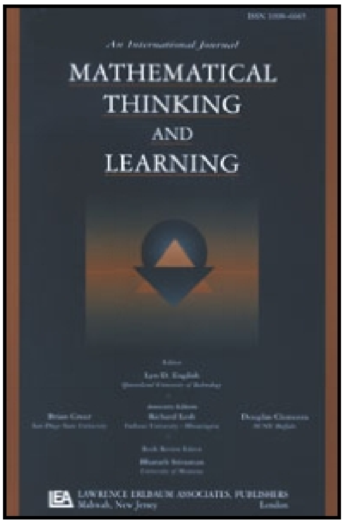

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The Overgeneralization of Linear Models among University Students' Mathematical Productions: A Long-Term Study

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The Overgeneralization of Linear Models among University Students' Mathematical Productions: A Long-Term Study

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Over the past several years, we have been exploring and researching a phenomenon that occurs among undergraduate students that we called extension of linear models to non-linear contexts or overgeneralization of linear models. This phenomenon appears when some students use linear representations in situations that are non-linear. In a first phase, we recognized and characterized the overgeneralization among agronomy majors while working with different kinds of problems. This process let us raise some conjectures about the statement of the problems that were studied in a second phase. In a third phase we conducted some interviews in order to deepen our understanding. The study allowed us to discuss: (a) the robustness and broadness of the linear models as tacit models among university students, (b) the emergence of a tension between unrealistic tasks posed by the teacher/researcher and the students' realistic interpretation of it, and (c) some considerations about the teaching environment at university level.

INTRODUCTION

The main purpose of this paper is to present and discuss some results coming from a 5-year long study of undergraduate students. In this study we focus on the documentation, description, and analysis of a phenomenon that occurs among those students, which we decided to call *extension of linear models to non-linear contexts* or *overgeneralization of linear models.*[1] We use the

[1]Sometimes, in the text, we will use the word *overgeneralization* instead of *overgeneralization of linear models* for simplicity.

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expression linear model to refer to different manifestations of mathematical linearity: a direct proportionality function $y = m \cdot x$, an affine function[2] $y = m \cdot x + k$, or the diagram of the rule of three. The phenomenon of overgeneralization arises when we can observe some manifestations of mathematical linearity in the students' solutions of mathematical problems that present situations in which two variables are not related linearly.

The way we name the phenomenon should be understood as our description of the observed students' responses and not necessarily indicative of students' decisions while solving a problem. We don't assume the students to ask themselves "Is this a situation that can be appropriately solved using a linear model?" (although it would of course be a desirable attitude among university students). We assume that many times the students give answers just to satisfy the teacher/researcher or to pass an examination. But we have also observed that some of the students' usual strategies are compatible with the use of linear models. But why? Is it just because it is the easiest way to cope with the situation and give an answer? Do the statements of the problems induce such solutions? Can we find other reasons for those answers?

In the literature, the overgeneralization is known as linear misconception, illusion of proportionality, or linearity (Freudenthal, 1983) and also proportionality trap. Studies with pupils from primary school have been developed using a variety of "missing value problems" and research procedures (see, for example, Van Dooren, De Bock, Vleugels, & Verschaffel, in this volume; Kontoyianni et al., 2006; Behr, Harel, Post, & Lesh, 1992). The tendency of overgeneralizing the use of linear models beyond its range of validity is also present in secondary school pupils. The extensive studies of De Bock, Van Dooren, Janssens, and Verschaffel (2002), De Bock, Van Dooren, Verschaffel, and Janssens (2001), and De Bock, Verschaffel, and Janssens (1998), carried out with 12 to 16-year-old students, reveal a strong presence of linear models to solve proportional and non-proportional word problems relating lengths and perimeters/areas/volumes of similar figures. These researchers have also studied the illusion of linearity in probabilistic problems (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). These authors carried out research based on written tests and some in-depth interviews. The explanation for the illusion of linearity has been investigated mainly from the perspective of the error as a learners' deficiency. For example, De Bock et al. (2002) pointed out a set of factors associated with the use of linear models in non-linear geometrical contexts: an intuitive approach toward mathematical problems, shortcomings in geometrical knowledge, non-adaptive beliefs, and attitudes or weak use of heuristics. The inappropriate use of linear models to solve non-linear problems has also been attributed to a superficial or deficient modelling process among the students.

The studies about the illusion of linearity report mainly on primary and secondary school students, and even though they present different theoretical and methodological approaches as shown in this volume, there exists agreement in describing the phenomenon as persistent and resistant to change. Studies on the overgeneralization of linear models among university students are not frequent, even though its presence and persistence have been frequently observed at that level within diverse types of problems and contexts. For instance, in a course of topography for agronomy majors, the students learn the concept of slope. Figure 1 shows a ground with a slope of 2%.

[2] In Argentina, the function $y = m \cdot x + k$ is called linear function.

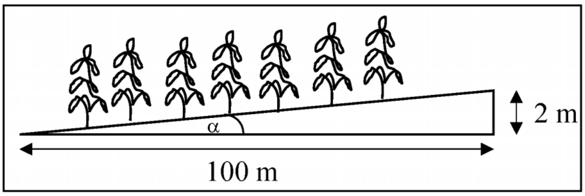

FIGURE 1 A slope of 2%.

The angle α that corresponds to the slope of 2% can be calculated as $\tan^{-1}(2/100)$. Therefore, a slope of 100% corresponds to $\alpha = \tan^{-1}(100/100) = \tan^{-1}(1) = 45°$. Inversely, the slope s that corresponds to an angle α can be calculated as $s = 100 \cdot \tan(\alpha)\%$. During a class the teacher asked which slope corresponds to an angle of 5º. The correct answer is $s = 100 \cdot \tan(5°)\% = 8.74\%$. An answer offered by one student was:

$$45° \to 100\%$$
$$5° \to s = \frac{100 \cdot 5}{45} = 11.11\%.$$

The student posed a "rule of three" using a usual diagram that is learned in Argentinean primary school to solve missing value proportional problems. In this way a linear relationship between the slope and the angle was implicitly assumed.

Another interesting example, in the same university context, comes from a first calculus course. The students were solving the following problem:

> In an experiment of orientation and navigation some pigeons were released 72 km away from the dovecot. If we consider the dovecot as being the origin of a coordinates system, the freedom point determines an angle of 241º measured in the clockwise sense from the north direction with respect to the dovecot. How many kilometres to the south of the dovecot is the freedom point? (Extracted from a list of exercises.)

The students didn't interpret the statement of the problem and the teacher made the sketch shown in Figure 2.

Trigonometrical relationships were expected to be used in order to solve the problem. For instance $x = 72\ km \cdot \cos(61°) = 34.91\ km$. Instead of this, one of the students noted that the south direction corresponds to an angle of 180º measured from the North direction, and proposed to use a rule of three to calculate the required distance

$$241° \to 72\ km$$
$$180° \to x = \frac{72 \cdot 180}{241} = 53.77\ km.$$

Again the relationship between the angle and the distance to the point of freedom was implicitly assumed as being linear.

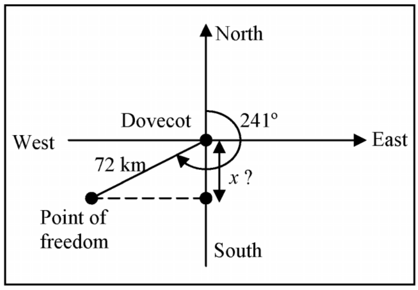

FIGURE 2 The teacher's sketch.

It is worth noting an important difference between this example and the one about the slope. Both uses of the rule of three give answers that are different from the correct ones. However, in the case of the problem about slope, the linear approach would work over a certain range, at least as a rough approximation. In the case of the problem about the dovecot, the approach yields a clearly wrong answer for 270°, for example. In the first case the linear approximation makes sense and is reasonable for small values of α or *s*. Of course, it would be naive to assume that this kind of thinking was done by the student who used the rule of three to find the slope that corresponded to $\alpha = 5^{o}$, but what we want to emphasize is that in some cases the linear approximation makes sense and the ability to analyze the conditions and domain of applicability of a linear model deserves to be developed among university students.

The solutions to the previous problems are some examples of the overgeneralization of linear models to non-linear contexts that we have observed among agronomy majors. Although we will not analyze them, the recurrent appearance of these factual data in classroom motivated us to start a systematic study. The main aim of our long-term study was to investigate and provide evidence for the phenomenon of extension of linear models to non-linear contexts among university students. For reasons that we explain below, we decided to look for explanations beyond the notion of the error as students' deficiency or fault. We assume the notion of error as stated in Balacheff (1984):

> An error is not only the consequence of ignorance, or of uncertainty, or of an accident. An error could be the consequence of a previous knowledge which has its own interest, its successes, but which appears to be false under some new circumstances, or more simply not adapted. Thus in the didactical analysis errors are not understood as mere failures of pupils but rather as symptoms of nature of conceptions which underlie their mathematical activity (p. 36, emphasis in the original).

We also recognize that some errors could come from the student's necessity to give an answer in a test or in an interview. As researchers we try to understand the reasons by which the students decide to present a particular answer. The studies of Confrey (1991, 1994) and Confrey and Smith (1994) addressing questions about the epistemological value of the students' mathematical constructions are compatible with the previous notion of error. Confrey (1991) argued that in order to understand the students' actions one needs to be introduced into their perspective and not to presuppose that it coincides with that of the teacher/researcher. The students' answers that stray from the expectation of the teacher/researcher can be legitimate as alternative or valid and effective in other contexts. If the students' voices are listened to, the teacher/researcher has an opportunity to glimpse at the students' perspectives and to question her/his own perspective, examining it through the ideas of the students. This was our epistemological option to study the students' mathematical productions.

With these assumptions in mind and according to our main purpose, we chose to do a descriptive and interpretative study. The methodological route through which we conducted our study followed an emergent design (Lincoln & Guba, 1985). The decisions on some of the activities or methodological procedures were taken based on the results of prior activities or with the aim of studying some of the conjectures that arose along the study. First, we worked with agronomy majors, and then we decided to extend our field of inquiry working with pre-service mathematics teachers. During the whole inquiry process we identified five phases. Phases 1 to 3 were conducted with agronomy majors, and phases 4 and 5 were conducted with pre-service mathematics teachers. In the first phase we looked for evidence of the phenomenon and raised some conjectures. In the second phase we examined the conjectures raised in the first phase. In the third phase we started an in-depth study. In the fourth phase we searched for evidence of the phenomenon among pre-service mathematics teachers, and in the fifth phase we deepened our study in this new context. The following sections present detailed descriptions of the methodological procedures, analysis, and results of Phases 1 to 3.

PHASE 1: EXPLORING AND RAISING CONJECTURES

We carried out an exploratory study to document, describe, and analyze the presence of the phenomenon among 18- to 20-year-old agronomy majors who were attending their first calculus course. It was a descriptive study based on the analysis of students' written solutions to mathematical problems coming from three sources:

1. An assessment test of the year 2000 first semester calculus course solved by 300 students from the University of Córdoba (Argentina),
2. A set of two word problems specially selected for this study and worked out by 17 students during the year 2000 (these 17 students had failed in the first semester calculus course and were attending the same course during the second semester), and
3. An assessment test of the 2001 first calculus course worked out by 53 students from the University of la Frontera (Chile). (It was possible to work with these students since one of the researchers was working as a visiting professor at the University of la Frontera at Temuco during 2001.)

Note that the assessment tests mentioned in sources (1) and (3) were not created with any inquiry purpose; among all the problems in the test we just took those in which the students' written solutions showed the presence of the overgeneralization of linear models. In Figure 3 we present those problems.

The two word problems mentioned in source (2) are shown in Figure 4. They came from the university entrance exam in mathematics and are typical in the introductory mathematics course for the agronomy majors at the University of Córdoba. We selected them because we had observed the presence of the phenomenon in many students' written solutions.

All the problems presented are typical for the examples and exercises that can be found in the textbooks and that are posed by the teachers in the first calculus course for agronomy majors. As

The tree-problem: After its first month of life, a tree grows according to the equation $h(t)=8\cdot\log_2 t+70$ where the height is given in cm and the time is given in months.
a) Calculate the height of the tree after 6 months.
b) Find the time it will take the tree to reach a height of 1 m.

The micro-organism-problem: In a biological experiment it was found that the population of a kind of micro-organisms follows the law where $N(x) = -2.x^2 + 360.x + 8000$ is the number of micro-organisms after x days.
a) How many micro-organisms will be after 10 days?
b) In which day does the population reach its maximum?
c) Which is the maximum number of micro-organisms?
d) In which day is the number of micro-organisms equal to 21000?

FIGURE 3 The tree-problem and the micro-organism-problem.

The insect-problem: Say if the following statement is true or false and justify your answer. An insect, that weighs 30 g at birth, increases its weight at 20% monthly. Then, its weight after two months is 43.2 g

The plant-problem: If a plant measures 30 cm of height, at the beginning of an experiment, and its height increases 50% monthly, how much will it measure after 3 months?

FIGURE 4 The insect-problem and the plant-problem.

the reader might note, those problems are fictitious in the sense that the mathematical models behind them are not representing “real” biological facts or the functions used do not fit real data. For instance, an insect cannot weigh 30 g, a plant cannot grow exponentially all the time, the height of a tree does not follow a logarithmic model, etc. Although the models could be approximations for specific ranges, this fact does not seem to be a concern for the creators of the problems since no specification is stated in any of the problems. We note that the problems could be characterized as didactical creations: A mathematical model or concept is first selected and then a story is constructed to offer a framework (biological) related to the institutional/educational environment (agronomy majors). All the problems are in fact disguised exercises in computation with respect to non-linear relations. Those characteristics are shared for both sets of problems in Figure 3 or 4, but we also note that there are differences between them. Problems in Figure 3 present an explicit non-linear model through an algebraic expression. Problems in Figure 4 represent non-linear situations, without an explicit non-linear model, and can be solved using strategies that describe the situation properly without employing a non-linear algebraic expression.

In order to study the students’ solutions while solving those problems, we determined the percentage of manifestations of the phenomenon in the each type of problems. A total of 10% of the written solutions, coming from Argentinean as well as Chilean students, showed the overgeneralization while solving problems in Figure 3. Eight of the 17 students (47%) solved the insect-problem or the plant-problem (Figure 4) in a linear way. After obtaining those percentages, we made a list with those students’ written productions showing uses of linear models and, finally, we started their analysis. The main results[3] coming from this analysis are related to the

[3]A complete report of the results of the first phase was published in Villarreal, Esteley, & Alagia (2005).

students' linear representations for each type of problems, the students' strategies, and the difficulties of interpretation that could be associated with the statements of the problems.

The mathematical representations that showed clearly linearity in the students' written productions were the algebraic model $y = m \cdot x + k$ (*x* and *y* are the variables related in the problem) or the diagram for the rule of three we have illustrated with some examples in the introduction:

$$a \to b$$
$$c \to z = \frac{b.c}{a} \cdot$$

This diagram is usually read as follows: if the value that corresponds to *a* is *b*, then the value that corresponds to *c* can be calculated as $z = \frac{b.c}{a}$. In this case, the linear model behind the diagram is $y = m \cdot x$, in which *a* and *c* are particular values of *x*, $b = y(a)$ and *z* is the unknown value for *y*(*c*) (we can also consider the possibility of *a* and *c* being particular values of *y*(*x*)). This is the mathematical interpretation of the so-called direct rule of three. These representations will be illustrated with some examples coming from students' written solutions for each type of problem; i.e., problems with and without an explicit non-linear model.

Students' Strategies in Problems with an Explicit Non-Linear Model

In the tree-problem, the explicit linear model is the logarithmic function $h(t) = 8 \cdot \log_2 t + 70$. Figure 5 shows a student's solution to this problem.

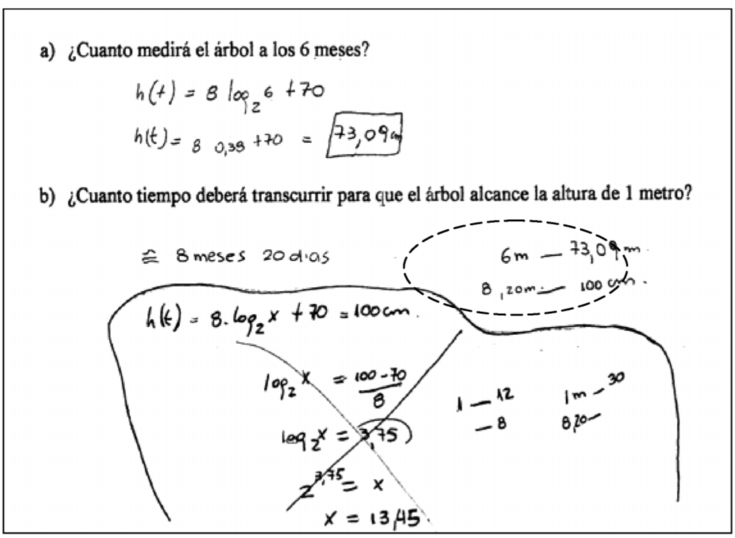

FIGURE 5 Original student's solution.

While solving item (a), this student obtained 73.09 cm as the height of the tree after 6 months by calculating $\log_2 6$ as log 2/log 6 instead of log 6/log 2 (the right answer is 90.68 cm). In item (b), she correctly solved the equation $8.\log_2 x + 70 = 100$ but this procedure was crossed out by the student. She finally gave an answer involving a rule of three in which she used the height of the tree after 6 months that she had previously obtained (see the numerical relations inside the dotted line in Figure 5).

In the micro-organism problem, the explicit non-linear model is the quadratic function $N(x) = -2x^2 + 360x + 8000$. A student indicated, in the answer to item (b), that the population reaches its maximum after 90 days and that the maximum number of micro-organisms (item c) would be $N(90) = 40220$ (the right value for $N(90)$ was 24200). When he solved item (d), he found the solution shown in Figure 6.

This student tried to solve the equation $21000 = -2x^2 + 360x + 8000$; he crossed out his attempt and put forward a rule of three using $N(90) = 40220$.

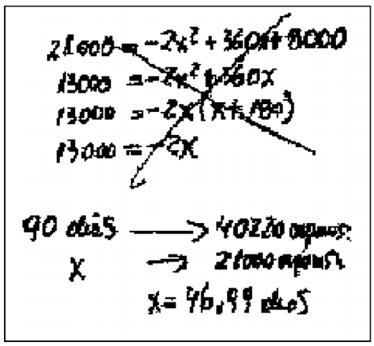

FIGURE 6 The student's solution.

Both students in the previous examples tried to solve the equations that led to the right answer. The first one solved it correctly; meanwhile the second one couldn't finish it. Finally, both of them crossed out those approaches and chose the rule of three to give their final answers.

If we carefully analyze the statements of the tree-problem and the micro-organism-problem we can observe that item (a) in the tree-problem is easily solved since we just have to evaluate the function for a given value. In the same way, items (b) and (c) in the micro-organism-problem can be easily solved using the formula for the vertex of a parabola. In both problems we obtain particular pairs $(t_1, h(t_1))$ or $(x_1, N(x_1))$. With those particular pairs in mind, when the student has to solve item (b) in the tree-problem or (d) in the micro-organism-problem he/she comes to get the data of the problem as having the structure of a missing value problem with "three known data and a fourth unknown." Such structure allows him/her to use the diagram of the rule of three. As a result of this observation we note that this structure of "three known data and a fourth unknown" may encourage the use of a representation of direct proportionality, tacitly assuming the existence of this type of relationship between the variables. By "tacitly" we mean that the student is not aware of the presence of a proportional relationship and its domain of applicability.

From the previous analysis we conjecture that the order of the items in the problems may provoke the appearance of such structure influencing the student's decision. This conjecture, that we named *conjecture of order*, was studied in Phase 2.

Students' Strategies in Problems without an Explicit Non-Linear Model

The expected answer for the insect-problem is that the statement is true and the expected answer for the plant-problem is that the height of the plant after three months will be 101.25 cm. Figure 7 presents two students' solutions for each problem, showing the expected answers.

It is worth noting that the students' solutions for the insect-problem or the plant-problem show, respectively, the presence of a recursive approach and the concatenation of rules of three. With those approaches the solutions of the problems displayed in Figure 7 do not require the creation of a function. However, some students constructed a (linear) function as shown in Figure 8. This decision could be related to the necessity of producing an explanation or justification for the solutions of the problems appealing to the mathematical contents studied in their regular calculus course.

For example, to solve the insect-problem (see Figure 8 on the left), a student proposed the linear function $y = 6 \cdot x + 30$, where y represents the weight of the insect and x the time in months. The slope of the line is calculated using a rule of three as shown in the right upper corner of

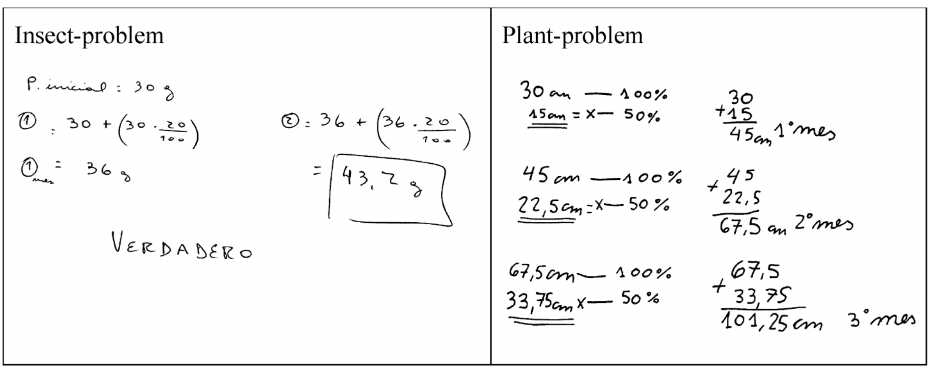

FIGURE 7 Expected answers for the tree and the insect-problems.

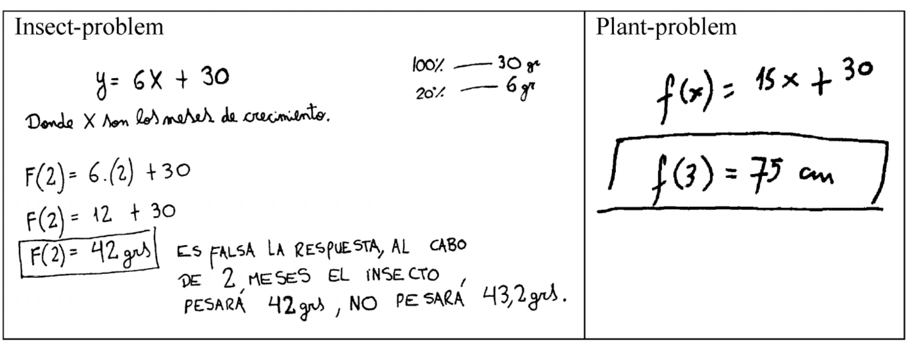

FIGURE 8 Linear functions for the insect or the plant-problems.

the written solution. Finally he concluded that the statement was false as he explained at the bottom of his solution: “After 2 months the insect will weigh 42 g, it will no weigh 43.2 g.” In the plant-problem (see Figure 8 on the right), another student put forward the linear function $f(x) = 15 \cdot x + 30$, where *f(x)* is the height of the plant and it seems that *x* represents the time in months and that the slope is the 50% of 30 cm. In both solutions the students implicitly assumed a constant growth: 6 g/month (20% of 30 g) in the insect-problem, or 15 cm/month (50% of 30 cm) in the plant-problem. This same assumption is present in the students’ solutions shown in Figure 9 where a single rule of three is posed to solve the insect-problem or the plant-problem, respectively.

In the insect-problem, the student wrote that the statement is *False* and explained that after 2 months the insect will grow up 12 g, so the insect “will weigh 42 g.” For the plant-problem the student indicated that “if after 3 months the plant augments 150%, then “its height will be 75 cm at that time.”

Considering the students’ strategies such as the ones shown in Figures 8 and 9, and considering that 50% of the students in the group solved this type of problems linearly, we conjectured that the students’ interpretation of the problems may be associated with the syntax of the statements. The plant-problem says that the height of the plant “increases 50% monthly” and the insect-problem says that the insect “increases its weight 20% monthly.” From this information, it is evident that the increment in height for the first month is 50% of 30 cm, or that the increment in weight for the first month is 20% of 30 g. In this way, for the first month, the height of the plant is 45 cm and the weight of the insect is 36 g. What about the following month? Is it equally evident how to calculate the increment in height or weight for the next month? We may think that the statement clearly suggests that the increment in height or weight for the second month is obtained computing 50% of 45 or 20% of 36. But the fact that 50% of the students used a linear approach (a constant increment of 15 cm—that is 50% of 30 cm; or 6 g—that is 20% of 30g) suggest otherwise. Because of this, we based the conjecture that the syntax of the statement was playing an important role in the appearance of a linear model in the problem’s solution. This conjecture, *conjecture of syntax*, was studied in Phase 2.

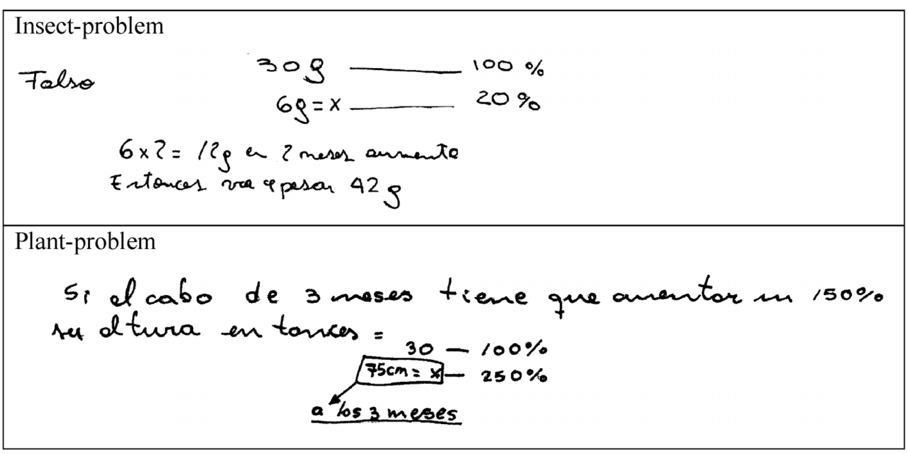

FIGURE 9 Rules of three to solve the insect or the plant-problems.

PHASE 2: EXAMINING CONJECTURES

With the aim of testing the two conjectures raised in the first phase, during 2002 we carried out two studies. Both studies were conducted with agronomy majors attending their first calculus course from the University of Córdoba.

Studying the Conjecture of Order

For studying the *conjecture of order*, we worked with 427 students. We posed, in the first assessment test of the calculus course, the potatoes-problem[4] shown in Figure 10.

The potatoes-problem is similar to the tree-problem, but the order of the items was inverted: while, in the tree-problem we first asked to compute $h(6)$ and then to find t such that $h(t) = 100$ *cm*. In the potatoes-problem we first asked to find d such that $P(d) = 6$, and then to compute $P(30)$.

A total of 427 students completed the test; 2.3% of them showed overgeneralization of linear model (in Phase 1, we reported a percentage of 10%). We could still identify students' written productions showing overgeneralization. In those cases we found two different approaches:

1. To solve item (b), and then, with that result, to solve item (a) using a rule of three as shown in Figure 11.
2. To solve item (a), and then, with that result, to solve item (b) using a rule of three as shown in Figure 12.

In the solution shown in Figure 11, the student used the answer he had obtained in item (b) ($P(0.3) = 6.5$) to find the missing value (x = distance between the plants) that corresponds with 7 potatoes. The value the student gave for x was 0.32 m, which can be calculated as $\frac{7 \times 0.3}{6.5}$. This procedure to obtain 0.32 m is compatible with the application of a rule of three.

In the solution shown in Figure 12, in item (a), the student tried to solve the equation $P(d) = 1$, but what he really calculated was $P(1) = 7.28$. In item (b), he used this information, but he first

The potatoes-problem: The quantity of potatoes of a plant changes depending on the distance between the plants in the same line. From experimental data it was observed that the mean quantity of potatoes (P) given by a plant depends on the distance (d, measured in meters) between the plants in the same line according to the function

$$P(d) = 8{,}5 - 2^{(7-20.d)}$$

a) If we want a mean quantity of 7 potatoes, how far apart should the plants be?
b) How many potatoes should a plant produce if the distance between two plants is 30 cm?

FIGURE 10 The potatoes-problem.

[4]It is interesting to note that the algebraic model in the potatoes-problem is a realistic one. It came from a modelling process aiming to solve a real problem related to the optimization of potatoes production (see Bassanezi, 2002). In the didactical adaptation, the creator of the potatoes-problem took the final algebraic model and transformed it in a disguised calculation exercise, loosing its original realistic flavor.

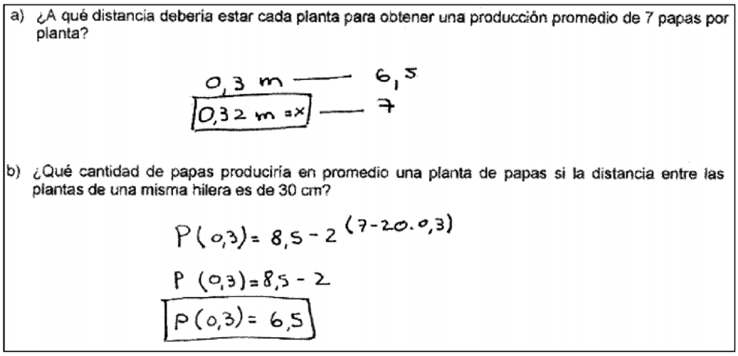

FIGURE 11 Solution with the sequence item b–item a.

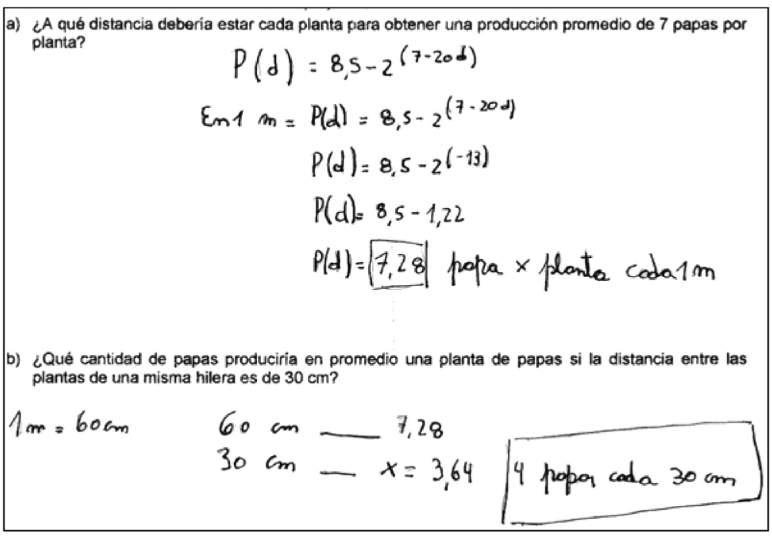

FIGURE 12 Solution with the sequence item a–item b.

wrote (without any explanation) 1 *m* = 60 *cm* and using the pair (60, 7.28), he calculated the value that corresponded with 30 cm through a procedure that is compatible with a rule of three: $x = 3.64 = \frac{30 \times 7.28}{60}$. Finally, his answer was "4 potatoes each 30 cm" ("4 papas cada 30 cm").

Notice that regardless of the approach the students appeal to, a problem relating two variables with the structure of "three known data and a forth unknown" emerges. And we argue that this structure influenced, once again, the appearance of the diagram of a rule of three and computations compatible with it.

Studying the Conjecture of Syntax

In order to explore the *conjecture of syntax* we worked with 84 students. A total of 42 of them solved the insect-problem as originally stated. The other 42 solved a reformulated plant-problem as shown in Figure 13.

In this case, the statement stated explicitly that the height of the plant would measure, in the next month, 50% of the height of the plant in the previous month.

Table 1 shows the results for each group. From the figures in Table 1 we note that the percentage of students that solved the insect-problem as originally stated by a linear approach (52.4%) is considerably greater than the percentage of students that solve the reformulated plant-problem by a linear approach (28.6%).

The results of the study of the *syntax conjecture* as well as the *order conjecture* showed that students are very sensitive to certain aspects of the problem formulation. Consequently the design of problem statements for research or educational purposes needs to be done carefully. The plant-problem was reformulated in order to clearly present a non-linear phenomenon, at least from the teacher's or researcher's point of view. But, despite that reformulation, the overgeneralization of linear models still appeared: The students finally solved the problem working under the hypothesis of linearity, transforming the original problem in a new one. From this state of affairs we wonder: When a student solves this new problem linearly, is she/he aware of the implicit assumption of a hypothesis of linearity? That is, does she/he know that the rule of three is a procedure that can be applied when a proportional relationship between two variables exists? Does she/he know that the linear function $y = m \cdot x$ represents a proportional relationship between the variables y and x? The results of the previous phases and the questions just raised suggested us the necessity to deepen our understanding.

Reformulated plant-problem: If a plant measures 30 cm of height, at the beginning of an experiment, and each month it measures 50% more than the previous month, how much will it measure after 3 months?

FIGURE 13 The reformulated plant-problem.

TABLE 1
Distribution of Answers for Each Problem

The Problems / *Answers*	*Expected Answer*	*Linear Answer*	*Other Answers*	*Total*
The insect-problem as originally stated	17 (40.4%)	22 (**52.4%**)	3 (7.2%)	42 (100%)
The reformulated plant-problem	24 (57.1%)	12 (**28.6%**)	6 (14.3%)	42 (100%)

PHASE 3: DEEPENING THE UNDERSTANDING

With the aim of getting more information about the students' thinking and decision processes when non-linear problems were solved using linear models, in the year 2002, we decided to perform individual semi-structured interviews. We interviewed four students that used a linear approach to solve the insect-problem (Figure 4) during the study of the conjecture of syntax in Phase 2. We carried out a single 1-hour-long interview with each student; all of them were audio-taped. The interviewer was not the students' teacher and the interviewees could use paper, pencil, or a scientific calculator if they wanted to use it while solving the problem. The interviews were structured around activities and aims as we will describe next.

During the first activity of the interview, we asked the student to explain the way she/he had solved the insect-problem with the aim of eliciting the student's strategies and reasons to apply a linear model in a non-linear problem. In this activity, the interviewer just listened to the student's descriptions and asked more details about the written solution without provoking any conflict. This decision was taken to avoid a possible influence in the solution of a new problem in the next activity. As a second activity, we asked the student to solve the reformulated plant-problem (see Figure 14) in order to bring forth the student's strategies while solving a new non-linear word problem. Finally, we asked to calculate the height of the plant in the reformulated plant-problem at 12 months with the aim of challenging the student to produce a general model and to verify the consistency of the student's strategies. In all the activities we asked the students to think aloud (Ginsburg, Kossan, Schwartz, & Swanson, 1982) while explaining or solving the problems.

Among the four interviews we carried out, we present the results related to the interviews with Santiago and Clelia. In the selected excerpts, we underlined some of the students' assertions to indicate the words we believe support our analysis; we include, between brackets, explanations to better understand the students' expressions or words that give continuity to the text; [. . .] indicates long pauses. Short pauses are represented with simple dots.

Santiago

In Phase 2, Santiago had solved the insect-problem using an affine function relating the time *t* (in months) and the weight (*y*) of the insect as shown in Figure 14.

The function offered by Santiago led him to conclude that the statement was false. During the first activity in the interview, and after looking at his written solution, Santiago explained his reasons for selecting the affine function:

Santiago:	Well, what I did, let's see . . . anyway.
Interviewer:	What do you mean by anyway?
Santiago:	I did it quickly, without thinking.

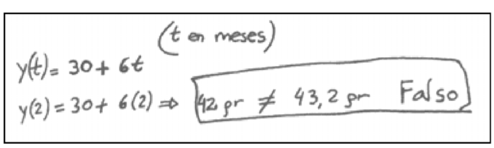

FIGURE 14 Santiago's written solution of the insect-problem. "t en meses" means "t in months."

Interviewer: And what you wrote here [see Figure 14] is what you did without thinking?

Santiago: Yes, I wanted to do it using a function . . . and not with a rule of three […] Then, I realized that it wasn't a linear function because that [he refers to the weight of the insect] hasn't an unlimited growth, it was an exponential or something like that.

Although Santiago realized that the growth of the insect's weight was not linear, he stated: "When I did realize that it was an exponential, I also realized that I wouldn't know how to do it, because, I don't know […] it seems to me that I don't have the tools yet." From these excerpts, we note that Santiago had decided to solve the problem using a function $y(t) = m \cdot t + k$ instead of applying the procedure of the rule of three. But at this moment he did not give any reason for his preferences.

We can also point out that the student used biological reasons ["that hasn't an unlimited growth"] to reject his initial linear solution, considering conditions not explicitly given in the statement of the problem and, at the same time, explicitly contradicted in it. When he proposed an exponential model [". . . it was an exponential or something like that"] he didn't realize that this new model could be rejected using the same biological reasons he had just stated for rejecting the linear model.

During the second activity and, after reading the reformulated plant-problem, the following dialogue occurred:

Santiago: The height of the plant is 30 cm at the beginning of the experiment, so, that is the "base," and each month it grows up 50% of the height it had the previous month. . . .

Interviewer: Yes.

Santiago: Well, I do it with the rule . . . well in a sort of mechanical way, the first month it would be 30 . . . plus the 50% of 30. [He used the calculator and wrote the first line in Figure 15.] In the second month I start at 45 plus 50% of 45, that would be . . . [he wrote the second line in Fig. 15] and in the third month I start at 67.5 and I do the same [he wrote the third line in Figure 15].

In this activity it became clear the reasons by which Santiago decided to solve the insect-problem appealing to a function instead of a rule of three. He considers the rule of three as a mechanical procedure ("I do it with the rule . . . well in a sort of mechanical way") and probably with a low mathematical status, at least, for him as a university student.

When the interviewer asked how to calculate the height of the plant after 12 months, the next dialogue took place:

Santiago: I should do it with a function, to make it easier […] it has to be an exponential and . . . because the growth has to be like this with a limit [he makes a gesture with his hand indicating an increasing graph with a horizontal asymptote when $t \longrightarrow \infty$], it can not grow up indefinitely . . . besides, because the variable is changing.

30 cm $\rightarrow$ 30 + 15 = 45 1 mes
45 cm $\rightarrow$ 45 + 22.5 = 67.5 2 mes
67.5 cm $\rightarrow$ 67.5 + 33.75 = 101.25 3 mes

FIGURE 15 Santiago's written solution of the reformulated plant-problem. "mes" means month.

Interviewer: What do you mean when you say "the variable is changing"?
Santiago: Because when I get to the first month, I see that it changes, I stop working with 30 and I start working with 45 and then, I change from 45 to 67.5. . . . It is like that it is always moving beyond. I should see if there is any . . . , I don't know if it is possible to have a relationship of growth . . . if the 45 has the same proportion of increment with this . . . but not . . . that from 30 to 45 it jumps the same, no, neither here, no, I don't know.
Interviewer: What do you mean with: "it jumps"?
Santiago: Maybe there exists some relationship . . . I was thinking.
Interviewer: Between what?
Santiago: Between the step from 30 to 45 . . . there is one . . . for example there is a parameter from 30 to 45 that is the same between 45 and 67.5 and the same from 67.5 to 101.25. . . . It could be that . . .

Santiago explained that he was trying to find a relationship between the heights of the plant for each month. He talked about a "parameter" that would show the growth. In order to get it, he drew a line as the one sketched in Figure 16.

The student started searching for that "parameter" calculating the following differences: 45–30; 67.5–45, and 101.25–67.5. Santiago wrote those differences in the second row of numbers in Figure 16. He indicated that the increment from 30 to 45 was not the same as that from 45 to 67.5 and from 67.5 to 101.25. After that, he continued searching for the "parameter" but, this time, computing the differences between the values in the second row. He calculates 22.5–15 and wrote 7.5 on third row (see Figure 16). While working with the calculator, he realized that adding 7.5 to 22.5 he wouldn't obtain 33.75. We should point out that his option for searching an additive constant (the "parameter") to model the variation of growth finally became an obstacle for him. After a while, he gave up this strategy and then he tried with some functions he had studied in his calculus course. First, he drew an upward pointing parabola and immediately rejected it because of its "unlimited growth" (see the graph in Figure 17). Then, he proposed $y = a^x + b$ and when the interviewer asked him about the value of *b*, he said it would be "the initial 30 cm." He finally said: "That one [he referred to $y = a^x + b$] doesn't help me, either, since it also has an unlimited growth, it must be something tending to a number . . . something that bends down," and he drew the increasing graph shown in Figure 17.

The graphical representation that Santiago considered is consistent with the biological conditions of limited growth he added to the problem. In this case, he realized that the exponential function was not the appropriate model to represent the growth of the plant. Although he couldn't generate an algebraic model, the graphical model he proposed (compatible with a logistic

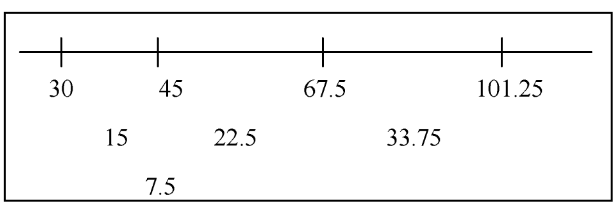

FIGURE 16 Sketch of Santiago's line.

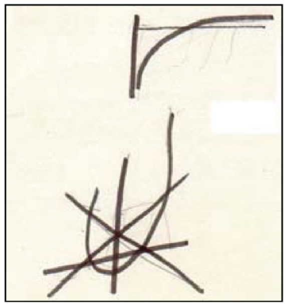

FIGURE 17 Santiago's graphs.

function) represents a realistic model of the growth of a plant. But, he was solving a realistic problem that was not posed.

Clelia

In Phase 2, Clelia had solved the insect-problem as follows. She wrote the initial weight of the insect as $p_i = 30$ gr, she calculated the 20% of p_i using a rule of three and continued working as it is shown in Figure 18. We observe that her written representation of 20% of 30 is (30 + 20/100) and then equivalent to *6*. In this way and considering the last line of her solution (12 gr + 30 = 42 gr), we can infer that the implicit model for the insect weight, is $w(t) = 6 \cdot t + 30$. Finally, she concluded that the statement was false "since after two months it [the insect] would weigh 42 g." During the first activity in the interview, after observing her written solution shown in Figure 18, Clelia said: "I thought on it and then, talking with my classmates, we realized that we had to calculate the 20% of the weight, month by month, while the weight was increasing, not to the initial weight. . . ." Clelia referred to the fact that she assumed that the insect always grows 6 gr every month.

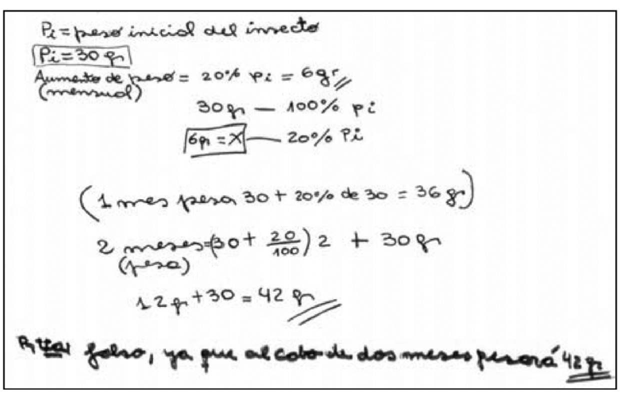

FIGURE 18 Clelia's written solution of the insect problem. "Peso inicial del insecto" means "initial weight of the insect."

During the second activity, after Clelia had read the reformulated plant-problem, the following dialogue occurred:

Clelia: After a month it will measures 30 […] plus 50% . . . of 30. . . . This is for one month, and for three months […] <u>all this times three</u> [she laughed].
Interviewer: Please, continue.
Clelia: I should calculate one by one to make it easier . . . 50% of 30 [she made a pause while whispering something inaudible and stopped].
Interviewer: You have just said "all this times three," what do you mean by "all this"?
Clelia: No, it just seemed to me.
Interviewer: It doesn't matter, when you said "all this," what were you referring to?
Clelia: To the 50% of 30 plus 30.

At this point, Clelia gave up her approach, probably because it was similar to the one she used to solve the insect-problem, and she knew that it wasn't right. After a while, she started using a calculator and finding the height of the plant for each month as it is shown in Figure 19.

Although her solution was not correct, at this point, the interviewer decided not to question it in order to avoid an influence in the next activity. When the interviewer asked Clelia to calculate the height of the plant after 12 months she answered: "I don't know. We know that plants don't grow without limit, at some moment they stop growing. . . . Well, I would calculate the 50% of the height from the previous month and add [such percentage] to it [referring to the height of the plant in the previous month]." After talking about this particular case, Clelia wrote the formula shown in Figure 20 as a model for the growth of the plant.

In this way, Clelia wrote down an expression in which the height of the plant was calculated adding 30 to the 50% of the height of the plant in the previous month. Even though Clelia couldn't get a height vs. time function her model was consistent with her previous numerical work.

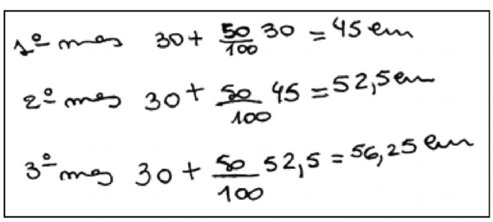

FIGURE 19 Clelia's written solution of the reformulated plant-problem.

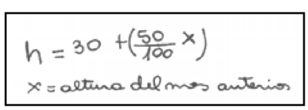

FIGURE 20 Model proposed by Clelia.

Finally, the interviewer asked Clelia what she was thinking about when she spontaneously took the decision of always adding 30 cm to the percentages she had been calculating while solving the reformulated plant-problem (see Figures 19 and 20), and she answered:

> Clelia: The existence of a constant [with emphasis in her voice]. It begins here, and to this one. . . . I have to add.
> Interviewer: Is that what you were thinking on?
> Clelia: Yes, I always notice that [with strong emphasis in her voice].

The strong presence of an initial value (30 in both problems) to which the student always adds the variation in weight and height was used consistently, but it became an obstacle for finding a general model.

Contrasting the Interviews

The interviews provided us with relevant information about the strategies and thinking processes followed by the interviewees while working with non-linear problems. In this sense we point out that Santiago and Clelia share some important aspects. Both students chose a linear model to solve the insect-problem, but while Santiago explicitly said that he had applied a linear function, Clelia didn't say anything about the kind of model she had applied. When the interviewer challenged them to find a general model both students regarded the growth (for the insect or the plant) as an "additive model" in the form of 30 + [something variable]. We called it an additive model since the initial value (either the weight or the height) is always added to the next variation. For example, in the case of Santiago he considered $y(t) = 30 + 6.t$ in the insect-problem or $y(t) = a^t + 30$ in the reformulated plant-problem. In the case of Clelia she proposed $h = 30 + \left(\frac{50}{100} \cdot x\right)$ where x represents the height of the plant in the previous month. It is also important to notice some differences between the approaches followed by the students. Clelia used arithmetical and algebraic resources, but Santiago used several other resources in solving the problem. He proposed graphical or algebraic models and he also tried, arithmetically, to find patterns relating the data when he looked for a "parameter." Both students referred to the fact that a plant cannot grow unlimited, but this realistic observation was at odds with the unrealistic problem that they had to solve. While Clelia did not go further with this constraint, Santiago used it meaningfully, but it became an obstacle to find a model for this unrealistic situation.

CONCLUSIONS AND DISCUSSION

The focus of our long-term study was the documentation, description, and analysis of the phenomenon of overgeneralization of linear models that occurs at university level, extending the well-documented and studied cases of the primary and secondary level. The results coming from the three phases previously described let us recognize the presence of the overgeneralization of linear models and the ways in which linearity appears among university students. Next, we summarize and discuss the main results coming from the three phases of the inquiry process while working with agronomy majors.

In Phase 1, we provided evidence for the presence of the overgeneralization of linear models in different university contexts (Argentina and Chile) and in diverse problems. In our research context, the phenomenon appears as persistent among students from the same institution at different times or among students from different institutions. Most of the students that showed the presence of the phenomenon privileged the use of the rule of three as it was learned in primary school without verifying the conditions for using it. This predilection seems to be related to the structure of the problem and not to the fact that an explicit non-linear algebraic model appears or does not in the statement of the problem. In this phase we also analyzed the structure of the problems posed. From this analysis we raised some conjectures related to such structure, which were studied in Phase 2.

The results of Phase 2, in which we studied the *conjectures of order* and *syntax*, provided evidence for the influence of the order of the tasks in the student's strategies and confirmed the necessity of a careful design of the problems for instruction, in textbooks, for tests, or even for research.

From the analysis performed in Phases 1 and 2, the necessity emerged for deepening the understanding of the overgeneralization of linear models among agronomy majors. Thus, in Phase 3 we carried out individual semi-structured interviews. Such interviews made us aware of students' strategies and ideas that we had not noticed in their written solutions. For example, we observed that the students established connections with reality, introducing biological constraints (limited growth) not explicitly stated in the unrealistic plant-problem. We also noted the strong use of "additive models" of growth in the form of 30 + [something variable] in the case of the plant-problem or the insect-problem. The biological restrictions and the "additive models" posed by the students finally became obstacles for obtaining a mathematical model for the situation described in the problems.

We have observed that when the students recognize in a problem relating two variables a structure of "three given data and a fourth unknown" or transform the problem to get such structure, most of them used the rule of three, implicitly assuming the existence of a direct proportional relationship between the variables. This observation is compatible with some of the conclusions presented by De Bock, Verschaffeld, and Janssens (2002) in a study about the illusion of linearity among secondary school students: "what lured them into the trap of proportional reasoning was not their confidence in an overused mathematical model as such (in this case, the linear function) but rather a gradually and implicitly grown association between that model and a particular kind of problem formulation (in this case, the missing-value type)" (p. 85).

The use of the rule of three in problems with the structure previously described acts as a *tacit model* in the sense that the students are not aware of its influence and domain of applicability. According to Fischbein (1989): "Many of the difficulties students are facing in science and mathematics education are due to the influence of tacit intuitive models acting uncontrolled in the reasoning process" (p. 9). As a tacit model, the rule of three has characteristics that make it robust, due to its practical nature, simplicity, and economy of action in terms of the possibility for solving problems with the structure previously described. The possibility of recognizing the structure of the rule of three as a tacit model enables us to understand the students' decisions when they use it in contexts in which it is not valid (non-linear contexts) as well as its extension and persistence. This particular tacit model enters in scene under certain conditions in such a way that students immediately appeal to it without casting doubt on it and with a noticeable confidence on it. This was evident in those students' solutions in which they crossed out their initial non-linear approaches, even though they were mathematically correct, to then apply a rule of three (see Figures 5 and 6). Or they altered the order of the tasks to transform the problem so that they were "allowed" to use the rule of three (see Figure 11).

We should keep in mind that in our study the overgeneralization of linear models was evidenced not only through the overuse of the rule of three but also through the overuse of the linear function $y = a \cdot x$ or the affine function $y = a \cdot x + b$. Such functions were learned in the primary or secondary school and also used in the university level. In this way, the understanding of the overgeneralization of linear models based on the notion of tacit models should be complemented with some explorations about the teaching environment. When the students arrive to the university courses, many of them enter with a bunch of mathematical facts with weak connections. This is the case of direct proportionality, the use of the rule of three, and the function $y = a \cdot x$ that represents a proportional variation that enables the solution of arithmetical problems of direct proportionality. Although the rule of three is a technique widely used in our schools, its domain of applicability is rarely discussed and, when the students enter the university, it is assumed well-known. In this way, the analysis of pertinence of application or the relations between the rule of three and proportional functions $y = a \cdot x$ are up to the students. Regarding the rule of three, our studies evidenced that an important amount of university students didn't control the conditions of applicability of such a rule, and they finally came to a decision "under the unaware control" of the tacit model. Clearly, some students didn't create the ways to control the use of the rule of three, but it also seems that the teaching environments don't usually offer the means to create them. It is true that the students won't be able to get rid of the influence of this model spontaneously; it is necessary to think on teaching actions that permit a deep analysis of the model.

The rule of three as a robust technique learned in our primary school context should be revisited in the university context. It is necessary: (a) to make the students aware of the proportional relationship behind the procedure of the rule of three, (b) to reflect with the students about the existence of linear and non-linear worlds analyzing similarities and differences, and (c) to encourage the students to use some retrospective strategies or procedures of analytical control. In order to do so it is necessary to create learning environments compatible with a "landscape of investigation serving as invitation for students to be involved in processes of exploration and explanation" (Skovsmose, 2000): a landscape of modelling.

These didactical proposals could help to overcome the phenomenon of overgeneralization of linear models to non-linear contexts among university students, but, at this moment, they should be considered as conjectures that deserve to be studied. There are some possible teaching actions to carry out, but it is also important to decide what to avoid. For example, if we focus on the tasks proposed in our research, the approaches posed by our students evidence some trouble with the problems. One of those troubles was related to the sequence of the tasks, another one to the syntax of the statements. More troubles have to do with the fictitious character of the problems or unrealistic situations created with didactical purposes, as it was noted before. In this kind of creation the teachers/researchers use references to a semi-reality in order to provide meaning for the mathematical activity. According to Skovsmose (2000),

> Some of the principles guiding the resolution of exercises with references to a semi-reality are: a semi-reality is totally described by the statement of the problem; no other information is relevant to solve the exercise; more information is totally irrelevant; the only purpose of the exercise is to be solved (p. 76).

However, the interviews showed that the students introduced biological constraints with references to reality breaking the above principles. Such constraints, external to the problems, became obstacles for obtaining the expected non-linear model, but it is worth noting that the ability to establish connections with reality is considered a positive habit for agronomy students. The students tried to solve a problem with reference to a reality, but the teacher/researcher posed the problem as an exercise with reference to a semi-reality. A tension emerges between the teacher's proposal and the students' realistic interpretation of an unrealistic task.

The research results and the factual data presented in the Introduction of this article put in evidence the strong presence of linearity among university students' approaches to solve non-linear problems. We provided evidence for the overconfidence of our students in linear models as well as the lack of control of the domain of applicability of such models. At the same time, we also know that a local linearization could be the first mathematical approximation and the first strategy when searching a solution to a given problem. For instance, this could be the case of the problem about the slope that we presented in the Introduction. Under particular conditions, the linear approximation is the first one that mathematicians try. How can we develop among our students the ability to recognize such conditions? We have made some didactical suggestions, but would they overcome the overgeneralization of linear models? Is it really possible to avoid this phenomenon?

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