Tiling the plane with different hexagons and triangles

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Abstract. We prove that the plane can be tiled with equilateral triangles and regular hexagons of integer sides using exactly one of each family.

1. INTRODUCTION AND RESULTS. This paper deals with hybrid tilings of the plane using certain polygons. We write S_n, T_n, H_n for a square, an equilateral triangle and a regular hexagon respectively of sides n. Also we write TI_n for a triangle of sides $\{n, n, n\sqrt{2}\}$.

A result of W.T. Tutte [**6**] says that it is impossible to tile an equilateral triangle using unequal equilateral triangles, and E. Buchman [**1**] proved that a convex region cannot be tiled by a finite union of unequal equilateral triangles. K. Scherer [**5**] proved that the plane cannot be tiled with equilateral triangles of different sizes if one triangle is the smallest. At the other hand, a square can be *squared*: it is possible to tile a square using a finite number of different squares of integer side. The smallest such configuration was given by A.J.W. Duijvestijn (see [**3**, p. 78], [**2**]). The minimum number of squares required is 21. This a bidimensional result as a cube can not be *cubed*.

In a similar vein, the following remarkable result is due to Frederick and James Henle (see [**4**]).

Theorem. *The plane can be tiled with squares S_n, $n = 1, 2, 3, \ldots$ using exactly one of each.*

This theorem answers a question originally posed by S. Golomb with contributions of different authors; for its history and contributors see [**3**] and [**4**]. Thus the mentioned result of Scherer says that the above theorem is false if S_n is replaced by T_n. It is an easy exercise to see that it is impossible to tile the plane using different hexagons H_n. The following theorems show that certain hybrid tilings are possible.

Theorem A. *The plane can be tiled with triangles T_n, $n = 1, 2, 3, \ldots$ and hexagons H_n, $n = 1, 2, 3, \ldots$ using exactly one of each.*

Theorem B. *The plane can be tiled with triangles TI_n, $n = 1, 2, 3, \ldots$ and squares S_n, $n = 1, 2, 3, \ldots$ using exactly one of each.*

Theorem B will follow from results of [**4**]. Our main contribution is Theorem A which was inspired by and contains ideas of [**4**].

2. PROOF OF THEOREM A. We need the following definitions.

Definition. A figure is called **perfect** if it is composed of a finite number of different polygons (here the word different means that no rigid movement or a reflection followed by a rigid movement can transform one polygon into another).

Definition. When we add certain polygons to a perfect figure A to form a new perfect figure B we will say that we ‘puff figure A up to figure B.’

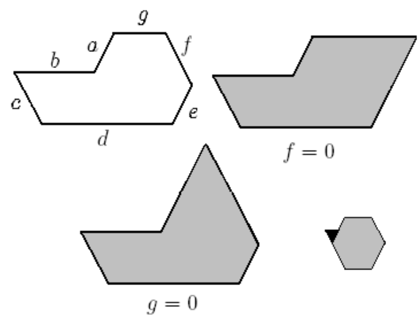

Figure 1.

We show in Figure 1 a heptagon with integer sides a, b, c, d, e, f, g; interior angles $\hat{ab} = 240°$, $\hat{bc} = 60°$ and the rest all equal to $120°$. Such a heptagon will be described as a 5-tuple (a, b, c, e, g). This is so because

$$\begin{aligned} a + c &= e + f, \quad \text{and} \\ a + b + g &= e + d. \end{aligned} \tag{1}$$

Hint: the first (respectively second) equation follows by projecting to a line perpendicular to the side d (respectively c).

Sometimes our heptagon will have side $f = 0\,(g \neq 0)$ or $g = 0\,(f \neq 0)$ and indeed it will be a hexagon (this is the case if $a + c = e$ or $a + b = e + d$ respectively). See Figure 1. We will keep our notation as a 5-tuple (a, b, c, e, g) if such is the case. In what follows a reflected heptagon or hexagon will be considered in the same way and all the figures will have integer sides.

Definition. A heptagon (or hexagon) defined by (a, b, c, e, g) is called **standard** if it is perfect (i.e. it is tiled by different regular hexagons and equilateral triangles of integer sides), $c \neq e$, and such that neither T_c nor T_e belong to its tiling.

Example 1. A standard heptagon defined by $(1, 1, 3, 2, 2)$ and tiled by T_1, H_2 is shown in Figure 1.

The following two lemmas are preparatory for Lemma 3, from which Theorem A follows easily. Lemma 3 says, roughly speaking, that given a standard heptagon (or hexagon) $\mathcal{H}_0$ such that T_{n_0} (or H_{n_0}) does not belong to its tiling (and with certain conditions), it is possible to puff it to a new standard heptagon (or hexagon) $\mathcal{H}_1$ using T_{n_0} (or H_{n_0}). Repeating this, one tiles the plane exhausting all the hexagons and triangles.

In Lemma 1 and Lemma 2, a principal role is played by the first entry (which we distinguish) of the 5-tuple of a heptagon (or hexagon) defined by $(a, *, *, *, *)$; such lemmas show that one may puff conveniently such a heptagon (or hexagon) to a new heptagon (or hexagon) **without** changing this entry 'a.' This permits us to lower this entry and to eventually prove Lemma 3. Therefore, our arguments may be seen as inductive in that first entry (as in [**4**]).

Lemma 1. *Assume that one has a standard heptagon (or hexagon), which we call* $\boldsymbol{H}_0$*, defined by* (a, b_0, c_0, e_0, g_0)*. Also let* $N \geq 2$ *be an integer number. Then one can puff* $\boldsymbol{H}_0$ *up to a new standard heptagon (or hexagon) defined by* (a, b, c, e, g)*, which we call* $\boldsymbol{H}$*, such that:*

I) $b = ka + g + i$, *with* $0 \le i < a$, $N \le k$;
II) $c_0 < c$, $e_0 < e$;
III) *to puff* $\boldsymbol{H}_0$ *up to* $\boldsymbol{H}$ *one uses* T_j *or* H_j *with* $j \ge \min\{c_0, e_0\}$;
IV) *the tiling of* $\boldsymbol{H}$ *contains neither the triangle* T_{n_1} *nor the hexagon* H_{n_1} *for some* $\min\{c_0, e_0\} < n_1 < \min\{c, e\}$.

Proof. The perfect heptagon (hexagon) $\boldsymbol{H}_0$ of the hypothesis is shown shadowed in Figure 2. As shown in that figure we add

$$T_{c_0}, T_{e_0}, T_{2c_0+d_0+e_0}, H_{c_0+d_0+e_0},$$

giving a new heptagon described by (a, b_1, c_1, e_1, g_0) with

$$b_1 = 2c_0 + d_0 + e_0 + b_0 > b_0.$$

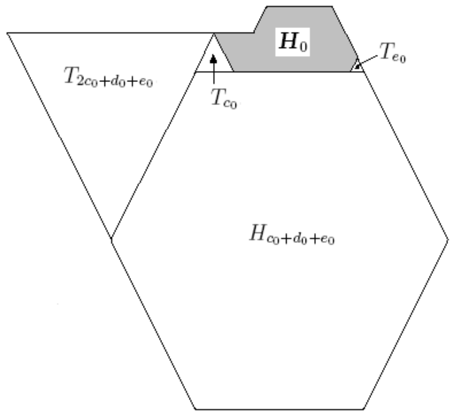

Figure 2.

This new heptagon (hexagon) is perfect for $\boldsymbol{H}_0$ contains no polygon of side $\ge c_0 + d_0 + e_0$ and T_{c_0}, T_{e_0} ($c_0 \ne e_0$) do not belong to its tiling because $\boldsymbol{H}_0$ is standard.

Also this new heptagon is standard for it contains no triangles T_{c_1}, T_{e_1} where $c_1 > e_1 > e_0$ and $c_1 > c_0$. Thus (II) and (III) are true in this case.

Repeating this procedure an appropriate number of times gives the required $\boldsymbol{H}$. Assume that (a, b_M, c_M, e_M, g_0) is the tuple which defines the heptagon (hexagon) obtained iterating M times. Then $\frac{b_M - g_0}{a} > N$ for suitably large M and (I) follows. Finally the sides (other than a and g_0) grow exponentially. From this (IV) follows taking, perhaps, a larger M. ■

Lemma 2. *Assume that one has a standard heptagon (or hexagon), which we call* $\boldsymbol{H}$, *defined by* (a, b, c, e, g) *such that* $b = ka + g + i$ *with* $0 \le i < a$ *and* $2 \le k$. *Then one can puff* $\boldsymbol{H}$ *up to a new standard hexagon* $\boldsymbol{H}'$ *defined by* (a, b', c', e', g') *and such that:*

I) $b' = (k-1)a + g' + i$;
II) $\boldsymbol{H}'$ *contains no hexagon* $H_{b'}$ *and no triangles* $T_{b'}, T_{d'}$ *or* T_j *with* $j \ge \frac{a+b'+g'}{2}$; *also if* $g \ne 0$ *then it does not contain* $T_{g'}$;

III) *to puff* $\boldsymbol{H}$ *up to* $\boldsymbol{H}'$ *one uses* T_j *or* H_j *with* $j \geq \min\{c, e\}$*;*
IV) $c < c'$, $e < e'$ *and* $b < b'$*;*
V) $\min\{c, e\} < \min\{b', c', d', e', g'\}$ *and* $f' = 0$*;*
VI) $d' < g' < b' < e'$.

Remark. Notice that by (I), one may repeatedly use the lemma $k-1$ times. It is clear by (VI) and (II) that if one uses the lemma at least twice (i.e. $k \geq 3$) then $\boldsymbol{H}'$ contains no triangle $T_{g'}$.

Proof. (I). The perfect heptagon (hexagon) $\boldsymbol{H}$ of the hypothesis is shown shadowed in Figure 3.

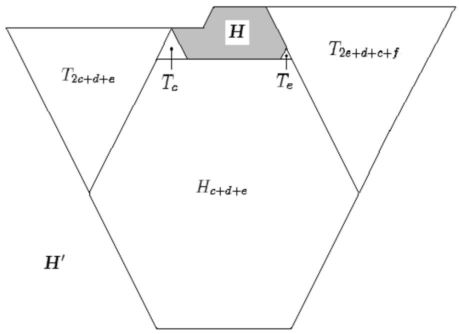

Figure 3.

We add four triangles and one hexagon as shown in the same figure. That is, we add

$$T_c,\ T_e,\ T_{2c+d+e},\ T_{2e+d+c+f},\ H_{c+d+e}.$$

The new hexagon $\boldsymbol{H}'$ is perfect because $\boldsymbol{H}$ can not contain figures of sides $\geq c+d+e$ and by hypothesis T_c, T_e $(c \neq e)$ are not in the tiling of $\boldsymbol{H}$. Observe that $(2c+d+e)+a = (2e+d+c+f)$, thus $T_{2c+d+e}, T_{2e+d+c+f}$ are different. Finally our new hexagon $\boldsymbol{H}'$ has

$$b' = 2c+d+e+b, \quad \text{and}$$
$$g' = g+2e+d+c+f.$$

Equation (1) and our hypothesis give

$$c = -a+e+f,$$
$$b = ka+g+i, \quad \text{and}$$
$$c+d+e = c+d+e.$$

Adding these equations term wise yields $b' = (k-1)a + g' + i$.

(II) and (VI). Notice that

$$b' = 2c+d+e+b,$$

$$\begin{aligned}
c' &= 3c + 2d + 2e < e' = c' + a,\\
d' &= c + d + e, \quad \text{and}\\
g' &= g + 2c + d + e + a,
\end{aligned}$$

(for the last equality use identity (1)). As $\boldsymbol{H}$ can not contain figures of sides $\geq c + d + e$, then $T_{b'}, T_{c'}, T_{e'}, T_{d'}, H_{b'}$ do not belong to the tiling of $\boldsymbol{H}$.

We can show that

$$\{e \neq c\} < d' < 2c + d + e < 2e + d + c + f \leq g' < b' < c' < e',$$

and therefore all the polygons

$$T_e,\ T_c,\ T_{d'},\ T_{2c+d+e},\ T_{2e+d+c+f},\ T_{g'},\ T_{b'},\ T_{c'},\ T_{e'},\ H_{c+d+e},\ H_{b'},$$

are different except possibly for $T_{2e+d+c+f}$ and $T_{g'}$ which depends on the fourth inequality, which is equivalent to $e + f \leq g + c + a$. Using (1) this is equivalent to $0 \leq g$. Thus if $g \neq 0$, then $\boldsymbol{H}'$ does not contain $T_{g'}$ as stated. Also $\boldsymbol{H}'$ is standard and (VI) follows.

The first, second, and seventh of the inequalities above are trivial. The third one is equivalent to $c < e + f$ which follows from (1). The fifth one is equivalent to $b > a + g$ which is true by hypothesis. Lastly, the sixth inequality is equivalent to $b < c + d + e$.

Finally, to show that $\boldsymbol{H}'$ does not contain triangles T_j with $j \geq \frac{a+b'+g'}{2}$, it suffices to notice that

$$\frac{a + b' + g'}{2} > 2e + d + c + f,$$

and that the triangle $T_{2e+d+c+f}$ is the biggest triangle we use to puff $\boldsymbol{H}$ up to $\boldsymbol{H}'$. The above inequality is equivalent to $a + b + c + g > e + f$, which follows from (1).

(III-IV-V). These properties are easy and they are left to the reader. ■

Lemma 3. *Assume that one has a standard heptagon (or hexagon), which we call $\mathcal{H}_0$, defined by $(A_0, B_0, C_0, E_0, G_0)$. Also assume that T_{n_0} (or H_{n_0}) does not belong to the tiling for some $n_0 < \min\{C_0, E_0\}$. Then one can puff $\mathcal{H}_0$, using T_{n_0} (or H_{n_0}), up to a new standard heptagon (hexagon) $\mathcal{H}_1$ defined by $(A_1, B_1, C_1, E_1, G_1)$. There exists n_1 such that neither T_{n_1} nor H_{n_1} belong to its tiling and $n_0 < n_1 < \min\{C_1, E_1\}$. Moreover the distance from $\mathcal{H}_0$ to the boundary of $\mathcal{H}_1$ is at least 1.*

Proof. **Step 1.** We write

$$\mathcal{H}_0 = \boldsymbol{H}_0, \quad \text{and}$$
$$(A_0, B_0, C_0, E_0, G_0) = (a, b_0, c_0, e_0, g_0).$$

Our hypothesis is equivalent to the following assertion:

> *$\boldsymbol{H}_0$ is standard and T_{n_0} (or H_{n_0}) does not belong to its tiling for some $n_0 < \min\{c_0, e_0\}$.*

We use Lemma 1 to put $\boldsymbol{H}_0$ in the hypothesis of Lemma 2. We choose N of that lemma large enough such that

$$\left(2-\frac{5n_0}{N}\right)>1. \tag{2}$$

The inequality forces $N \geq 6$.

Indeed using Lemma 1 one can puff $\boldsymbol{H}_0$ up to a new perfect heptagon (or hexagon) $\boldsymbol{H}$, defined by (a,b,c,e,g), such that

$$b=ka+g+i, \qquad k\geq N, \quad 0\leq i<a, \tag{3}$$

and by (III), T_{n_0} (or H_{n_0}) do not belong to the tiling of $\boldsymbol{H}$.

Notice that by (IV) and the hypothesis there exists n_1 such that

$$n_0<\min\{c_0,e_0\}<n_1<\min\{c,e\}, \tag{4}$$

and neither T_{n_1} nor H_{n_1} belong to the tiling of $\boldsymbol{H}$.

Step 2. Now $\boldsymbol{H}$ satisfies the hypothesis of Lemma 2. By the remark after Lemma 2 we apply it $k-1$ times ($\geq N-1\geq 5$ by (3)) to produce a puffing from $\boldsymbol{H}$ up to $\boldsymbol{H}'$ defined by (a,b',c',e',g') and

$$\begin{aligned} b' &= a+g'+i, \quad \text{with} \\ b' &\geq N. \end{aligned} \tag{5}$$

This last inequality follows from (IV).

Notice that by (III), (IV) and (4) this puffing does not add T_{n_0} (or H_{n_0}) or T_{n_1} or H_{n_1}. Also by (V) one has

$$\min\{c,e\}<\min\{b',c',d',e',g'\}, \quad \text{and} \tag{6}$$

$$f'=0. \tag{7}$$

Finally, observe that by the same remark, the triangle $T_{g'}$ does not belong to the tiling of $\boldsymbol{H}'$.

Step 3. Noticing that $\boldsymbol{H}'$ is standard and using (VI), (II), (4) and (6) one notices the following remark.

One may add freely the following different figures

$$T_{d'},\ T_{g'},\ T_{b'},\ T_{e'},\ H_{b'},\ T_{n_0}(\textit{or } H_{n_0}),\ T_{n_1},H_{n_1}, \tag{8}$$

to $\boldsymbol{H}'$ *to form a new figure keeping it perfect.*

Now we distinguish two cases: either $i>0$ or $i=0$.

If $0<i$ we puff $\boldsymbol{H}'$ (which by (7) is a hexagon) up to the perfect heptagon $\boldsymbol{H}''$ defined by (i,b'',c'',e'',g'') adding $H_{b'},T_{g'},T_{e'}$, see Figure 4.

But T_{n_0} (or H_{n_0}), T_{n_1} or H_{n_1} do not belong to the tiling of $\boldsymbol{H}''$ and

$$n_0<n_1<\min\{c'',e''\}. \tag{9}$$

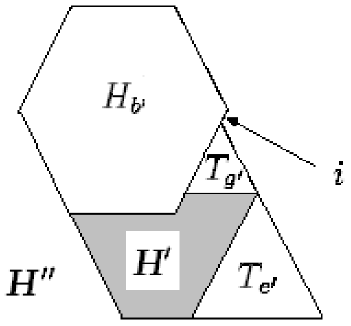

Figure 4.

By (3) one has $i < a$ and therefore in the tuple (i, b'', c'', e'', g'') we have lowered the first entry. Notice that the triangles $T_{c''=d'+e'}, T_{e''=b'}$ ($e'' < c''$ by (VI)) do not belong to the tiling of $\boldsymbol{H}''$ (the first triangle is too big), i.e. $\boldsymbol{H}''$ is standard. Finally notice that the puffing from $\boldsymbol{H}_0$ up to $\boldsymbol{H}''$ uses figures of sides $\geq \min\{c_0, e_0\}$.

Gathering what we have yields the following:

$\boldsymbol{H}''$ is standard and neither T_{n_0} (or H_{n_0}) nor T_{n_1} nor H_{n_1} belong to its tiling with $n_0 < n_1 < \min\{c'', e''\}$.

Therefore, one may go to Step 1 (taking the same N) and repeat until we have $i = 0$.

If $i = 0$ then using (5) we have

$$b' = a + g', \quad b' \geq N, \tag{10}$$

and then we puff $\boldsymbol{H}'$ up to the perfect hexagon ***He*** adding $H_{b'}, T_{g'}$; but now ***He*** is a hexagon whose interior angles are all equal to $120°$. See Figure 5.

We add T_{n_0} (or H_{n_0}) as shown in Figure 5 and Figure 6 respectively. In any case it will give the desired perfect figure $\mathcal{H}_1$ of our lemma defined by $(A_1, B_1, C_1, E_1, G_1)$ such that:

i) neither T_{C_1} nor T_{E_1} belong to its tiling and $C_1 \neq E_1$, i.e. $\mathcal{H}_1$ is standard;
ii) neither T_{n_1} nor H_{n_1} belong to its tiling and $n_0 < n_1 < \min\{C_1, E_1\}$;
iii) $\mathcal{H}_1$ contains T_{n_0} (or H_{n_0}) in its tiling.

Note: if one follows this process, a ‘belt’ of thickness at least one is formed around our original $\boldsymbol{H}_0 = \mathcal{H}_0$.

The case for the triangle T_{n_0} is clear from Figure 5: we add to ***He*** the triangles $T_{e'}$ and T_{n_0}. The so built hexagon $\mathcal{H}_1$ (here $A_1 = n_0$) is standard by remark (8).

The case for the H_{n_0} is as follows. The inequalities

$$d' < g' < b' < 2b' - 5n_0 < 2b' - 4n_0 < \cdots$$
$$\cdots < 2b' - n_0 < 3b' + g' - n_0 < 5b' + g' - n_0,$$

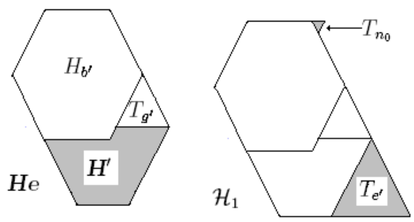

Figure 5.

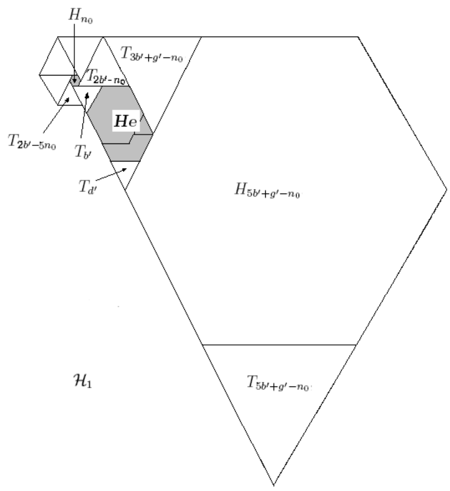

Figure 6.

hold and we add all triangles with the above sides (except for $T_{g'}$) and the hexagons $H_{5b'+g'-n_0}, H_{n_0}$ to ***He*** as shown in Figure 6, where ***He***, H_{n_0} are shadowed. All these inequalities follow from (VI) but the third one, which follows from (2) and (10):

$$2b' - 5n_0 = b'\Big(2 - \frac{5n_0}{b'}\Big) \geq b'\Big(2 - \frac{5n_0}{N}\Big) > b'.$$

It is clear from remark (8) that $T_{d'}$, $T_{g'}$, $T_{b'}$ do not belong to the tiling of $\boldsymbol{H}'$. The triangles

$$T_{3b'+g'-n_0},\ T_{5b'+g'-n_0},\ T_{2b'-in_0}$$

$i = 1, \dots, 5$ do not belong to the tiling of $\boldsymbol{H}'$ by (II) of Lemma 2. Indeed by (10) one has $\frac{a+b'+g'}{2} = b'$. Therefore, the heptagon $\mathcal{H}_1$ is perfect and it is easy to see that it is standard. ■

Proof of Theorem A. The proof follows easily from Lemma 3 taking n_0 to be the minimum with the property so stated. The lemma can be used again repeatedly and, in this way, we exhaust all the possible triangles and hexagons.

One may start the process with the heptagon of Example 1: H_1 does not belong to its tiling and $1 < \min\{C_0, E_0\} = \min\{3, 2\} = 2$. ■

3. PROOF OF THEOREM B. An **ell** is a polygon as shown in Figure 7, whose interior angles are all equal to $90°$ except one which is equal to $270°$.

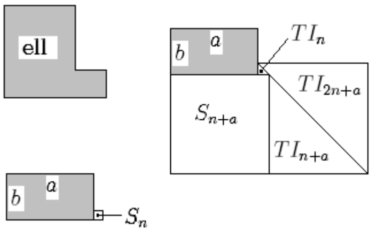

Figure 7.

One may summarize the key results of [**4**] as follows.

Lemma 4. *Every ell with integer side lengths can be puffed up to a perfect rectangle using squares of different integer sizes such that each new square is so large that no translate of it is a subset of the ell. Moreover the distance of the ell from the boundary of the rectangle is at least one.*

Based on this lemma one can prove the theorem as follows.

Step 0. Start with S_1.

Step 1. Assume that one has a perfect rectangle R of sides $a \geq b$. Let $n = \min\{m : S_m$ or TI_m does not belong to the tiling of $R\}$. If $n < b$ we puff R up to a perfect ell by adding either S_n or TI_n, TI_{n+a}, TI_{2n+a}, and S_{n+a} as shown in Figure 7. Otherwise we puff R up by adding a square S_m with $m > a$.

Step 2. We puff the given perfect ell up to a rectangle according to Lemma 4. Then go back to Step 1.

ACKNOWLEDGMENT. We are very grateful to the referees for their very careful reading and their constructive remarks.

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