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An upper bound for the distance to finitely generated ideals in Douglas algebras

by

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Abstract. Let f be a function in the Douglas algebra A and let I be a finitely generated ideal in A. We give an estimate for the distance from f to I that allows us to generalize a result obtained by Bourgain for H^∞ to arbitrary Douglas algebras.

1. Introduction. The theory of division and multiplication in H^∞, the algebra of bounded analytic functions, is well understood. One may also view these results as contributing to our understanding of the behavior of the ideals in this algebra. For example, if f and g are bounded analytic functions and $|f(z)| \le |g(z)|$ for all points z in the open unit disk, $\mathbb{D}$, then f is divisible by g; in other words, f is in the principal ideal generated by g. One can consider the corona theorem [3] to be a statement about ideals as well: if $f_1, \ldots, f_n$ are bounded analytic functions and

$$|f_1(z)| + \ldots + |f_n(z)| \ge \delta > 0 \quad \text{for all } z \in \mathbb{D},$$

then 1 is in the ideal generated by $f_1, \ldots, f_n$. Wolff's generalization of the corona theorem ([6, p. 329]) states that if f and $f_1, \ldots, f_n$ are bounded analytic functions satisfying

$$|f(z)| \le |f_1(z)| + \ldots + |f_n(z)| \quad \text{on } \mathbb{D},$$

then f^3 is in the ideal generated by $f_1, \ldots, f_n$. In [2], Bourgain proved that if α is a real-valued function satisfying $\alpha(t)/t \to 0$ as $t \to 0$ and if $f, f_1, \ldots, f_n \in H^\infty$ are such that

$$|f| \le \alpha(|f_1| + \ldots + |f_n|) \quad \text{on } \mathbb{D},$$

then f is in the closed ideal I generated by $f_1, \ldots, f_n$. More recently, attention has turned to similar questions in algebras other than H^∞. While these results are about ideals in the algebras under consideration, they offer insight into the structure of the algebra, division and factorization of functions.

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Let A be a uniform algebra. The maximal ideal space of A is defined to be

$$M_A = \{\varphi : A \to \mathbb{C} : \varphi \text{ is linear, multiplicative, and } \varphi \neq 0\}.$$

Provided with the weak-star topology, M_A is a compact Hausdorff space. The Gelfand transform, defined by $\widehat{f}(\varphi) = \varphi(f)$ for $f \in A$ and $\varphi \in M_A$, embeds A isometrically and isomorphically onto a closed subalgebra of $C(M_A)$. In what follows, we will identify a function f with its Gelfand transform and consider A as a uniform algebra on M_A. For H^∞, the multiplicative linear functional which is evaluation at a point z of the open unit disk is identified with the point and we think of the disk as contained in M_{H^∞}. The corona theorem is equivalent to the statement that the disk is dense in the maximal ideal space of H^∞. Thus, we may reformulate the theorems above in terms of the maximal ideal space. For example, we know that if $|\varphi(f)| \leq |\varphi(g)|$ for all $\varphi \in M_{H^\infty}$, then f is divisible by g in H^∞.

In this paper, we consider closed subalgebras A of L^∞ on the unit circle containing H^∞. Such algebras are called *Douglas algebras* (see [6, IX] for definitions and general background). We then ask the questions above for these algebras. The techniques involved in generalizing results for H^∞ are necessarily different, as there is no space as natural and as easy to work with as the disk for a general Douglas algebra. A simple illustration of the differences that we can find is that a formally valid version of Bourgain's theorem for a general Douglas algebra requires imposing the additional condition $\alpha(0) = 0$. The density of the disk in M_{H^∞} makes the last condition completely irrelevant when dealing with H^∞.

If $f, g \in A$ and $|f| \leq |g|$ on M_A, is f divisible by g in A? Such questions were first studied by Guillory and Sarason [8], who gave an example to show that one can have $|f| \leq |g|$ on M_A, but f is not divisible by g in A. On the other hand, for the algebra $H^\infty + C$, consisting of sums of bounded analytic functions and continuous functions, they found the existence of an integer N (independent of the function f) such that the condition $|f| \leq |g|$ on $M_{H^\infty + C}$ implies that f^N is divisible by g. Though their example showed that N cannot be chosen so that $N = 1$, K. Izuchi and Y. Izuchi [10] showed that $N = 2$ does indeed work. In this same vein, one may ask the following question. If $f, f_1, \ldots, f_n \in A$ and $|f| \leq |f_1| + \ldots + |f_n|$ on M_A, how far is f from the ideal generated by $f_1, \ldots, f_n$? From the comments above, it is clear that f need not be in the ideal generated by $f_1, \ldots, f_n$. In this paper we carefully examine Bourgain's proof and extend it to Douglas algebras. Our examination reveals that Bourgain's theorem can be stated in a more quantitative form. In particular, we will provide answers to questions about closed ideals by determining an upper bound for the distance from the function f to the ideal I in the Douglas algebra.

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2. Carleson measures and regions. In this section we state several known results that are required during the rest of the paper.

The *pseudohyperbolic metric* is defined for $z, \omega \in \mathbb{D}$ by

$$\varrho(z,\omega) = |(z-\omega)/(1-\overline{\omega}z)|.$$

For $0 < r \le 1$ and $\theta_0 \in [0, 2\pi)$ let

$$Q = \{z \in \mathbb{D} : 1 - r \le |z| < 1 \text{ and } \theta_0 \le \arg z \le \theta_0 + 2\pi r\}.$$

If μ is a complex measure on $\mathbb{D}$ such that there is a positive constant C with $|\mu|(Q) \le Cr$ for all such Q, then μ is called a *Carleson measure*. The smallest constant C will be denoted by $\|\mu\|_{\mathcal{C}}$. Consider the Cauchy–Riemann operator $\overline{\partial} = 2^{-1}(\partial/\partial x + i\partial/\partial y)$. In [11] Jones constructed a special solution in the distributional sense of the equation $\overline{\partial}G = \mu$, where μ is a Carleson measure (see also [6, pp. 358–361]). We summarize his result in the next lemma.

LEMMA 1. *Let μ be a Carleson measure on $\mathbb{D}$. Then there is an absolute constant $K > 0$ and a function $G(z)$ defined for every $z \in \mathbb{D}$ and for almost every $z \in \partial\mathbb{D}$, such that $\overline{\partial}G = \mu$, $\|G\|_{L^\infty(\partial\mathbb{D})} \le K\|\mu\|_{\mathcal{C}}$, and*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(z)G(z)\,dz = \int_{\mathbb{D}} f(z)\,d\mu(z) \quad \text{for all } f \in H^1. \tag{1}$$

The function G of the lemma is given constructively. When $\mu = g\lambda_\Gamma$, where $\Gamma \subset \mathbb{D}$ is a rectifiable curve whose arclength induces a Carleson measure λ_Γ, and g is a bounded function on Γ, the proof in [6, pp. 358–363] shows that if $0 < a < 1$ then G is bounded and analytic on $\{z \in \mathbb{D} : \varrho(z, \Gamma) > a\}$.

The next result was proved by Marshall (see [12] or [6, VIII. 4]) in transit to proving his part of the Chang–Marshall theorem.

LEMMA 2. *Given $0 < \alpha < 1$ there exists $\beta(\alpha)$, with $\alpha < \beta(\alpha) < 1$, such that for any inner function u there is a set $R \subset \mathbb{D}$ with rectifiable boundary such that*

$$\{|u(z)| \ge \beta(\alpha)\} \subset R \subset \{|u(z)| \ge \alpha\}$$

and $\|\lambda_{\partial R}\|_{\mathcal{C}} \le K$, where $K > 0$ is an absolute constant.

Clearly, if $0 < \alpha_1 < \alpha_2 < 1$ then the parameter $\beta(\alpha_2)$ in the lemma also works for α_1. This makes the lemma most interesting when α is close to 1. The lemma holds under the more general assumption that u is a harmonic function on $\mathbb{D}$ with $|u(e^{it})| = 1$ almost everywhere on $\partial\mathbb{D}$. This result allowed Marshall to show that if A is a Douglas algebra and $U \subset M_{H^\infty}$ is an open neighborhood of M_A, then there is an inner function b such that $|b| \equiv 1$ on M_A and $\sup\{|b(z)| : z \in \mathbb{D} \setminus U\} < 1$.

At the opposite extreme there is a result of Bourgain [2] stating that if b is a Blaschke product and $0 < \varepsilon < 1$, then there is a region $R \subset \mathbb{D}$ with rectifiable boundary ∂R such that $\|\lambda_{\partial R}\|_{\mathcal{C}} \le C$ (an absolute constant), and

$$\{|b(z)| \le \delta(\varepsilon)\} \subset R \subset \{|b(z)| \le \varepsilon\}, \tag{2}$$

where $0 < \delta(\varepsilon) < \varepsilon$. The main difference between this result and Carleson's original construction for the corona theorem is that $\|\lambda_{\partial R}\|_{\mathcal{C}}$ is bounded independently of ε. In [2, p. 166] it is stated without proof that (2) holds for every function in the unit ball of H^∞. We briefly sketch below a proof of this fact that is based on a standard argument given in [6, p. 334]. Factor $f = Fb$, where F is zero free on $\mathbb{D}$, $\|F\| \le 1$ and b is a Blaschke product. Changing $\delta(\varepsilon)$ it is enough to show that (2) holds separately for F and b. Fix an arbitrary $\varepsilon_0 \in (0,1)$ and let $0 < \varepsilon < 1$. Now let $p = p(\varepsilon) > 0$ be such that $\varepsilon_0^p = \varepsilon$. Applying Carleson's result to the function $F^{1/p}$ we see that there exists a γ_0 depending on ε_0 and a region $S \subset \mathbb{D}$ with rectifiable boundary such that

$$\{|F^{1/p}(z)| \le \gamma_0\} \subset S \subset \{|F^{1/p}(z)| \le \varepsilon_0\},$$

and $\|\lambda_{\partial S}\|_{\mathcal{C}} \le C(\varepsilon_0)$, a constant independent of ε. This clearly means that (2) holds for F with $\delta(\varepsilon) = \gamma_0^{p(\varepsilon)}$. Summing up, we can restate Bourgain's result as

Lemma 3. *For $0 < \varepsilon < 1$ there exists $0 < \delta(\varepsilon) < \varepsilon$ with the following property. If $f \in H^\infty$ has norm 1 then there is a region $R \subset \mathbb{D}$ such that*

$$\{|f(z)| \le \delta(\varepsilon)\} \subset R \subset \{|f(z)| \le \varepsilon\}, \tag{3}$$

and $\|\lambda_{\partial R}\|_{\mathcal{C}} \le C$, where $C > 0$ is an absolute constant.

We can assume without loss of generality that the region R in either of Lemma 2 or 3 is open or closed (in the topological space $\mathbb{D}$). Also, for technical reasons, it will be useful to assume that the function $\delta(\varepsilon)$ of Lemma 3 is continuous and strictly monotone. This can be achieved using the following elementary argument:

Suppose that a function $\delta(\varepsilon)$ satisfying the lemma has already been given and choose a sequence $\{r_k\}_{k\ge1}$ such that $0 < r_k < \delta(1/2^k)$ and $r_{k+1} < r_k$. Let δ^* be the function defined in each interval $[1/2^{k+1}, 1/2^k]$, with $k \ge 0$, by

$$\delta^*(t/2^k + (1-t)/2^{k+1}) = t r_{k+1} + (1-t) r_{k+2} \qquad (0 \le t \le 1).$$

As easily verified, δ^* is continuous, strictly increasing and

$$\delta^*(\varepsilon) \le r_{k+1} < \delta(1/2^{k+1}) \qquad \text{when } \varepsilon \in [1/2^{k+1}, 1/2^k].$$

Since δ satisfies the lemma, the two inequalities $\delta^*(\varepsilon) < \delta(1/2^{k+1})$ and $\varepsilon \ge 1/2^{k+1}$ immediately imply that δ^* does also.

3. The distance estimate. The next theorem generalizes a result discovered by Bourgain about the algebra H^∞ to arbitrary Douglas algebras [2]. The proof is based on Bourgain's proof; the essential difference is that some estimates involving $\overline{\partial}$-equations are no longer valid on $\mathbb{D}$, but rather on regions of $\mathbb{D}$ that are asymptotically close to the maximal ideal space of the Douglas algebra. In the next theorem, $\delta(\varepsilon)$ is the function of Lemma 3 and C denotes a positive absolute constant, not necessarily the same in each occurrence.

THEOREM 4. *Let A be a Douglas algebra and let $f, f_1, \ldots, f_n \in A$ be such that $\|f\| = 1$ and $\|f_j\| \leq 1$ for $j = 1, \ldots, n$. Let $I \subset A$ be the ideal generated by $f_1, \ldots, f_n$ and $0 < \varepsilon < 1$. Suppose that $|f_1| + \ldots + |f_n| > 0$ on the set $\{x \in M_A : |f(x)| \geq \delta(\varepsilon)\}$. Then there is an absolute constant $C > 0$ such that*

$$\operatorname{dist}(f, I) \leq \varepsilon + Cn^2 \sup\left\{\frac{|f(x)|}{|f_1(x)| + \ldots + |f_n(x)|} : x \in M_A,\ \delta(\varepsilon) \leq |f(x)| \leq \varepsilon\right\}.$$

Proof. In what follows, we will write δ in place of $\delta(\varepsilon)$. Also, in order to simplify the proof we have divided it in four steps, and present these below.

STEP 1: *Preliminary estimates.* Assume first that $f, f_1, \ldots, f_n \in H^\infty$. By Lemma 3 there exists an open set $R \subset \mathbb{D}$ such that

$$\{|f(z)| \leq \delta\} \subset R \subset \{|f(z)| \leq \varepsilon\} \tag{4}$$

and $\|\lambda_{\partial R}\|_{\mathcal{C}} \leq C$. Write $F = |f_1| + \ldots + |f_n|$ and take $\gamma > 0$ such that

$$\gamma < \inf\{F(x) : x \in M_A,\ |f(x)| \geq \delta\}.$$

Then there exists an open neighborhood U of M_A in M_{H^∞} such that

$$F(x) > \gamma \quad \text{whenever } x \in U \text{ and } |f(x)| \geq \delta. \tag{5}$$

Notice that $\gamma \leq n$ since $\|f_j\| \leq 1$ for all j. By a result of Dahlberg [5] (see also [6, VIII, Thm. 6.1]), for $\tau > 0$ there exist $v_j \in C^\infty(\mathbb{D})$, for $1 \leq j \leq n$, such that

$$\|f_j - v_j\|_{L^\infty(\mathbb{D})} < \tau \quad \text{and} \quad \|\,|\nabla v_j| dxdy\|_{\mathcal{C}} \leq C\tau^{-1}. \tag{6}$$

An elementary estimate yields $|f_1\overline{v}_1 + \ldots + f_n\overline{v}_n| \geq n^{-1}(|f_1| + \ldots |f_n|)^2 - n\tau$. Hence, if we take $\tau = \gamma^2/(2n^2)$, (5) gives

$$|f_1\overline{v}_1 + \ldots + f_n\overline{v}_n| \geq \left(\frac{F^2}{n} - \frac{\gamma^2}{2n}\right) \geq \frac{F^2}{2n} \quad \text{on } U \cap \mathbb{D} \cap \{|f| \geq \delta\}. \tag{7}$$

Since U is an open neighborhood of M_A, using Marshall's half of the Chang–Marshall theorem we obtain an inner function u such that $|u| \equiv 1$ on M_A and $|u| < \alpha$ on $\mathbb{D} \setminus U$ for some $0 < \alpha < 1$. Let $\beta(\alpha)$ be the parameter given by Lemma 2 and choose α_2 with $\beta(\alpha) < \alpha_2 < 1$. Therefore, applying Lemma 2 we obtain a closed region $R_1 \subset \mathbb{D}$ such that

$$R_2 := \{|u(z)| > \alpha_2\} \subset \{|u(z)| > \beta(\alpha)\} \subset R_1 \subset \{|u(z)| > \alpha\} \subset U \cap \mathbb{D}, \tag{8}$$

and $\|\lambda_{\partial R_1}\|_{\mathcal{C}} \le C$. Observe that $R_1 \setminus R \subset U \cap \mathbb{D} \cap \{|f| \ge \delta\}$ by (8) and (4). Let

$$g_j = \overline{v}_j (f_1 \overline{v}_1 + \ldots + f_n \overline{v}_n)^{-1} \chi_{R_1 \setminus R}, \qquad 1 \le j \le n,$$

where, as usual, χ_E denotes the characteristic function of the set E. By (7) and (6), on $U \cap \mathbb{D} \cap \{|f| \ge \delta\}$ we have

$$|f g_j| \le 2n \frac{|f v_j|}{F^2} \le 2n \frac{|f|}{F} \left(\frac{|f_j| + \tau}{F} \right) \le 2n \frac{|f|}{F} \left(1 + \frac{\tau}{F} \right) \le \frac{4n|f|}{F}, \tag{9}$$

where the last inequality holds because $\tau = \gamma^2/(2n^2) \le \gamma/2 < F$ on $U \cap \mathbb{D} \cap \{|f| \ge \delta\}$. Since the support of g_j is contained in $U \cap \mathbb{D} \cap \{|f| \ge \delta\}$, (5) and (9) yield

$$\|f g_j\|_{L^\infty(\mathbb{D})} \le 4n\gamma^{-1}. \tag{10}$$

Step 2: *Bounded solutions of some $\overline{\partial}$-equations.* We will use Lemma 1 to find solutions with $L^\infty(\partial\mathbb{D})$-norm control of the $\overline{\partial}$-equations

$$\overline{\partial} a_{j,k} = f g_j \overline{\partial} g_k \tag{11}$$

and

$$\overline{\partial} b_j = f g_j \chi_{R_2} \overline{\partial} \chi_{R_1 \setminus R}, \tag{12}$$

where $1 \le j, k \le n$. Here $\overline{\partial} \chi_{R_1 \setminus R} dx dy$ is a complex measure whose variation is essentially $\lambda_{\partial(R_1 \setminus R)}$. Since

$$\begin{aligned} \overline{\partial} g_k = {} & \overline{\partial} v_k \Big(\sum_{i=1}^n f_i \overline{v}_i \Big)^{-1} \chi_{R_1 \setminus R} \\ & - \overline{v}_k \Big(\sum_{i=1}^n f_i \overline{v}_i \Big)^{-2} \Big(\sum_{i=1}^n f_i \overline{\partial} \overline{v}_i \Big) \chi_{R_1 \setminus R} + \overline{v}_k \Big(\sum_{i=1}^n f_i \overline{v}_i \Big)^{-1} \overline{\partial} \chi_{R_1 \setminus R}, \end{aligned}$$

we find that (6), (7) and our choice of τ lead to

$$\begin{aligned} \|\overline{\partial} g_k \, dx dy\|_{\mathcal{C}} &\le C\tau^{-1} \frac{2n}{\gamma^2} + C(1+\tau) n \tau^{-1} \frac{4n^2}{\gamma^4} \\ &\quad + (1+\tau) \frac{2n}{\gamma^2} (\|\lambda_{\partial R_1}\|_{\mathcal{C}} + \|\lambda_{\partial R}\|_{\mathcal{C}}) \\ &\le C \left(\frac{n^3}{\gamma^4} + \frac{n^5}{\gamma^6} + \frac{n}{\gamma^2} \right) \le C \left(\frac{n}{\gamma} \right)^6, \end{aligned}$$

where the last inequality holds because $n/\gamma \geq 1$. Using this estimate, (10) and Lemma 1 we see that there exists a solution of (11) such that

$$\|a_{j,k}\|_{L^\infty(\partial\mathbb{D})} \leq C\left(\frac{n}{\gamma}\right)^7. \tag{13}$$

On the other hand, by (8) the measure $\mu = |fg_j|\chi_{R_2}\lambda_{\partial(R_1\setminus R)}$ is majorized by $|fg_j|\lambda_{R_2\cap\partial R}$. Since (4) and (8) imply that $R_2\cap\partial R \subset \{z\in U\cap\mathbb{D} : \delta \leq |f(z)| \leq \varepsilon\}$, (9) yields

$$\begin{aligned}\|\mu\|_{\mathcal{C}} &\leq \sup\{|fg_j| : z\in U\cap\mathbb{D},\ \delta\leq|f(z)|\leq\varepsilon\}\|\lambda_{\partial R}\|_{\mathcal{C}}\\ &\leq 4n\sup\{|f|F^{-1} : z\in U,\ \delta\leq|f(z)|\leq\varepsilon\}\|\lambda_{\partial R}\|_{\mathcal{C}}.\end{aligned} \tag{14}$$

Let $S = \sup\{|f|F^{-1} : x\in M_A,\ \delta\leq|f(x)|\leq\varepsilon\}$. Then given any $\zeta > 0$ we can choose the open neighborhood U of M_A so small that

$$\sup\{|f|F^{-1} : z\in U,\ \delta\leq|f(z)|\leq\varepsilon\} \leq S+\zeta.$$

Putting this estimate together with $\|\lambda_{\partial R}\|_{\mathcal{C}} \leq C$ in (14) we get

$$\|\mu\|_{\mathcal{C}} \leq Cn(S+\zeta).$$

Now Lemma 1 tells us that (12) admits a solution satisfying

$$\|b_j\|_{L^\infty(\partial\mathbb{D})} \leq Cn(S+\zeta), \tag{15}$$

where $\zeta > 0$ can be taken as small as we wish.

STEP 3: *Correcting the functions g_j.* Consider the functions

$$h_j = fg_j + \sum_{1\leq k\leq n}(a_{j,k}-a_{k,j})f_k - b_j \qquad (1\leq j\leq n). \tag{16}$$

For applications we need to show that h_j has a bounded boundary function a.e. on $\partial\mathbb{D}$ that satisfies an equality like (1). This clearly reduces to proving the same for fg_j.

PROPOSITION. *The function fg_j has a radial limit at almost every point of $\partial\mathbb{D}$, such that*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} k(z)(fg_j)(z)\,dz = \int_{\mathbb{D}} k(z)\bar\partial(fg_j)(z)\,dx\,dy \qquad \textit{for } k\in H^\infty. \tag{17}$$

Proof. Using the fact that the H^∞ functions f_i are uniformly continuous with respect to ϱ and (7), we see that there is some $0 < \xi_0 < 1$ such that $|\sum_{i=1}^n f_i\overline{v}_i|$ is bounded below away from zero on the set $V = \{z\in\mathbb{D} : \varrho(z, R_1\setminus R) < \xi_0\}$. Therefore

$$q := f\overline{v}_j(f_1\overline{v}_1+\ldots+f_n\overline{v}_n)^{-1} \in C^\infty(V)\cap L^\infty(V).$$

Let $0 < \xi < \min\{\xi_0, 1/4\}$. We can modify $\chi_{R_1\setminus R}$ on a pseudohyperbolic ξ-neighborhood of $\partial(R_1\setminus R)$ to obtain a function $\phi_\xi \in C^\infty(\mathbb{D},[0,1])$ such that $(1-|\omega|)|\nabla\phi_\xi(\omega)| \le c\xi^{-1}$ for some $c>0$. Since $\lambda_{\partial(R_1\setminus R)}$ is a Carleson measure, so is $|\nabla\phi_\xi|dxdy$.

Thus, $q\phi_\xi \in C^\infty(\mathbb{D}) \cap L^\infty(\mathbb{D})$, $q\phi_\xi \equiv q\chi_{R_1\setminus R}$ outside a pseudohyperbolic ξ-neighborhood of $\partial(R_1\setminus R)$, and $|\overline{\partial}(q\phi_\xi)|dxdy$ is a Carleson measure. Clearly, the measures $\overline{\partial}(q\phi_\xi)dxdy$ converge weak-star to $\overline{\partial}(q\chi_{R_1\setminus R})dxdy$ as $\xi\to 0^+$. Hence,

$$\int_{\mathbb{D}} k\overline{\partial}(q\chi_{R_1\setminus R})\,dx\,dy = \lim_{\xi\to 0^+}\int_{\mathbb{D}} k\overline{\partial}(q\phi_\xi)\,dx\,dy \quad \text{for } k\in H^\infty. \tag{18}$$

We claim that both functions ϕ_ξ and $\chi_{R_1\setminus R}$ have nontangential limits a.e. on $\partial\mathbb{D}$ and that they coincide. There is a standard procedure to pick an interpolating sequence $\{\omega_n\}$ in $\partial(R_1\setminus R)$ (see [6, p. 341]) such that $\varrho(z,\{\omega_n\})<\xi$ for every $z\in\partial(R_1\setminus R)$. Let b be the associated Blaschke product. If $z\in\mathbb{D}$ is such that $\varrho(z,z_0)<\xi$ for some $z_0\in\partial(R_1\setminus R)$ then

$$\varrho(z,\{\omega_n\}) \le \varrho(z,z_0)+\varrho(z_0,\{\omega_n\}) \le 2\xi < 1/2.$$

This means that $|b|<1/2$ on the set $W=\{z\in\mathbb{D} : \varrho(z,\partial(R_1\setminus R))<\xi\}$. The claim follows because b has nontangential limits of modulus 1 a.e. on $\partial\mathbb{D}$, and $\phi_\xi(z)=\chi_{R_1\setminus R}(z)$ when $\varrho(z,\partial(R_1\setminus R))>\xi$.

Since $|\nabla v_i|dxdy$ is Carleson by (6), the proof of Corollary 6.2 in [6, pp. 348–349] shows that v_i has radial limit a.e. on $\partial\mathbb{D}$ (for $1\le i\le n$). Hence, the same holds for $q\chi_{R_1\setminus R}$, and $q\phi_\xi = q\chi_{R_1\setminus R}$ in $L^\infty(\partial\mathbb{D})$. Therefore

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} k(z)(q\phi_\xi)(z)\,dz = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} k(z)(q\chi_{R_1\setminus R})(z)\,dz \quad \text{for } k\in H^\infty. \tag{19}$$

For a fixed ξ let $0<r<1$ and write $G_{\xi,r}(z)=(q\phi_\xi)(rz)$. By Green's theorem

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} k(rz)G_{\xi,r}(z)\,dz = \int_{\mathbb{D}} k(rz)(\overline{\partial}G_{\xi,r})(z)\,dx\,dy \quad \text{for } k\in H^\infty. \tag{20}$$

By changing the variable $\omega=rz$ (with $\omega=u+iv$), the second integral becomes

$$\int_{\mathbb{D}} k(\omega)r\overline{\partial}(q\phi_\xi)(\omega)\chi_{r\mathbb{D}}(\omega)r^{-2}\,du\,dv.$$

Since $|\overline{\partial}(q\phi_\xi)|\in L^1(dudv)$, we can apply the dominated convergence theorem to this integral as $r\to 1^-$, and since $\|G_{\xi,r}\|_{L^\infty(\mathbb{D})}\le\|q\|_{L^\infty(V)}$ for every r, we can do the same with the first integral in (20). Then

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} k(z)(q\phi_\xi)(z)\,dz = \int_{\mathbb{D}} k(z)\overline{\partial}(q\phi_\xi)(z)\,dx\,dy \quad \text{for } k\in H^\infty.$$

Since $q\chi_{R_1\setminus R}=fg_j$, the proposition follows from the above equality, (18) and (19). ■

By (17) and Lemma 1,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} k(z)h_j(z)\,dz = \int_{\mathbb{D}} k(z)\bar\partial h_j(z)\,dx\,dy \qquad \text{for } k \in H^\infty. \tag{21}$$

By (10) and (13) the $L^\infty(\partial\mathbb{D})$-norms of the functions h_j are bounded by $K_0 + \|b_j\|_{L^\infty(\partial\mathbb{D})}$, where $K_0 > 0$ is a constant that only depends on n and γ. Since $S \le 1/\gamma$ by (5), taking $\zeta < 1$ in (15) we obtain a function $K(n,\gamma) > 0$ such that

$$\|h_j\|_{L^\infty(\partial\mathbb{D})} \le K(n,\gamma). \tag{22}$$

Since

$$\sum_{k=1}^{n} f_k g_k = \chi_{R_1\setminus R} \tag{23}$$

we see that $\sum_{k=1}^n f_k\bar\partial g_k = \bar\partial\chi_{R_1\setminus R}$, and since R_1 is closed and R is open, $f\bar\partial g_j$ is supported on $R_1\setminus R$. This means that $f\bar\partial g_j = \chi_{R_1\setminus R} f\bar\partial g_j$. Therefore (11), (12) and (23) yield

$$\begin{aligned} \bar\partial h_j &= f\bar\partial g_j + \sum_{k=1}^{n}(fg_j\bar\partial g_k - fg_k\bar\partial g_j)f_k - \bar\partial b_j \\ &= f\bar\partial g_j\Big[\chi_{R_1\setminus R} - \sum_{k=1}^{n} f_k g_k\Big] + fg_j\sum_{k=1}^{n} f_k\bar\partial g_k - \bar\partial b_j \\ &= fg_j\bar\partial\chi_{R_1\setminus R} - \bar\partial b_j = fg_j\chi_{\mathbb{D}\setminus R_2}\bar\partial\chi_{R_1\setminus R}. \end{aligned} \tag{24}$$

By (10) the measure

$$\nu := fg_j\chi_{\mathbb{D}\setminus R_2}\lambda_{\partial(R_1\setminus R)}$$

is majorized by $Cn\gamma^{-1}\chi_{\mathbb{D}\setminus R_2}(\lambda_{\partial R_1} + \lambda_{\partial R})$. Therefore ν is a Carleson measure of the type considered in the comment that follows Lemma 1. It is supported on a curve contained in $\mathbb{D}\setminus R_2 = \{z \in \mathbb{D} : |u(z)| \le \alpha_2\}$. Choose α_3 with $\alpha_2 < \alpha_3 < 1$. If $z \in \mathbb{D}$ is such that $|u(z)| > \alpha_3$, then for every ω in the support set of ν, denoted by $\operatorname{supp}\nu$, the Schwarz–Pick inequality implies that

$$\varrho(z,\omega) \ge \varrho(u(z),u(\omega)) > \frac{\alpha_3 - \alpha_2}{1 - \alpha_3\,\alpha_2} := a > 0.$$

That is, $\varrho(z,\operatorname{supp}\nu) \ge a$ whenever $|u(z)| > \alpha_3$. By Lemma 1 and the remark that follows it, there exists a function G_j on $\mathbb{D}$ such that

(i) $\bar\partial G_j = fg_j\chi_{\mathbb{D}\setminus R_2}\bar\partial\chi_{R_1\setminus R}$,
(ii) $\|G_j\|_{L^\infty(\partial\mathbb{D})} \le C\|\nu\|_{\mathcal{C}}$, and
(iii) G_j is bounded and analytic on the set $\{z \in \mathbb{D} : |u(z)| > \alpha_3\}$.

By (i) and (24) the function $d_j := h_j - G_j$ satisfies the equation $\bar\partial d_j = 0$, and therefore is analytic. By (ii) and (22), $\|d_j\|_{L^\infty(\partial\mathbb{D})} < \infty$. In order to conclude that $d_j \in H^\infty$ we observe that by (1) and (21),

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} k(z)d_j(z)\,dz = \int\limits_{\mathbb{D}} k(z)\bar\partial d_j(z)\,dx\,dy = 0 \quad \text{for } k \in H^\infty.$$

Hence $h_j = d_j - G_j$ is bounded and analytic on $\{z \in \mathbb{D} : |u(z)| > \alpha_3\}$. Since $|u| \equiv 1$ on M_A, Theorem 5.2 of [6, p. 392] says that $h_j \in A$. By (16) and (23),

$$f\chi_{R_1} - \sum_{j=1}^n f_j h_j = f\,(\chi_{R_1} - \chi_{R_1\setminus R}) + \sum_{j=1}^n f_j b_j = f\chi_{R_1\cap R} + \sum_{j=1}^n f_j b_j. \tag{25}$$

Since $R_1 \cap R \subset \{|f| \le \varepsilon\}$ by (4), we have $\|f\chi_{R_1\cap R}\|_{L^\infty(\mathbb{D})} \le \varepsilon$. Finally, since (8) implies that $\chi_{R_1} \equiv 1$ in $L^\infty(\partial\mathbb{D})$, going back to (25) we see that

$$\begin{aligned}\Big\|f - \sum_{j=1}^n f_j h_j\Big\|_{L^\infty(\partial\mathbb{D})} &\le \|f\chi_{R_1\cap R}\|_{L^\infty(\mathbb{D})} + \sum_{j=1}^n \|b_j\|_{L^\infty(\partial\mathbb{D})} \\ &\le \varepsilon + n^2C(S+\zeta) \le \varepsilon + Cn^2S + \zeta',\end{aligned}$$

where $\zeta' = Cn^2\zeta > 0$, and the second inequality follows from (15). Since ζ can be taken arbitrarily small, we have $\operatorname{dist}(f, I) \le \varepsilon + Cn^2S$, as claimed.

STEP 4: *The general case.* Now suppose that $f, f_1, \ldots, f_n \in A$ are any functions satisfying the hypothesis of the theorem, and let $0 < \gamma < \inf\{|f_1| + \ldots + |f_n| : |f| \ge \delta(\varepsilon)\}$. By the Chang–Marshall theorem, for $\eta > 0$ there are $g, g_1, \ldots, g_n \in H^\infty$ with norm ≤ 1 and $\|g\| = 1$, and an inner function u with $|u| \equiv 1$ on M_A, so that

$$\|f - \overline{u}g\| < \eta \quad \text{and} \quad \|f_j - \overline{u}g_j\| < \eta \quad (1 \le j \le n). \tag{26}$$

If we fix some $\varepsilon_0 > \varepsilon$, our assumption that δ is a strictly increasing continuous function allows us to take η so small that $\delta(\varepsilon) < \delta(\varepsilon_0) - \eta$. Therefore (26) implies

$$\{\delta(\varepsilon_0) \le |g|\} \subset \{\delta(\varepsilon_0) - \eta \le |f|\} \subset \{\delta(\varepsilon) \le |f|\},$$

where the sets are considered as subsets of M_A. Hence, if η is sufficiently small, the above inclusions and (26) yield

$$|g_1| + \ldots + |g_n| \ge \sum_{j=1}^n (|f_j| - |f_j - \overline{u}g_j|) \ge \gamma - n\eta \ge \gamma/2 \quad \text{on } \{\delta(\varepsilon_0) \le |g|\}.$$

Let $G = |g_1| + \ldots + |g_n|$. In the previous case we proved that under these conditions, for an arbitrary $\zeta' > 0$ there exist $h_j \in A$ with $\|h_j\| \le K(n, \gamma/2)$, for $1 \le j \le n$, such that

$$(27)\qquad \Big\| g - \sum_{j=1}^{n} h_j g_j \Big\| \le \varepsilon_0 + Cn^2 \sup\Big\{ \frac{|g(x)|}{G(x)} : x \in M_A,\ \delta(\varepsilon_0) \le |g(x)| \le \varepsilon_0 \Big\} + \zeta'.$$

On the other hand, (26) implies that

$$\Big\| f - \sum_{j=1}^{n} h_j f_j \Big\| \le \|f - \overline{u}g\| + \Big\| \overline{u}(g - \sum_{j=1}^{n} h_j g_j) \Big\| + \sum_{j=1}^{n} \|h_j\| \cdot \|\overline{u}g_j - f_j\|$$
$$\le \eta + \Big\| g - \sum_{j=1}^{n} h_j g_j \Big\| + nK(n, \gamma/2)\eta.$$

Since $K(n, \gamma/2)$ does not depend on η or ε_0 and $\{\delta(\varepsilon_0) \le |g| \le \varepsilon_0\} \subset \{\delta(\varepsilon) \le |f| \le \varepsilon_0 + \eta\}$, by letting $\eta \to 0$, $\varepsilon_0 \to \varepsilon$ and applying both (26) and (27), we obtain the desired result. ∎

The technicalities involved in the proof of Theorem 4 are specific for a Douglas algebra other than H^∞. A simplified version of the proof also works for H^∞. However, in this particular case, this "simplified version" reduces to a careful examination of Bourgain's proof. While ideal theory for H^∞ has been widely examined, many questions remain open (including a complete description of the closed ideals [7]). Ideal theory for Douglas algebras other than H^∞ remains even more elusive due to the sort of study presented in the proof above.

4. Consequences and examples

COROLLARY 5. *Let A be a Douglas algebra and $f, f_1, \ldots, f_n \in A$. Suppose that there is a real-valued function α such that $\alpha(0) = 0$, $\alpha(t)/t \to 0$ as $t \to 0$ and*

$$(28)\qquad |f| \le \alpha(|f_1| + \ldots + |f_n|) \quad \text{on } M_A.$$

Then f belongs to the closed ideal I generated by $f_1, \ldots, f_n$.

Proof. The result above is clear if $f = 0$. If this is not the case, we may assume without loss of generality that $\|f\| = 1$ and $\|f_j\| \le 1$ for all j. Indeed, if $a > 0$ is a number such that $\|af_j\| \le 1$ for $1 \le j \le n$, then $|f|/\|f\| \le \widetilde{\alpha}(|af_1| + \ldots + |af_n|)$, where $\widetilde{\alpha}(t) = \|f\|^{-1}\alpha(a^{-1}t)$. Now, condition (28) clearly implies that the hypothesis of Theorem 4 is satisfied for every $\varepsilon > 0$. Let $\varepsilon > 0$ and write $F = |f_1| + \ldots + |f_n|$. If $\delta(\varepsilon) \le |f(x)| \le \varepsilon$, then

$$\frac{|f(x)|}{F(x)} \le m(x) := \min\Big\{ \frac{\varepsilon}{F(x)}, \frac{\alpha(F(x))}{F(x)} \Big\}.$$

By hypothesis $\sup\{m(x) : \delta(\varepsilon) \le |f(x)| \le \varepsilon\} \to 0$ as $\varepsilon \to 0$. The result now follows from Theorem 4. ∎

It is clear that the above corollary holds for $A = L^\infty$ under the relaxed hypothesis $\alpha(t) \le Ct$ for some $C > 0$. We will see that this is not the case for any other Douglas algebra. When $A = H^\infty$, by modifying an example of Rao [13], Bourgain showed that there are two Blaschke products b_1, b_2 such that $b_1 b_2$ is not in the closed ideal generated by b_1^2 and b_2^2, though clearly $|b_1 b_2| \le |b_1^2| + |b_2^2|$. In [7] it is shown that a modification of Bourgain's construction works for any Douglas algebra other than L^∞. That is, for every Douglas algebra $A \ne L^\infty$ there are Blaschke products b_1, b_2 (depending on A) such that $b_1 b_2$ is not in the closed ideal of A generated by b_1^2 and b_2^2. A more dramatic example can be given for a Douglas algebra different from L^∞ and H^∞, as we show below.

EXAMPLE. Let μ_x be the representing measure of $x \in M_{H^\infty}$. If $f \in L^\infty$, the formula

$$f(x) := \int f \, d\mu_x$$

determines a natural continuous extension of f to M_{H^∞}. When f belongs to a Douglas algebra A and $x \in M_A$ then this value of $f(x)$ coincides with the usual value given by the Gelfand transform.

Let A be a Douglas algebra different from L^∞ and H^∞. Then there exists a clopen set $E \subset M_{L^\infty}$ such that $\chi_E \in L^\infty \setminus A$. By Axler's theorem [1, Thm. 1] there is a Blaschke product b such that $b\chi_E \in H^\infty + C$. Since $H^\infty + C$ is regular on M_{L^∞} (see [1, Cor. 1]) and $E^c = M_{L^\infty} \setminus E$ is a clopen set, the theorem of [9, p. 190] says that

$$\begin{aligned} (29) \qquad b\chi_E \equiv 0 \quad \text{on } & \{x \in M_{H^\infty + C} : \operatorname{supp} \mu_x \cap E^c \ne \emptyset\} \\ & = \{x \in M_{H^\infty + C} : \chi_E(x) < 1\}. \end{aligned}$$

On the other hand, if $x \in M_{H^\infty + C}$ is such that $\chi_E(x) = 1$, then $\operatorname{supp} \mu_x \subset E$. Consequently,

$$(30) \qquad (b\chi_E)(x) = \int_{\operatorname{supp} \mu_x} b\chi_E \, d\mu_x = \int_{\operatorname{supp} \mu_x} b \, d\mu_x = b(x).$$

By (29) and (30) we have $|b\chi_E| \le |b|$ on $M_{H^\infty + C}$, and hence on M_A. However, if $f \in A$ then

$$\|b\chi_E - bf\| = \|\chi_E - f\| \ge \operatorname{dist}(\chi_E, A) > 0,$$

implying that $b\chi_E$ does not belong to the closed ideal of A generated by b.

Let I be an ideal in a Douglas algebra A and $f \in A$. Our next result shows how to use Theorem 4 to provide a sufficient condition for $f \in \bar{I}$, even when I is not finitely generated. The result, which cannot be deduced from Corollary 5, illustrates the main advantage of Theorem 4 over Corollary 5.

COROLLARY 6. *Let A be a Douglas algebra, $I \subset A$ an ideal and $f \in A$ be a function of norm* 1. *Suppose that there exist $\gamma > 0$ and a positive integer n*

such that for every $0 < \delta < 1$ *there are* $f_1, \ldots, f_n \in I$ *of norm at most* 1 *satisfying*

$$|f_1| + \ldots + |f_n| \geq \gamma \quad \text{on } \{x \in M_A : |f(x)| \geq \delta\}.$$

Then f *is in the closure of* I.

Proof. Given $0 < \varepsilon < 1$, let $\delta(\varepsilon)$ as in Theorem 4. By hypothesis there are $f_1, \ldots, f_n \in I$ such that $|f_1| + \ldots + |f_n| \geq \gamma$ on $\{|f| \geq \delta(\varepsilon)\}$. Theorem 4 then says that $\operatorname{dist}(f, I) \leq \varepsilon + Cn^2\varepsilon\gamma^{-1} \to 0$ as $\varepsilon \to 0$. ∎

For an ideal I in the Douglas algebra A let

$$Z(I) = \{x \in M_A : f(x) = 0 \text{ for all } f \in I\}, \qquad J = \{f \in A : f \equiv 0 \text{ on } Z(I)\}.$$

Then J is the largest ideal of A with the property that $Z(J) = Z(I)$, or equivalently, it is the intersection of all the maximal ideals that contain I.

It is clear that if $\bar{I}$ denotes the closure of I then $\bar{I} \subset J$, and we will see that Corollary 6 provides a sufficient condition for the reverse inclusion. In fact, suppose that there are $\gamma > 0$ and a positive integer n such that for every open neighborhood V of $Z(I)$ there exist $f_1, \ldots, f_n \in I$ of norm at most one with $|f_1| + \ldots + |f_n| \geq \gamma$ on $M_A \setminus V$. Therefore, if $f \in A$ is a normalized function with $f \equiv 0$ on $Z(I)$, then the set $V_\delta = \{x \in M_A : |f(x)| < \delta\}$ is an open neighborhood of $Z(I)$ for every $0 < \delta < 1$. Thus, the corollary says that $f \in \bar{I}$.

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