is a Carleson measure and $\chi(f(z), g(z)) < \varepsilon$ for every $z \in \mathbb{D}$.*

Proof. (1) ⇒ (2) By Theorem 2.1 f is uniformly (ρ, χ) continuous. Let s: $\hat{\mathbb{C}} \to S_2$ be the stereographic projection, and write $s \circ f = (f_1, f_2, f_3)$. Then $f_j \in C(M_{H^\infty}, \mathbb{R})$ for $j = 1, 2, 3$. By the corresponding implication of Theorem 3.1, for each j there is a rectifiable curve $\Lambda_j \subset \mathbb{D}$ whose arc length is a Carleson measure, and a locally constant function h_j: $\mathbb{D} \backslash \Lambda_j \to \mathbb{C}$ such that ess sup $|f_j - h_j| < \varepsilon/6$. Clearly, we can assume that the functions h_j are real-valued. Setting $\Lambda = \bigcup_{j=1}^{3} \Lambda_j$, the three functions h_j are well defined and constant on each connected component of $\mathbb{D} \backslash \Lambda$. Define H: $\mathbb{D} \backslash \Lambda \to \mathbb{R}^3$ by $H = (h_1, h_2, h_3)$.

Let $z \in \mathbb{D} \backslash \Lambda$. Assuming that $\varepsilon < 1$ then $H(z) \neq (0, 0, 1/2)$, and consequently there is a unique point $P(z) \in S_2$ such that $\|H(z) - P(z)\|_{\mathbb{R}_3} = \text{dist}_{\mathbb{R}_3}(H(z), S_2)$. Thus, if $z \in \mathbb{D} \backslash \Lambda$,

$$\begin{aligned} \|P(z) - (s \circ f)(z)\|_{\mathbb{R}^3} &\leqslant \|P(z) - H(z)\|_{\mathbb{R}^3} + \|H(z) - (s \circ f)(z)\|_{\mathbb{R}^3} \\ &\leqslant 2\, \|H(z) - (s \circ f)(z)\|_{\mathbb{R}^3} \\ &\leqslant 2 \sum_{j=1}^{3} |h_j(z) - f_j(z)| < 6\varepsilon/6, \end{aligned}$$

where the second inequality holds because $(s \circ f)(z) \in S_2$. Since s is an isometry from $(\hat{\mathbb{C}}, \chi)$ onto $(S_2 \, \| \, \|_{\mathbb{R}^3})$ then $\chi(s^{-1}(P(z)), f(z)) < \varepsilon$ for $z \in \mathbb{D} \backslash \Lambda$. That is, the function $h(z) = s^{-1}(P(z))$ satisfies (2).

(2) ⇒ (3) Assume that $\varepsilon < 1$. Let h be a function as in (2) with $\varepsilon/30$ instead of ε. Then $\|s \circ f(z) - s \circ h(z)\|_{\mathbb{R}^3} = \chi(f(z), h(z)) < \varepsilon/30$ for every $z \in \mathbb{D} \backslash \Lambda$. Since each of the coordinate functions f_j $(j = 1, 2, 3)$ is $(\rho, |\,|)$ uniformly continuous, there is an η in $(0, 1)$ such that

$$\sup \{ |h_j(z) - h_j(\omega)| : z, \omega \in K(z_0, \eta) \backslash \Lambda \} < 3\varepsilon/30$$
$$\text{for every} \quad z_0 \in \mathbb{D} \quad \text{and} \quad j = 1, 2, 3.$$

Consider the operator $T = T_\eta$ of the previous subsection, and write $t_j \overset{\text{def}}{=} Th_j$, where $j = 1, 2, 3$. If $z \in \mathbb{D} \backslash \Lambda$, then Lemma 3.2 (b) and the above inequality yield

$$|t_j(z) - f_j(z)| \leqslant |t_j(z) - h_j(z)| + |h_j(z) - f_j(z)| < 3\varepsilon/30 + \varepsilon/30.$$

Since t_j and f_j are continuous and Λ has null area, then $\|t_j - f_j\|_\infty \leqslant 4\varepsilon/30$ for every j. In addition, by Lemma 3.2(c), $(1 - |z|^2) \, |\nabla t_j(z)|$ is bounded, and by Lemma 3.3 $|\nabla t_j(z)| \, dA(z)$ is a Carleson measure. In order to pull back the vector $(t_1(z), t_2(z), t_3(z))$ with s^{-1} we need to "project" this vector on the sphere S_2. The projection gives

$$(g_1(z), g_2(z), g_3(z)) = \frac{1}{\|(2t_1(z), 2t_2(z), 2t_3(z) - 1)\|} \times (t_1(z), t_2(z), t_3(z) - 2^{-1}) + (0, 0, 2^{-1}).$$

Hence, for every $z \in \mathbb{D}$,

$$\begin{aligned} &\|((g_1 - f_1)(z), (g_2 - f_2)(z), (g_3 - f_3)(z))\| \\ &\leqslant 2 \, \|((t_1 - f_1)(z), (t_2 - f_2)(z), (t_3 - f_3)(z))\| \\ &\leqslant 2 \sum_{j=1}^{3} |(t_j - f_j)(z)| \leqslant 6 \, \frac{4\varepsilon}{30} < \varepsilon. \end{aligned} \tag{3.7}$$

A straightforward calculation shows that there is a constant $C > 0$ such that

$$|\nabla g_j| \leqslant C \sum_{k=1}^{3} |\nabla t_k| \tag{3.8}$$

for $j=1, 2, 3$. Define

$$g(z)=\begin{cases}(1-g_3(z))^{-1}\,(g_1(z)+ig_2(z)) & \text{if } \; g_3(z)\neq 1\\ \infty & \text{if } \; g_3(z)=1.\end{cases}$$

Since $g=s^{-1}\circ(g_1, g_2, g_3)$ then (3.7) says that $\chi(g(z), f(z))<\varepsilon$ for $z\in\mathbb{D}$. It is elementary to check that $g\in C^1(\mathbb{D}, \hat{\mathbb{C}})$,

$$\frac{1}{1+|g|^2}\,\nabla g=\nabla(g_1+ig_2)+\frac{g_1+ig_2}{1-g_3}\,\nabla g_3 \qquad \text{when} \quad g_3<\frac{3}{4}$$

and

$$\frac{1}{1+(1/|g|^2)}\,\nabla\left(\frac{1}{g}\right)=\nabla(g_1-ig_2)-\frac{g_1-ig_2}{g_3}\,\nabla g_3 \qquad \text{when} \quad g_3>\frac{1}{4}.$$

Since $0\leqslant|g_j|\leqslant 1$ for $j=1, 2, 3$, the above formula and (3.8) yield

$$g^{\#}\leqslant(|\nabla g_1|+|\nabla g_2|+8\,|\nabla g_3|)\leqslant C'\sum_{k=1}^{3}|\nabla t_k|.$$

Therefore g satisfies (3).

(3) ⇒ (1). It will enough to prove that any function $g\in C^1(\mathbb{D}, \hat{\mathbb{C}})$ that satisfies (3) is in $C(M_{H^\infty}, \hat{\mathbb{C}})$. Writing $s\circ g=(g_1, g_2, g_3)$, it is enough to see that each g_j can be extended continuously from M_{H^∞} into $\mathbb{R}$, or equivalently, into $\mathbb{C}$. We will prove this for g_1, since the same argument works for g_2 and g_3. If $z=x+iy$, put $\partial_x=\partial/\partial x$ and $\partial_y=\partial/\partial y$. Then

$$\begin{aligned}|\partial_x g_1|&=\left|\frac{\partial_x \operatorname{Re} g}{1+|g|^2}-\frac{\operatorname{Re} g}{1+|g|^2}\,\frac{2\operatorname{Re}(\bar{g}\,\partial_x g)}{1+|g|^2}\right|\\ &\leqslant\frac{|\nabla g|}{1+|g|^2}+\frac{2\,|g|^2\,|\nabla g|}{(1+|g|^2)^2}\leqslant 3\,\frac{|\nabla g|}{1+|g|^2}.\end{aligned}$$

Since the same inequality holds for $|\partial_y g_1|$, our hypothesis (3) implies that $(1-|z|^2)\,|\nabla g_1(z)|$ is bounded and that $|\nabla g_1(z)|\,dA(z)$ is a Carleson measure. Theorem 3.1 then tells us that $g_1(z)$ can be extended by continuity from M_{H^∞} into $\mathbb{C}$. ∎

Remark. In the proof of (2) ⇒ (3) the existence of functions playing the role of t_j for $j=1, 2, 3$ is guaranteed by Bishop's theorem. That is, we could avoid the use of the regularization operator T. I prefer to use T because it provides a constructive relation between the function h of (2) and the function g of (3). Besides, T will appear again in the next two sections.

4. INTEGRAL TRANSFORMS OF CARLESON MEASURES

In [9, pp. 348–358] Garnett showed that the Poisson integral of a function in $L^\infty(\partial\mathbb{D})$ can be uniformly approximated by functions g such that $(1-|z|^2)\,|\nabla g(z)|$ is bounded and $|\nabla g(z)|\,dA(z)$ is a Carleson measure. The present section is devoted to proving two different area versions of this result. Given the length of the proof, we have divided it into several lemmas distributed in 5 subsections, with a spinal argument between lemmas. At an early stage the proof splits in two parts; the first part uses an idea of Jones [11, pp. 199–200] and the second part is modelled after Garnett's arguments. However, some results that are easier to prove or can be taken for granted in the case of a bounded harmonic function require more work in our situation.

THEOREM 4.1. *Let p be a complex-valued function on $\mathbb{D}$ such that $(1-|\xi|^2)\,p(\xi)\,dA(\xi)$ is a Carleson measure and $(1-|\xi|^2)^2\,p(\xi)$ is bounded. If $q(z,\xi)$ is either of the kernels* $\log|\varphi_z(\xi)|^{-2}$ *or* $1-|\varphi_z(\xi)|^2$ *then the function*

$$f(z)=\int_{\mathbb{D}} q(z,\xi)\,p(\xi)\,dA(\xi)$$

can be continuously extended from M_{H^∞} into $\mathbb{C}$.

4.1. *Decomposing $f(z)=f_1(z)+f_2(z)$*

We will use repeatedly the well known fact [9, p. 239] that a positive measure m on $\mathbb{D}$ is Carleson if and only if

$$\|m\|_{**}\overset{\text{def}}{=}\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\frac{1-|z|^2}{|1-\bar{\xi}z|^2}\,dm(\xi)<\infty.$$

In this case there is an absolute constant $C>1$ such that

$$C^{-1}\,\|m\|_*\leqslant\|m\|_{**}\leqslant C\,\|m\|_*. \tag{4.1}$$

There is no loss of generality if we assume that $p(\xi)\geqslant 0$, $\|(1-|\xi|^2)\,p(\xi)\,dA(\xi)\|_*\leqslant 1$ and $\|(1-|\xi|^2)^2\,p(\xi)\|_\infty\leqslant 1$. These assumptions will be implicit in every lemma of this section. Also, let us fix at once an ε in $(0,1)$. We shall show with the aid of Bishop's theorem that there is an absolute constant $K>0$ and a function $g\in C(M_{H^\infty},\mathbb{C})$ such that $\|f(z)-g(z)\|_\infty<K\varepsilon$. Put $dv(\xi)=p(\xi)\,dA(\xi)$.

LEMMA 4.2. *If* $0<\gamma\leqslant r<1$ *then*

$$\int_{K(z,r)} \log \frac{1}{|\varphi_z(\xi)|^2}\, dv(\xi) \leqslant (1+\log \gamma^{-2}) \left[\frac{\gamma^2}{(1-\gamma^2)^2} + v(K(z,r)\backslash K(z,\gamma)) \right]. \tag{4.2}$$

Proof. First we remove the singularity of the logarithm in the integral. This is achieved by the simple change of variables $u=\varphi_z(\xi)$.

$$\begin{aligned} \int_{K(z,\gamma)} \log \frac{1}{|\varphi_z(\xi)|^2}\, p(\xi)\, dA(\xi) &\leqslant \int_{|\varphi_z(\xi)|\leqslant\gamma} \log \frac{1}{|\varphi_z(\xi)|^2} \frac{dA(\xi)}{(1-|\xi|^2)^2} \\ &= \int_{|u|\leqslant\gamma} \log \frac{1}{|u|^2} \frac{dA(u)}{(1-|u|^2)^2} \\ &\leqslant \frac{1}{(1-\gamma^2)^2} \int_{|u|\leqslant\gamma} \log \frac{1}{|u|^2}\, dA(u) \\ &= \frac{\gamma^2}{2(1-\gamma^2)^2} (1+\log\gamma^{-2}), \end{aligned}$$

where the first inequality holds because $\|(1-|\xi|^2)^2\, p(\xi)\|_\infty \leqslant 1$, and the first equality because the measure $(1-|\xi|^2)^{-2}\, dA(\xi)$ is conformally invariant.

To obtain the estimate for the integral on $K(z,r)\backslash K(z,\gamma)$ we simply notice that $\log|\varphi_z(\xi)|^{-2} \leqslant \log\gamma^{-2}$ when $\rho(\xi,z)>\gamma$. ∎

By (4.1) the function $\int (1-|\varphi_z(\xi)|^2)\, dv(\xi)$ is bounded. So, by (1.2) and the case $r=\gamma$ of (4.2), $\int \log|\varphi_z(\xi)|^{-2}\, dv(\xi)$ is also bounded. A straightforward calculation shows that

$$\begin{aligned} (1-|z|^2) &\left| \nabla \int \frac{1-|z|^2}{|1-\bar{\xi}z|^2} (1-|\xi|^2)\, dv(\xi) \right| \\ &\leqslant C \int \frac{1-|z|^2}{|1-\bar{\xi}z|^2} (1-|\xi|^2)\, dv(\xi) \leqslant C', \end{aligned}$$

implying that $f(z)$ is Lipschitz with respect to the metric ρ in the case $q(z,\xi)=1-|\varphi_z(\xi)|^2$. For the other kernel observe that if $\rho(z_1,z_2)<1/2$ then (1.3) yields

$$\begin{aligned}
&|\rho(z_1,\xi)-\rho(z_2,\xi)|\\
&\quad\leqslant [1-\rho(z_1,\xi)\,\rho(z_2,\xi)]\,\rho(z_1,z_2)\\
&\quad= [(1-\rho(z_1,\xi)^2)+\rho(z_1,\xi)(\rho(z_1,\xi)-\rho(z_2,\xi))]\,\rho(z_1,z_2)\\
&\quad\leqslant (1-\rho(z_1,\xi)^2)\,\rho(z_1,z_2)+\tfrac{1}{2}\,|\rho(z_1,\xi)-\rho(z_2,\xi)|.
\end{aligned}$$

Therefore, when ξ belongs to the set $E=\{\xi\in\mathbb{D}:\rho(z_1,\xi),\ \rho(z_2,\xi)>\gamma\}$ we have

$$\begin{aligned}
\left|\log\frac{1}{|\varphi_{z_1}(\xi)|^2}-\log\frac{1}{|\varphi_{z_2}(\xi)|^2}\right| &\leqslant \frac{2}{\gamma}\,|\rho(z_1,\xi)-\rho(z_2,\xi)|\\
&\leqslant \frac{4}{\gamma}\,(1-\rho(z_1,\xi)^2)\,\rho(z_1,z_2). \qquad (4.3)
\end{aligned}$$

Integrating the above inequality against the measure $dv(\xi)$ on E, and using (4.2) with $r=\gamma$ small to estimate the integral on $\mathbb{D}\backslash E$, we see that $f(z)$ is uniformly continuous with respect to ρ also in the case $q(z,\xi)=\log|\varphi_z(\xi)|^{-2}$.

For $\delta<1/4$ let $\{D_j\colon j\geqslant 1\}$ be a pairwise disjoint decomposition of $\mathbb{D}$ such that every D_j contains some pseudohyperbolic ball of radius $\delta/4$ and is contained in a pseudohyperbolic ball of radius $\delta/2$. Hence, $\operatorname{diam}_\rho D_j\leqslant\delta$ for all j. Put $p_j=\int_{D_j}p\,dA$.

Let N be a positive integer to be determined later. We write D_j^+ for those sets D_j such that $p_j\geqslant 1/N$ and D_j^- for the sets D_j such that $p_j<1/N$. If $L^+=\bigcup_j D_j^+$ and $L^-=\bigcup_j D_j^-$, decompose the measure $p\,dA$ as

$$p(\xi)\,dA(\xi)=\chi_{L^+}(\xi)\,p(\xi)\,dA(\xi)+\chi_{L^-}(\xi)\,p(\xi)\,dA=dv^+(\xi)+dv^-(\xi).$$

The measures v^+ and v^- depend on δ and N. It is clear that both measures satisfy the same properties as the measure v, including Lemma 4.2 and the restrictions for the Carleson and sup norms. So, if $q(z,\xi)$ is one of the kernels $\log|\varphi_z(\xi)|^{-2}$ or $1-\rho(\xi,z)^2$, then

$$\begin{aligned}
\int_{\mathbb{D}} q(z,\xi)\,p(\xi)\,dA(\xi) &= \int_{\mathbb{D}} q(z,\xi)\,dv^+(\xi)+\int_{\mathbb{D}} q(z,\xi)\,dv^-(\xi)\\
&= f_1(z)+f_2(z),
\end{aligned}$$

where $f_1(z)$ and $f_2(z)$ satisfy all the properties that we have seen so far for $f(z)$. We will study the behaviors of $f_1(z)$ and $f_2(z)$ separately.

4.2. *The Behavior of* $f_1(z)$

PROPOSITION 4.3. *Let* $q(z, \xi)$ *be either of our kernels. Then there are an* r *in* $(0, 1)$ *and a Blaschke product* b *whose zeros* $\{\omega_k\}$ *form a finite union of interpolating sequences* (*both depending on* N *and* δ), *such that*

$$\sup_{\rho(z, \{\omega_k\}) > r} \left| \int_{\mathbb{D}} q(z, \xi)\, dv^+(\xi) - \frac{1}{N^2} \log \frac{1}{|b(z)|^2} \right| \leqslant C\left(\delta + \frac{1}{N}\right), \tag{4.4}$$

where $C > 0$ *is an absolute constant.*

Proof. Consider first the case $q(z, \xi) = \log |\varphi_z(\xi)|^{-2}$. For each D_j^+ we fix a point $\xi_j \in D_j^+$. Let $[t]$ be the integer defined by $[t] \leqslant t < [t] + 1$. Then

$$0 \leqslant p_j - \frac{[N^2 p_j]}{N^2} \leqslant \frac{1}{N^2} \leqslant \frac{p_j}{N}. \tag{4.5}$$

If $\xi \in D_j$ then $\rho(\xi, \xi_j) \leqslant \delta < 1/4$, and consequently there is an absolute constant $c > 0$ such that $(1 - |\xi_j|^2) \leqslant c(1 - |\xi|^2)$. Also, if Q is a circular square such that $\xi_j \in Q$ then $D_j^+ \subset 2Q$. Therefore

$$\begin{aligned} \sum_{\xi_j \in Q} (1 - |\xi_j|^2) \frac{[N^2 p_j]}{N^2} &\leqslant \sum_{\xi_j \in Q} (1 - |\xi_j|^2)\, p_j \\ &\leqslant c \int_{2Q} (1 - |\xi|^2)\, p(\xi)\, dA(\xi) \leqslant C_0 \ell(Q). \end{aligned} \tag{4.6}$$

If we write $\{\omega_k\}$ for the sequence $\{\xi_j\}$ with each ξ_j repeated $[N^2 p_j]$ times, then (4.6) says that

$$\sum_k (1 - |\omega_k|^2)\, \delta_{\omega_k} = \sum_j (1 - |\xi_j|^2)[N^2 p_j]\, \delta_{\xi_j}$$

is a Carleson measure of constant bounded by $N^2 C_0$. This means that $\{\omega_k\}$ is a finite union of interpolating sequences (see [9, p. 314]). Denote by $b(z)$ the Blaschke product with zero sequence $\{\omega_k\}$.

Suppose that $1/2 < r < 1$. If $z \in \mathbb{D}$ is such that $\rho(z, \xi_j) > r$ then (1.3) and the fact that $\operatorname{diam}_\rho D_j^+ \leqslant \delta < 1/4$ imply that $\rho(z, D_j^+) > r_0(r)$, where $r_0 \to 1$ as $r \to 1$. Thus, when $\rho(z, \xi_j) > r$ for every j,

$$\left|\int_{\mathbb{D}} \log \frac{1}{|\varphi_z(\xi)|^2}\, dv^+(\xi) - \frac{1}{N^2} \log \frac{1}{|b(z)|^2}\right|$$

$$\leqslant \sum_j \int_{D_j^+} \left|\log \frac{1}{|\varphi_z(\xi)|^2} - \log \frac{1}{|\varphi_z(\xi_j)|^2}\right| p(\xi)\, dA(\xi)$$

$$+ \sum_j \log \frac{1}{|\varphi_z(\xi_j)|^2} \left(p_j - \frac{[N^2 p_j]}{N^2}\right)$$

$$= I_1 + I_2,$$

where $I_2 \geqslant 0$ by (4.5). By (4.5), (1.2), (4.6), and (4.1),

$$I_2 \leqslant \frac{1}{N} \sum_j p_j \log \frac{1}{|\varphi_z(\xi_j)|^2}$$

$$\leqslant \frac{1}{N}\left(1 + \log \frac{1}{r^2}\right) \sum_j p_j (1 - |\varphi_z(\xi_j)|^2) \leqslant \frac{C_2}{N}, \tag{4.7}$$

where $C_2 > 0$ is an absolute constant.

In order to estimate I_1 we notice that if $\xi \in D_j^+$ then $\rho(\xi, \xi_j) \leqslant \delta < 1/2$. So, if $z \in \mathbb{D}$ is such that $\rho(\xi_j, z)$, $\rho(\xi, z) > r_0$ for every $j \geqslant 1$ then we can apply inequality (4.3), where the roles of z and ξ are interchanged and r_0 takes the place of γ. That is,

$$\left|\log \frac{1}{|\varphi_z(\xi_j)|^2} - \log \frac{1}{|\varphi_z(\xi)|^2}\right| \leqslant \frac{4}{r_0} (1 - \rho(\xi, z)^2)\, \rho(\xi_j, \xi)$$

$$\leqslant \frac{4\delta}{r_0} (1 - |\varphi_z(\xi)|^2). \tag{4.8}$$

In addition, we can assume that r is close enough to 1 so that $r_0 > 1/2$. Therefore when $\rho(z, \xi_j) > r$ for all j, (4.8) gives

$$I_1 \leqslant 8\delta \sum_j \int_{D_j^+} \frac{1 - |z|^2}{|1 - \bar{\xi} z|^2} (1 - |\xi|^2)\, p(\xi)\, dA(\xi) \leqslant C_1 \delta, \tag{4.9}$$

where the last inequality follows from (4.1). By (4.7) and (4.9) the lemma holds for $\log |\varphi_z(\xi)|^{-2}$.

Since (1.2) tells us that $1 - |\varphi_z(\xi)|^2$ behaves like $\log |\varphi_z(\xi)|^{-2}$ when $\rho(\xi, z) > r_0$ for r_0 close to 1, we can take r so close to 1 (and consequently

r_0 so close to 1) that for either of our kernels $q(z, \xi)$ there is an absolute constant $C_3 > 0$, such that

$$\left| \int_{\mathbb{D}} q(z, \xi)\, dv^+(\xi) - \frac{1}{N^2} \log \frac{1}{|b(z)|^2} \right|$$

$$\leqslant C_3 \left(\delta + \frac{1}{N} \right) \qquad \text{when} \quad z \notin \bigcup_j K(\xi_j, r),$$

as the proposition states. ∎

It will follow from our study of the function f_2 that the size of δ has no further influence on the proof of Theorem 4.1. Hence, we can fix now a value of $\delta < \varepsilon/8C$ in (4.4). Also, although N will be determined in the next subsection, we can assume that $N > 8C/\varepsilon$. By Proposition 4.3 there is some $r = r(\delta, N)$ in (0, 1) such that (4.4) holds for $C(\delta + (1/N)) < \varepsilon/4$.

Put $B \stackrel{\text{def}}{=} \bigcup_k K(\omega_k, r)$. Since $\{\omega_k\}$ is the zero sequence of b, and a finite union of interpolating sequences, by Lemma 1.1 there is $\alpha > 0$ such that $|b(z)| > \alpha$ for every $z \in \mathbb{D} \backslash B$. Define

$$h_1(z) = \begin{cases} f_1(z) & \text{if} \quad z \in B \\ N^{-2} \log |b(z)|^{-2} & \text{if} \quad z \notin B. \end{cases}$$

Then (4.4) says that $\|f_1 - h_1\|_\infty \leqslant \varepsilon/4$. By the uniform continuity of f_1 with respect to ρ there is some η in (0, 1/2) such that

$$\rho(z_1, z_2) \leqslant \eta \Rightarrow |h_1(z_1) - h_1(z_2)| < 3\varepsilon/4.$$

Consider the regularization operator $T = T_\eta$. The last condition and (b) of Lemma 3.2 imply that $|Th_1(z) - h_1(z)| < (3/4)\, \varepsilon$ for every $z \in \mathbb{D}$. Consequently $\|Th_1 - f_1\|_\infty \leqslant \varepsilon$. We want to prove that Th_1 can be continuously extended to M_{H^∞}, which by (c) of Lemma 3.2 and Theorem 2.1 reduces to proving that it can be extended to some neighborhood of Γ, the set of trivial points. By (1.3) the set $B_\eta \stackrel{\text{def}}{=} \{z \in \mathbb{D} : \rho(z, B) \leqslant \eta\}$ is contained in $\bigcup_k K(\omega_k, R)$, where $R = (r + \eta)/(1 + r\eta)$. Thus, if $z_0 \in \mathbb{D}$ satisfies $\rho(z_0, B) > R$ then $K(z_0, \eta) \subset \mathbb{D} \backslash B$, and consequently $h_1(z)$ coincides with the harmonic function $N^{-2} \log |b(z)|^{-2}$ on $K(z_0, \eta)$. By Lemma 3.2(a) then $Th_1(z) = N^{-2} \log |b(z)|^{-2}$ on $\mathbb{D} \backslash B_\eta$. Since $\{\omega_k\}$ is a finite union of interpolating sequences, Lemma 1.2 says that the closure of B_η in M_{H^∞} is contained in $\mathcal{G}$. Therefore $M_{H^\infty} \backslash \overline{B_\eta}$ is an open neighborhood of Γ. Since $\log |b(z)|^{-2}$ extends continuously to $M_{H^\infty} \backslash \overline{B_\eta}$, so does Th_1. Therefore

$$\operatorname{dist}(f_1, C(M_{H^\infty}, \mathbb{C})) \leqslant \varepsilon \tag{4.10}$$

for both possible choices of $q(z, \xi)$.

4.3. *Getting Close to $f_2(z)$*

In the previous subsection we fixed a value of δ and set a lower bound for N. We will ask for N to satisfy several further requirements, beginning with the following lemma.

LEMMA 4.4. *Let $\eta > 0$ and $0 < r < 1$. Then there is $N = N(r, \eta)$ big enough so that $\nu^-(K(z, r)) < \eta$ for every $z \in \mathbb{D}$.*

Proof. Since by construction the sets D_j^- are pairwise disjoint and each of them contains a pseudohyperbolic ball of radius $\delta/4$, there is a positive integer $M = M(\delta, r)$ such that every ball $K(z, r)$ meets at most M of the sets D_j^-. Since $\nu^-(D_j^-) < 1/N$ for all j, then

$$\nu^-(K(z, r)) \leqslant \sum_{D_j^- \cap K(z, r) \neq \varnothing} \nu^-(D_j^-) < M/N$$

for every $z \in \mathbb{D}$. So, it is enough to take $N > M/\eta$. ∎

We show now that the study of the function $f_2(z) = \int q(z, \xi)\, d\nu^-(\xi)$, when $q(z, \xi)$ is either of our two kernels, reduces to the case $q(z, \xi) = 1 - |\varphi_z(\xi)|^2$. In fact, by (4.1) $\int (1 - |\varphi_z(\xi)|^2)\, d\nu^-(\xi) \leqslant C\, \|(1 - |\xi|^2) d\nu^-\|_* \leqslant C$. Hence (1.2) implies that

$$\left| \int_{\rho(z, \xi) > r} \log \frac{1}{|\varphi_z(\xi)|^2}\, d\nu^-(\xi) - \int_{\rho(z, \xi) > r} 1 - |\varphi_z(\xi)|^2\, d\nu^-(\xi) \right| \leqslant C \log \frac{1}{r^2} < \frac{\varepsilon}{2} \tag{4.11}$$

if $r = r(\varepsilon)$ is close enough to 1. Applying Lemma 4.2 to the measure ν^- we conclude that there is a $\gamma = \gamma(\varepsilon)$ in $(0, r)$ sufficiently small that

$$\int_{K(z, r)} \log \frac{1}{|\varphi_z(\xi)|^2} + (1 - |\varphi_z(\xi)|^2)\, d\nu^-(\xi) < \frac{\varepsilon}{4} + \left(2 + \log \frac{1}{\gamma^2}\right) \nu^-(K(z, r)). \tag{4.12}$$

With r and γ fixed we can take $N = N(r, \gamma, \varepsilon)$ according to Lemma 4.4 so big that the right member of (4.12) is bounded by $\varepsilon/2$. This estimate and (4.11) then give

$$\left| \int_{\mathbb{D}} \log \frac{1}{|\varphi_z(\xi)|^2}\, d\nu^-(\xi) - \int_{\mathbb{D}} 1 - |\varphi_z(\xi)|^2\, d\nu^-(\xi) \right| < \varepsilon. \tag{4.13}$$

That is, from now on we can assume that f_2 corresponds to the kernel $q(z, \xi) = 1 - |\varphi_z(\xi)|^2$.

Write $d\mu(\xi) = (1 - |\xi|^2)\, dv^-(\xi)$. By our initial assumptions μ is a Carleson measure with $\|\mu\|_* \leqslant 1$.

LEMMA 4.5. *Let $n \geqslant 2$ be an integer. Then there is $N = N(n, \varepsilon)$ big enough so that*

$$\left| \int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-\bar{\xi}z|^2}\, d\mu(\xi) - \int_{(1-|\xi|) \leqslant (1-|z|)/n} \frac{(1-|z|^2)}{|1-\bar{\xi}z|^2}\, d\mu(\xi) \right| < \frac{\varepsilon}{4}. \quad (4.14)$$

Proof. Suppose that $|z| > 3/4$, and for $k \geqslant 2$ integer let

$$E_k = \{\xi \in \mathbb{D} : |\xi - (z/|z|)| < 2^k (1 - |z|)\}.$$

Then $\mu(E_k) \leqslant 2^{k+1}(1 - |z|)$ and

$$\frac{1-|z|^2}{|1-\bar{\xi}z|^2} < \frac{1}{4^{k-2}(1-|z|)} \qquad \text{for} \quad \xi \notin E_k.$$

So, if $k_0 \geqslant 2$ is an integer,

$$\begin{aligned} \int_{\mathbb{D} \setminus E_{k_0}} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\, d\mu(\xi) &\leqslant \sum_{k=k_0}^{\infty} \int_{E_{k+1} \setminus E_k} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\, d\mu(\xi) \\ &\leqslant \sum_{k=k_0}^{\infty} \frac{2^{k+2}}{4^{k-2}} = \frac{4^3}{2^{k_0-1}} < \frac{\varepsilon}{8} \end{aligned} \quad (4.15)$$

if we take $k_0 = k_0(\varepsilon)$ big enough.

The pseudohyperbolic diameter of the set $E = \{\xi \in E_{k_0} : 1 - |\xi| > (1 - |z|)/n\}$ is bounded away from 1 by a constant depending only on k_0 and n (i.e.: independent of z). So, by Lemma 4.4 we can take N so big that

$$\int_E \frac{(1-|z|^2)}{|1-\bar{\xi}z|^2}\, d\mu(\xi) = \int_E (1 - \rho(z, \xi)^2)\, dv^-(\xi) \leqslant v^-(E) \leqslant \frac{\varepsilon}{8}. \quad (4.16)$$

From (4.15) and (4.16) we get

$$\int_{(1-|\xi|) > (1-|z|)/n} \frac{(1-|z|^2)}{|1-\bar{\xi}z|^2}\, d\mu(\xi) < \frac{\varepsilon}{4} \qquad \text{when} \quad |z| > 3/4,$$

as claimed. If $|z| \leqslant 3/4$ and $1 - |\xi| > (1 - |z|)/n$, then $|\xi| < 1 - (1/4n)$, meaning that $\xi \in K(0, 1 - \frac{1}{4n})$. A new application of Lemma 4.4 says that there is $N = N(n, \varepsilon)$ such that $v^-(K(0, 1 - \frac{1}{4n})) < \varepsilon/8$. Hence, when $|z| \leqslant 3/4$ we have an inequality like (4.16), with $K(0, 1-\frac{1}{4n})$ in the place of E, and the lemma follows. ∎

Let $I=[\theta_0, \theta_0+2\pi r]$ be an interval, where $0 \leqslant r \leqslant 1$ and $0 \leqslant \theta_0 \leqslant 2\pi$. The normalized Lebesgue measure of I is $|I|=r$. In [9, p. 350] it is shown that if $u(z)$ is the Poisson integral of a bounded function $u(\theta)$ and $n_0 \geqslant 2$ is an integer then

$$\left| \frac{1}{|I|} \int_I u(\theta) \frac{d\theta}{2\pi} - \frac{1}{|I|} \int_I u\left(\left(1-\frac{|I|}{n_0}\right) e^{i\theta}\right) \frac{d\theta}{2\pi} \right| \leqslant C \, \|u\|_\infty \frac{\log n_0}{n_0}, \quad (4.17)$$

where $C>0$ is an absolute constant. Our next lemma shows that $f_2(z)$ satisfies a similar inequality when N is sufficiently large. The proof reduces our situation to the harmonic case and then applies (4.17). Let $n_0 \geqslant 2$ be an integer to be determined later, but large enough so that in (4.17) we have

$$C \frac{\log n_0}{n_0} < \frac{\varepsilon}{4}. \quad (4.18)$$

Consider the family of intervals

$$\mathscr{I} = \left\{ \left[2\pi \frac{j-1}{n_0^k}, 2\pi \frac{j}{n_0^k} \right] : 1 \leqslant j \leqslant n_0^k, k \geqslant 0 \right\},$$

and for each $I \in \mathscr{I}$, $I \neq [0, 2\pi]$, let

$$Q(I) = \{ z \in \mathbb{D} : 1-|I| \leqslant |z| < 1, z/|z| = e^{i\theta} \text{ with } \theta \in I \},$$

and $Q([0, 2\pi]) = \mathbb{D}$.

Lemma 4.6. *There are $n_0 = n_0(\varepsilon)$ and $N = N(n_0, \varepsilon)$ big enough so that*

$$\left| \frac{1}{|I|} \int_I f_2\left(\left(1-\frac{|I|}{n_0}\right) e^{i\theta}\right) \frac{d\theta}{2\pi} - \frac{\mu(Q(I))}{|I|} \right| < \varepsilon \quad (4.19)$$

for every $I \in \mathscr{I}$.

Proof. Although n_0 will be determined during the present proof, we can take $N = N(n_0, \varepsilon)$ according to Lemma 4.5 big enough so that (4.14) holds for $n = n_0$.

Given $I \in \mathscr{I}$, let $I_j \in \mathscr{I}$ be the intervals of length $|I_j| = |I|/n_0^2$, where $1 \leqslant j \leqslant n_0^2/|I|$. Suppose that $|z| = 1-(|I|/n_0)$. If $\xi \in \mathbb{D}$, then $\xi \in Q(I_j)$ for some j if and only if $1-|\xi| \leqslant |I_j| = |I|/n_0^2 = (1-|z|)/n_0$. Therefore $\{\xi \in \mathbb{D} : (1-|\xi|) \leqslant (1-|z|)/n_0\} = \bigcup_j Q(I_j)$, and (4.14) says that

$$\left| f_2(z) - \int_{\bigcup_j Q(I_j)} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\, d\mu(\xi) \right| < \frac{\varepsilon}{4}. \tag{4.20}$$

When $\xi \in Q(I_j)$ and $t \in I_j$ we have

$$|e^{it}-\xi| \leqslant 2\,|I_j| = \frac{2}{n_0}\,\frac{|I|}{n_0} = \frac{2}{n_0}(1-|z|) \leqslant \frac{2}{n_0}\,|e^{it}-z|, \tag{4.21}$$

and consequently $|1-\bar{\xi}z| \leqslant |1-e^{-it}z| + |e^{it}-\xi|\,|z| \leqslant 2\,|1-e^{-it}z|$. This yields

$$\begin{aligned} &\big|\,|1-e^{-it}z|^2 - |1-\bar{\xi}z|^2\big| \\ &\quad = \big|\,|1-e^{-it}z| - |1-\bar{\xi}z|\,\big|\,(|1-e^{-it}z| + |1-\bar{\xi}z|) \\ &\quad \leqslant |e^{it}-\xi|\; 3\,|1-e^{-it}z| \overset{\text{by (4.21)}}{\leqslant} (6/n_0)\,|1-e^{-it}z|^2. \end{aligned}$$

Multiplying the above inequality by $(1-|z|^2)\,|1-e^{-it}z|^{-2}\,|1-\bar{\xi}z|^{-2}$ we get

$$\left| \frac{1-|z|^2}{|1-\bar{\xi}z|^2} - \frac{1-|z|^2}{|1-e^{-it}z|^2} \right| \leqslant \frac{6}{n_0}\,\frac{1-|z|^2}{|1-\bar{\xi}z|^2}.$$

Therefore,

$$\begin{aligned} &\left| \int_{\bigcup Q(I_j)} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\, d\mu(\xi) - \int_0^{2\pi} \frac{1-|z|^2}{|1-e^{-it}z|^2} \left(\sum_j \frac{\mu(Q(I_j))}{|I_j|}\, \chi_{I_j}(t) \right) \frac{dt}{2\pi} \right| \\ &\quad = \left| \sum_j \int_{Q(I_j)} \left[\frac{1-|z|^2}{|1-\bar{\xi}z|^2} - \frac{1}{|I_j|} \int_0^{2\pi} \frac{1-|z|^2}{|1-e^{-it}z|^2}\, \chi_{I_j}(t)\, \frac{dt}{2\pi} \right] d\mu(\xi) \right| \\ &\quad \leqslant \sum_j \int_{Q(I_j)} \left[\frac{1}{|I_j|} \int_{I_j} \left| \frac{1-|z|^2}{|1-\bar{\xi}z|^2} - \frac{1-|z|^2}{|1-e^{-it}z|^2} \right| \frac{dt}{2\pi} \right] d\mu(\xi) \\ &\quad \leqslant \frac{6}{n_0} \sum_j \int_{Q(I_j)} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\, d\mu(\xi) \leqslant \frac{6}{n_0} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\, d\mu(\xi) \\ &\quad \leqslant \frac{C_1}{n_0}\, \|\mu\|_* < \frac{\varepsilon}{4} \end{aligned} \tag{4.22}$$

if $n_0 > 4C_1/\varepsilon$. So, we fix a value of n_0 satisfying (4.18) and (4.22).

The second integral in the first member of (4.22) is the Poisson integral $u(z)$ of the bounded function $u(t)=\sum_j (\mu(Q(I_j))/|I_j|)\,\chi_{I_j}(t)$. Since $\|u\|_{L^\infty(\partial\mathbb{D})}\leqslant \|\mu\|_*\leqslant 1$ and

$$\frac{1}{|I|}\int_I u(t)\,\frac{dt}{2\pi}=\frac{1}{|I|}\sum_{I_j\subset I}\mu(Q(I_j))=\frac{\mu(\bigcup_{I_j\subset I}Q(I_j))}{|I|},$$

then (4.17) and (4.18) say that

$$\left|\frac{\mu(\bigcup_{I_j\subset I}Q(I_j))}{|I|}-\frac{1}{|I|}\int_I u((1-(|I|/n_0))\,e^{i\theta})\,\frac{d\theta}{2\pi}\right|<\frac{\varepsilon}{4}. \tag{4.23}$$

Let $S_I=Q(I)\backslash\bigcup_{I_j\subset I}Q(I_j)$. The pseudohyperbolic diameter of S_I is bounded away from 1 by a constant depending only on n_0 (i.e., not depending on I). Thus, by Lemma 4.4 we can choose N so big that $v^-(S_I)<\varepsilon/8$. This is the last restriction that we impose on N. Since for every $\xi\in S_I$ we have $1-|\xi|\leqslant|I|$ then

$$0\leqslant\frac{\mu(Q(I))}{|I|}-\frac{\mu(\bigcup_{I_j\subset I}Q(I_j))}{|I|}=\frac{\mu(S_I)}{|I|}$$
$$=\frac{1}{|I|}\int_{S_I}(1-|\xi|^2)\,dv^-(\xi)\leqslant 2v^-(S_I)<\frac{\varepsilon}{4}.$$

Hence, the lemma follows from the above inequality, (4.23), (4.22), and (4.20). ∎

4.4. *Generations*

We have fixed n_0 and N such that (4.19) holds for every $I\in\mathcal{I}$. Our next key result, Lemma 4.8, will make use of the following elementary lemma.

LEMMA 4.7. *Let m be a positive integer, $0<\alpha<1$ and $0\leqslant\alpha_j\leqslant\alpha$, for $j=1,\ldots,m$. Then*

$$\alpha_1+\alpha_2(1-\alpha_1)+\alpha_3(1-\alpha_2)(1-\alpha_1)+\cdots$$
$$+\alpha_m(1-\alpha_{m-1})\cdots(1-\alpha_1)\leqslant 1-(1-\alpha)^m.$$

Proof. Write A for the left expression in the above inequality. Regrouping terms we have

$$A=\alpha_1+(1-\alpha_1)[\alpha_2+(1-\alpha_2)[\cdots[\alpha_{m-1}+(1-\alpha_{m-1})\,\alpha_m]\cdots]].$$

Consider the functions $\beta_m(x_m)=x_m$ and $\beta_j(x_j, ..., x_m)=x_j+(1-x_j)\,\beta_{j+1}(x_{j+1}, ..., x_m)$, where $j=1, ..., m-1$. We will use induction to prove that

$$\beta_j(\alpha_j, ..., \alpha_m)\leqslant\beta_j(\alpha, ..., \alpha)<1$$

for every $1\leqslant j\leqslant m$. The inequalities hold for $j=m$ because $\alpha_m\leqslant\alpha<1$. Assuming that the inequalities hold for some j with $1<j\leqslant m$, we obtain

$$\begin{aligned}\beta_{j-1}(\alpha_{j-1}, ..., \alpha_m)&=\alpha_{j-1}+(1-\alpha_{j-1})\,\beta_j(\alpha_j, ..., \alpha_m)\\&\leqslant\alpha_{j-1}+(1-\alpha_{j-1})\,\beta_j(\alpha, ..., \alpha)\\&\leqslant\alpha+(1-\alpha)\,\beta_j(\alpha, ..., \alpha)<1.\end{aligned}$$

Since $A=\beta_1(\alpha_1, ..., \alpha_m)$, then

$$A\leqslant\alpha+\alpha(1-\alpha)+\alpha(1-\alpha)^2+\cdots+\alpha(1-\alpha)^{m-1}=1-(1-\alpha)^m,$$

as the lemma states. ∎

It will be convenient to write $\{Q_k\colon k\geqslant 0\}$ for the family $\mathscr{F}=\{Q(I)\colon I\in\mathscr{I}\}$, with $Q_0=\mathbb{D}$. So, $\ell(Q_k)=|I|$ when $Q_k=Q(I)$ for some $I\in\mathscr{I}$.

Let $m>1/\varepsilon$ be a positive integer. Beginning with Q_0 we define successive generations of squares in $\mathscr{F}$ by a stopping time argument. The zero generation is $G_0=G_0(Q_0)=\{Q_0\}$,

$$G_1=G_1(Q_0)=\left\{Q_k\subset Q_0\colon\left|\frac{\mu(Q_k)}{\ell(Q_k)}-\frac{\mu(Q_0)}{\ell(Q_0)}\right|>\frac{1}{m},\ \text{with } Q_k \text{ maximal}\right\},$$

and for $p\geqslant 2$, $G_p=G_p(Q_0)=\bigcup\{G_1(Q_k) : Q_k\in G_{p-1}\}$. For $Q_k\in G_p$ $(p\geqslant 0)$ write $R(Q_k)=Q_k\setminus\bigcup G_1(Q_k)$. Then the family

$$\{R(Q_k)\colon Q_k\in G_p,\ p\geqslant 0\}$$

covers $\mathbb{D}$ and has pairwise disjoint interiors.

Lemma 4.8. *The arc length of the curve*

$$\Lambda\overset{\text{def}}{=}\bigcup\{\partial R(Q_k)\colon Q_k\in G_p,\ p\geqslant 0\}$$

is a Carleson measure.

Proof. Let λ be the measure induced by the arc length of Λ. Since every square in $\mathbb{D}$ can be covered by at most two squares in $\mathscr{F}$ of comparable size, it will be enough to prove that $\lambda(Q)/\ell(Q)$ is bounded for every $Q\in\mathscr{F}$. Moreover, since for such a square Q we have that $\lambda(Q)$ is bounded by $3\ell(Q)+\sum\lambda(Q_k)$, where the sum runs over all the maximal $Q_k\subset Q$ that belong to some generation G_p, it is enough to show that $\lambda(Q_k)/\ell(Q_k)$ is bounded for all the squares Q_k in some generation. Finally, if $Q_k \in G_q$ then

$$\lambda(Q_k) \leqslant \sum_{p \geqslant 0} \sum_{Q_n \in G_p(Q_k)} 3\ell(Q_n), \tag{4.24}$$

which reduces the proof of the lemma to proving that the last sum is bounded by $C\ell(Q_k)$, where $C > 0$ does not depend on Q_k.

Let Q be a square in some generation G_q. We will group together portions of the subgenerations $G_p(Q)$ (for $p \geqslant 1$) by using a refinement of the initial stopping time process. If $P \in G_1(Q)$ then either

$$\frac{\mu(P)}{\ell(P)} > \frac{\mu(Q)}{\ell(Q)} + \frac{1}{m} \tag{4.25}$$

or

$$\frac{\mu(P)}{\ell(P)} < \frac{\mu(Q)}{\ell(Q)} - \frac{1}{m}. \tag{4.26}$$

Let $G_1^+(Q) = \{P \in G_1(Q) : P \text{ satisfies (4.25)}\}$ and $G_1^-(Q) = \{P \in G_1(Q) : P \text{ satisfies (4.26)}\}$. Assuming recursively that $G_{p-1}^-(Q)$ is given, define

$$G_p^+(Q) = \bigcup_{P \in G_{p-1}^-(Q)} G_1^+(P) \qquad \text{and} \qquad G_p^-(Q) = \bigcup_{P \in G_{p-1}^-(Q)} G_1^-(P).$$

Figure 1 shows some of the squares in the signed generations for Q when $n_0 = 3$. Shadowed squares belong to plus generations.

Observe that when we pass from some $P_1 \in G_{p-1}^-(Q)$ to some $P_2 \in G_p^-(Q)$, with $P_2 \in G_1^-(P_1)$, (4.26) gives

$$0 \leqslant \frac{\mu(P_2)}{\ell(P_2)} < \frac{\mu(P_1)}{\ell(P_1)} - \frac{1}{m} < \left(\frac{\mu(Q)}{\ell(Q)} - \frac{p-1}{m}\right) - \frac{1}{m} \leqslant 1 - \frac{p}{m},$$

which means that $p \leqslant m$. That is, the minus generations are exhausted after at most m steps, and consequently the plus generations are exhausted after at most $m+1$ steps. In particular, all the signed generations are contained in $\bigcup_{j=1}^{m+1} G_j(Q)$. By accepting the empty set as a possible $+$ or $-$ generation we can assume that there are m minus generations and $m+1$ plus generations. This will avoid inessential considerations of particular cases. Hence,

$$\ell(Q) + \sum_{p=1}^{m} \sum_{P \in G_p^-(Q)} \ell(P) \leqslant (m+1)\, \ell(Q). \tag{4.27}$$

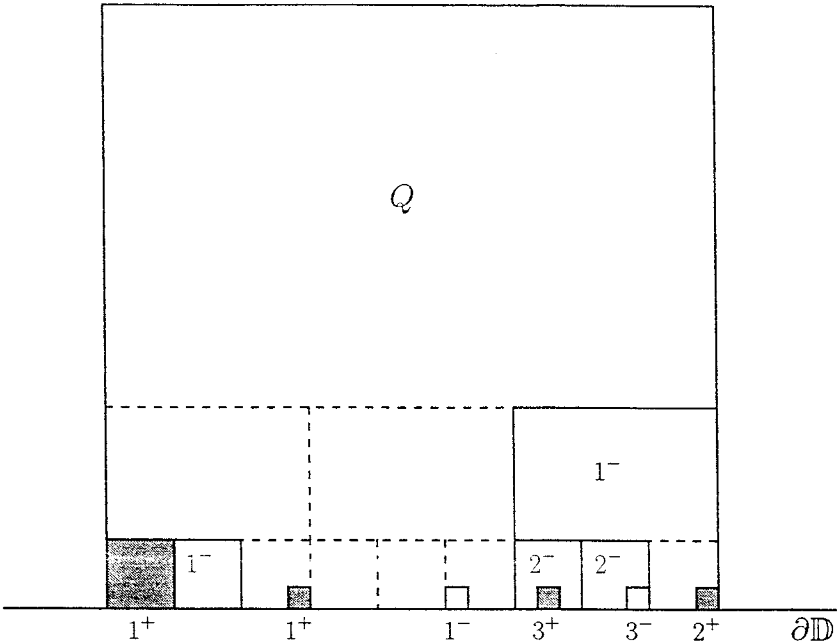

FIG. 1. Here p^+ and p^- indicate that the corresponding square is in $G_p^+(Q)$ or $G_p^-(Q)$.

If $G_1^+(Q) \neq \emptyset$ then it is clear that $\mu(Q) > 0$. Since $\|\mu\|_* \leqslant 1$, (4.25) implies that

$$\mu(Q) \geqslant \sum_{P \in G_1^+(Q)} \mu(P) > \sum_{P \in G_1^+(Q)} \left(\frac{\mu(Q)}{\ell(Q)} + \frac{1}{m}\right) \ell(P)$$
$$\geqslant \frac{\mu(Q)}{\ell(Q)}\left(1 + \frac{1}{m}\right) \sum_{P \in G_1^+(Q)} \ell(P).$$

So, if $\alpha \stackrel{\text{def}}{=} m/(m+1) < 1$, then

$$\sum_{P \in G_1^+(Q)} \ell(P) \leqslant \alpha \ell(Q), \tag{4.28}$$

and this inequality also holds when $G_1^+(Q) = \emptyset$. Hence, $\sum_{P \in G_1^+(Q)} \ell(P) = \alpha_1 \ell(Q)$ with $\alpha_1 \leqslant \alpha$, and since the squares in $G_1^+(Q)$ are essentially disjoint (i.e.: except for sets of null area) from the squares in $G_1^-(Q)$, $\sum_{P \in G_1^-(Q)} \ell(P) \leqslant (1 - \alpha_1)\, \ell(Q)$. So, since every square in $G_2^+(Q)$ is in $G_1^+(P)$ for some $P \in G_1^-(Q)$, there is $\alpha_2 \leqslant \alpha$ such that $\sum_{P \in G_2^+(Q)} \ell(P) = \alpha_2(1 - \alpha_1)\, \ell(Q)$. This comes from applying (4.28) to the appropriate squares.

Let $2 \leqslant j \leqslant m$. Since the squares in $G_j^-(Q)$ are essentially disjoint from the squares in $G_j^+(Q)$, and both signed generations are contained in $G_{j-1}^-(Q)$ then

$$\sum_{P \in G_j^-(Q)} \ell(P) \leqslant \sum_{P \in G_{j-1}^-(Q)} \ell(P) - \sum_{P \in G_j^+(Q)} \ell(P), \tag{4.29}$$

which for $j=2$ yields

$$\sum_{P \in G_2^-(Q)} \ell(P) \leqslant (1-\alpha_2)(1-\alpha_1)\, \ell(Q).$$

This process keeps going by alternating (4.28) and (4.29) one after the other, therefore obtaining $m+1$ numbers $0 \leqslant \alpha_1, ..., \alpha_{m+1} \leqslant \alpha$ such that

$$\sum_{P \in G_j^+(Q)} \ell(P) = \alpha_j (1-\alpha_{j-1}) \cdots (1-\alpha_1)\, \ell(Q)$$

for every $j \leqslant m+1$, and

$$\sum_{P \in G_j^-(Q)} \ell(P) \leqslant (1-\alpha_j)(1-\alpha_{j-1}) \cdots (1-\alpha_1)\, \ell(Q)$$

for every $j \leqslant m$. Consequently, Lemma 4.7 gives

$$\begin{aligned} &\sum_{j=1}^{m+1} \sum_{P \in G_j^+(Q)} \ell(P) \\ &\quad = [\alpha_1 + \alpha_2(1-\alpha_1) + \cdots + \alpha_{m+1}(1-\alpha_m) \cdots (1-\alpha_1)]\, \ell(Q) \\ &\quad \leqslant [1-(1-\alpha)^{m+1}]\, \ell(Q) = \beta \ell(Q), \end{aligned} \tag{4.30}$$

where $\beta = 1-(m+1)^{-(m+1)} < 1$.

If $P \in G_p(Q)$ for some $p \geqslant 1$ then there are two mutually exclusive possibilities: either $P \in G_j^-(Q)$ for some $1 \leqslant j \leqslant m$, or P is contained in some $Q' \in G_j^+(Q)$ for some $1 \leqslant j \leqslant m+1$. In the first case P is one of the squares in the sum of (4.27). Otherwise P is one of the squares in the sum

$$\sum_{p \geqslant 0} \sum_{P' \in G_p(Q')} \ell(P'),$$

where $Q' \in G_j^+(Q)$ for some $1 \leqslant j \leqslant m+1$. Since the same argument used for Q (decomposition into signed generations) now can be applied to every square $Q' \in G_j^+(Q)$ (for $1 \leqslant j \leqslant m+1$), then (4.27) and (4.30) yield

$$\sum_{p \geqslant 0} \sum_{Q_n \in G_p(Q)} \ell(Q_n)$$
$$\leqslant (m+1)\, \ell(Q) + (m+1)\, \beta\ell(Q) + (m+1)\, \beta^2\ell(Q) + \cdots$$
$$= \frac{m+1}{1-\beta}\, \ell(Q) = (m+1)^{m+2}\, \ell(Q).$$

Thus, (4.24) says that $\lambda(Q) \leqslant 3(m+1)^{m+2}\, \ell(Q)$, proving the lemma. ∎

If $Q = Q(I) \in \mathscr{F}$, put $S(Q) = \{z \in Q : |z| \leqslant 1 - (\ell(Q)/n_0)\}$ for the "top" $(n_0 - 1)/n_0$ portion of Q, and

$$a(Q) \stackrel{\text{def}}{=} \frac{1}{|I|} \int_I f_2\left(\left(1 - \frac{|I|}{n_0}\right) e^{i\theta}\right) \frac{d\theta}{2\pi}$$

for the average of f_2 over the "base" of $S(Q)$. Thus, (4.19) says that $|a(Q) - \mu(Q)/\ell(Q)| < \varepsilon$.

Define a function h_1 on $\bigcup\{R(Q_k)^\circ : Q_k \in G_p,\ p \geqslant 0\}$ by $h_1(z) = \mu(Q_k)/\ell(Q_k)$ when $z \in R(Q_k)^\circ$ (here E° denotes the interior of E). Then h_1 is constant on each connected component of $\mathbb{D} \backslash \Lambda$, where Λ is the curve of Lemma 4.8. Suppose that $z \in S(Q)^\circ$ for some $Q \in \mathscr{F}$, and that $S(Q)^\circ$ is contained in the region $R(Q_k)$. Then $h_1(z) = \mu(Q_k)/\ell(Q_k)$, and by construction of the region $R(Q_k)$ we have $|h_1(z) - \mu(Q)/\ell(Q)| < 1/m < \varepsilon$. So, $|h_1(z) - a(Q)| < 2\varepsilon$.

Let $Q \in \mathscr{F}$. As in [9, p. 353], we say that the (circular) rectangle $S(Q)$ is blue if $\sup_{S(Q)} |f_2(z) - f_2(\omega)| \leqslant \varepsilon$. Otherwise we say that $S(Q)$ is red. Therefore,

$$|h_1(z) - f_2(z)| \leqslant 3\varepsilon \qquad \text{when} \quad z \in S(Q)^\circ \quad \text{with } S(Q) \text{ blue.} \tag{4.31}$$

Now we turn our attention to the red rectangles.

4.5. *The Region of Red Rectangles*

We need some preliminary results. The next lemma follows immediately from the proof of Theorem 3.1 in [13].

LEMMA 4.9 (Nicolau–Xiao). *Let* $g(z) = \int |1 - \bar{\xi}z|^{-2}\, G(\xi)\, dA(\xi)$, *where* G *is a positive function on* $\mathbb{D}$ *such that*

$$G(\xi)\, dA(\xi) \qquad \textit{and} \qquad (1 - |\xi|^2)\, G(\xi)^2\, dA(\xi) \textit{ are Carleson measures.}$$

Then $(1 - |z|^2)\, g(z)^2\, dA(z)$ *is a Carleson measure.*

A straightforward calculation shows that

$$\frac{|\nabla f_2(z)|}{10} \leqslant v(z) \stackrel{\text{def}}{=} \int \frac{1}{|1-\bar{\xi}z|^2}\, d\mu(\xi). \tag{4.32}$$

Since $d\mu(\xi) = (1-|\xi|^2)\, p(\xi)\, \chi_{L^-}(\xi)\, dA(\xi)$ is a Carleson measure and $(1-|\xi|^2)^2\, p(\xi)$ is bounded, Lemma 4.9 implies that $(1-|z|^2)\, v(z)^2\, dA(z)$ is a Carleson measure. In addition, we will see in the next lemma that v^2 satisfies a type of subharmonicity condition.

LEMMA 4.10. *There is an absolute constant $C>0$ such that*

$$v(z_0)^2 \leqslant \frac{C}{(1-|z_0|)^2} \int_{|z-z_0|<(1-|z_0|)/2} v(z)^2\, dA(z). \tag{4.33}$$

Proof. If z_0, $z \in \mathbb{D}$ are such that $|z-z_0| < (1-|z_0|)/2$, and $\xi \in \mathbb{D}$, then

$$|1-\bar{\xi}z| \leqslant |1-\bar{\xi}z_0| + |\xi|\, |z_0 - z| \leqslant |1-\bar{\xi}z_0| + \frac{(1-|z_0|)}{2} \leqslant \frac{3}{2}\, |1-\bar{\xi}z_0|.$$

We recall that $dA(\rho e^{it}) = \rho\, d\rho\, \frac{dt}{2\pi}$. Therefore, if $r = (1-|z_0|)/2$,

$$\frac{1}{|1-\bar{\xi}z_0|^2} \leqslant \frac{9}{2r^2} \int_{|z-z_0|<r} \frac{1}{|1-\bar{\xi}z|^2}\, dA(z).$$

Integrating the above inequality with respect to $d\mu(\xi)$ and using Fubini's theorem, we get

$$v(z_0) \leqslant \frac{C'}{(1-|z_0|)^2} \int_{|z-z_0|<(1-|z_0|)/2} v(z)\, dA(z).$$

The lemma now follows from the Cauchy–Schwarz inequality. ∎

Lemma 4.10 and the comments that follow Lemma 4.9 allow us to apply Garnett's argument to our situation almost word for word. Let $Q_k \in \mathscr{F}$, and suppose that $S(Q_k)$ is red, so,

$$\sup_{S(Q_k)} |f_2(z) - f_2(\omega)| > \varepsilon.$$

Since every pair of points of $S(Q_k)$ can be connected by three segments inside $S(Q_k)$, then there is a segment $[z_1, z_2] \subset S(Q_k)$ such that $|f_2(z_1) - f_2(z_2)| > \varepsilon/3$. Then there is a point z_0 in $[z_1, z_2]$ such that

$$|z_1 - z_2|\, |\nabla f_2(z_0)| > \varepsilon/3.$$

Since $|z_1 - z_2| \leqslant 2\pi\ell(Q_k) \leqslant 2\pi n_0(1-|z_0|)$ and by (4.32) $10v(z_0) \geqslant |\nabla f_2(z_0)|$, then

$$20\pi n_0(1-|z_0|)\, v(z_0) > \varepsilon/3. \tag{4.34}$$

Writing as before $r = (1-|z_0|)/2$, we have

$$\begin{aligned}\int_{|z-z_0|<r} (1-|z|)\, v(z)^2\, dA(z) &\geqslant \tfrac{1}{2}(1-|z_0|) \int_{|z-z_0|<r} v(z)^2\, dA(z) \\ &\overset{\text{by (4.33)}}{\geqslant} C_0(1-|z_0|)^3\, v(z_0)^2 \\ &\overset{\text{by (4.34)}}{\geqslant} C_1 n_0^{-2}(1-|z_0|)\, \varepsilon^2.\end{aligned}$$

Letting $\tilde{S}(Q_k) = \{z\colon |z-z_0| < (1-|z_0|)/2 \text{ for some } z_0 \in S(Q_k)\}$, we obtain

$$\int_{\tilde{S}(Q_k)} (1-|z|)\, v(z)^2\, dA(z) \geqslant Cn_0^{-2}\ell(Q_k)\, \varepsilon^2. \tag{4.35}$$

Let $Q \subset \mathbb{D}$ be any circular square. Since every point lies in at most four regions $\tilde{S}(Q_k)$ and $(1-|z|)\, v(z)^2\, dA(z)$ is a Carleson measure, (4.35) yields

$$\sum \{\ell(Q_k)\colon Q_k \subset Q,\ S(Q_k) \text{ red}\} \leqslant C'\varepsilon^{-2} n_0^2 \ell(Q).$$

In particular, if $\mathscr{R} \overset{\text{def}}{=} \bigcup\{S(Q_k)\colon S(Q_k) \text{ red}\}$, the arc length of $\partial\mathscr{R}$ is a Carleson measure. On the other hand, since $(1-|z|^2)\, v(z) = f_2(z)$ then $\|(1-|z|)\, v(z)\|_\infty \leqslant c$ (and absolute constant), and consequently

$$\int_{S(Q_k)} v(z)\, dA(z) \leqslant \|(1-|z|)\, v(z)\|_\infty \int_{S(Q_k)} \frac{1}{1-|z|}\, dA(z) \leqslant Cn_0 \ell(Q_k)$$

for any $S(Q_k)$. For $S(Q_k)$ red, this inequality and (4.35) give

$$\int_{S(Q_k)} v\, dA \leqslant C\varepsilon^{-2} n_0^3 \int_{\tilde{S}(Q_k)} (1-|z|)\, v(z)^2\, dA(z),$$

implying that $v(z)\, \chi_{\mathscr{R}}(z)\, dA(z)$ is a Carleson measure. Hence, (4.32) tells us that the measure $|\nabla(f_2|_{\mathscr{R}^\circ})|\, \chi_{\mathscr{R}}\, dA$ is Carleson.

Let $h = f_2\chi_{\mathscr{R}} + h_1\chi_{\mathbb{D}\setminus\mathscr{R}} \in L^\infty(dA)$, where h_1 is the function in (4.31). If Λ is the curve of Lemma 4.8, then the arc lengths of Λ and $\partial\mathscr{R}$ are Carleson measures. Therefore so is the arc length of

$$\Lambda' \overset{\text{def}}{=} \partial\mathscr{R} \cup (\Lambda\setminus\mathscr{R}).$$

Clearly h is continuously differentiable on any connected component of $\mathbb{D}\backslash\Lambda'$, and since

$$|\nabla(h|_{\mathbb{D}\backslash\Lambda'})| \leqslant |\nabla(f_2|_{\mathbb{D}\backslash\Lambda'})|\,\chi_{\mathscr{R}},$$

then $|\nabla(h|_{\mathbb{D}\backslash\Lambda'})|\,dA$ is a Carleson measure. Let $z\in S(Q_k)^\circ$. If $S(Q_k)$ is blue then (4.31) says that $|h(z)-f_2(z)|=|h_1(z)-f_2(z)|\leqslant 3\varepsilon$, and if $S(Q_k)$ is red then $h(z)=f_2(z)$. Therefore

$$\operatorname*{ess\,sup}_{\mathbb{D}} |h-f_2| \leqslant 3\varepsilon. \tag{4.36}$$

Since f_2 is uniformly continuous with respect to the metric ρ there is η in $(0, 1/2)$ such that $|f_2(z)-f_2(\omega)|<\varepsilon$ when $\rho(z,\omega)<\eta$. By (4.36) then

$$\operatorname{ess}\sup_{\omega\in K(z,\eta)} |h(z)-h(\omega)| \leqslant 7\varepsilon \qquad \text{for all} \quad z\in\mathbb{D}. \tag{4.37}$$

Let $g(z)=Th(z)$, where $T=T_\eta$ is the regularization operator. Item (b) of Lemma 3.2 in conjunction with (4.36) and (4.37) gives $\|g-f_2\|_\infty \leqslant 10\varepsilon$. Besides, $(1-|z|^2)\,|\nabla g(z)|$ is bounded by (c) of Lemma 3.2, and $|\nabla g|\,dA$ is a Carleson measure by Lemma 3.3. So, Bishop's theorem asserts that $g\in C(M_{H^\infty},\mathbb{C})$ and then

$$\operatorname{dist}(f_2, C(M_{H^\infty},\mathbb{C}))\leqslant 10\varepsilon. \tag{4.38}$$

Putting together (4.38), (4.13), and (4.10), we obtain $\operatorname{dist}(f, C(M_{H^\infty},\mathbb{C}))\leqslant 12\varepsilon$ for any of the kernels $q(z,\xi)$. Since ε is arbitrary, this proves Theorem 4.1. ▌

5. CONTINUOUS EXTENSIONS OF MEROMORPHIC AND HARMONIC FUNCTIONS

Let g be a function of class C^2 on some neighborhood of the ball $|\omega|\leqslant r$. A standard application of Green's theorem gives

$$g(0)=\int_0^{2\pi} g(re^{it})\,\frac{dt}{2\pi}-\int_{|\omega|\leqslant r} \Delta g(\omega)\log\frac{r}{|\omega|}\,dA(\omega).$$

So, if g is C^2 on some neighborhood of the pseudohyperbolic ball $K(z,r)$, replacing g by $g\circ\varphi_z$ in the above expression and changing variables in the area integral, we obtain

$$g(z)=\int_0^{2\pi} (g\circ\varphi_z)(re^{it})\,\frac{dt}{2\pi}-\frac{1}{2}\int_{K(z,r)} \Delta g(\omega)\log\frac{r^2}{|\varphi_z(\omega)|^2}\,dA(\omega). \tag{5.1}$$

In particular, if $g \in C^\infty(\mathbb{D})$ has compact support we can take $r=1$, leading to

$$g(z) = -\frac{1}{2}\int_{\mathbb{D}} \Delta g(\omega) \log \frac{1}{|\varphi_z(\omega)|^2}\, dA(\omega). \tag{5.2}$$

LEMMA 5.1. *Let F be a bounded complex function of class C^2 on $\mathbb{D}$ such that $(1-|z|^2)^2\, |\Delta F(z)|$ is bounded and $(1-|z|^2)\, |\Delta F(z)|\, dA(z)$ is a Carleson measure. Then $F \in C(M_{H^\infty}, \mathbb{C})$.*

Proof. By Theorem 4.1 the function

$$H(z) = -\frac{1}{2}\int_{\mathbb{D}} \Delta F(\omega) \log \frac{1}{|\varphi_z(\omega)|^2}\, dA(\omega)$$

is in $C(M_{H^\infty}, \mathbb{C})$. In particular, H is bounded. If $g \in C^\infty(\mathbb{D})$ has compact support, then (5.2) and Fubini's theorem yield

$$\int_{\mathbb{D}} H(z)\, \Delta g(z)\, dA(z) = \int_{\mathbb{D}} \Delta F(\omega) \left[-\frac{1}{2}\int_{\mathbb{D}} \Delta g(z) \log \frac{1}{|\varphi_\omega(z)|^2}\, dA(z) \right] dA(\omega)$$

$$= \int_{\mathbb{D}} \Delta F(\omega)\, g(\omega)\, dA(\omega) = \int_{\mathbb{D}} F(\omega)/\Delta g(\omega)\, dA(\omega).$$

Thus, $\Delta(F-H)=0$ in the distributional sense. Then $F-H$ is a (bounded) harmonic function, and consequently it can be extended continuously to M_{H^∞}. Hence, the same holds for $F=(F-H)+H$. ∎

Let $h \in C(M_{H^\infty}, \hat{\mathbb{C}})$. Since the euclidean and the chordal metrics are equivalent on bounded subsets of $\mathbb{C}$, we can use (1) ⇒ (2) of Theorem 3.4 to produce an open region $L \subset \mathbb{D}$ such that

$$\{z \in \mathbb{D} : |h(z)| < 4\} \subset L \subset \{z \in \mathbb{D} : |h(z)| < 5\}$$

and the arc length of ∂L is a Carleson measure. The uniform (ρ, χ) continuity of h, together with a new use of the equivalence between χ and the modulus on bounded sets imply that there is some η, with $0 < \eta < 1/2$, such that

$$\rho(z, \partial L) \leqslant \eta \Rightarrow 3 < |h(z)| < 6. \tag{5.3}$$

Define the function $\phi = T\chi_L$, where $T = T_\eta$ is the regularization operator and χ_L is the characteristic function of L. Therefore, $\phi \in C^\infty(\mathbb{D}, [0,1])$. If $z \in \mathbb{D}$ is such that $|h(z)| \leqslant 2$ then (5.3) says that $K(z, \eta) \cap \partial L = \varnothing$, and then

$\chi_L \equiv 1$ on $K(z, \eta)$. By Lemma 3.2(a) then $\phi(z) = 1$. Analogously, $\phi(z) = 0$ if $|h(z)| \geqslant 7$. It will be convenient to have a reference for these facts,

$$\phi(z) = \begin{cases} 0 & \text{if} \quad |h(z)| \geqslant 7 \\ 1 & \text{if} \quad |h(z)| \leqslant 2 \\ \in [0, 1] & \text{otherwise.} \end{cases} \tag{5.4}$$

In addition, Lemmas 3.2 and 3.3 say that the norms

$$\|(1-|z|^2)\,|\nabla\phi|\,\|_\infty \qquad \|(1-|z|^2)^2\,|\Delta\phi|\,\|_\infty \qquad \|\,|\nabla\phi|\;dA\|_*, \qquad \text{and}$$
$$\|(1-|z|^2)\,|\Delta\phi|\;dA\|_* \tag{5.5}$$

are bounded, where the last two conditions mean that the respective measures are Carleson.

LEMMA 5.2. *Let $h \in C(M_{H^\infty}, \hat{\mathbb{C}}) \cap C^2(\mathbb{D}, \hat{\mathbb{C}})$ with $(1-|z|^2)\,h^{\#}(z)$ bounded. Suppose that $\phi \in C^\infty(\mathbb{D}, [0, 1])$ is the function associated to h by the above process. Then there is $C = C(h, \phi) = C(h) > 0$ such that*

$$\int_{K(z,r)} \log \frac{r^2}{|\varphi_z(\omega)|^2}\, \varphi(\omega)(\Delta\,|h|^2)(\omega)\, dA(\omega) \leqslant C \tag{5.6}$$

for every $z \in \mathbb{D}$ and $0 < r < 1$.

Proof. Since $h \in C^2(\mathbb{D}, \hat{\mathbb{C}})$ it is clear from (5.4) that $\phi\,|h|^2 \in C^2(\mathbb{D}, \mathbb{R})$ and $\|\phi\,|h|^2\|_\infty \leqslant 49$. Then (5.1) implies that

$$\left|\int_{K(z,r)} \Delta(\phi\,|h|^2)(\omega) \log \frac{r^2}{|\varphi_z(\omega)|^2}\, dA(\omega)\right| \leqslant 4\,\|\phi\,|h|^2\|_\infty$$
$$< 200 = C_0 \tag{5.7}$$

for every $z \in \mathbb{D}$ and $0 < r < 1$. A straightforward calculation shows that

$$\begin{aligned}\Delta(\phi\,|h|^2) &= \phi\Delta(|h|^2) + 4(\operatorname{Re} h)\langle \nabla\phi, \nabla \operatorname{Re} h\rangle \\ &\quad + 4(\operatorname{Im} h)\langle \nabla\phi, \nabla \operatorname{Im} h\rangle + |h|^2\,\Delta\phi \\ &= \phi\Delta(|h|^2) + P, \end{aligned} \tag{5.8}$$

where $\langle\ ,\ \rangle$ is the usual pairing in $\mathbb{R}^2$. Since by (5.4), $\phi(z) = |\nabla\phi(z)| = \Delta\phi(z) = 0$ when $|h(z)| > 7$, then

$$\begin{aligned} |P| &\leqslant 4\,|h|\,|\nabla\phi|\,(|\nabla\operatorname{Re} h| + |\nabla\operatorname{Im} h|) + |h|^2\,|\Delta\phi| \\ &\leqslant 4\,|h|\,(1+|h|^2)\,|\nabla\phi|\,2\frac{|\nabla h|}{(1+|h|^2)} + |h|^2\,|\Delta\phi| \\ &\leqslant 10^4(|\nabla\phi|\,h^{\#} + |\Delta\phi|). \end{aligned}$$

The above inequality, the boundedness of $(1-|z|^2)\,h^{\#}(z)$ and (5.5) imply that $(1-|z|^2)^2\,|P(z)|$ is bounded and $(1-|z|^2)\,|P(z)|\,dA(z)$ is a Carleson measure. So, because for every $z, \omega \in \mathbb{D}$ and $0 < r < 1$,

$$\chi_{K(z,r)}(\omega)\log\frac{r^2}{|\varphi_z(\omega)|^2} \leqslant \log\frac{1}{|\varphi_z(\omega)|^2}$$

we obtain

$$\begin{aligned} &\int_{K(z,r)} \log\frac{r^2}{|\varphi_z(\omega)|^2}\,|P(\omega)|\,dA(\omega) \\ &\qquad\leqslant \int_{\mathbb{D}} \log\frac{1}{|\varphi_z(\omega)|^2}\,|P(\omega)|\,dA(\omega) \leqslant C_1 \end{aligned} \tag{5.9}$$

for every $z \in \mathbb{D}$ and $0 < r < 1$, where the last inequality comes from Theorem 4.1.

By (5.9), (5.8), and (5.7) then

$$\int_{K(z,r)} \log\frac{r^2}{|\varphi_z(\omega)|^2}\,\phi(\omega)\,(\Delta\,|h|^2)(\omega)\,dA(\omega) \leqslant C_0 + C_1$$

for every $z \in \mathbb{D}$ and $0 < r < 1$. ∎

Remark 5.3. In the proof of Lemma 5.2 we have not used the full strength of the condition $(1-|z|^2)\,h^{\#}(z) < \infty$. Actually, the lemma only uses that $(1-|z|^2)\,|\nabla h(z)|$ is bounded on the set $\{|h(z)| \leqslant 7\}$. However, because of the invariance of the spherical gradient under reciprocals, the condition of the lemma will be meaningful when we apply it to h and h^{-1}.

LEMMA 5.4. *Let h as in Lemma 5.2, and suppose that $\Delta\,|h|^2 \geqslant 0$ on the set $\{|h| \leqslant 7\}$. Then*

$$(1-|\omega|^2)(\Delta\,|h|^2)(\omega)\,\chi_{\{|h|\leqslant 2\}}(\omega)\,dA(\omega)$$

is a Carleson measure.

Proof. Since by (5.4) the support set of ϕ is contained in $\{|h| \leqslant 7\}$, the hypothesis on $\Delta(|h|^2)$ implies that the expression inside the integral in (5.6) is nonnegative. So, if we fix an arbitrary $z \in \mathbb{D}$ and let $r \to 1^-$ in (5.6), the monotone convergence theorem gives

$$\int_{\mathbb{D}} \log \frac{1}{|\varphi_z(\omega)|^2} \phi(\omega)(\Delta\, |h|^2)(\omega)\, dA(\omega) \leqslant C$$

for every $z \in \mathbb{D}$. Since by (5.4) $\chi_{\{|h| \leqslant 2\}}(\omega) \leqslant \phi(\omega)$ and by (1.2) $1 - |\varphi_z(\omega)|^2 \leqslant \log |\varphi_z(\omega)|^{-2}$, then

$$\int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{\omega}z|^2} (1-|\omega|^2)\, \chi_{\{|h| \leqslant 2\}}(\omega)(\Delta\, |h|^2)(\omega)\, dA(\omega) \leqslant C.$$

Thus, $(1-|\omega|^2)(\Delta\, |h|^2)\, \chi_{\{|h| \leqslant 2\}}\, dA$ is a Carleson measure. ∎

There is nothing special about the numbers 2 and 7 in Lemma 5.4 or in (5.4). With obvious modifications, any pair of positive numbers $a < b$ work. In future applications it will be important though that $a \geqslant 1$, but this is all we need. We can prove now the main results of the paper.

THEOREM 5.5. *Let f be a meromorphic function on $\mathbb{D}$. Then $f \in C(M_{H^\infty}, \hat{\mathbb{C}})$ if and only if*

(1) $(1-|z|^2)\, f^{\#}(z)$ *is bounded and*

(2) $(1-|z|^2)(f^{\#}(z))^2\, dA(z)$ *is a Carleson measure.*

Proof. Suppose that (1) and (2) hold. The function $F = |f|^2 (1 + |f|^2)^{-1}$ is the third coordinate of the vector $s \circ f$, where s: $\hat{\mathbb{C}} \to S_2$ is the stereographic projection. Therefore the implication (3) ⇒ (1) of Theorem 2.3 reduces the problem to checking that $F \in C(M_{H^\infty}, \mathbb{C})$. Clearly F is C^2 on $\mathbb{D}$, and a simple calculation shows that

$$\Delta F = 4 \frac{|f'|^2}{(1+|f|^2)^2} - 8 \frac{|f|^2\, |f'|^2}{(1+|f|^2)^3} = 4(f^{\#})^2\, (1-2F).$$

Since $0 \leqslant F \leqslant 1$ then $|\Delta F| \leqslant 4(f^{\#})^2$. Hence, (1) and (2) imply that $(1-|z|^2)^2\, |\Delta F|$ is bounded and $(1-|z|^2)\, |\Delta F|\, dA$ is a Carleson measure. The result follows from Lemma 5.1.

Conversely, suppose that $f \in C(M_{H^\infty}, \hat{\mathbb{C}})$. The theorem of Brown and Gauthier then says that f satisfies (1). Since $\Delta\, |f|^2 = |f'|^2$ when $|f| \leqslant 7$ and $\Delta\, |f|^{-2} = |f'|^2/|f|^4$ when $|f| \geqslant 1/7$, then f and f^{-1} satisfy the hypotheses of Lemma 5.4. Consequently the lemma says that

$$(1-|\omega|^2)\, \chi_{\{|f| \leqslant 2\}}\, |f'|^2\, dA \qquad \text{and} \qquad (1-|\omega|^2)\, \chi_{\{|f| \geqslant 1/2\}}\, |(f'/f^2)|^2\, dA$$

are Carleson measures. Since clearly

$$(1-|\omega|^2)\frac{|f'|^2}{(1+|f|^2)^2}$$
$$\leqslant (1-|\omega|^2)\,|f'|^2\,\chi_{\{|f|\leqslant 2\}} + (1-|\omega|^2)\,\frac{|f'|^2}{|f|^4}\,\chi_{\{|f|\geqslant 1/2\}},$$

then $(1-|\omega|^2)(f^{\#})^2\,dA$ is a Carleson measure, and the theorem is proved. ▮

Remark 5.6. The proof that (1) and (2) imply $f \in C(M_{H^\infty}, \hat{\mathbb{C}})$ in Theorem 5.5 works with almost no modification for f harmonic. We only have to use Corollary 2.8 instead of Theorem 2.3. This fact is not stated in the theorem because we can obtain a result that is formally stronger (see Theorem 5.8 below).

As mentioned before, condition (1) of the theorem means that the function f is normal. In [6] it is shown that a Schwarz triangle function whose initial triangle is strictly interior to the disk is not in $C(M_{H^\infty}, \hat{\mathbb{C}})$. Since such a function is normal (1) does not imply (2). A more dramatic example is given in [2], where the authors prove that there is a function in the little Bloch space (i.e., f analytic and $(1-|z|^2)\,|f'(z)| \to 0$ as $|z| \to 1^-$) that is not in $C(M_{H^\infty}, \hat{\mathbb{C}})$. In the other direction, it is shown in [1] that if $0<\beta<1$ and $b(z)=\prod_{n=1}^{\infty}\varphi_{z_n}(z)$ is the Blaschke product with zeros $z_n = 1-\beta^n$, then the function $f(z)=b(z)\log(1-z)$ satisfies (2) but not (1). Therefore the conditions of the theorem are independent of each other.

Specialists in meromorphic functions have studied both conditions in a different context (see for instance, [18, 19]). In [18] the class of meromorphic functions that satisfy (2) is denoted by $M_1^{\#}$, and the subclass of functions that satisfy (1) and (2) by $Q_1^{\#}$. The latter class coincides with UBC, the meromorphic functions of uniformly bounded characteristic on $\mathbb{D}$ (see [19]). Among other things, [18] contains several characterizations of both classes and an extensive bibliography. We only mention here that for a meromorphic function f the two conditions of the theorem amount to the single condition

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(\xi))^2 \log\frac{1}{|\varphi_z(\xi)|^2}\,dA(\xi) < \infty. \tag{5.10}$$

Indeed, by the second inequality in (1.2) and Lemma 4.2, the conditions of Theorem 5.5 imply (5.10). For the other direction we need a result of Yamashita [19, Theorem 3.1], stating that (5.10) ⇒ (1); while the remaining implication (5.10) ⇒ (2) is an immediate consequence of the first inequality in (1.2). With this terminology, Theorem 5.5 says that a meromorphic function f is in $C(M_{H^\infty}, \hat{\mathbb{C}})$ if and only if $f \in Q_1^{\#}$ (or UBC).

Let u be a real-valued harmonic function on $\mathbb{D}$. As usual, we denote by $\tilde{u}$ the harmonic conjugate of u normalized by the condition $\tilde{u}(0) = 0$.

THEOREM 5.7. *Let u be a real-valued harmonic function. Then the following conditions are equivalent.*

(a) $u \in C(M_{H^\infty}, [-\infty, +\infty])$.

(b) $\mathrm{Cl}_{[-\infty, +\infty]}(u, x) \neq [-\infty, +\infty]$ *for every* $x \in M_{H^\infty}$.

(c) $u \in C(M_{H^\infty}, \hat{\mathbb{C}})$ (*i.e., when identifying* $-\infty$ *with* $+\infty$).

(d) $e^{u + i\tilde{u}} \in C(M_{H^\infty}, \hat{\mathbb{C}})$.

(e) $(1 - |z|^2)\, u^{\#}$ *is bounded and* $(1 - |z|^2)(u^{\#})^2\, dA$ *is a Carleson measure.*

(f) $(1 - |z|^2)\, e^{-|u|}\, |\nabla u|$ *is bounded and* $(1 - |z|^2)\, e^{-2|u|}\, |\nabla u|^2\, dA$ *is a Carleson measure.*

If any of the above conditions holds then

(g) $u + i\tilde{u} \in C(M_{H^\infty}, \hat{\mathbb{C}})$.

Proof. The equivalence of (a), (b), and (c) is an immediate consequence of Corollary 2.6. Since the exponential is a homeomorphism from $[-\infty, +\infty]$ with the two-point topology onto $[0, \infty]$ with the topology induced by $\hat{\mathbb{C}}$, then $u \in C(M_{H^\infty}, [-\infty, +\infty])$ if and only if $e^u \in C(M_{H^\infty}, [0, \infty])$. Besides, by Remark 2.4 the analytic function $e^{u+i\tilde{u}}$ is in $C(M_{H^\infty}, \hat{\mathbb{C}})$ if and only if $e^u \in C(M_{H^\infty}, [0, \infty])$. Therefore (a) is equivalent to (d).

We show now that (a) $\Rightarrow$ (e). If (a) holds then (d) holds, and the theorem of Brown and Gauthier implies that the analytic function $e^{u+i\tilde{u}}$ is normal. Hence, so is u. By [12, Theorem 4] then $(1 - |z|^2)\, u^{\#}(z)$ is bounded. By (c), $u \in C(M_{H^\infty}, \hat{\mathbb{C}})$ and then so is u^{-1}. Since $\Delta u^2 = 2\, |\nabla u|^2$ when $|u| \leqslant 7$ and $\Delta u^{-2} = 6u^{-4}\, |\nabla u|^2$ when $|u^{-1}| \leqslant 7$, then u and u^{-1} satisfy the hypotheses of Lemma 5.4. The lemma then tells us that

$$(1 - |z|^2)\, \chi_{\{|u| \leqslant 2\}}\, |\nabla u|^2\, dA \quad \text{and} \quad (1 - |z|^2)\, \chi_{\{|u| \geqslant 1/2\}} (|\nabla u|^2 / u^2)^2\, dA$$

are Carleson measures. Since

$$(1 - |z|^2)(u^{\#})^2 \leqslant (1 - |z|^2) \left[\chi_{\{|u| \leqslant 2\}}\, |\nabla u|^2 + \chi_{\{|u| \geqslant 1/2\}} \left(\frac{|\nabla u(z)|}{u(z)^2} \right)^2 \right],$$

then $(1 - |z|^2)(u^{\#})^2\, dA$ is a Carleson measure. So, (e) holds.

The elementary inequality $e^{-|u|} \leqslant (1+u^2)^{-1}$ immediately proves that (e) $\Rightarrow$ (f). We shall close the loop of equivalences by showing that (f) and (d) are equivalent. Writing $f = e^{u+i\tilde{u}}$, we have

$$f^{\#} = \frac{|\nabla u|\, e^u}{1+e^{2u}} = \frac{|\nabla u|}{e^{-u}+e^u},$$

and since clearly $e^{|u|} \leqslant e^{-u}+e^u \leqslant 2e^{|u|}$, then

$$\tfrac{1}{2} e^{-|u|}\, |\nabla u| \leqslant f^{\#} \leqslant e^{-|u|}\, |\nabla u|.$$

The equivalence of (d) and (f) now follows from Theorem 5.5.

Finally, (a) implies that $\mathrm{Cl}_{\hat{\mathbb{C}}}(u+i\tilde{u}, x) \neq \hat{\mathbb{C}}$ for all $x \in M_{H^\infty}$, which together with Theorem 2.3 gives (g). ▮

THEOREM 5.8. *Let f be a complex-valued harmonic function on $\mathbb{D}$ such that*

$$(1-|z|^2)\, e^{-|f|}\, |\nabla f| \text{ is bounded and } (1-|z|^2)\, e^{-2|f|}\, |\nabla f|^2\, dA$$

is a Carleson measure.

Then $f \in C(M_{H^\infty}, \hat{\mathbb{C}})$.

Proof. Write $f(z) = u(z) + iv(z)$, where u and v are real-valued. Consider the analytic functions

$$g_1 = e^{(u+v)+i(\tilde{u}+\tilde{v})}, \quad g_2 = e^{-(u+v)-i(\tilde{u}+\tilde{v})}, \quad g_3 = e^{(u-v)+i(\tilde{u}-\tilde{v})}, \quad \text{and}$$

$$g_4 = e^{(v-u)+i(\tilde{v}-\tilde{u})}.$$

Then

$$e^{2(|u|+|v|)} \leqslant |g_1|^2 + \cdots + |g_4|^2 \leqslant 4e^{2(|u|+|v|)} \tag{5.11}$$

and

$$\begin{aligned}\sum |g_j'|^2 &= |\nabla(u+v)|^2\, |g_1|^2 + |\nabla(u+v)|^2\, |g_2|^2 + |\nabla(u-v)|^2\, |g_2|^2 \\ &\quad + |\nabla(v-u)|^2\, |g_4|^2 \\ &\leqslant (|\nabla(u+v)|^2 + |\nabla(u-v)|^2) \sum |g_j|^2 \\ &= 2\, |\nabla f|^2 \sum |g_j|^2. \end{aligned} \tag{5.12}$$

Here and below the index j of the sum runs from 1 to 4. Let

$$G = \frac{\sum |g_j|^2}{1 + \sum |g_j|^2}.$$

It is easy to check that

$$\Delta G = -8 \frac{|\sum g_j \overline{g_j'}|^2}{(1 + \sum |g_j|^2)^3} + 4 \frac{\sum |g_j'|^2}{(1 + \sum |g_j|^2)^2}.$$

Since $|\sum g_j \overline{g_j'}|^2 \leqslant (\sum |g_j|^2)(\sum |g_j'|^2)$ then (5.11) and (5.12) yield

$$|\Delta G| \leqslant 12 \frac{\sum |g_j'|^2}{(1 + \sum |g_j|^2)^2} \leqslant 24 \frac{|\nabla f|^2}{(1 + \sum |g_j|^2)}$$
$$\leqslant 24 \frac{|\nabla f|^2}{1 + e^{2(|u| + |v|)}} \leqslant 24 \frac{|\nabla f|^2}{e^{2|f|}}.$$

The hypotheses on $|\nabla f|$ then imply that $(1 - |z|^2)^2 |\Delta G|$ is bounded and $(1 - |z|^2) |\Delta G| \, dA$ is a Carleson measure. Therefore $G \in C(M_{H^\infty}, \mathbb{C})$ by Lemma 5.1, and consequently $\mathrm{Cl}_{[0, \infty]}(\sum |g_j|^2, x)$ is a singleton for every $x \in M_{H^\infty}$. By (5.11), $\mathrm{Cl}_{[0, \infty]}(|u| + |v|, x)$ is either bounded or $\{\infty\}$. Since the same holds for $|f|$, and f is harmonic, Corollary 2.8 says that $f \in C(M_{H^\infty}, \hat{\mathbb{C}})$. ∎

I believe that a complex-valued harmonic function f in $C(M_{H^\infty}, \hat{\mathbb{C}})$ must fulfill the conditions of Theorem 5.8. Maybe even the formally stronger conditions

$$\|(1 - |z|^2) f^{\#}\|_\infty < \infty \qquad \text{and} \qquad \|(1 - |z|^2)(f^{\#})^2 \, dA\|_* < \infty \quad (5.13)$$

are necessary. My only reason for this statement is that by Theorem 5.7 the conjecture is true when the real or the imaginary part of f is bounded. This is a very particular situation, because such $f = u + iv$ is in $C(M_{H^\infty}, \hat{\mathbb{C}})$ if and only if the (possibly) unbounded function u or v is in $C(M_{H^\infty}, [-\infty, +\infty])$. Another good indication is that by Theorems 5.5 and 5.7 the conjecture is true for f analytic. In all these cases, the function also satisfies (5.13).

Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the classes of analytic functions in $C(\mathcal{G}, \hat{\mathbb{C}})$ and $C(M_{H^\infty}, \hat{\mathbb{C}})$, respectively. It is said in [2] that we have no reason to expect that $\mathcal{A}_1$ is closed under addition. Since $\mathcal{A}_0$ is the class of normal analytic functions and $\mathcal{A}_1$ is the class of analytic functions of uniformly bounded characteristic, it is known that neither of them is closed under addition or multiplication. That is, bearing in mind the above identifications we can browse the literature on meromorphic functions and find all kind of oddities. We give below very simple examples to illustrate that the sum or the product of two functions in $\mathscr{A}_1$ does not need to be in $\mathscr{A}_\theta$.

EXAMPLE. Take $z_n = 1 - 2^{-n}$, with $n = 1, 2, \ldots$, and consider the inner functions

$$s(z) = \exp\left(\frac{z+1}{z-1}\right) \qquad \text{and} \qquad b(z) = \prod_{n \geqslant 1} \frac{z_n - z}{1 - \bar{z}_n z}.$$

The Blaschke product b is interpolating, meaning that there is $\delta > 0$ such that $(1 - |z_n|^2)\,|b'(z_n)| > \delta$. It is clear that s has no zeros on $\mathbb{D}$ and easy to check that $s(z) \to 0$ as $z \in [0, 1]$ tends to 1. Since b, $s \in H^\infty$ and $1/\omega\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is continuous, then b and $1/s$ are in $\mathscr{A}_1$. By Brown and Gauthier's theorem, the meromorphic function b/s is in $\mathscr{A}_0$ if and only if $(1 - |z|)(b/s)^{\#}(z)$ is bounded. Evaluating at the zeros of b we obtain

$$(1 - |z_n|^2)\left(\frac{b}{s}\right)^{\#}(z_n) = (1 - |z_n|^2)\left|\frac{b'(z_n)}{s(z_n)}\right| > \frac{\delta}{|s(z_n)|} \to \infty$$

as $n \to \infty$. Therefore $b/s \notin \mathscr{A}_0$. Now write $g = 1/s$. Since b, $b^2 \in H^\infty$ and g, $g^2 \in \mathscr{A}_1$ then Theorem 2.3 implies that $b + g$ and $b^2 + g^2$ are in $\mathscr{A}_1$. Then so is $(b+g)^2$. Therefore, $2bg = (b+g)^2 - (b^2 + g^2)$ is the sum of two functions in $\mathscr{A}_1$, but it is not in $\mathscr{A}_0$.

In [17] Sundberg proved that if $f \in$ BMOA and α is any positive number then f can be uniformly approximated on the set $\{z \in \mathbb{D} : |f(z)| < \alpha\}$ by functions in H^∞. This is no longer true for $\mathscr{A}_1$, as our all-purpose example $1/s$ will show. For this, suppose that there is $\alpha > 1$ and $h \in H^\infty$ such that

$$\sup_{z \in E} |h(z) - 1/s(z)| \leqslant 1/2 \qquad \text{where} \quad E = \{z \in \mathbb{D} : |s(z)| > 1/\alpha\}.$$

Then $hs \in H^\infty$ and $\sup_{x \in E} |h(z)\,s(z) - 1| < 1/2$. Let $\hat{s}$ and $\hat{h}$ denote the continuous extensions to M_{H^∞} of s and h, respectively. Since s is an inner function, Newman's theorem (see [9, p. 194]) asserts that $|\hat{s}| \equiv 1$ on the Shilov boundary, $S(H^\infty)$. Hence, the density of $\mathbb{D}$ in M_{H^∞} implies that $S(H^\infty)$ is in the closure of E. Therefore $\|hs - 1\|_\infty = \sup_{S(H^\infty)} |\hat{h}\hat{s} - 1| < 1/2$, and consequently s must be invertible in H^∞, which is false. In particular, the example shows that if we metrize $\mathscr{A}_1$ by letting $d(f, g) = \sup_{z \in \mathbb{D}} \chi(f(z), g(z))$, H^∞ is not dense in $\mathscr{A}_1$. The theorem of Sundberg comes very close to proving that the closure of H^∞ in $\mathscr{A}_1$ contains BMOA. So, although we do not know whether BMOA is closed in $\mathscr{A}_1$, it is possible that BMOA is actually the closure of H^∞.

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