On the analytic structure of the H^∞ maximal ideal space

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Abstract

We characterize the algebra $H^\infty \circ L_m$, where m is a point of the maximal ideal space of H^∞ with nontrivial Gleason part $P(m)$ and $L_m : \mathbb{D} \to P(m)$ is the coordinate Hoffman map. In particular, it is shown that for any continuous function $f : P(m) \to \mathbb{C}$ with $f \circ L_m \in H^\infty$ there exists $F \in H^\infty$ such that $F|_{P(m)} = f$.

Introduction - Preliminaries

The Gelfand transform represents H^∞ as an algebra of continuous functions on its maximal ideal space $\mathcal{M}$ (provided with the weak star topology) via the formula $\hat{f}(\varphi) \stackrel{\text{def}}{=} \varphi(f)$, where $f \in H^\infty$ and $\varphi \in \mathcal{M}$. We will not write the hat of f unless the contrary is stated. Beginning with a seminal paper of Hoffman [6], many papers have studied the analytic behavior of H^∞ on parts of $\mathcal{M}$ others than the disk (see [1], [3], [4], [7] and [8]). While the whole picture seems to be unreachable, the present paper intends to throw some light into this never-ending program. A more precise statement of our result (Thm. 2.2 and Coro. 2.3) will require to develop some notation and machinery.

The pseudohyperbolic metric for $x, y \in \mathcal{M}$ is defined by

$$\rho(x,y) = \sup\{|f(y)| : f \in H^\infty,\ \|f\| = 1 \text{ and } f(x) = 0\},$$

which for $z, \omega \in \mathbb{D}$ reduces to $\rho(z,\omega) = |z - \omega|/|1 - \overline{\omega} z|$. The Gleason part of $m \in \mathcal{M}$ is $P(m) \stackrel{\text{def}}{=} \{x \in \mathcal{M} : \rho(m,x) < 1\}$. Clearly $\mathbb{D}$ is a Gleason part. If $z_0 \in \mathbb{D}$, we can think of the analytic function

$$L_{z_0}(z) = \frac{z + z_0}{1 + \overline{z}_0 z},\ z \in \mathbb{D}$$

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as mapping $\mathbb{D}$ into $\mathcal{M}$. In [6] Hoffman proved that if $m \in \mathcal{M}$ and (z_α) is a net in $\mathbb{D}$ converging to m, then the net L_{z_α} tends in the space $\mathcal{M}^{\mathbb{D}}$ (i.e., pointwise) to some analytic map L_m from $\mathbb{D}$ onto $P(m)$ such that $L_m(0) = m$. Here 'analytic' means that $f \circ L_m \in H^\infty$ for every $f \in H^\infty$. The map L_m does not depend on the particular choice of the net (z_α) that tends to m. A Blaschke product b with zero sequence $\{z_n\}$ satisfying

$$\delta(b) = \delta(\{z_n\}) \overset{\text{def}}{=} \inf_n \prod_{j\,:\,z_j \neq z_n} \rho(z_n, z_j) > 0$$

is called an interpolating Blaschke product and $\{z_n\}$ is called an interpolating sequence. Let $\mathcal{G}$ denote the set of points in $\mathcal{M}$ that lie in the closure of some interpolating sequence. If $m \in \mathcal{M} \setminus \mathcal{G}$ then $P(m) = \{m\}$ and hence L_m is a constant map. If $m \in \mathcal{G}$ then L_m is one-to-one, meaning that $P(m)$ is an analytic disk in $\mathcal{M}$. Hoffman also realized that even when $P(m)$ is a disk, there are cases in which L_m is a homeomorphism and cases in which it is not.

By an abstract version of Schwarz's lemma [9, p. 162], any connected portion of $\mathcal{M}$ provided with a nontrivial analytic structure must be contained in some $P(m)$ with $m \in \mathcal{G}$. In order to understand the analytic structure of $\mathcal{M}$ it is then fundamental to study the Hoffman algebras $H^\infty \circ L_m$, where $m \in \mathcal{G}$.

In [8] it is proved that $H^\infty \circ L_m$ is a closed subalgebra of H^∞ and that they coincide when $P(m)$ is a homeomorphic disk (i.e., L_m is a homeomorphism). Particular versions of the last result were obtained in [6, pp. 106-107] and [4, Coro. 3.3]. On the other hand, when $P(m)$ is not a homeomorphic disk it is well known that the identity function is not in $H^\infty \circ L_m$, meaning that this algebra is properly contained in H^∞. But, what is it? We provide an answer to this question by giving several characterizations of $H^\infty \circ L_m$, the most natural being

$$H^\infty \circ L_m = H^\infty \cap [C(P(m), \mathbb{C}) \circ L_m],$$

where $C(P(m), \mathbb{C})$ is the algebra of continuous maps from $P(m)$ into $\mathbb{C}$. The inclusion $\subseteq$ is trivial, but the proof of the other inclusion turned out to be very difficult. We can look at the equality as an extension result; it says that for every continuous function f on $P(m)$ such that $f \circ L_m \in H^\infty$ there is an extension $F \in H^\infty$ of f (i.e., $F|_{P(m)} = f$). By the above comments, the result is new only for non-homeomorphic disks, but the argument here works in general. However, the technical complications introduced by considering non-homeomorphic disks make the proof much more difficult and longer than in [8].

1 Algebraic properties of Hoffman maps

Let $\tau : \mathbb{D} \to \mathcal{M}$ be an analytic function. As before, this means that $f \circ \tau \in H^\infty$ for all $f \in H^\infty$. We can extend τ to a continuous map $\tau^* : \mathcal{M} \to \mathcal{M}$ by the formula $\tau^*(\varphi)(f) \overset{\text{def}}{=} \varphi(f \circ \tau)$, where $\varphi \in \mathcal{M}$. Two particular cases will be of interest here. If τ is an analytic self-map of

$\mathbb{D}$ then we can think of τ as mapping $\mathbb{D}$ into $\mathcal{M}$ and consider its extension τ^*. We can do this with any automorphism of $\mathbb{D}$, which therefore induces a homeomorphism from $\mathcal{M}$ onto $\mathcal{M}$. In particular, if $\lambda \in \partial\mathbb{D}$, the rotation $z \mapsto \lambda z$ ($z \in \mathbb{D}$) extends to $\mathcal{M}$ in this way. From now on, for $\varphi \in \mathcal{M}$ we simply write $\lambda\varphi$ for this 'λ-rotation' in $\mathcal{M}$. We point out that even when each such rotation is a homeomorphism, the action of the group $\partial\mathbb{D}$ into $\mathcal{M}$ is not continuous [5, pp. 164-165]. The other relevant case for the paper is L_m (for $m \in \mathcal{G}$). The extension L_m^* maps $\mathcal{M}$ onto $\overline{P(m)}$, and $P(m)$ is a homeomorphic disk if and only if L_m^* is one-to-one [8, Sect. 3]. We will also denote this extension by L_m, where the meaning will be clear from the context.

The inclusion of the disk algebra in H^∞ induces a natural projection $\pi : \mathcal{M} \to \overline{\mathbb{D}}$. The fiber of a point $\omega \in \partial\mathbb{D}$ is $\pi^{-1}(\omega) \subset \mathcal{M}$. Let $x, y \in \mathcal{M}$ and let (z_α) be a net in $\mathbb{D}$ so that $y = \lim z_\alpha$. We claim that the limit of $(1 + \overline{\pi(x)} z_\alpha)/(1 + \pi(x)\overline{z_\alpha})$ always exists (in $\overline{\mathbb{D}}$) and it is independent of the net (z_α). A rigorous statement would say that the above limit exists when $\pi(z_\alpha)$ is in place of z_α; but since π identifies $\mathbb{D}$ with $\pi(\mathbb{D})$, no harm is done with the appropriate mind adjustment.

It is clear that the limit exists whenever the denominator does not tend to zero. So, suppose that $|\pi(x)| = 1$ and $\pi(y) = -\pi(x)$. Write $z_\alpha = -\pi(x)(1 - r_\alpha e^{i\theta_\alpha})$, with $-\pi/2 < \theta_\alpha < \pi/2$ and $0 < r_\alpha \to 0$. The point $y \in \mathcal{M}$ is either a nontangential point, in which case $\theta_\alpha \to \theta \in (-\pi/2, \pi/2)$ (see [6, pp. 107-108]), or it is a tangential point and θ_α accumulates in $\{-\pi/2,\ \pi/2\}$. In both cases

$$\lim \frac{1 + \overline{\pi(x)} z_\alpha}{1 + \pi(x)\overline{z_\alpha}} = \lim \frac{1 - (1 - r_\alpha e^{i\theta_\alpha})}{1 - (1 - r_\alpha e^{-i\theta_\alpha})} = \lim e^{i2\theta_\alpha},$$

proving our claim.

Definition. Let $\lambda : \mathcal{M} \times \mathcal{M} \to \partial\mathbb{D}$ be the function

$$\lambda(x, y) = \lim_\alpha \frac{1 + \overline{\pi(x)} z_\alpha}{1 + \pi(x)\overline{z_\alpha}} \quad (x, y \in \mathcal{M}),$$

where (z_α) is any net in $\mathbb{D}$ that tends to y.

Observe that if $x, y \in \mathcal{M}$ do not satisfy the extreme conditions $|\pi(x)| = 1$ and $\pi(x) = -\pi(y)$ then

$$\lambda(x, y) = \frac{1 + \overline{\pi(x)}\pi(y)}{1 + \pi(x)\overline{\pi(y)}},$$

and this expression reduces to $\overline{\pi(x)}\pi(y)$ when $|\pi(x)| = 1 = |\pi(y)|$. We will use indistinctly the notations $\lambda(x, y)$ or $\lambda_{x,y}$ to denote this function. A word of warning: the function λ was introduced by Budde in [1] for the purpose of proving the same result given in Proposition 1.1 below. However, the value of $\lambda(x, y)$ stated in [1] when $|\pi(x)| = |\pi(y)| = 1$ is $\overline{\pi(x)}\pi(y)$, therefore overlooking the pathological behavior of λ when $\pi(x) = -\pi(y)$. Fortunately, all the proofs and results in [1] remain valid by only adjusting λ to its right value.

In [4, Lemma 1.8] Gorkin, Lingerberg and Mortini proved that if $m \in \mathcal{G}$ and b is an interpolating Blaschke product then $b \circ L_m = Bf$, where B is an interpolating Blaschke product and $f \in (H^\infty)^{-1}$. This fact will be used frequently along the paper. We will also need a result of Budde [1] stating that if $\varphi \in \mathcal{M}$ has trivial Gleason part then $L_m(\varphi)$ also has trivial Gleason part. In symbols, $L_m^{-1}(\mathcal{G}) \subset \mathcal{G}$.

Proposition 1.1 *Let* $m, y \in \mathcal{M}$. *Then*

$$L_m \circ L_y(\lambda(m,y)z) = L_{L_m(y)}(z) \quad \textit{for all} \quad z \in \mathbb{D}. \tag{1.1}$$

Proof. Suppose first that m or y (or both) is not in $\mathcal{G}$. Then one of the maps L_m or L_y is constant and the left member of (1.1) is the constant map $L_m(y)$. If $y \notin \mathcal{G}$ then Budde's result asserts that $L_m(y) \notin \mathcal{G}$, which is trivially the case if $m \notin \mathcal{G}$, too. Therefore also the right member of (1.1) is the constant map $L_m(y)$.

Suppose now that $m, y \in \mathcal{G}$ and let ω, ξ and z in $\mathbb{D}$. An elementary calculation shows that

$$L_\omega(L_\xi(z)) = L_{L_\omega(\xi)}(\overline{\lambda(\omega,\xi)}z). \tag{1.2}$$

Replace ω in (1.2) by a net (ω_α) in $\mathbb{D}$ tending to m. Then the first member of (1.2) tends to $L_m(L_\xi(z))$ and $L_{\omega_\alpha}(\xi) \to L_m(\xi)$. Consequently

$$L_{L_{\omega_\alpha}(\xi)} \to L_{L_m(\xi)} \quad \text{pointwise on} \quad \mathbb{D}. \tag{1.3}$$

It is clear that the constants $\lambda_\alpha = \lambda(\omega_\alpha, \xi)$ tend to $\lambda(m, \xi)$. Using that for $x \in \mathcal{G}$ the map L_x is an isometry on $\mathbb{D}$ with respect to ρ [6, p. 105] we get

$$\begin{aligned} \rho(L_{L_{\omega_\alpha}(\xi)}(\overline{\lambda}_\alpha z)\,,\, L_{L_{\omega_\alpha}(\xi)}(\overline{\lambda(m,\xi)}z)) &= \rho(\overline{\lambda}_\alpha z, \overline{\lambda(m,\xi)}z) \\ &\leq \frac{|\overline{\lambda}_\alpha - \overline{\lambda(m,\xi)}|}{1-|z|^2} \to 0. \end{aligned} \tag{1.4}$$

Hence, the lower semicontinuity of ρ (see [6, Thm. 6.2]) together with (1.2), (1.3) and (1.4) yields

$$\rho(L_m(L_\xi(z)), L_{L_m(\xi)}(\overline{\lambda(m,\xi)}z)) \leq \lim \rho(L_{L_{\omega_\alpha}(\xi)}(\overline{\lambda}_\alpha z), L_{L_{\omega_\alpha}(\xi)}(\overline{\lambda(m,\xi)}z)) = 0. \tag{1.5}$$

That is, $L_m(L_\xi(z)) = L_{L_m(\xi)}(\overline{\lambda(m,\xi)}z)$ for every $\xi, z \in \mathbb{D}$. Now replace ξ by a net (ξ_α) in $\mathbb{D}$ tending to y. By the continuity of L_m on $\mathcal{M}$ then $L_m(L_{\xi_\alpha}(z)) \to L_m(L_y(z))$ and $L_m(\xi_\alpha) \to L_m(y)$. Since the map $x \mapsto L_x$ is continuous from $\mathcal{M}$ into $\mathcal{M}^{\mathbb{D}}$, then we also have

$$L_{L_m(\xi_\alpha)} \to L_{L_m(y)} \quad \text{pointwise on} \quad \mathbb{D}.$$

Since $L_m(\xi_\alpha) \in P(m) \subset \mathcal{G}$ for every ξ_α, then $L_{L_m(\xi_\alpha)}$ are isometries with respect to ρ. In addition, $\lambda(m, \xi_\alpha) \to \lambda(m, y)$ by definition. Therefore the same argument as in (1.4) and (1.5) yields

$$L_m(L_y(z)) = L_{L_m(y)}(\overline{\lambda(m, y)} z).$$

The proposition follows replacing z by $\lambda(m, y)z$. □

Corollary 1.2 (Budde) *Let $m \in \mathcal{G}$ and $\xi \in \mathcal{M}$ such that $L_m(\xi) \in P(m)$. Then L_m maps $P(\xi)$ onto $P(m)$ in a one-to-one fashion.*

Proof. By hypothesis there is $\omega \in \mathbb{D}$ such that $L_m(\xi) = L_m(\omega)$. Hence, by Proposition 1.1 $L_m \circ L_\xi(\lambda_{m,\xi} z) = L_m \circ L_\omega(\lambda_{m,\omega} z)$ for every $z \in \mathbb{D}$. The result follows because $L_m : \mathbb{D} \to P(m)$, $L_\omega(\lambda_{m,\omega}\, \_) : \mathbb{D} \to \mathbb{D}$ and $L_\xi(\lambda_{m,\xi}\, \_) : \mathbb{D} \to P(\xi)$ are onto and one-to-one. □

Lemma 1.3 *Let $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ and let $y \in \mathcal{M}$. Then*

$$L_{\gamma y}(z) = \gamma L_y(\overline{\gamma} z). \tag{1.6}$$

Proof. Let $f \in H^\infty$ and $\{z_\alpha\}$ be a net in $\mathbb{D}$ tending to y. Thus $\gamma z_\alpha \to \gamma y$ and for every $z \in \mathbb{D}$,

$$f(L_{\gamma y}(z)) = \lim_\alpha f(L_{\gamma z_\alpha}(z)) = \lim_\alpha f(\gamma L_{z_\alpha}(\overline{\gamma} z)) = f(\gamma L_y(\overline{\gamma} z)),$$

as desired. □

2 A characterization of Hoffman algebras

Definition. Let $m \in \mathcal{G}$. The m-saturation of a set $E \subset \mathcal{M}$ is defined as $L_m^{-1}(L_m(E))$, and E will be called m-saturated if it coincides with its m-saturation. We also write $\mathcal{L}_m(y) \overset{\text{def}}{=} L_m^{-1}(L_m(y))$ for the m-saturation of $y \in \mathcal{M}$.

It is clear that $\mathcal{L}_m(0) \cap \mathbb{D} = \{0\}$. For $f \in H^\infty$ write

$$Z_D(f) = \{z \in \mathbb{D} : f(z) = 0\} \quad \text{and} \quad Z(f) = \{\varphi \in \mathcal{M} : f(\varphi) = 0\}.$$

It is well known that if f is an interpolating Blaschke product then $Z(f)$ is the closure of $Z_D(f)$. This immediately implies that if $m \in \mathcal{G}$ and $y \in \mathcal{M}$ are different points then there is an interpolating Blaschke product f such that $f(m) = 0 \neq f(y)$. As a consequence we obtain that if $m \in \mathcal{G}$ then

$$\mathcal{L}_m(0) = \bigcap \{Z(b \circ L_m) : b \text{ is an interpolating Blaschke product with } b(m) = 0\}.$$

Since $Z(b \circ L_m)$ is the zero set of an interpolating Blaschke product then $\mathcal{L}_m(0)$ is an intersection of closures of interpolating sequences, which in a sense is quite small. Furthermore, [4, Thm. 1.4] implies that $P(m)$ is a homeomorphic disk if and only if $\mathcal{L}_m(0) = \{0\}$.

Lemma 2.1 *Let $m \in \mathcal{G}$. Then for $\omega \in \mathbb{D}$ we have*

$$\mathcal{L}_m(\omega) = \{L_x(\lambda_{m,x}\omega) : x \in \mathcal{L}_m(0)\}.$$

Proof. Since $L_m(x) = m$ for all $x \in \mathcal{L}_m(0)$ then by Proposition 1.1 $L_m(L_x(\lambda_{m,x}z)) = L_m(z)$ on $\mathbb{D}$. So, $L_x(\lambda_{m,x}\omega) \in \mathcal{L}_m(\omega)$. Let $\xi \in \mathcal{M}$ such that $L_m(\xi) = L_m(\omega)$. The 'onto' part of Corollary 1.2 implies that there is $x \in P(\xi)$ such that $L_m(x) = m$. So, $L_m \circ L_x(\lambda_{m,x}z) = L_m(z)$ for $z \in \mathbb{D}$. In particular, $L_m(L_x(\lambda_{m,x}\omega)) = L_m(\omega) = L_m(\xi)$. That is, L_m takes the same value on the points $L_x(\lambda_{m,x}\omega)$ and ξ, which belong to $P(\xi)$. The 'one-to-one' part of Corollary 1.2 implies that $L_x(\lambda_{m,x}\omega) = \xi$. $\square$

Theorem 2.2 *Let $m \in \mathcal{G} \setminus \mathbb{D}$ and let $f \in H^\infty$ such that*

$$f(L_x(\lambda_{m,x}z)) = f(z) \quad \text{for all} \quad x \in \mathcal{L}_m(0) \quad \text{and all} \quad z \in \mathbb{D}.$$

Then there is $F \in H^\infty$ such that $F \circ L_m = f$.

For $m \in \mathcal{G}$ the theorem and Lemma 2.1 provide a description of the algebra $H^\infty \circ L_m$ as the functions $f \in H^\infty$ such that f is constant on $L_m^{-1}(m')$ for every $m' \in P(m)$. Only the sufficiency needs to be proved. We devote the next two sections to prove Theorem 2.2. For the sake of clarity it is convenient to rescue the hat for the Gelfand transform in the next corollary.

Corollary 2.3 *Let $m \in \mathcal{G} \setminus \mathbb{D}$ and $h : P(m) \to \mathbb{C}$ such that $h \circ L_m = f \in H^\infty$. Then the following conditions are equivalent.*

(a) *h is continuous on $P(m)$ with the topology induced by $\mathcal{M}$,*

(b) *$\hat{f} \circ L_x(\lambda_{m,x}z) = f(z)$ for every $x \in \mathcal{L}_m(0)$ and $z \in \mathbb{D}$,*

(c) *there exists $F \in H^\infty$ such that $\hat{F} \circ L_m(z) = h \circ L_m(z)$ for every $z \in \mathbb{D}$, and*

(d) *there exists $F \in H^\infty$ such that $\hat{F}|_{P(m)} = h$.*

Proof. We assume first that (a) holds. If (b) fails then there are $x \in \mathcal{L}_m(0)$ and $z_0 \in \mathbb{D}$ such that $\alpha = |\hat{f}(L_x(\lambda_{m,x}z_0)) - f(z_0)| > 0$. By the density of $\mathbb{D}$ in $\mathcal{M}$ the point $L_x(\lambda_{m,x}z_0)$ is in the closure of the set $U = \{z \in \mathbb{D} : |\hat{f}(L_x(\lambda_{m,x}z_0)) - f(z)| < \alpha/2\}$. Thus, by Lemma 2.1 $L_m(z_0) = L_m(L_x(\lambda_{m,x}z_0)) \in L_m(\overline{U}) \subset \overline{L_m(U)}$. The continuity of h on $P(m)$ now implies that $h(L_m(z_0)) \in \overline{h(L_m(U))}$. But this contradicts the fact that for $z \in U$,

$$\begin{aligned} |h(L_m(z_0)) - h(L_m(z))| &= |f(z_0) - f(z)| \\ &\geq |f(z_0) - \hat{f}(L_x(\lambda_{m,x}z_0))| - |\hat{f}(L_x(\lambda_{m,x}z_0)) - f(z)| \\ &> \alpha - \alpha/2 = \alpha/2. \end{aligned}$$

The implication (b)$\Rightarrow$(c) is Theorem 2.2. Since $P(m) = L_m(\mathbb{D})$ then (d) is just a rephrasing of (c). Now suppose that (d) holds. Since $F \in H^\infty$ then $\hat{F}$ is continuous on $\mathcal{M}$, and consequently $\hat{F}|_{P(m)} = h$ is continuous on $P(m)$. $\square$

3 Technical lemmas

The hyperbolic metric for $z, \omega \in \mathbb{D}$ is

$$h(z,\omega) = \log \frac{1+\rho(z,\omega)}{1-\rho(z,\omega)}.$$

So, h and ρ are increasing functions of each other and $h(z,\omega)$ tends to infinity if and only if $\rho(z,\omega)$ tends to 1. We will use alternatively one metric or the other according to convenience. The hyperbolic ball of center $z \in \mathbb{D}$ and radius $r > 0$ will be denoted by $\Delta(z,r)$.

The next two lemmas are easy consequences of similar results in Hoffman's paper [6, pp. 82 and 86-88]. (or see [2, pp. 404-408]).

Lemma 3.1 *Let S be an interpolating sequence and let $m \in \overline{S}$. Then for any $0 < \delta_0 < 1$ there is a subsequence S' of S such that $m \in \overline{S'}$ and $\delta(S') > \delta_0$.*

Lemma 3.2 *Let b be an interpolating Blaschke product with $\delta(b) \geq \delta$ and let $\omega \in \mathbb{D}$. Then there is $0 < c = c(\delta) < 1$ such that $c \to 1$ as $\delta \to 1$, and*

$$|b(\omega)| \geq c\rho(\omega, Z_D(b)).$$

Our next lemma is a trivial consequence of Lemma 3.2; we state it for convenience.

Lemma 3.3 *Let $0 < \sigma < 1$. Then there are functions $0 < \delta(\sigma) < 1$ and $s(\sigma) > 0$ such that if b is an interpolating Blaschke product with $\delta(b) > \delta(\sigma)$ then*

$$|b(z)| \geq \sigma \quad \textit{for} \quad z \notin \bigcup \{\Delta(z_n, s(\sigma)) : \; z_n \in Z_D(b)\}.$$

Lemma 3.4 *Let $\{z_n\}$ be an interpolating sequence and (z_α) be a subnet with $z_\alpha \to m \in \mathcal{M}$. Suppose that $F \in C(\mathcal{M})$. Then $F \circ L_{z_\alpha} \to F \circ L_m$ uniformly on compact subsets of $\mathbb{D}$.*

Proof. Since $z_\alpha \to m$ in $\mathcal{M}$ then $L_{z_\alpha} \to L_m$ in $\mathcal{M}^{\mathbb{D}}$, and by the continuity of F on $\mathcal{M}$ then $F \circ L_{z_\alpha} \to F \circ L_m \stackrel{\text{def}}{=} f$ pointwise on $\mathbb{D}$. We will see that $F(L_{z_\alpha}(z)) \to f(z)$ uniformly on $|z| \leq r$ for any $0 < r < 1$. In fact, otherwise there is $\varepsilon > 0$, a subnet (z_β) of (z_α) and points ω_β with $|\omega_\beta| \leq r$ such that

$$|F(L_{z_\beta}(\omega_\beta)) - f(\omega_\beta)| > \varepsilon \quad \text{for all} \quad \beta. \tag{3.1}$$

Taking a subnet of (z_β) if necessary, we can also assume that $\omega_\beta \to \omega$, with $|\omega| \leq r$. Since f is continuous at ω and $F(L_{z_\beta}(\omega)) \to f(\omega)$, then there is β_0 such that for every $\beta \geq \beta_0$,

$$|f(\omega) - f(\omega_\beta)| < \varepsilon/4 \quad \text{and} \quad |F(L_{z_\beta}(\omega)) - f(\omega)| < \varepsilon/4.$$

These inequalities together with (3.1) give

$$|F(L_{z_\beta}(\omega_\beta)) - F(L_{z_\beta}(\omega))| > \varepsilon/2 \quad \text{for all} \quad \beta \geq \beta_0.$$

This will contradict the continuity of F if we prove that $L_{z_\beta}(\omega_\beta)$ tends to $L_m(\omega)$. Let $(L_{z_\gamma}(\omega_\gamma))$ be an arbitrary convergent subnet of $(L_{z_\beta}(\omega_\beta))$, say to $y \in \mathcal{M}$. Then by the lower semicontinuity of ρ,

$$\rho(y, L_m(\omega)) \leq \lim \rho(L_{z_\gamma}(\omega_\gamma), L_{z_\gamma}(\omega)) = \lim \rho(\omega_\gamma, \omega) = 0,$$

meaning that $y = L_m(\omega)$. So, every convergent subnet of $(L_{z_\beta}(\omega_\beta))$ tends to y, and consequently the whole net tends to y. $\square$

An immediate consequence of Lemma 3.4 is that if $F \in C(\mathcal{M})$ and S is an interpolating sequence with $m \in \overline{S} \setminus S$, then for every $0 < r < 1$ and $\varepsilon > 0$,

$$\left\{ z_n \in S : \sup_{|z| \leq r} |F \circ L_{z_n}(z) - F \circ L_m(z)| < \varepsilon \right\}$$

is a subsequence of S having m in its closure.

Lemma 3.5 *Let $\xi, \omega \in \mathbb{D}$ and $m \in \mathcal{M} \setminus \mathbb{D}$. Then for any $0 < r < 1$,*

$$\sup_{|z| \leq r} \rho(L_\xi(\lambda_{m,\xi} z), L_\omega(\lambda_{m,\omega} z)) < \frac{30}{(1-r)^2} \rho(\xi, \omega).$$

Proof. We can assume that $\pi(m) = 1$. Also, since the desired inequality is obvious for $\rho(\xi, \omega) > 1/30$, we can assume that $\rho(\xi, \omega) < 1/2$. So, for $z \in \mathbb{D}$ with $|z| \leq r$,

$$\begin{aligned} \rho(L_\xi(\lambda_{m,\xi} z), L_\omega(\lambda_{m,\omega} z)) &\leq \rho(L_\xi(\lambda_{m,\xi} z), L_\xi(\lambda_{m,\omega} z)) + \rho(L_\xi(\lambda_{m,\omega} z), L_\omega(\lambda_{m,\omega} z)) \\ &= \varrho_1 + \varrho_2. \end{aligned}$$

Since $L_\omega : \mathbb{D} \rightarrow \mathbb{D}$ is an onto isometry with respect to ρ, then there exists $v \in \mathbb{D}$ such that $\xi = L_\omega(v)$ and $|v| = \rho(\xi, \omega)$. The elementary formula (for $\pi(m) = 1$)

$$\lambda(m, L_\omega(v)) = \frac{1 + L_\omega(v)}{1 + \overline{L_\omega(v)}} = \left(\frac{1 + \omega \overline{v}}{1 + \overline{\omega} v} \right) \left(\frac{1 + v \overline{\lambda(m, \omega)}}{1 + \overline{v} \lambda(m, \omega)} \right) \lambda(m, \omega)$$

yields

$$\begin{aligned} |\lambda(m, L_\omega(v)) - \lambda(m, \omega)| &= \frac{|v(\overline{\lambda} - \overline{\omega}) + \overline{v}(\omega - \lambda) + |v|^2(\omega \overline{\lambda} - \overline{\omega} \lambda)|}{|(1 + \overline{\omega} v)(1 + \overline{v} \lambda)|} \\ &< \frac{6|v|}{(1 - |v|)^2} < 24|v|, \end{aligned} \tag{3.2}$$

where $\lambda = \lambda(m, \omega)$, and the last inequality holds because $|v| < 1/2$. Thus, (3.2) and the isometric property of L_ξ give

$$\varrho_1 = \rho(\lambda_{m,\xi} z, \lambda_{m,\omega} z) \leq \frac{|\lambda(m,\xi) - \lambda(m,\omega)|}{1 - r^2} < \frac{24|v|}{1 - r^2} < \frac{24|v|}{(1-r)^2}. \tag{3.3}$$

Put $z' = \lambda_{m,\omega} z$. Then by (1.1)

$$\begin{aligned} \varrho_2 &= \rho(L_{L_\omega(v)}(z'), L_\omega(z')) = \rho(L_\omega \circ L_v(\lambda_{\omega,v} z'), L_\omega(z')) \\ &= \rho(L_v(\lambda_{\omega,v} z'), z') \leq \frac{|L_v(\lambda_{\omega,v} z') - z'|}{1 - |z'|} \\ &= \frac{|z'(\lambda_{\omega,v} - 1) + v - \overline{v}(z')^2 \lambda_{\omega,v}|}{(1 - |z'|)\,|1 + \overline{v}\lambda_{\omega,v} z'|} \leq \frac{|\lambda_{\omega,v} - 1| + 2|v|}{(1-r)^2}. \end{aligned}$$

This inequality together with

$$|\lambda_{\omega,v} - 1| = \frac{|\overline{\omega} v - \omega \overline{v}|}{|1 + \omega \overline{v}|} < \frac{2|v|}{1 - |v|} < 4|v|$$

yields $\varrho_2 < 6|v|(1-r)^{-2}$. So, adding this estimate to (3.3) we obtain

$$\varrho_1 + \varrho_2 < (24 + 6)|v|(1-r)^{-2} = 30\rho(\xi, \omega)(1-r)^{-2},$$

as promised. □

Lemma 3.6 *Let $m \in \mathcal{G}$ and $E \subset \mathcal{M}$ be a closed m-saturated set. If V is an open neighborhood of E then there exists an open m-saturated set W such that $E \subset W \subset V$.*

Proof. Since $\mathcal{M} \setminus V$ is closed then so is $L_m(\mathcal{M} \setminus V)$. Since E is m saturated then the closed m-saturated set $F \overset{\text{def}}{=} L_m^{-1}(L_m(\mathcal{M} \setminus V))$ does not meet E. Additionally, $F \supset \mathcal{M} \setminus V$ and then the open set $W \overset{\text{def}}{=} \mathcal{M} \setminus F$ satisfies the lemma. □

Lemma 3.7 *Let S be an interpolating sequence and $m \in \overline{S}$. If $W \subset \mathcal{M}$ is an open m-saturated neighborhood of $\mathcal{L}_m(0)$ then there is a subsequence $S_0 \subset S$ such that $m \in \overline{S_0} \cap \overline{P(m)} \subset L_m(W)$.*

Proof. The hypothesis on W implies that $L_m(W)$ is open in $\overline{P(m)}$ and $m \in L_m(W)$. That is, the set $E = \overline{P(m)} \setminus L_m(W)$ is closed and $m \notin E$. By compactness then there is an open set $U \subset \mathcal{M}$ such that $m \in U$ and $\overline{U} \cap E = \emptyset$. Defining $S_0 = S \cap U$ we have that $m \in \overline{S_0} \subset \overline{U}$. Hence, $\overline{S_0} \cap E = \emptyset$ and then $\overline{S_0} \cap \overline{P(m)} \subset \overline{P(m)} \setminus E = L_m(W)$. □

Lemma 3.8 *Let $m \in \mathcal{G}$ and $f \in H^\infty$ such that $f \circ L_x(\lambda_{m,x} z) = f(z)$ for all $x \in \mathcal{L}_m(0)$ and $z \in \mathbb{D}$. For $\varepsilon > 0$ and $0 < r < 1$ consider the set*

$$U = \{\omega \in \mathbb{D} : |f \circ L_\omega(\lambda_{m,\omega} z) - f(z)| < \varepsilon \ \text{ for } \ |z| \le r\}.$$

Then $\overline{U}$ is a neighborhood of $\mathcal{L}_m(0)$.

Proof. If the lemma fails then there is $x \in \mathcal{L}_m(0)$ in the closure of $V = \mathcal{M} \setminus \overline{U}$. Since V is open and $\mathbb{D}$ is dense in $\mathcal{M}$, a simple topological argument shows that $\overline{V} = \overline{V \cap \mathbb{D}}$. Therefore $x \in \overline{V \cap \mathbb{D}}$, where

$$V \cap \mathbb{D} \subset \{\omega \in \mathbb{D} : |f \circ L_\omega(\lambda_{m,\omega} z_\omega) - f(z_\omega)| \ge \varepsilon \ \text{ for some } \ z_\omega \ \text{ with } \ |z_\omega| \le r\}.$$

Let (ω_α) be a net in $V \cap \mathbb{D}$ that tends to x, and write $z_\alpha \overset{\text{def}}{=} z_{\omega_\alpha}$. By taking a suitable subnet we can also assume that $z_\alpha \to z_0$, where $|z_0| \le r$. Thus,

$$|f \circ L_{\omega_\alpha}(\lambda_{m,\omega_\alpha} z_\alpha) - f(z_\alpha)| \ge \varepsilon \ \text{ for every } \ \alpha. \tag{3.4}$$

We can assume $\|f\|_\infty \le 1$. By the Schwarz-Pick inequality [2, p. 2],

$$\begin{aligned} \rho(f \circ L_{\omega_\alpha}(\lambda_{m,\omega_\alpha} z_\alpha), f \circ L_{\omega_\alpha}(\lambda_{m,x} z_0)) &\le \rho(\lambda_{m,\omega_\alpha} z_\alpha, \lambda_{m,x} z_0) \\ &\le \rho(\lambda_{m,\omega_\alpha} z_\alpha, \lambda_{m,x} z_\alpha) + \rho(\lambda_{m,x} z_\alpha, \lambda_{m,x} z_0) \\ &\le \frac{1}{1 - |z_\alpha|^2} |\lambda_{m,\omega_\alpha} - \lambda_{m,x}| + \rho(z_\alpha, z_0), \end{aligned}$$

which tends to zero. Since $L_{\omega_\alpha} \to L_x$ then the last inequality gives

$$\lim_\alpha f \circ L_{\omega_\alpha}(\lambda_{m,\omega_\alpha} z_\alpha) = \lim_\alpha f \circ L_{\omega_\alpha}(\lambda_{m,x} z_0) = f \circ L_x(\lambda_{m,x} z_0) = f(z_0),$$

which contradicts (3.4). □

The next lemma is in [8, Lemma 2.1].

Lemma 3.9 *Let u be an inner function and $0 < \beta < 1$. Put $V = \{z \in \mathbb{D} : |u(z)| < \beta\}$ and suppose that $f \in H^\infty(V)$. Then there are $0 < \gamma = \gamma(\beta) < \beta$, $C = C(\beta) > 0$ and $F \in H^\infty$ such that*

(i) $\|F\|_\infty \le C \|f\|_{H^\infty(V)}$, *and*

(ii) $|F(z) - f(z)| \le A \|f\|_{H^\infty(V)} |u(z)|$ *when* $|u(z)| < \gamma$, *where* $A = \gamma^{-1}(C + 1)$.

4 Proof of Theorem 2.2

Given $m \in \mathcal{G} \setminus \mathbb{D}$ and $f \in H^\infty$ that satisfy the hypotheses of the theorem, we are going to construct a function $F \in H^\infty$ such that $F \circ L_m = f$. We can assume without loss of generality that $\pi(m) = 1$ and $\|f\| = 1$. Let $\{\sigma_k\} \subset (0,1)$ be a sequence satisfying $\prod_{k\geq 1} \sigma_k > 0$ and let $s(\sigma_k)$ be the associated parameters given by Lemma 3.3. Take $s_k > s(\sigma_k)$ tending *increasingly* to ∞, and put $r_k \overset{\text{def}}{=} 4\,(2^k s_1 + 2^{k-1} s_2 + \cdots + 2 s_k)$.

Given an arbitrary interpolating sequence S such that $m \in \overline{S}$, and $\{\varepsilon_k\} \subset (0,1)$ a decreasing sequence that tends to 0, we will construct a decreasing chain of subsequences $S_k = \{z_{k,n} : n \geq 1\}$, $S \supset S_1 \supset S_2 \supset \cdots$, such that for every $k \geq 1$,

(1) $m \in \overline{S}_k$,
(2) $h(\{|z-1| > \varepsilon_k\}, S_k) > r_k$,
(3) $\sum_{n\geq 1}(1 - |z_{k,n}|) < 2^{-k}$,
(4) $h(z_{k,n_1}, z_{k,n_2}) > r_k$ for $n_1 \neq n_2$,
(5) if b_k is the interpolating Blaschke product with zero sequence S_k, then

$$|b_k(z)| \geq \sigma_k \quad \text{for} \quad z \notin \bigcup_{n\geq 1} \Delta(z_{k,n}, s_k),$$

(6) if $l \geq k$ and $h(z_{l,p}, z_{k,n}) < r_l$ then $\rho(L_{-z_{l,p}}(z_{k,n}), T_k) < \varepsilon_k$, where $T_k = L_m^{-1}(\overline{S}_k \cap \overline{P(m)})$, and
(7) $|f \circ L_\omega(\lambda_{m,\omega} z) - f(z)| \leq \varepsilon_k$ when $h(z,0) \leq s_k$ and $\omega \in T_k \cap \mathbb{D}$.

The first construction. The argument will be inductive. By Lemmas 3.8 and 3.6, for every $k \geq 1$ there is an open m-saturated neighborhood of $\mathcal{L}_m(0)$, $W_k \subset \mathcal{M}$, such that $W_{k+1} \subset W_k$ and for all $\omega \in W_k$,

$$|f \circ L_\omega(\lambda_{m,\omega} z) - f(z)| \leq \varepsilon_k \ \text{ if } \ h(z,0) \leq s_k. \tag{4.1}$$

Step 1. By Lemma 3.7 there is $S_1' \subset S$ such that (1) holds and $\overline{S_1'} \cap \overline{P(m)} \subset L_m(W_1)$. Since W_1 is m-saturated then $T_1' \overset{\text{def}}{=} L_m^{-1}(\overline{S_1'} \cap \overline{P(m)}) \subset L_m^{-1}(L_m(W_1)) = W_1$. Then (4.1) tells us that S_1' satisfies (7), and then so does any subsequence of S_1' that contains m in its closure. By Lemmas 3.1 and 3.3 we can assume that $\delta(S_1')$ is so close to 1 that (4) and (5) hold. Furthermore, since $\pi(m) = 1$ we can easily achieve conditions (2) and (3) by taking as S_1 the subsequence of S_1' whose elements are contained in a sufficiently small Euclidean ball centered at 1.

Condition (6) only makes sense for $k = l = 1$. If $z_{1,n}, z_{1,p} \in S_1$ are such that $h(z_{1,n}, z_{1,p}) < r_1$ then (4) implies that $z_{1,n} = z_{1,p}$. Therefore $L_{-z_{1,p}}(z_{1,n}) = 0 \in T_1 \cap \mathbb{D}$, because $L_m(0) = m \in \overline{S}_1 \cap \overline{P(m)}$.

Step l. Let $l \geq 2$ and suppose that we already have $S \supset S_1 \supset \ldots \supset S_{l-1}$ satisfying (1)…(7). By Lemma 3.7 there exists $S'_l \subset S_{l-1}$ such that (1) holds and $\overline{S'_l} \cap \overline{P(m)} \subset L_m(W_l)$. Since W_l is m-saturated then $T'_l \overset{\text{def}}{=} L_m^{-1}(\overline{S'_l} \cap \overline{P(m)}) \subset L_m^{-1}(L_m(W_l)) = W_l$. By (4.1) the sequence S'_l satisfies (7), and the same holds for any subsequence of S'_l having m in its closure. As in the case $l = 1$, by Lemma 3.1 and 3.3 we can assume that S'_l satisfies (4) and (5), and by taking the points of S'_l that are close enough to 1 we can also assume that S'_l satisfies (2) and (3). Clearly, any subsequence S_l of S'_l such that $m \in \overline{S_l}$ will satisfy all the above properties. Therefore we will be done if we can pick the sequence S_l so that it also satisfies (6).

Let $k \leq l-1$ and let $\eta_k > 0$ to be chosen later. By [4, Lemma 1.8] we have $b_k \circ L_m = B_k g_k$, where $g_k \in (H^\infty)^{-1}$ and B_k is an interpolating Blaschke product with zero sequence $Z_{\mathbb{D}}(B_k)$ contained in T_k. The inclusion holds because if $B_k(z_0) = 0$ then $b_k(z_0) = 0$, and consequently $L_m(z_0) \in P(m) \cap Z(b_k) = P(m) \cap \overline{S}_k$.

Since $m \in \overline{S'_l}$ then by the remark following Lemma 3.4 there is a subsequence $\Lambda_k \subset S'_l$ such that $m \in \overline{\Lambda}_k$ and

$$|b_k \circ L_\nu(z) - B_k(z) g_k(z)| < \eta_k \quad \text{for} \quad \nu \in \Lambda_k \quad \text{and} \quad h(z, 0) < r_l. \tag{4.2}$$

If $\nu \in \Lambda_k$ and $z_{k,n} \in S_k$ satisfy $h(z_{k,n}, \nu) < r_l$ then $h(L_{-\nu}(z_{k,n}), 0) < r_l$. Applying (4.2) to $z = L_{-\nu}(z_{k,n})$ we get

$$|B_k(L_{-\nu}(z_{k,n}))|\ |g_k(L_{-\nu}(z_{k,n}))| = |b_k \circ L_\nu(L_{-\nu}(z_{k,n})) - (B_k g_k)(L_{-\nu}(z_{k,n}))| < \eta_k.$$

We are using here that $L_{-\nu} = L_\nu^{-1}$ and $b_k(z_{k,n}) = 0$. Then $|B_k(L_{-\nu}(z_{k,n}))| < \eta_k \|g_k^{-1}\|_\infty$. Since B_k is interpolating, Lemma 3.2 implies that for small values of η_k the point $L_{-\nu}(z_{k,n})$ must be close to the zero sequence of B_k in the ρ-metric. That is, choosing η_k small enough we obtain

$$\rho(L_{-\nu}(z_{k,n}), T_k) \leq \rho(L_{-\nu}(z_{k,n}), Z_{\mathbb{D}}(B_k)) < \varepsilon_k \tag{4.3}$$

for every $\nu \in \Lambda_k$ such that $h(\nu, z_{k,n}) < r_l$ for some $z_{k,n}$. Doing this process for $k = 1, \ldots, l-1$ we obtain the respective subsequences $\Lambda_k \subset S'_l$ satisfying (4.3), and such that $m \in \overline{\Lambda}_k$ for $k = 1, \ldots, l-1$. Since disjoint subsequences of an interpolating sequence have disjoint closures then m is in the closure of

$$S_l \overset{\text{def}}{=} \bigcap_{1 \leq k \leq l-1} \Lambda_k,$$

and by (4.3) S_l satisfies (6) for $k = 1, \ldots, l-1$. Finally, the same argument used in Step 1 shows that S_l satisfies (6) also for $k = l$.

The second construction. Observe that condition (4) implies that for a fixed value of k,

$$\Delta(z_{k,n_1}, s_k) \cap \Delta(z_{k,n_2}, s_k) = \emptyset \quad \text{if} \quad n_1 \neq n_2.$$

Now we define recursively some sets made of unions of the balls $\Delta(z_{k,n}, s_k)$, which we call 'swarms'. For $n \geq 1$ the swarm of height 1 and center $z_{1,n}$ is defined as

$$E_{1,n} = \Delta(z_{1,n}, s_1).$$

Once we have the swarms of height $j = 1, \dots, k-1$, we define the swarm of height k and center $z_{k,n}$ (for $n \geq 1$) as

$$E_{k,n} = \Delta(z_{k,n}, s_k) \bigcup \{E_{j,p} : \ j \leq k-1, \ p \geq 1 \text{ and } E_{j,p} \cap \Delta(z_{k,n}, s_k) \neq \emptyset\}.$$

We write $\mathrm{diam}_h E = \sup\{h(x,y) : x, y \in E\}$ for the hyperbolic diameter of a set $E \subset \mathbb{D}$. The next three properties will follow by induction.
(I) $\mathrm{diam}_h E_{k,n} \leq 2^k s_1 + 2^{k-1} s_2 + \cdots + 2 s_k$,
(II) $E_{k,n_1} \cap E_{k,n_2} = \emptyset$ if $n_1 \neq n_2$, and
(III) each swarm of height $j \leq k-1$ meets (and then it is contained in) at most one swarm of height k.
Proof of (I). This is trivial for $k = 1$. By the definition of swarms and inductive hypothesis,

$$\begin{aligned}
&\mathrm{diam}_h E_{k,n} \leq \\
&\quad \leq \ \mathrm{diam}_h \Delta(z_{k,n}, s_k) + 2 \max\{\mathrm{diam}_h E_{j,p} : \ j \leq k-1, \ E_{j,p} \cap \Delta(z_{k,n}, s_k) \neq \emptyset\} \\
&\quad \leq \ 2 s_k + 2\,(2^{k-1} s_1 + 2^{k-2} s_2 + \cdots + 2 s_{k-1}) = 2^k s_1 + \cdots + 2 s_k.
\end{aligned}$$

Proof of (II) *and* (III). Suppose that $E_{j,p}$ (with $j \leq k$) meets E_{k,n_1} and E_{k,n_2}, where $n_1 \neq n_2$. Then by (I)

$$\begin{aligned}
h(z_{k,n_1}, z_{k,n_2}) \ &\leq \ \mathrm{diam}_h E_{k,n_1} + \mathrm{diam}_h E_{j,p} + \mathrm{diam}_h E_{k,n_2} \\
&\leq \ 3\,(2^k s_1 + 2^{k-1} s_2 + \cdots + 2 s_k) < r_k,
\end{aligned}$$

which contradicts condition (4). When $j = k$ this proves (II), and for $j < k$ this proves (III) except for the statement between brackets. So, suppose that $E_{j,p}$ meets $E_{k,n}$, where $j \leq k-1$. If $E_{j,p}$ meets $\Delta(z_{k,n}, s_k)$ then $E_{j,p} \subset E_{k,n}$ by definition. Otherwise there is some swarm E of height at most $k-1$ such that

$$E \cap \Delta(z_{k,n}, s_k) \neq \emptyset \quad \text{and} \quad E \cap E_{j,p} \neq \emptyset.$$

If height $E \geq j =$ height $E_{j,p}$ then by inductive hypothesis (II) for the equality and (III) for the strict inequality, we have $E_{j,p} \subset E$. Hence, $E_{j,p}$ is contained in $E_{k,n}$. Similarly, if height $E < j$ then inductive hypothesis (III) implies that $E \subset E_{j,p}$. Therefore

$$E_{j,p} \cap \Delta(z_{k,n}, s_k) \supset E \cap \Delta(z_{k,n}, s_k) \neq \emptyset,$$

and then $E_{j,p} \subset E_{k,n}$ by definition.

Some remarks are in order. Condition (II) says that two swarms of the same height are either the same (with the same n) or they are disjoint, and condition (III) says that if two different swarms have non-void intersection, then the one of smaller height is contained into the other. Also, observe that by (II), $E_{k,n} \cap S_k = \{z_{k,n}\}$.

We will see that if $l \geq k$ then

$$\{|z-1| > \varepsilon_k\} \cap E_{l,p} = \emptyset \quad \text{for all} \quad p \geq 1. \tag{4.4}$$

In fact, suppose that for some $l \geq k$ and $p \geq 1$ there is $\omega \in E_{l,p}$ with $|\omega - 1| > \varepsilon_k$. Then, since $\{\varepsilon_j\}$ is a decreasing sequence, (2) and (I) yield

$$h(\omega, z_{l,p}) \geq h(\{|z-1| > \varepsilon_k\}, S_l) \geq h(\{|z-1| > \varepsilon_l\}, S_l) > r_l \geq 4 \operatorname{diam}_h E_{l,p},$$

which is not possible. By (4.4) every strictly increasing chain of swarms $E^{(1)} \subset E^{(2)} \subset \ldots$ is finite. Because if $\omega \in E^{(1)}$ then there is k such that $|\omega - 1| > \varepsilon_k$ (since ε_j tends to 0), and therefore $E^{(1)}$ cannot lie in any swarm of height $\geq k$. Roughly speaking, we could say that there is no swarm of infinite height. Consequently, every swarm is contained in a unique maximal swarm, and

$$\Omega \stackrel{\text{def}}{=} \bigcup_{k,n \geq 1} \Delta(z_{k,n}, s_k) = \bigcup \{E_{l,p} \text{ maximal}\}.$$

Choosing ε_k. Define a function $g \in H^\infty(\Omega)$ by $g(\omega) = f \circ L_{z_{l,p}}^{-1}(\omega)$ for $\omega \in E_{l,p}$, with $E_{l,p}$ a maximal swarm. The only requirements that we have imposed so far to the sequence $\{\varepsilon_k\}$ are that it is contained in $(0,1)$ and decreases to zero. We claim that there is a choice of the sequence $\{\varepsilon_k\}$ so that

$$\lim_k g(L_{z_{k,n}}(z)) = f(z), \tag{4.5}$$

where the limit is uniform on n and on compact subsets of $\mathbb{D}$. Fix $0 < r < 1$ and let $z \in \mathbb{D}$ with $|z| \leq r < 1$. Since $\lim_k s_k = \infty$ then for k big enough we have

$$|z| \leq r < (e^{s_k} - 1)/(e^{s_k} + 1). \tag{4.6}$$

This means that $z \in \Delta(0, s_k)$. The point $z_{k,n}$ is in some maximal swarm $E_{l,p}$ with $l \geq k$. Hence by (4.6)

$$L_{z_{k,n}}(z) \in \Delta(z_{k,n}, s_k) \subset E_{l,p} \subset \Omega$$

Since g is defined on Ω then $g(L_{z_{k,n}}(z))$ makes sense, and

$$g(L_{z_{k,n}}(z)) = f \circ L_{z_{l,p}}^{-1}(L_{z_{k,n}}(z)) = f \circ L_{L_{-z_{l,p}}(z_{k,n})}(\overline{\lambda(-z_{l,p}, z_{k,n})} z), \tag{4.7}$$

where the last equality comes from the identity $L_{z_{l,p}}^{-1} = L_{-z_{l,p}}$ and (1.1). A simple calculation shows that $\overline{\lambda(-z_{l,p}, z_{k,n})} = \lambda(z_{k,n}, L_{-z_{l,p}}(z_{k,n}))$. So, if $\xi \stackrel{\text{def}}{=} L_{-z_{l,p}}(z_{k,n})$ we can rewrite (4.7) as

$$g(L_{z_{k,n}}(z)) = f \circ L_\xi(\lambda_{z_{k,n},\xi} z). \tag{4.8}$$

Since $z_{k,n}, z_{l,p} \in E_{l,p}$ then by (I), $h(z_{k,n}, z_{l,p}) \leq \text{diam}_h E_{l,p} < r_l$. Thus, (6) implies that there is $\omega \in T_k$ such that $\rho(\xi, \omega) < \varepsilon_k$. Since $h(z,0) \leq s_k$ (by (4.6)) and $\|f\|_\infty = 1$, then successive applications of (4.8), (7) and the Schwarz-Pick inequality yield

$$\begin{aligned} |g(L_{z_{k,n}}(z)) - f(z)| &\leq |f \circ L_\xi(\lambda_{z_{k,n},\xi} z) - f \circ L_\omega(\lambda_{m,\omega} z)| \\ &+ |f \circ L_\omega(\lambda_{m,\omega} z) - f(z)| \\ &\leq 2\rho(f \circ L_\xi(\lambda_{z_{k,n},\xi} z), f \circ L_\omega(\lambda_{m,\omega} z)) + \varepsilon_k \\ &\leq 2\rho(L_\xi(\lambda_{z_{k,n},\xi} z), L_\omega(\lambda_{m,\omega} z)) + \varepsilon_k, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} \rho(L_\xi(\lambda_{z_{k,n},\xi} z), L_\omega(\lambda_{m,\omega} z)) &\leq \rho(L_\xi(\lambda_{z_{k,n},\xi} z), L_\xi(\lambda_{m,\xi} z)) + \rho(L_\xi(\lambda_{m,\xi} z), L_\omega(\lambda_{m,\omega} z)) \\ &= \varrho_1 + \varrho_2. \end{aligned}$$

Using the isometric property of L_ξ, a straightforward calculation shows that

$$\begin{aligned} \varrho_1 &= \rho(\lambda_{z_{k,n},\xi} z, \lambda_{m,\xi} z) \leq \frac{|\lambda_{z_{k,n},\xi} - \lambda_{m,\xi}|}{1 - |z|^2} \\ &= \left| \frac{(1 - z_{k,n})(1+\xi)\overline{\xi} - (1 - \overline{z}_{k,n})(1+\overline{\xi})\xi}{(1 - |z|^2)(1 + \overline{\xi} z_{k,n})(1 + \overline{\xi})} \right| \\ &\leq \frac{2}{(1 - r^2)} \frac{|1 - z_{k,n}|}{(1 - |\xi|)}. \end{aligned} \tag{4.10}$$

By (2) and since $\varepsilon_j \to 0$, there is $j > 1$ such that

$$\varepsilon_j < |1 - z_{k,n}| \leq \varepsilon_{j-1}, \tag{4.11}$$

and since $z_{k,n} \in E_{l,p}$ then (4.4) says that $l = \text{height}\, E_{l,p} < j$. By (I) then $h(z_{k,n}, z_{l,p}) \leq \text{diam}_h E_{l,p} < r_l < r_j$. Thus,

$$|\xi| = |L_{-z_{l,p}}(z_{k,n})| = \rho(L_{-z_{l,p}}(z_{k,n}), 0) = \rho(z_{k,n}, z_{l,p}) < \frac{e^{r_j} - 1}{e^{r_j} + 1},$$

and consequently $(1 - |\xi|) \geq e^{-r_j}$. Choosing $\varepsilon_{q-1} = (2\, q\, e^{r_q})^{-1}$ for all $q > 1$, we can insert the last inequality and (4.11) in (4.10), thus obtaining

$$\varrho_1 \leq \frac{2e^{r_j}\, \varepsilon_{j-1}}{1 - r^2} = \frac{j^{-1}}{(1-r)^2} < \frac{k^{-1}}{(1-r)^2}.$$

The last inequality holds because $j > l \geq k$. On the other hand, since $|z| \leq r$ and $\rho(\xi, \omega) < \varepsilon_k$ then Lemma 3.5 says that $\varrho_2 < (1-r)^{-2} 30 \varepsilon_k$. Putting all this together in (4.9) we obtain that whenever $|z| \leq r$ and k is large enough so that (4.6) holds, then

$$|g(L_{z_{k,n}}(z)) - f(z)| \leq 2(\varrho_1 + \varrho_2) + \varepsilon_k < \frac{C}{(1-r)^2} \frac{1}{k}$$

for some absolute constant $C > 0$. Since this estimate is independent of n then (4.5) follows.

The construction of F. We recall that b_k is a Blaschke product with zero sequence S_k. Since S is an interpolating sequence, then $a = \inf\{|b_k(z)| : z \in S \setminus S_k\} > 0$, and since $m \in \overline{S}_k$ then $\{x \in \mathcal{M} : |b_k(x)| < a/2\}$ is an open neighborhood of m. So, if (z_α) is a net in S converging to m then there is $\alpha(k)$ such that the tail $(z_\alpha)_{\alpha \geq \alpha(k)}$ is completely contained in $\{z \in \mathbb{D} : |b_k(z)| < a/2\} \cap S = S_k$. Therefore (4.5) implies that $\lim_\alpha g(L_{z_\alpha}(z)) = f(z)$.

By (3), $\sum_{k,n\geq 1}(1 - |z_{k,n}|) < \sum_{k\geq 1} 1/2^k = 1$, and consequently the Blaschke product $b = \prod_{k\geq 1} b_k$ converges. Furthermore, if we write $\beta = \prod_{k\geq 1} \sigma_k$, condition (5) tells us that

$$|b(z)| \geq \beta \quad \text{when} \quad z \notin \bigcup_{k\geq 1} \bigcup_{n\geq 1} \Delta(z_{k,n}, s_k) = \Omega.$$

That is, $V = \{z \in \mathbb{D} : |b(z)| < \beta\} \subset \Omega$. In addition, since each b_k vanishes on m (because $m \in \overline{S}_k$ for every $k \geq 1$) then b vanishes on m with infinite multiplicity. So, $b \equiv 0$ on $P(m)$.

Thus, $g \in H^\infty(\Omega) \subset H^\infty(V)$. By Lemma 3.9 then there are $0 < \gamma = \gamma(\beta) < \beta$, a constant $C > 0$ and $F \in H^\infty$ such that

$$|F(z) - g(z)| \leq C|b(z)| \quad \text{when} \quad |b(z)| < \gamma. \tag{4.12}$$

Let (z_α) be any net in S that tends to m and let $z \in \mathbb{D}$. Then $L_{z_\alpha}(z) \to L_m(z) \in P(m)$, and since $b \equiv 0$ on $P(m)$ then $b \circ L_{z_\alpha}(z) \to b \circ L_m(z) = 0$. So, there is α_0 (depending on z) such that $L_{z_\alpha}(z) \in \{|b| < \gamma\}$ for every $\alpha \geq \alpha_0$. Thus, by (4.12)

$$|F \circ L_{z_\alpha}(z) - g \circ L_{z_\alpha}(z)| \leq |b \circ L_{z_\alpha}(z)| \quad \text{for} \quad \alpha \geq \alpha_0,$$

where the last term tends to zero when we take limit in α. Henceforth

$$F \circ L_m(z) = \lim_\alpha F \circ L_{z_\alpha}(z) = \lim_\alpha g \circ L_{z_\alpha}(z) = f(z),$$

and we are done. □

The proof above shows that if $y \in \bigcap_{k\geq 1} \overline{S_k}$ is any point and $F \in H^\infty$ is the function constructed in the last step, then $F \circ L_y(z) = f(z)$. This does not mean that $\mathcal{L}_y(0) = \mathcal{L}_m(0)$, because the chain of interpolating sequences constructed depends on the function f.

5 Examples

A point $m \in \mathcal{M} \setminus \mathbb{D}$ is called *oricycular* if it is in the closure of a region limited by two circles in $\mathbb{D}$ that are tangent to $\partial\mathbb{D}$ at the same point. Every oricycular point is in $\mathcal{G}$ and it is in the closure of some tangent circle to $\partial\mathbb{D}$ (see [6, pp. 107-108]). We are going to search for the possible fibers that meet $\mathcal{L}_m(0)$ when m is a nontangential or an oricycular point, and we shall determine $\lambda(m, x)$ for all possible $x \in \mathcal{L}_m(0)$.

Every point in $\mathcal{M} \setminus \mathbb{D}$ has the form γm, where $\pi(m) = 1$ and $\gamma \in \mathbb{C}$ has modulus 1. Since $\lambda(m, x) = \lambda(\gamma m, \gamma x)$ for every $x \in \mathcal{M}$, and by Lemma 1.3 $\mathcal{L}_{\gamma m}(0) = \gamma \mathcal{L}_m(0)$, then there is no loss of generality by considering $\pi(m) = 1$.

Let b be an interpolating Blaschke product with zero sequence $\{z_k\}$ such that $b(m) = 0$. If the point $\omega \in \mathbb{D}$ is a zero of $b \circ L_m$ then there is a subsequence $\{z_{k_j}\}$ of $\{z_k\}$ such that $b \circ L_{z_{k_j}}(\omega) \to 0$. By Lemma 3.2 then

$$\lim_j \rho(\omega, \{L_{-z_{k_j}}(z_k)\}_{k \geq 1}) = \lim_j \rho(L_{z_{k_j}}(\omega), \{z_k\}) \leq c(\delta(b))^{-1} \lim_j |b(L_{z_{k_j}}(\omega))| = 0.$$

Consequently, ω is an accumulation point of

$$A_N = \{L_{-z_n}(z_k) : k \geq 1, \ n \geq N\}$$

for every positive integer N.

If m is a nontangential point we can assume that there is some fixed $-\pi/2 < \theta < \pi/2$, such that $\{z_n\}$ lies in the straight segment

$$S = \{z \in \mathbb{D} : 1 - z = re^{i\theta}, \ r > 0\}.$$

A straightforward calculation shows that the closure of S in $\mathbb{C}$ meets $\partial\mathbb{D}$ when $r = 0$ and $r = 2\cos\theta$. Therefore, $S = (1 - 2\cos\theta\, e^{i\theta}, 1)$ and $z_n = 1 - r_n e^{i\theta}$, with $0 < r_n < 2\cos\theta$. We can also assume that $z_n \to 1$. The conformal map L_{-z_n} sends S into a circular segment $C_n \cap \mathbb{D}$, where C_n is the circle that pass through the points $L_{-z_n}(z_n) = 0$, $L_{-z_n}(1) = e^{i2\theta}$ and

$$L_{-z_n}(1 - 2\cos\theta\, e^{i\theta}) = \frac{(r_n - 2\cos\theta)e^{i\theta}}{2\cos\theta\, e^{i\theta} + r_n(e^{-i\theta} - 2\cos\theta)}. \tag{5.1}$$

We are including here the extreme case when C_n is a straight line (i.e., $\theta = 0$). Since $r_n \to 0$, taking limits in (5.1) we see that the limit curve of C_n is the circular segment $C \cap \mathbb{D}$, where C is the circle that pass through 0, $e^{i2\theta}$ and -1. Therefore the zero sequence of $b \circ L_m$ lies in $C \cap \mathbb{D}$, and since $b \circ L_m$ vanishes on $\mathcal{L}_m(0)$, only the fibers of -1 and $e^{i2\theta}$ can have points of $\mathcal{L}_m(0)$.

Clearly, if $x \in \mathcal{L}_m(0)$ is in the fiber of $e^{i2\theta}$ then $\lambda(m, x) = e^{i2\theta}$. Suppose that $x \in \mathcal{L}_m(0)$ is in the fiber of -1. Since the straight segment $-1 + Re^{-i\theta}$, with $0 < R < 2\cos\theta$, is tangent to C at the point -1 then x is a nontangential point lying in the closure of this segment. So, $\lambda(m, x) = e^{-i2\theta}$ by Section 1.

If m is a oricycular point (with $\pi(m) = 1$) a similar but easier analysis shows that $\mathcal{L}_m(0)$ lies in the closure of $C \cap \mathbb{D}$, where C is the tangent circle to $\partial\mathbb{D}$ that pass through 0 and -1. Therefore $\mathcal{L}_m(0)$ only can meet the fiber of -1, and indeed it does unless $P(m)$ is a homeomorphic disk. Since C is tangent to $\partial\mathbb{D}$ then every $x \in \mathcal{L}_m(0)$ with $\pi(x) = -1$ is a tangential point, which by Section 1 yields $\lambda(m, x) = -1$.

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