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Formulas for the Riemann Zeta-function and certain Dirichlet series

Pablo Andrés Panzone[a]

[a]Departamento e Instituto de Matematica, Universidad Nacional del Sur, Av. Alem 1253, (8000) Bahia Blanca, Argentina

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ABSTRACT
Using known theta identities and formulas of S. Ramanujan and G. Hardy among others we prove several formulas for the Riemann zeta-function and two Dirichlet series.

KEYWORDS
Riemann zeta-function; Dirichlet series; Fourier transforms

Disclosure statement

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1. Introduction, notation and results

The following two formulas are due to B. Riemann and they were stated in his famous and epoch-making memoir *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, see [1] and [2]. As usual we write $\Gamma(s)$ for the Gamma function and $\zeta(s) = \sum_1^\infty \frac{1}{n^s}$ for the Riemann zeta-function. They are

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \psi(x)(x^{s/2} + x^{(1-s)/2})\frac{dx}{x}, \tag{1.1}$$

with $s \in \mathbb{C}$ where

$$\psi(x) := \sum_{n=1}^\infty e^{-n^2\pi x}, \tag{1.2}$$

and if $s = \frac{1}{2} + it$, $t \in \mathbb{C}$ then

$$\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty \chi(u)cos(ut)du, \tag{1.3}$$

CONTACT Pablo Andrés Panzone. Email: pablopanzone@hotmail.com

where

$$\chi(u) := 4\sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2})e^{-\pi n^2 e^{2u}}.$$

These powerful integral representations are quite useful. We recall just a few examples on the use of these formulas.

Among the many things that Riemann proved in the cited memoir, he showed that the reflection formula follows using formula (1.1) which is unchanged under $s \to 1-s$, see Chapter II of [2].

G. H. Hardy proved in 1914 that an infinite number of zeros of the zeta function are on the critical line. G. Pólya gave a proof of this fact using formula (1.3). In fact, the series defining the kernel $\chi(t)$ in formula (1.3) converges very rapidly. Pólya considered a cosine transform with a different but asymptotically equivalent kernel:

$$16\pi^2 \int_0^{\infty} \cosh(9u/2)e^{-2\pi \cosh(2u)} \cos(ut)du.$$

He showed that the last integral can be written in terms of Bessel functions and it has only real zeros. Furthermore, its zero distribution is the same as what the Riemann Hypothesis predicts, see Chapter X of [2]. In this line of research a far-reaching generalization was given by D. Hejhal [3] and Haseo Ki [4] proved zero-distribution theorems of certain approximations of Epstein zeta-functions.

G. Csordas, T. S. Norfolk and Richard S. Varga [5] proved a fifty-eight year-old conjecture of Pólya, on a necessary condition for the Riemann Hypothesis. They used again formula (1.3). Its proof is very technical and depends on a delicate analysis on the kernel $\chi(u)$. See [6] pp. 92.

P. Walker [7] proved an approximating formula, which we recall in Theorem 3.1, related to formula (1.1). He shows that the term $\frac{1}{s-1} - \frac{1}{s}$ in that formula can be absorbed, so to say, into a finite cosine transform. We will use his approach to prove formulas of the same type.

G. Pólya seems to be the first who emphasized the importance on giving necessary and sufficient conditions on a general kernel $\chi(u)$ to secure that the integral $\int_0^{\infty} \chi(u)\cos(uz)du$ have only real zeros. There is now a vast literature concerning the location of the zeros of sine or cosine transforms with contributions of many masters of the classical analysis. The reader may consult the very comprehensive paper [6]. But despite that many important, necessary and/or suficient conditions upon a kernel $\chi(u)$ are known for the above integral to have only real zeros, the results are so involved that it is impossible to apply them to Riemann's formula (1.3).

So it seems to be of interest and desirable to have other formulas involving the Riemann zeta-function as cosine transform of certain kernels. This is the aim of this paper.

Theorems 2.1, 3.2, 4.1, 4.2 contain formulas in the above spirit. There we also give formulas for the following two Dirichlet series defined for $\Re s > 0$ (the first series is Dirichlet beta-function which corresponds to the non-trivial character *mod* 4 and the second one corresponds to the non-trivial character *mod* 3):

$$\begin{aligned} L_1(s) &= \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots, \\ L_2(s) &= \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} \cdots. \end{aligned} \tag{1.4}$$

In particular we believe that the formulas (2.4), (4.1), (4.8) and the approximation formulas (3.4), (4.7) are new.

Our proofs depend on known Lambert series, on certain formulas appearing in P. Walker's paper [7] and on formulas given by S. Ramanujan in his paper **Some Definite Integrals**, *Messenger of Mathematics*, (1915), see [8] pp. 56 (some were also shown by G. H. Hardy). Several integrals of Ramanujan in the cited paper have been proved by C. T. Preece [9] (see also [10] pp. 291).

Section 6 contains a curious formula.

We start recalling the definition of the theta function (see [11])

$$\theta_3(q) := \sum_{-\infty}^{\infty} q^{n^2}. \tag{1.5}$$

Briefly, if $q = e^{-\pi x}$, then

$$\theta_3(e^{-\pi x}) = 1 + 2\psi(x). \tag{1.6}$$

Observe that the transformation formula for the theta function

$$\sqrt{x}\theta_3(e^{-x\pi}) = \theta_3(e^{-\pi/x}), \tag{1.7}$$

can be written in terms of $\psi(x)$ as

$$1 + 2\psi(x) = \frac{1 + 2\psi(\frac{1}{x})}{\sqrt{x}}.$$

2. Formulas for the Riemann zeta-function

Using formula (1.2) set

$$\psi_0(x) := 4\psi(x)\left\{1 + \psi(x)\right\}. \tag{2.1}$$

The following formulas are similar to formulas (1.1) and (1.3).

Theorem 2.1. *a) If* $s = \frac{1}{2} + it$, $t \in \mathbb{C}$, *then*

$$s(s-1)2\pi^{-s}\zeta(s)\Gamma(s)L_1(s) = \int_0^{\infty} \left\{2e^{\frac{3u}{2}}\psi_0'(e^u) + e^{\frac{5u}{2}}\psi_0''(e^u)\right\}\cos(tu)du. \tag{2.2}$$

(Note: here $\psi_0'(e^u)$ *is* $\frac{d}{dx}\psi_0(x)|_{x=e^u}$ *and* $\psi_0''(e^u)$ *is* $\frac{d^2}{dx^2}\psi_0(x)|_{x=e^u}$*.)*

b) If $s \in \mathbb{C}$, *then*

$$4\pi^{-s}\zeta(s)\Gamma(s)L_1(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (x^{-s} + x^{s-1})\psi_0(x)dx. \tag{2.3}$$

c) If $s = \frac{1}{2} + it$, $t \in \mathbb{C}$, then

$$\zeta(s)\left\{4\pi^{-s}\Gamma(s)L_1(s) - \pi^{-s/2}\Gamma(s/2)\right\}$$

$$\begin{aligned} &= 4\int_0^\infty \left\{2\psi(e^u) + 2\psi^2(e^u) - \psi(e^{2u})\right\} e^{u/2}\cos(tu)du \\ &= 2\int_0^\infty \left\{\theta_3^2(e^{-\pi e^u}) - \theta_3(e^{-\pi e^{2u}})\right\} e^{u/2}\cos(tu)du. \end{aligned} \tag{2.4}$$

In the last two integrals the kernel is an even function of u.

Proof. Let us follow Riemann's footsteps. We have ([11] pp. 285)

$$\frac{\theta_3^2(q) - 1}{4} = \sum_1^\infty \frac{q^n}{1 + q^{2n}}. \tag{2.5}$$

Next we take $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$, and $1 < \sigma$; thus our variable s belongs to an open half plane which we call H_*.

The integration of the right-hand side of the above formula for $q = e^{-x}$ with respect to $x^{s-1}dx$ yields

$$\int_0^\infty x^{s-1}\sum_1^\infty \frac{e^{-nx}}{1 + e^{-2nx}}dx = \zeta(s)\int_0^\infty y^{s-1}\frac{e^{-y}}{1 + e^{-2y}}dy = \zeta(s)\Gamma(s)L_1(s),$$

where the last formula follows after expanding $\frac{e^{-y}}{1+e^{-2y}}$ on a the geometric series and using the following well-known formula

$$\int_0^\infty y^{s-1}e^{-ny}dy = \frac{\Gamma(s)}{n^s}.$$

Observe that $\sum_1^\infty \frac{e^{-nx}}{1+e^{-2nx}}$ is $O(1/x)$ as $x \to 0+$ and is $O(e^{-x})$ as $x \to +\infty$; thus the integrals above are absolutely convergent if $s \in H_*$. One may use Lebesgue's dominated convergence theorem with $f_N(x) \to f(x)$ and $f_N(x) := x^{s-1}\sum_1^N \frac{e^{-nx}}{1+e^{-2nx}}, f(x) := x^{s-1}\sum_1^\infty \frac{e^{-nx}}{1+e^{-2nx}}$ noticing that $|f_N(x)| \le x^{\sigma-1}\sum_1^\infty e^{-nx}$.

Using the left hand side of (2.5) one gets

$$\zeta(s)\Gamma(s)L_1(s) = \frac{1}{4}\int_0^\infty x^{s-1}\{\theta_3^2(e^{-x}) - 1\}dx = \frac{\pi^s}{4}\int_0^\infty x^{s-1}\{\theta_3^2(e^{-\pi x}) - 1)\}dx =$$

$$\frac{\pi^s}{4}\Big\{\int_0^1+\int_1^\infty\Big\},$$

where we have changed variables $x=\pi X$ in the second equality.

Thus, multiplying by $s(s-1)4\pi^{-s}$, we obtain, if $s\in H_*$,

$$s(s-1)4\pi^{-s}\zeta(s)\Gamma(s)L_1(s)=s(s-1)\Big\{\int_0^1+\int_1^\infty\Big\}. \tag{2.6}$$

Now we use the theta transformation formula for the first integral in (2.6):

$$\int_0^1=\int_0^1 x^{s-1}\{\theta_3^2(e^{-\pi x})-1\}dx=\int_0^1 x^{s-1}\Big\{\frac{1}{x}\theta_3^2(e^{-\pi/x})-1\Big\}dx=(y=1/x)=$$

$$\int_1^\infty y^{-1-s}\Big\{y\theta_3^2(e^{-\pi y})-1\Big\}dy=\int_1^\infty y^{-s}\Big\{\theta_3^2(e^{-\pi y})-1\Big\}dy+\int_1^\infty (y-1)y^{-1-s}dy=$$

$$\int_1^\infty y^{-s}\Big\{\theta_3^2(e^{-\pi y})-1\Big\}dy+\frac{1}{s-1}-\frac{1}{s}.$$

The identity is true if $s\in H_*$ but the last integral is defined for $s\in\mathbb{C}$.

Observe that $\psi_0(x)=\theta_3^2(e^{-\pi x})-1=O(e^{-\pi x})$ as $x\to+\infty$. Therefore by analytical continuation one has the following identity valid for $s\in\mathbb{C}$

$$s(s-1)4\pi^{-s}\zeta(s)\Gamma(s)L_1(s)=1+s(s-1)\int_1^\infty (x^{-s}+x^{s-1})\psi_0(x)dx, \tag{2.7}$$

which proves part (b) of the theorem.

Let $s=1/2+it$, $t\in\mathbb{R}$. Then

$$1+s(s-1)\int_1^\infty (x^{-s}+x^{s-1})\psi_0(x)dx=$$

$$1+s(s-1)\int_1^\infty \frac{d}{dx}\Big\{\psi_0(x)\Big(\frac{x^{-s+1}}{1-s}+\frac{x^s}{s}\Big)\Big\}dx-s(s-1)\int_1^\infty \psi_0'(x)\Big(\frac{x^{-s+1}}{1-s}+\frac{x^s}{s}\Big)dx=$$

$$1+\psi_0(1)+\int_1^\infty \psi_0'(x)\Big(sx^{1-s}+(1-s)x^s\Big)dx=$$

$$1+\psi_0(1)-\int_1^\infty \frac{d}{dx}\Big(x^2\psi_0'(x)(x^{-s}+x^{s-1})\Big)dx+\int_1^\infty \Big\{x^2\psi_0'(x)\Big\}'\Big(x^{-s}+x^{s-1}\Big)dx=$$

$$1+\psi_0(1)+2\psi_0'(1)+\int_1^\infty \left\{x^2\psi_0'(x)\right\}'\left(x^{-s}+x^{s-1}\right)dx =$$

$$\int_1^\infty \left\{x^2\psi_0'(x)\right\}'\left(x^{-s}+x^{s-1}\right)dx.$$

In the last equality we have used the fact that the transformation formula for the theta function gives $x\psi_0(x)+x=\psi_0(1/x)+1$ and the derivative at $x=1$ yields

$$1+\psi_0(1)+2\psi_0'(1)=0.$$

Observe that if $s=1/2+it$, then

$$\left(x^{-s}+x^{s-1}\right)=\frac{2\cos(t\ln x)}{\sqrt{x}}.$$

Thus,

$$\int_1^\infty \left\{x^2\psi_0'(x)\right\}'\left(x^{-s}+x^{s-1}\right)dx = 2\int_1^\infty \left\{2\sqrt{x}\psi_0'(x)+x\sqrt{x}\psi_0''(x)\right\}\cos(t\ln x)dx.$$

Formula (2.2), i.e. part (a) of the theorem, follows changing variables $x=e^u$.

To prove the formula (2.4) substract the formula (2.3), namely,

$$4\pi^{-s}\zeta(s)\Gamma(s)L_1(s)=\frac{1}{s(s-1)}+\int_1^\infty (x^{-s}+x^{s-1})\psi_0(x)dx,$$

from Riemann's formula

$$\pi^{-s/2}\Gamma(s/2)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty (x^{s/2}+x^{(1-s)/2})\psi(x)\frac{dx}{x}.$$

getting, if $s=1/2+it$,

$$\zeta(s)\left\{4\pi^{-s}\Gamma(s)L_1(s)-\pi^{-s/2}\Gamma(s/2)\right\}=$$

$$8\int_1^\infty \{\psi(x)+\psi^2(x)\}\cos(t\ln x)\frac{dx}{\sqrt{x}}-2\int_1^\infty \psi(x)\cos\left(\frac{t}{2}\ln x\right)\frac{dx}{x^{3/4}}.$$

The last integral is equal to $2\int_1^\infty \psi(x^2)\cos(t\ln x)\frac{dx}{\sqrt{x}}$ after changing variables. Thus the last expression is equal to

$$4\int_1^\infty \{2\psi(x)+2\psi^2(x)-\psi(x^2)\}\cos(t\ln x)\frac{dx}{\sqrt{x}}.$$

The first equality of (2.4) follows from this as before, after changing variables $x = e^u$. The second one is a consequence of

$$\{2\psi(x) + 2\psi^2(x) - \psi(x^2)\} = \left\{ \frac{\theta_3^2(e^{-\pi x}) - \theta_3(e^{-\pi x^2})}{2} \right\},$$

which follows from (1.6).

Finally, observe that because of (1.7) one has

$$\left\{\theta_3^2(e^{-\pi x}) - \theta_3(e^{-\pi x^2})\right\} \sqrt{x} = \left\{\theta_3^2(e^{-\pi/x}) - \theta_3(e^{-\pi/x^2})\right\} \frac{1}{\sqrt{x}},$$

which yields that the stated kernel is an even function of u whenever $x = e^u$.

The proof is complete. □

3. Remarks on a paper of P. Walker

In this section we follow closely P. Walker's paper [7]. Putting $s = 1/2 + it$, $0 < \Re(s) < 1$ and replacing the terms $-1/s, -1/(1-s)$ by the divergent integrals $1/2\int_1^\infty x^{s/2-1}dx$, $1/2\int_1^\infty x^{(1-s)/2-1}dx$ in formula (1.1) we get

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty \psi(x)(x^{s/2} + x^{(1-s)/2})\frac{dx}{x} \\
&= \int_1^\infty (1/2 + \psi(x))(x^{s/2} + x^{(1-s)/2})\frac{dx}{x} \\
&= \int_0^\infty 2(1 + 2\psi(e^{2u}))e^{u/2}\cos(tu)du,
\end{aligned}$$

(recall that $\psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x}$). This is a **formal** identity, indeed the inner function in the last integral is not integrable on the real positive axis.

Walker's idea is to modify the above integral by inserting a suitable kernel (one may see this as if the term $1/(s-1) - 1/s$ were *absorbed* into the integral). He proved the following theorem. Here the set H_1 is defined as follows:

$$H_1 := i(H - 1/2) \cap \{-i(H - 1/2)\},$$

where

$$H := \{z : \Re z < 0\} \cup \left\{z : \Re z \geq 0, |2(1 + \sqrt{1+z^2})^{-1} exp(\sqrt{1+z^2} - 1)| < 1\right\}.$$

Theorem 3.1. *(P. Walker [7]) Define for* $n = 1, 2, 3, \ldots$, $t = t_1 + it_2$, $t_1, t_2 \in \mathbb{R}$,

$$P_n(t) := \int_0^n \left(1 - \frac{u^2}{n^2}\right)^n 2(1 + 2\psi(e^{2u}))e^{u/2}\cos(tu)du. \tag{3.1}$$

Then for $s = 1/2 + it$

$$P_n(t) \to \pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{3.2}$$

uniformly on each compact subset (in the variable t) contained in H_1.

It is important to note that the set H_1 contains the closed critical strip

$$\{t| - 1/2 \le \Im(t) \le 1/2; \Re(t) \ge 2\}.$$

A first aim of this section is to give an analogous result. Recall that

$$\psi_0(x) := 4\psi(x)\{1 + \psi(x)\}.$$

Theorem 3.2. *Let $n = 1, 2, 3, \ldots$ and set*

$$Q_n(t) := \int_0^n \left(1 - \frac{u^2}{n^2}\right)^n 2(1 + \psi_0(e^u))e^{u/2}\cos(tu)du. \tag{3.3}$$

Then, if $s = 1/2 + it$,

$$Q_n(t) \to 4\pi^{-s}\zeta(s)\Gamma(s)L_1(s), \tag{3.4}$$

uniformly on each compact subset (in the variable t) contained in H_1.

One can summarize the results in [7] in the following lemma whose proof follows easily from the results given in that paper.

Lemma 3.3. *If $t \in H_1$ and $s = 1/2 + it$, then*

$$n\int_0^1 \{(1 - u^2)e^{su}\}^n\, du \to -\frac{1}{s},$$

uniformly on each compact set (in the variable t) contained in H_1.

Proof. (Theorem 3.2) Assume $s = \frac{1}{2} + it$. We slice the integral of $Q_n(t)$ in two parts: firstly, due to the rapid decay at infinity of $\psi_0(e^u)$ and the fact that $(1 - u^2/n^2)^n$ increases to unity as $n \to \infty$ one has

$$\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n 2\psi_0(e^u)e^{u/2}\cos(tu)du \to$$

$$\int_0^\infty 2\psi_0(e^u)e^{u/2}\cos(tu)du = \int_1^\infty (x^{-s} + x^{s-1})\psi_0(x)dx$$

if $n \to \infty$ for any complex t.

Next,

$$\int_0^n \left(1-\frac{u^2}{n^2}\right)^n 2e^{u/2}\cos(tu)du = \int_0^n \left(1-\frac{u^2}{n^2}\right)^n e^{u/2}(e^{itu}+e^{-itu})du = \tag{3.5}$$

$$\int_0^n \left(1-\frac{u^2}{n^2}\right)^n \left\{e^{su}+e^{(1-s)u}\right\} du =$$

$$n\int_0^1 \left\{(1-u^2)e^{su}\right\}^n du + n\int_0^1 \left\{(1-u^2)e^{(1-s)u}\right\}^n du \to \frac{1}{s-1}-\frac{1}{s}$$

if $n\to\infty$ and $t\in H_1$ where the limit follows from Lemma 3.3 (notice that $t\in H_1 \Leftrightarrow \bar{t}\in H_1$.) The same formula holds uniformly on compact sets contained in H_1. □

4. Some formulas

The following theorems give closed form formulas for the Dirichlet series (1.4) and for the Riemann zeta-function.

Theorem 4.1. *a) If*

$$\chi(x) := 2e^{x/2}\int_0^\infty \frac{e^{-y^2}}{\cosh(e^x y\sqrt{\pi})}dy,$$

and $t\in\mathbb{C}$, $|\Im(t)|<1/2$, *then*

$$\frac{1}{\pi^{\frac{it}{2}+\frac{1}{4}}}\Gamma\left(it+\frac{1}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)L_1\left(it+\frac{1}{2}\right) = \int_0^\infty \chi(x)\cos(tx)dx. \tag{4.1}$$

b) If

$$\chi(x) := 2e^{x/2}\int_0^\infty \frac{e^{-y^2}}{1+2\cosh(e^x y 2\sqrt{\frac{\pi}{3}})}dy,$$

and $t\in\mathbb{C}$, $|\Im(t)|<1/2$, *then*

$$\frac{1}{2}\left(2\sqrt{\frac{\pi}{3}}\right)^{-it-1/2}\Gamma\left(it+\frac{1}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)L_2\left(it+\frac{1}{2}\right) \tag{4.2}$$

$$= \int_0^\infty \chi(x)\cos(tx)dx.$$

c) If $\chi(x) := 2e^{x/2}\int_0^\infty \frac{1}{(1+2\cosh(\pi y))\cosh(\pi\frac{\sqrt{3}}{2}e^x y)}dy$, *and* $t\in\mathbb{C}$, $|\Im(t)|<1/2$, *then*

$$\frac{2}{\pi}\left(\frac{\sqrt{3}}{2}\right)^{-it-\frac{1}{2}} L_1\left(it+\frac{1}{2}\right) L_2\left(-it+\frac{1}{2}\right)\Gamma\left(it+\frac{1}{2}\right)\Gamma\left(-it+\frac{1}{2}\right)$$

$$= \int_0^\infty \chi(x)\cos(tx)dx. \tag{4.3}$$

For the next theorem we need the definition of the open set H_2 which is $H_2 := \{t : -3/2 < \Im(t) < 3/2\} \cap H_1$. The set H_2 contains the closed critical strip

$$\{t| - 1/2 \leq \Im(t) \leq 1/2; \Re(t) \geq 2\}.$$

Theorem 4.2. *Set*

$$I(\alpha) := \frac{2\pi}{\alpha^3}\int_0^\infty \frac{e^{-\pi x/\alpha^4}}{e^{2\pi\sqrt{x}}-1}dx. \tag{4.4}$$

If $-1 < \Re(s) < 2$, *then the following equality holds:*

$$\frac{\Gamma(1-\frac{s}{2})\Gamma(s)}{2^{s-1}\pi^{s/2}}\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \frac{2I(\alpha)}{\alpha}\left\{\alpha^{2s-1} + \alpha^{-(2s-1)}\right\}d\alpha. \tag{4.5}$$

Moreover, set

$$R_n(t) := \int_0^n \left(1-\frac{u^2}{n^2}\right)^n 2\left(1+\frac{I(e^{u/2})}{e^{u/2}}\right)e^{u/2}\cos(tu)du. \tag{4.6}$$

Then

$$R_n(t) \to \frac{\Gamma(1-\frac{s}{2})\Gamma(s)}{2^{s-1}\pi^{s/2}}\zeta(s) \tag{4.7}$$

where $s = \frac{1}{2} + it$, *uniformly on each compact set (in the variable* t*) contained in open set* H_2.

Also, if $s = \frac{1}{2} + it$ *and* $-3/2 < \Im(t) < 3/2$, *then*

$$\frac{s(s-1)\Gamma(1-\frac{s}{2})\Gamma(s)}{2^{s-1}\pi^{s/2}}\zeta(s) = \tag{4.8}$$

$$8\pi\int_0^\infty\left\{\int_0^\infty \frac{e^{-\pi x e^{-2u}}}{e^{2\pi\sqrt{x}}-1}\left\{e^{-3u/2} - e^{-7u/2}5\pi x + e^{-11u/2}2\pi^2x^2\right\}dx\right\}\cos(ut)du.$$

The kernel in the last integral is an even function of u.

In this section we provide tools for the proofs of these theorems which are given in the next section. It is desirable to prove all these formulas in a unified way. So, we recall first some well-known arguments.

Lemma 4.3. *Assume that $I(\alpha), J(\alpha)$ are defined for real $\alpha > 0$ and, for fixed s, that $\alpha^{s-1}I(\alpha)$, $\alpha^{s-1}J(\alpha)$ are absolutely integrable in α on $[0,\infty], [0,1]$, respectively. Moreover, assume that they satisfy with $c_0 > 0$:*

$$I(\alpha) = \pm c_0 I\left(\frac{1}{\alpha}\right) + J(\alpha).$$

Then,

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = \int_1^\infty \frac{I(\alpha)}{\alpha}\left\{\alpha^s \pm c_0\alpha^{-s}\right\} d\alpha + \int_0^1 \alpha^{s-1}J(\alpha)d\alpha.$$

Futhermore, if one has $J \equiv 0$, $c_0 = 1$ and $s = it$, $t \in \mathbb{R}$, then setting $\chi(x) := 2I(e^x)$, the following formula holds

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = \int_0^\infty \chi(x)\cos(tx)dx,$$

in case one takes the plus sign. In case one takes the minus sign the following formula holds

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = i\int_0^\infty \chi(x)\sin(tx)dx.$$

Proof. One has

$$\int_0^1 \alpha^{s-1} I(\alpha) d\alpha = \pm c_0 \int_0^1 \alpha^{s-1} I(1/\alpha) d\alpha + \int_0^1 \alpha^{s-1}J(\alpha)d\alpha$$

$$= \pm c_0 \int_1^\infty \beta^{-s-1} I(\beta) d\beta + \int_0^1 \alpha^{s-1}J(\alpha)d\alpha.$$

where in the last equality we changed variables $\alpha = 1/\beta$. Inserting this in

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = \int_0^1 + \int_1^\infty,$$

yields

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = \int_1^\infty \frac{I(\alpha)}{\alpha}\left\{\alpha^s \pm c_0\alpha^{-s}\right\} d\alpha + \int_0^1 \alpha^{s-1}J(\alpha)d\alpha,$$

and the first part of the lemma follows.

If $J \equiv 0$, $I(\alpha) = \pm I(\frac{1}{\alpha})$ and $s = it$, $t \in \mathbb{R}$, then

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = \int_1^\infty \frac{I(\alpha)}{\alpha} \left\{\alpha^s \pm \alpha^{-s}\right\} d\alpha.$$

If one takes the plus sign, making the change of variable $\alpha = e^x$, one obtains

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = \int_0^\infty \chi(x) \cos(tx) dx.$$

The other case is similar. □

In the next two lemmas we gather and reformulate some of G. Hardy and S. Ramanujan's formulas which will be suitable for our purposes.

Lemma 4.4. *Let $I(\alpha)$ be any of the following functions stated below. Then $I(\alpha) = I(\frac{1}{\alpha})$ if $\alpha > 0$.*

a) $\sqrt{\alpha} \int_0^\infty \frac{e^{-x^2}}{\cosh \alpha x \sqrt{\pi}} dx.$

b) $\sqrt{\alpha} \int_0^\infty \frac{e^{-x^2}}{1+2\cosh \alpha x 2\sqrt{\frac{\pi}{3}}} dx.$

c) $\sqrt{\alpha} \int_0^\infty \frac{1}{(1+2\cosh \pi x)\cosh \alpha x \frac{\pi}{2}\sqrt{3}} dx.$

Proof. Formulas (a), (b), (c) correspond respectively to formulas (12), (11), (10) of [8], pp. 55. Formulas similar to (a), (b) were given by G.H. Hardy. We have rescaled the formulas given by Ramanujan. For example formula (12) of [8] pp. 55 is: if $I_0(\alpha') := \sqrt{\alpha}' \int_0^\infty \frac{e^{-x^2}}{\cosh \alpha' x} dx$ and $\alpha'\beta' = \pi$ then $I_0(\alpha') = I_0(\beta')$. Formula (a) follows putting $\alpha' = \alpha\sqrt{\pi}$, $\beta' = \beta\sqrt{\pi}$ and $I(\alpha) := I_0(\alpha\sqrt{\pi})/\sqrt[4]{\pi}$. □

Lemma 4.5. *If $I(\alpha), J(\alpha)$ are defined by*

$$I(\alpha) = \frac{2\pi}{\alpha^3} \int_0^\infty \frac{e^{-\pi x/\alpha^4}}{e^{2\pi\sqrt{x}} - 1} dx, \quad J(\alpha) = -\left(\alpha - \frac{1}{\alpha}\right),$$

then $I(\alpha) = I(\frac{1}{\alpha}) + J(\alpha)$ if $\alpha > 0$.

Proof. Rescaling formula (13) of [8] pp. 56, one obtains: if

$$I_0(\alpha) = 2(\alpha\pi)^{3/4} \int_0^\infty \frac{e^{-\alpha\pi x}}{e^{2\pi\sqrt{x}} - 1} dx, \quad J_0(\alpha) = \frac{1}{\pi^{1/4}} \left(\alpha^{1/4} - \frac{1}{\alpha^{1/4}}\right),$$

then $I_0(\alpha) = I_0(\frac{1}{\alpha}) + J_0(\alpha)$ if $\alpha > 0$. The formula of the lemma follows changing α by α^4. See [9] and [10] pp. 291. □

5. Proofs of Theorems 4.1 and 4.2

Proof. Notice that if $I(\alpha)$ is any of the functions defined in (a), (b), (c) of Lemma 4.4 then $I(\alpha) = O(\sqrt{\alpha})$ as $\alpha \to 0+$. As $I(\alpha) = I(1/\alpha)$ one has $I(\alpha) = O(1/\sqrt{\alpha})$ as $\alpha \to +\infty$. Therefore, we may apply Lemma 4.3 with $s = it$, $t \in \mathbb{R}$ to get

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = \int_0^\infty \chi(x) \cos(tx) dx.$$

Applying this to the function defined in (a) of Lemma 4.4 yields:

$$\begin{aligned}
&\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = \int_0^\infty \alpha^{s-1/2} \int_0^\infty \frac{e^{-x^2}}{\cosh \alpha x \sqrt{\pi}} dx d\alpha \\
&= \int_0^\infty e^{-x^2} \int_0^\infty \frac{\alpha^{s-\frac{1}{2}}}{\cosh \alpha x \sqrt{\pi}} d\alpha dx = \frac{1}{\pi^{\frac{s}{2}+\frac{1}{4}}} \int_0^\infty e^{-x^2} \int_0^\infty \frac{\beta^{s-\frac{1}{2}}}{\cosh \beta x} d\beta dx \\
&= \frac{1}{\pi^{\frac{s}{2}+\frac{1}{4}}} \int_0^\infty e^{-x^2} \int_0^\infty 2\beta^{s-\frac{1}{2}} (e^{-\beta x} - e^{-3\beta x} + e^{-5\beta x} - \cdots) d\beta dx \\
&= \frac{1}{\pi^{\frac{s}{2}+\frac{1}{4}}} \Gamma\left(s + \frac{1}{2}\right) \int_0^\infty \frac{2e^{-x^2}}{x^{s+\frac{1}{2}}} \Big(1 - \frac{1}{3^{s+\frac{1}{2}}} + \frac{1}{5^{s+\frac{1}{2}}} - \cdots \Big) dx \\
&= \frac{1}{\pi^{\frac{s}{2}+\frac{1}{4}}} \Gamma\left(s + \frac{1}{2}\right) \Gamma\left(\frac{1}{4} - \frac{s}{2}\right) \Big(1 - \frac{1}{3^{s+\frac{1}{2}}} + \frac{1}{5^{s+\frac{1}{2}}} - \cdots \Big).
\end{aligned}$$

Thus, for $t \in \mathbb{R}$,

$$\int_0^\infty \chi(x) \cos(tx) dx = \frac{1}{\pi^{\frac{it}{2}+\frac{1}{4}}} \Gamma\left(it + \frac{1}{2}\right) \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \Big(1 - \frac{1}{3^{it+\frac{1}{2}}} + \frac{1}{5^{it+\frac{1}{2}}} - \cdots \Big),$$

where $\chi(x) = 2e^{x/2} \int_0^\infty \frac{e^{-y^2}}{\cosh e^x y \sqrt{\pi}} dy$. This proves part (a) of the theorem in case that t is real. The general case follows by analytical continuation: the integral $\int_0^\infty \chi(x) \cos(tx) dx$ defines an analytical function if $t = t_1 + it_2$ is a complex number with $|Im(t) = t_2| < 1/2$ since $\chi(x)$ is continuous on $[0, +\infty)$, $\chi(x) = O(e^{-x/2})$ as $x \to \infty$ and $|\cos(tx)| = O(e^{xt_2})$ as $x \to \infty$. This proves (a).

Proofs of the other formulas are similar.

Using Lemma 4.3 with the function defined in (b) of Lemma 4.4 with $s = it$, $t \in \mathbb{R}$

yields

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = \int_0^\infty \alpha^{s-\frac{1}{2}} \int_0^\infty \frac{e^{-x^2}}{1+2\cosh \alpha x 2\sqrt{\frac{\pi}{3}}} dx d\alpha$$

$$= \int_0^\infty e^{-x^2} \int_0^\infty \frac{\alpha^{s-\frac{1}{2}}}{1+2\cosh \alpha x 2\sqrt{\frac{\pi}{3}}} dx d\alpha$$

$$= \left(2\sqrt{\frac{\pi}{3}}\right)^{-s-\frac{1}{2}} \int_0^\infty e^{-x^2} \int_0^\infty \frac{\beta^{s-\frac{1}{2}}}{1+2\cosh \beta x} d\beta dx$$

$$= \left(2\sqrt{\frac{\pi}{3}}\right)^{-s-\frac{1}{2}} \int_0^\infty e^{-x^2} \int_0^\infty \frac{\beta^{s-\frac{1}{2}}}{e^{\beta x}(1+e^{-\beta x}+e^{-2\beta x})} d\beta dx$$

$$= \left(2\sqrt{\frac{\pi}{3}}\right)^{-s-\frac{1}{2}} \Gamma\left(s+\frac{1}{2}\right) \int_0^\infty \frac{e^{-x^2}}{x^{s+\frac{1}{2}}} \left(1 - \frac{1}{2^{s+\frac{1}{2}}} + \frac{1}{4^{s+\frac{1}{2}}} - \frac{1}{5^{s+\frac{1}{2}}} + \frac{1}{7^{s+\frac{1}{2}}} - \frac{1}{8^{s+\frac{1}{2}}} \cdots\right) dx$$

$$= \left(2\sqrt{\frac{\pi}{3}}\right)^{-s-\frac{1}{2}} \Gamma\left(s+\frac{1}{2}\right) \frac{1}{2}\Gamma\left(\frac{1}{4}-\frac{s}{2}\right) \left(1 - \frac{1}{2^{s+\frac{1}{2}}} + \frac{1}{4^{s+\frac{1}{2}}} - \frac{1}{5^{s+\frac{1}{2}}} + \frac{1}{7^{s+\frac{1}{2}}} - \frac{1}{8^{s+\frac{1}{2}}} \cdots\right).$$

Thus, if t is real and

$$\chi(x) = 2e^{x/2} \int_0^\infty \frac{e^{-y^2}}{1+2\cosh e^x y 2\sqrt{\frac{\pi}{3}}} dy,$$

then

$$\int_0^\infty \chi(x) \cos(tx) dx =$$

$$\frac{1}{2}\left(2\sqrt{\frac{\pi}{3}}\right)^{-it-1/2} \Gamma\left(it+\frac{1}{2}\right) \Gamma\left(\frac{1}{4}-\frac{it}{2}\right) \left(1 - \frac{1}{2^{it+\frac{1}{2}}} + \frac{1}{4^{it+\frac{1}{2}}} - \frac{1}{5^{it+\frac{1}{2}}} + \frac{1}{7^{it+\frac{1}{2}}} - \frac{1}{8^{it+\frac{1}{2}}} \cdots\right).$$

The rest of the proof goes as before.

Case (c) is similar and left to the reader. □

Proof. Let $I(\alpha), J(\alpha)$ be the functions of Lemma 4.5. It is clear from its integral definition that $I(\alpha) = O(1/\alpha^3)$ as $\alpha \to +\infty$. Also, from its transformation formula one gets $I(\alpha) = O(1/\alpha)$ as $\alpha \to 0+$.

Thus, if $1 < \Re(s) < 3$, one has that $\alpha^{s-1} I(\alpha)$ is absolutely convergent on $[0, \infty]$.

Also, (in the third equality in the formula that follows set $1/\alpha^4 = \beta$)

$$\int_0^\infty \alpha^{s-1} I(\alpha) d\alpha = 2\pi \int_0^\infty \alpha^{s-4} \int_0^\infty \frac{e^{-\pi x/\alpha^4}}{e^{2\pi\sqrt{x}}-1} dx d\alpha$$
$$= 2\pi \int_0^\infty \frac{1}{e^{2\pi\sqrt{x}}-1} \int_0^\infty \alpha^{s-4} e^{-\pi x/\alpha^4} d\alpha dx$$
$$= \frac{\pi}{2} \int_0^\infty \frac{1}{e^{2\pi\sqrt{x}}-1} \int_0^\infty \beta^{-(s+1)/4} e^{-\beta\pi x} d\beta dx$$
$$= \frac{\pi^{(s+1)/4}\Gamma\left(\frac{3-s}{4}\right)}{2} \int_0^\infty \frac{x^{(s-3)/4}}{e^{2\pi\sqrt{x}}-1} dx$$
$$= \pi^{(s+1)/4}\Gamma\left(\frac{3-s}{4}\right) \frac{\Gamma(\frac{s+1}{2})}{(2\pi)^{(s+1)/2}} \zeta\left(\frac{s+1}{2}\right).$$

Also, if $1 < \Re(s)$, then

$$\int_0^1 \alpha^{s-1} J(\alpha) d\alpha = -\frac{1}{s+1} + \frac{1}{s-1}.$$

These last two formulas and Lemma 2 yield for $1 < \Re(s) < 3$ the following equality:

$$\frac{\Gamma(\frac{3-s}{4})\Gamma(\frac{s+1}{2})}{2^{(s+1)/2}\pi^{(s+1)/4}} \zeta\left(\frac{s+1}{2}\right) = \int_1^\infty \frac{I(\alpha)}{\alpha} \left\{\alpha^s + \alpha^{-s}\right\} d\alpha - \frac{1}{s+1} + \frac{1}{s-1}.$$

Making the change $s \to 2s-1$ and multiplying by 2 yields that, for $1 < \Re(s) < 2$, the following holds:

$$\frac{\Gamma(1-\frac{s}{2})\Gamma(s)}{2^{s-1}\pi^{s/2}} \zeta(s) = \int_1^\infty \frac{2I(\alpha)}{\alpha} \left\{\alpha^{2s-1} + \alpha^{-(2s-1)}\right\} d\alpha - \frac{1}{s} + \frac{1}{s-1}.$$

But, as $I(\alpha) = O(1/\alpha^3)$ if $\alpha \to \infty$, the above formula is an identity for $-1 < \Re(s) < 2$ as can be seen by analytical continuation. This proves formula (4.5).

The proof of (4.7) is very similar to that given in Theorem 3.2: set $s = 1/2 + it$ and slice $R_n(t)$ in two parts. Firstly,

$$\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n 2\left(\frac{I(e^{u/2})}{e^{u/2}}\right) e^{u/2} \cos(tu) du \to$$

$$\int_0^\infty \left(\frac{I(e^{u/2})}{e^{u/2}}\right) e^{u/2} \cos(tu) du = \int_1^\infty \frac{2I(\alpha)}{\alpha} \left\{\alpha^{2s-1} + \alpha^{-(2s-1)}\right\} d\alpha,$$

for any complex t if $n \to \infty$. The limit follows due to the rapid decay at infinity of $I(e^{u/2})$ and the fact that $(1 - u^2/n^2)^n$ increases to unity as $n \to \infty$.

The other part is exactly (3.5) and formula (4.7) follows using (4.5). Notice that condition $1 < \Re(s) < 2$ is equivalent to $-3/2 < \Im(t) < 3/2$.

Finally, we prove (4.8). Then, multypling by $s(s-1)$ formula (4.5), we obtain

$$\frac{s(s-1)\Gamma(1-\frac{s}{2})\Gamma(s)}{2^{s-1}\pi^{s/2}}\zeta(s) = 1 + s(s-1)\int_1^\infty \frac{2I(\alpha)}{\alpha}\left\{\alpha^{2s-1}+\alpha^{-(2s-1)}\right\}d\alpha$$

$$= 1 + s(s-1)\int_1^\infty \frac{d}{d\alpha}\Big\{\frac{2I(\alpha)}{\alpha}\Big(\frac{\alpha^{-2s+2}}{2-2s}+\frac{\alpha^{2s}}{2s}\Big)\Big\}d\alpha$$

$$- s(s-1)\int_1^\infty \frac{d}{d\alpha}\left\{\frac{2I(\alpha)}{\alpha}\right\}\Big(\frac{\alpha^{-2s+2}}{2-2s}+\frac{\alpha^{2s}}{2s}\Big)d\alpha$$

$$= 1 + I(1) - s(s-1)\int_1^\infty \frac{d}{d\alpha}\left\{\frac{2I(\alpha)}{\alpha}\right\}\Big(\frac{\alpha^{-2s+2}}{2-2s}+\frac{\alpha^{2s}}{2s}\Big)d\alpha$$

$$= 1 + I(1) - \int_1^\infty \frac{d}{d\alpha}\left\{\frac{I(\alpha)}{\alpha}\right\}\Big((s-1)\alpha^{2s} - s\alpha^{-2s+2}\Big)d\alpha.$$

Changing variables $\alpha^2=\beta$ in the last integral one obtains that the last formula is equal to

$$1 + I(1) - \int_1^\infty \frac{d}{d\beta}\left\{\frac{I(\sqrt{\beta})}{\sqrt{\beta}}\right\}\Big((s-1)\beta^s - s\beta^{1-s}\Big)d\beta$$

$$= 1 + I(1) - \int_1^\infty \frac{d}{d\beta}\left\{\beta^2\frac{d}{d\beta}\left\{\frac{I(\sqrt{\beta})}{\sqrt{\beta}}\right\}\Big(\beta^{-s}+\beta^{s-1}\Big)\right\}d\beta$$

$$+ \int_1^\infty \frac{d}{d\beta}\left\{\beta^2\frac{d}{d\beta}\left\{\frac{I(\sqrt{\beta})}{\sqrt{\beta}}\right\}\right\}\Big(\beta^{-s}+\beta^{s-1}\Big)d\beta$$

$$= 1 + I'(1) + \int_1^\infty \frac{d}{d\beta}\left\{\beta^2\frac{d}{d\beta}\left\{\frac{I(\sqrt{\beta})}{\sqrt{\beta}}\right\}\right\}\Big(\beta^{-s}+\beta^{s-1}\Big)d\beta.$$

The last equality follows by noticing that $\int_1^\infty \frac{d}{d\beta}\left\{\beta^2\frac{d}{d\beta}\left\{\frac{I(\sqrt{\beta})}{\sqrt{\beta}}\right\}\Big(\beta^{-s}+\beta^{s-1}\Big)\right\}d\beta = I'(1) - I(1)$ which follows from $I(\alpha) = O(1/\alpha^3)$ as $\alpha \to +\infty$. Observe that the transformation formula for $I(\alpha)$ yields $1 + I'(1) = 0$ and therefore the last formula is equal to

$$\frac{s(s-1)\Gamma(1-\frac{s}{2})\Gamma(s)}{2^{s-1}\pi^{s/2}}\zeta(s) = \int_1^\infty \frac{d}{d\beta}\left\{\beta^2\frac{d}{d\beta}\left\{\frac{I(\sqrt{\beta})}{\sqrt{\beta}}\right\}\right\}\Big(\beta^{-s}+\beta^{s-1}\Big)d\beta.$$

Now, using

$$\frac{d}{d\beta}\left\{\beta^2\frac{d}{d\beta}\left\{\frac{I(\sqrt{\beta})}{\sqrt{\beta}}\right\}\right\} = 4\pi\int_0^\infty \frac{e^{-\pi x/\beta^2}}{e^{2\pi\sqrt{x}}-1}\left\{\frac{1}{\beta^2}-\frac{5\pi x}{\beta^4}+\frac{2\pi^2x^2}{\beta^6}\right\}dx,$$

in the last formula and changing variables $\beta = e^u$ one obtains the result.

One can see, using the transformation formula $I(\alpha) = I(1/\alpha) + 1/\alpha - \alpha$ (i.e. Lemma 4.5) that

$$\sqrt{\beta}\frac{d}{d\beta}\left\{\beta^2\frac{d}{d\beta}\left\{\frac{I(\sqrt{\beta})}{\sqrt{\beta}}\right\}\right\}$$

with $\beta = e^u$ is an even function of u.

The proof is complete. $\square$

6. Final remarks

The following theorem holds.

Theorem 6.1. *If $s \in \mathbb{C}$, then*

$$\frac{16}{\pi^{2s}}\zeta(2s)\zeta(2s-1)\Gamma(2s)\left(1-\frac{1}{4^{2s-1}}\right) = \tag{6.1}$$

$$\frac{1}{s-1} - \frac{1}{s} + 16\int_1^{\infty}\left\{\psi(x)+3\psi(x)^2+4\psi(x)^3+2\psi(x)^4\right\}(x^{2s-1}+x^{-(2s-1)})dx.$$

Proof. One has ([11] pp. 71)

$$\frac{\theta_3^4(q)-1}{8} = \sum_{n=1;4\nmid n}^{\infty}\frac{nq^n}{1-q^n}. \tag{6.2}$$

Integrating the right side of the above formula against x^{s-1} gives ($q = e^{-x}$):

$$\int_0^{\infty} x^{s-1}\sum_{n=1;4\nmid n}^{\infty}\frac{ne^{-nx}}{1-e^{-nx}}dx = \sum_{n=1;4\nmid n}^{\infty} n\Gamma(s)\left\{\frac{1}{n^s}+\frac{1}{(2n)^s}+\frac{1}{(3n)^s}+\cdots\right\} =$$

$$\Gamma(s)\zeta(s)\sum_{n=1;4\nmid n}^{\infty}\frac{1}{n^{s-1}} = \Gamma(s)\zeta(s)\zeta(s-1)\left(1-\frac{1}{4^{s-1}}\right),$$

Thus,

$$\Gamma(s)\zeta(s)\zeta(s-1)\left(1-\frac{1}{4^{s-1}}\right) = \int_0^{\infty} x^{s-1}\left\{\frac{\theta_3^4(e^{-x})-1}{8}\right\}dx =$$

$$\pi^s\int_0^{\infty} x^{s-1}\left\{\frac{\theta_3^4(e^{-x\pi})-1}{8}\right\}dx = \pi^s\Big\{\int_0^1 + \int_1^{\infty}\Big\}.$$

For the last integral $\int_0^1$ use the transformation formula

$$\frac{\theta_3^4(e^{-\pi x})-1}{8} = \frac{1}{x^2}\frac{\theta_3^4(e^{-\pi/x})-1}{8} + \frac{1}{8x^2} - \frac{1}{8}$$

(this follows from (1.7)) which yields the result

$$\int_0^1 = \int_1^\infty x^{1-s} \left\{ \frac{\theta_3^4(e^{-x\pi}) - 1}{8} \right\} dx + \frac{1}{8} \left\{ \frac{1}{s-2} - \frac{1}{s} \right\}.$$

Inserting this in the above formula gives

$$\frac{\Gamma(s)\zeta(s)\zeta(s-1)}{\pi^s} \left(1 - \frac{1}{4^{s-1}} \right) =$$

$$\frac{1}{8} \left\{ \frac{1}{s-2} - \frac{1}{s} \right\} + \int_1^\infty \left\{ \frac{\theta_3^4(e^{-\pi x}) - 1}{8} \right\} \left(x^{s-1} + x^{1-s} \right) dx.$$

Therefore, replacing in this formula s by $2s$, multiplying by 16 and using that $\theta_3(e^{-\pi x}) = 1 + 2\psi(x)$, one obtains (6.1). □

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