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Deterministic robust control design for an iron core linear drive including second order electrical dynamics

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Abstract

This work proposes a deterministic robust controller to improve tracking performance for a linear motor, taking into account the electrical dynamics imposed by a commercial current controller. The design is split in two parts by means of the backstepping technique, in which the first part corresponds to a typical deterministic robust controller neglecting the electrical dynamics. In the second part, a second order electrical dynamics is considered using a particular state transformation. There, the proposed control law is composed of a term to compensate the known part of the model, and a robust control term to impose a bound on the effect of uncertainties on tracking error. Stability and boundedness results for the complete controller are given. To this effect, a general result on boundedness and stability of nonlinear systems with conditionally bounded state variables is derived first. Finally, experimental results for the complete controller show an improvement on tracking error of up to $31.7\,\%$ when compared with the results from the typical controller that neglects the electrical dynamics.

Keywords

Linear motor, position control, robust control, non-linear control, electrical dynamics

Introduction

The search for improvement in production time and quality imposes specific requirements in machine-tool feed drives regarding speed and precision. An attractive alternative for the typical ball-screw drives are linear motors, which show high speed and acceleration capability while keeping a low wear due to the absence of mechanical coupling, as well as avoiding the limitations coming from vibration modes associated with the screw. However, this absence makes linear motors more sensitive to force disturbances. Also, depending on the constructive characteristics of a particular linear motor, some effects affecting motion performance can become more important, such as force ripple, cogging force, and electrical dynamics.

There have been many approaches for the control of linear motors, such as a feedforward and state feedback controller[1]; a technique called nominal characteristic trajectory following[2 3]; repetitive control[4]; a PID-based controller[5]; an improved sliding mode controller in the work by Xie[6]; and several model-based adaptive robust controllers[7–11] based on the work by Yao[12,13], just to cite a few. Model based controllers often have high robustness[14]. In particular, sliding mode control is a type of model based robust control, and its use has increased in precision positioning systems[14].

As can be seen on several of the mentioned references, a usual approach for the control of linear motors consists in neglecting the electrical dynamics, since it has a much faster response than the mechanical dynamics[7,8]. Then, a controller is set for the part of the system related to the mechanical dynamics assuming a proportionality between control action and motor current. This is most suitable in ironless linear motors, with reported bandwidths on the order of kilohertz[15], and where certainly low errors are achieved, on the order of micrometers for moderate to high speed trajectories. However, in iron core linear motors the time constant is not so small due to the large inductance associated with the iron core[9], and although particular iron-core linear motors are shown to have electrical bandwidths on the order of kilohertz[16], there are other cases in which bandwidths below the hundred hertz have been reported[17]. This has a negative impact on tracking error, as the existence of unmodeled actuator dynamics in a system is able to degrade control performance. For instance, for sliding mode controllers, unmodeled actuator dynamics can lead to chattering, with lower tracking accuracy, excitation of unmodeled high frequency dynamics, and even system instability[18].

One approach followed in the literature, which considers the electrical dynamics, is feasible when there is direct access to motor voltage[19]. However, in a setup including a commercial current controller, this is not usually possible. Also, these works are restricted to a first order dynamics, as in the study by Yao and Xu[9]. This might not be enough to represent current loop dynamics in a setup where the only possible control action is the current reference of a commercial current controller[17]. Therefore, the present work seeks to approach motion control for a setup including a commercial current controller and a permanent magnet linear

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synchronous motor (PMLSM) like the one in the work by Villegas et al.[17], where electrical dynamics is represented by a second order linear time-invariant (LTI) system.

As the system under consideration is nonlinear, there are quite a few possible approaches for control synthesis and stability analysis. First of all, the Lyapunov formalism is one of the most popular and is usually used in robust control, whether in its continuous-time version, as in the work by Zhao et al.[20], or in its discrete-time version, as in the work by Zhang et al.[21]. This formalism has several derivations, such as the Lyapunov-Krasovskii approach, which is extensively used for stability analysis of nonlinear systems with time delay[22–26]. Another branch of approaches comes from the input-output formalism with results as the small gain theorem or its extensions, like the scaled small-gain theorem that is used for the design of an output feedback controller in the work by Wei et al.[27].

Fuzzy and piecewise-affine systems are other interesting alternatives to deal with nonlinear systems, as they allow to represent nonlinear systems with arbitrary accuracy. Also, several approaches are available for stability analysis of these systems. Among these can be mentioned Markovian Lyapunov functions[28], mode-dependent piecewise Lyapunov functions[29], piecewise Markovian Lyapunov functions[30]. Even cases with time delay can be considered, by means of piecewise-Markovian Lyapunov-Krasovskii functionals[31]. However, when applying these system representations, care must be taken with the increase on the number of regions, as it comes with the corresponding increment in the number of local models and parameters. In fact, as accuracy increases, so does the number of regions, increasing also with the domain of approximation[32].

In view of the previous considerations, controller synthesis in this work is based in the nonlinear model of the system. In particular, the backstepping technique[33], which is extensively used in the literature as well as several of its variants[34,35], is applied. However, stability analysis is based on results derived in this work, developed from fundamental properties of nonlinear systems.

In this context, a controller is proposed using a novel approach to reduce the performance degradation imposed by the electrical dynamics. In this development, a result which allows to express the electrical dynamics in a more convenient form for controller synthesis is given. Also, a general Lyapunov-like result on stability of systems with conditionally bounded states is derived, which is then used to prove stability of the proposed controller. The distinct performance improvement achieved by the proposed controller is shown with experimental tests, where it is compared to a typical deterministic robust controller in which electrical dynamics is not considered for controller synthesis.

Thus, in the next section the system model and some general assumptions are presented. The third section is devoted to the development of the proposed controller, while the fourth section considers the stability issue. The experimental results are shown in the fifth section. Then, conclusions are presented in the sixth section. Finally, proofs of several results presented throughout the text are given in the appendix.

Linear feed drive model

The system under consideration in this paper consists of a permanent magnet linear synchronous motor driven by a commercial current amplifier. Following the work by Villegas et al.[17], the system is modeled as shown in Fig. 1. Here, $G(s)$ characterizes the electrical dynamics from the reference input of the current controller (u) to the motor current (x_3). The output of the block K_f represents the force developed by the motor for a particular current, and depends on the position of the moving part (x_1) due to the ripple effect[36]. The block F_{cg} is related to cogging force and F_f represents friction force. Finally, f_{dis} encompasses unmodeled disturbances and model uncertainties.

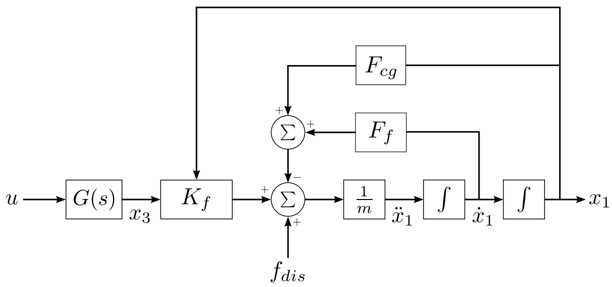

Figure 1. System model[17]

In particular, $G(s)$ is assumed to be a second order strictly minimum-phase LTI system with relative degree one, just like the model in the work by Villegas et al.[17]. Furthermore, it is considered that $G(s)$ might not necessarily be positive real. Thus, the state equation for the whole system can be represented as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ m\dot{x}_2 &= K_f(x_1)x_3 - F_{cg}(x_1) - F_f(x_2) + f_{dis}(t, \mathbf{x}) \\ \dot{x}_3 &= a_{33}x_3 + a_{34}x_4 + b_3 u \\ \dot{x}_4 &= a_{43}x_3 + a_{44}x_4 + b_4 u. \end{aligned} \tag{1}$$

Here, the states x_3 and x_4 correspond to the electrical dynamics, where x_3 represents the motor current, and where $b_3 \neq 0$ and $b_4 \neq 0$. This representation of the electrical dynamics is possible for any second order LTI system with relative degree one (corresponding to a transfer function with two poles and one zero), as shown in the next proposition.

Proposition 1. *Given a second order SISO LTI system with relative degree one*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}. \end{aligned} \tag{2}$$

Then there is a nonsingular matrix $\mathbf{T}_d$ such that by the state transformation $\mathbf{x} = \mathbf{T}_d\mathbf{z}$ the system can be expressed in the following form

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}_d\mathbf{z} + \mathbf{B}_d u \\ y &= \mathbf{C}_d\mathbf{z} \end{aligned} \tag{3}$$

for $\mathbf{C}_d = [1\ 0]$ and $\mathbf{B}_d = [b\ b]^T$ where $b = 1/\det \mathbf{T}_d$.

Proof: Proof is given in the appendix.

As for friction force, its behavior is complex and there are several models with different levels of sophistication to represent part of this behavior. In this particular case, the simple Stribeck-tanh friction model[37] is considered. This model is capable of representing the Stribeck curve for a range of speeds while keeping the formal analysis simple. Besides, as previously pointed out, deviations from this model will be considered as part of the disturbances. The Stribeck-tanh model is a static friction model in which the dependence of friction force with velocity can be expressed as

$$F_f(x_2) = \left(F_c + (F_s - F_c)e^{-\left|\frac{x_2}{V_s}\right|^{\delta_{V_s}}}\right)\tanh(k_{tanh}x_2) + k_v x_2 \tag{4}$$

being F_c the Coulomb force, F_s the static force, V_s the Stribeck velocity, δ_{V_s} the Stribeck shape factor, k_v the viscous friction coefficient, and k_{tanh} a coefficient that determines the behavior of the model in the transition between positive and negative velocities.

It is assumed that the system parameters are not perfectly known, but there are known bounds for the uncertainties and unmodeled disturbances on the dynamics for x_1 and x_2 in (1).

$K_f(x_1)$ depends on x_1 in order to represent the ripple effect. This is an electromagnetic effect by which force constant varies with position. The cogging force, represented by the term $F_{cg}(x_1)$, is a position-varying force due to the attraction of the motor translator to the permanent magnets in the secondary, and can be important for PMLSM with ferromagnetic core. Several models have been proposed in literature, either purely periodical or including possible aperiodical behavior to represent this variation along motor displacement. Whichever the case, it is assumed that there are bounds $K_{fmax} > 0$, $K_{fmin} > 0$ and $F_{cmax} > 0$ such that $K_{fmin} \leq K_f(x_1) \leq K_{fmax}$ and $|F_{cg}(x_1)| \leq F_{cmax}$ for all $x_1 \in \mathbb{R}$, and both $K_f(x_1)$ and $F_{cg}(x_1)$ are differentiable in $\mathbb{R}$.

It is also assumed that $f_{dis}(t, \mathbf{x})$ is bounded, piecewise continuous in t and locally Lipschitz in $\mathbf{x}$. Finally it is considered that the reference trajectory $x_d(t)$ is such that $x_d, \dot{x}_d, \ddot{x}_d$, and $\dddot{x}_d$ are bounded, equivalent to a jerk-limited trajectory.

Proposed controller

In this section, a deterministic robust controller is proposed including the electrical dynamics, seeking for an improvement in control performance. It should be noticed that although the controller will be designed for the whole system, the design process will be split in two stages using the backstepping technique[18,33]. Therefore, in the first stage, the control problem will be approached as if direct access to motor current could be achieved, like it is done when electrical dynamics is neglected. Then, in the second stage, electrical dynamics will be included in the design.

First stage

For the first stage, a typical deterministic robust controller is developed following a procedure similar to that for the design of the robust control law in the work by Xu and Yao[8], except that the system is augmented with a state representing the integral of the position error. Such augmentation is a common technique, used for instance with a sliding mode controller to achieve asymptotic tracking of a constant reference in presence of uncertainties[18].

Let $e_1 = x_1 - x_d$ be the position error, where x_d is the position reference, and $e_2 = x_2 - \dot{x}_d$ the velocity error. Let the state e_0 be defined as $\dot{e}_0 = e_1$, and define a variable $s = m\psi_0 e_0 + m\psi_1 e_1 + m e_2$ as could be done for a sliding mode controller. Thus, considering (1), $\dot{s}$ can be expressed as

$$\begin{aligned}\dot{s} &= m\psi_0 e_1 + m\psi_1 e_2 + K_f(x_1)x_3 \\ &\quad - F_{cg}(x_1) - F_f(x_2) + f_{dis}(t, \mathbf{x}) - m\ddot{x}_d.\end{aligned} \tag{5}$$

As a first step, a control law $\alpha = \alpha_a + \alpha_s$ is proposed for the virtual control variable x_3. As in the work by Xu and Yao[8] this control law is split in two parts: α_a, intended to compensate known terms; and α_s, a robust control term. This separation allows more clarity in the design process for the robust control term α_s, as α_a becomes part of a term encompassing model uncertainties and unmodeled disturbances.

Considering (5) and our prior assumptions, α_a is given by

$$\begin{aligned}\alpha_a = \frac{1}{\hat{K}_f(x_1)}\Big[&\hat{F}_{cg}(x_1) + \hat{F}_f(t) \\ &+ \hat{m}\left(\ddot{x}_d - \psi_0 e_1 - \psi_1 e_2\right)\Big].\end{aligned} \tag{6}$$

Here, the accent "^" stands for an approximate value, an estimation obtained from a previous identification of the system. In particular, given an actual value ξ and its estimate $\hat{\xi}$, its error is defined as $\tilde{\xi} = \hat{\xi} - \xi$. In the particular case of the functions $\hat{K}_f(x_1)$ and $\hat{F}_{cg}(x_1)$, the same assumptions about boundedness and differentiability explicited for $K_f(x_1)$ and $F_{cg}(x_1)$ when describing the model are assumed to apply.

Also, it is considered a feedforward compensation for friction, represented by a time function $\hat{F}_f(t)$ that is obtained solely from the reference trajectory. However, it is assumed that the friction model used for this is such that the feedforward signal is piecewise smooth and bounded under the boundedness assumption for $x_d, \ldots, \dddot{x}_d$.

The error variable for the virtual control x_3 is defined as $z_3 = x_3 - \alpha$, and considering that $K_f(x_1) = \hat{K}_f(x_1) - \tilde{K}_f(x_1)$, then

$$\begin{aligned}\dot{s} &= m\psi_0 e_1 + m\psi_1 e_2 + K_f(x_1)z_3 + K_f(x_1)(\alpha_a + \alpha_s) \\ &\quad - F_{cg}(x_1) - F_f(x_2) + f_{dis}(t, \mathbf{x}) - m\ddot{x}_d \\ &= K_f(x_1)(z_3 + \alpha_s) + d_m(t, \mathbf{x})\end{aligned} \tag{7}$$

where d_m includes unmodeled disturbances and model uncertainties, and it is given by

$$\begin{aligned}d_m(t, \mathbf{x}) &= -\tilde{K}_f(x_1)\alpha_a + \tilde{F}_{cg}(x_1) + \tilde{F}_f(t, x_2) \\ &\quad + f_{dis}(t, \mathbf{x}) + \tilde{m}\left(\ddot{x}_d - \psi_0 e_1 - \psi_1 e_2\right).\end{aligned} \tag{8}$$

Knowing a bound for the parametrical uncertainties and the unmodeled disturbances in $d_m(t, \mathbf{x})$, a smooth function $\delta_m(t, \mathbf{x})$ can be found such that $|d_m(t, \mathbf{x})| \leq \delta_m(t, \mathbf{x})$.

Let $V_1 = \frac{1}{2}w_1 s^2$ be an auxiliary function, where $w_1 > 0$. Thus,

$$\dot{V}_1 = w_1 s K_f(x_1)z_3 + w_1 s(K_f(x_1)\alpha_s + d_m(t, \mathbf{x})). \tag{9}$$

Let α_s be expressed as $\alpha_s = \alpha_{s1} + \alpha_{s2}$, where α_{s1} is a stabilizing feedback term which takes the form

$$\alpha_{s1} = -\frac{k_s}{K_{fmin}} s \tag{10}$$

where k_s is a positive constant.

Now a term α_{s2} is proposed to reduce the effect of the uncertainties and disturbances considered in the term $d_m(t, \mathbf{x})$. This term is chosen in order to verify the condition

$$s(K_f(x_1)\alpha_{s2} + d_m(t, \mathbf{x})) \leq \epsilon \tag{11}$$

for a constant $\epsilon > 0$ that constitutes a design parameter for the proposed controller. As described in the work by Yao and Xu[9], a suitable term can be

$$\alpha_{s2} = -\frac{\delta_m^2(t, \mathbf{x})}{4K_{fmin}\epsilon} s. \tag{12}$$

It can be shown that this function satisfies the inequality in (11). To see this, define

$$R(t, \mathbf{x}) = -K_f(x_1)\frac{\delta_m^2(t, \mathbf{x})}{4K_{fmin}\epsilon} s + d_m(t, \mathbf{x}) \tag{13}$$

then

$$\begin{aligned} sR(t, \mathbf{x}) &= -\frac{K_f(x_1)}{K_{fmin}}\frac{\delta_m^2(t, \mathbf{x})}{4\epsilon} s^2 + s d_m(t, \mathbf{x}) \\ &\leq -\frac{d_m^2(t, \mathbf{x})}{4\epsilon} s^2 + s \cdot d_m(t, \mathbf{x}) \\ &= -\left(\frac{d_m(t, \mathbf{x})}{2\sqrt{\epsilon}} s\right)^2 + s \cdot d_m(t, \mathbf{x}). \end{aligned} \tag{14}$$

Now consider $a = \frac{d_m(t,\mathbf{x})}{2\sqrt{\epsilon}} s$ and $b = \sqrt{\epsilon}$. As $(a - b)^2 \geq 0$, or equivalently $a^2 - 2ab + b^2 \geq 0$, then $-a^2 + 2ab \leq b^2 = \epsilon$, which combined with (14) leads to

$$sR(t, \mathbf{x}) \leq -\left(\frac{d_m(t, \mathbf{x})}{2\sqrt{\epsilon}} s\right)^2 + s \cdot d_m(t, \mathbf{x}) \leq \epsilon. \tag{15}$$

Thus, the control law α is proposed for the virtual control input x_3, which represents the motor current. In this manner, if electrical dynamics were to be neglected, α would constitute the proposed control law. However, as electrical dynamics is to be considered, it has to be included. This is done on the second stage of the design, from which a novel control law for the real input is proposed, based on the virtual control α.

Second stage

In the previous section, α, the desired value for the state x_3, has been obtained. Now, suppose an input u_0 is applied to the electrical dynamics such that x_3 would be able to track α. In such a case, while x_3 tracks α, x_4 should track some β, verifying the equations corresponding to the electrical dynamics, leading to

$$\begin{aligned} \dot{\alpha} &= a_{33}\alpha + a_{34}\beta + b_3 u_0 \\ \dot{\beta} &= a_{43}\alpha + a_{44}\beta + b_4 u_0. \end{aligned} \tag{16}$$

Therefore, given α, such a β could be obtained by the following LTI system

$$\dot{\beta} = \left(a_{44} - \frac{b_4}{b_3}a_{34}\right)\beta + \left(a_{43} - \frac{b_4}{b_3}a_{33}\right)\alpha + \frac{b_4}{b_3}\dot{\alpha} \tag{17}$$

which is a stable (and proper) LTI system, given that the electrical dynamics is minimum-phase and thus $a_{44} - \frac{b_4}{b_3}a_{34}$, which corresponds to the zero of the electrical dynamics, is negative.

Having defined the quantity β, now the states z_3 and z_4 are defined as $z_3 = x_3 - \alpha$ and $z_4 = x_4 - \beta$, i.e., the difference between each state and its desired value. Then, its dynamics can be expressed as

$$\begin{aligned} \dot{z}_3 &= a_{33}z_3 + a_{34}z_4 + b_3 u + a_{33}\alpha + a_{34}\beta - \dot{\alpha} \\ \dot{z}_4 &= a_{43}z_3 + a_{44}z_4 + b_4 u + \frac{b_4}{b_3}(a_{33}\alpha + a_{34}\beta - \dot{\alpha}) \end{aligned} \tag{18}$$

which can be written as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\left(u + \frac{1}{b_3}(a_{33}\alpha + a_{34}\beta - \dot{\alpha})\right) \tag{19}$$

where $\mathbf{z} = [z_3\ z_4]^T$, $\mathbf{B} = [b_3\ b_4]^T$ and

$$\mathbf{A} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}. \tag{20}$$

Here, $\mathbf{A}$ is Hurwitz, therefore $\exists!\mathbf{P}$ symmetric positive definite matrix such that $\mathbf{PA} + \mathbf{A}^T\mathbf{P} = -\mathbf{I}$. Considering now an auxiliary function $V = V_1 + w_2\mathbf{z}^T\mathbf{P}\mathbf{z}$, with $w_2 > 0$, then

$$\begin{aligned} \dot{V} = &-w_1\frac{K_f(x_1)}{K_{fmin}}k_s s^2 - w_2\mathbf{z}^T\mathbf{z} + w_1 s K_f(x_1) z_3 \\ &+ w_1 s(K_f(x_1)\alpha_{s2} + d_m(t, \mathbf{x})) \\ &+ 2w_2\mathbf{z}^T\mathbf{P}\mathbf{B}\left(u + \frac{1}{b_3}(a_{33}\alpha + a_{34}\beta - \dot{\alpha})\right). \end{aligned} \tag{21}$$

The particular form of the term $\dot{\alpha}$ in (21) is not developed in order to allow certain freedom on the way in which its compensation might be handled.

Now, the actual control law is proposed, taking the form $u = u_a + u_s$ where u_s is a robust control law to be defined later and

$$u_a = -\frac{1}{\hat{b}_3}\left(\hat{a}_{33}\alpha + \hat{a}_{34}\hat{\beta} - \hat{\dot{\alpha}}\right) \tag{22}$$

is a compensation term associated with the electrical dynamics. Here, approximate values are used for β and $\dot{\alpha}$, as these will depend on parameters that are assumed to be known only in an approximate way, as well as on unmodeled disturbances. It can also be noticed that in (16), for zero initial error, the u_0 would take the form of u_a except that it should be calculated with the exact values.

Using the proposed control, and defining $\mathbf{x}_a = [s\ z_3\ z_4]^T$, $\dot{V}$ is given by

$$\begin{aligned} \dot{V} = &-\mathbf{x}_a^T \begin{bmatrix} w_1\frac{K_f(x_1)}{K_{fmin}}k_s & -w_1\frac{K_f(x_1)}{2} & 0 \\ -w_1\frac{K_f(x_1)}{2} & w_2 & 0 \\ 0 & 0 & w_2 \end{bmatrix} \mathbf{x}_a \\ &+ w_1 s(K_f(x_1)\alpha_{s2} + d_m(t, \mathbf{x})) \\ &+ 2w_2\mathbf{z}^T\mathbf{P}\mathbf{B}(u_s + d_e(t, \mathbf{x})) \end{aligned} \tag{23}$$

where d_e is

$$d_e(t,\mathbf{x},\beta,\hat{\beta}) = \left(\frac{a_{33}}{b_3} - \frac{\hat{a}_{33}}{\hat{b}_3}\right)\alpha + \left(\frac{a_{34}}{b_3}\beta - \frac{\hat{a}_{34}}{\hat{b}_3}\hat{\beta}\right) - \left(\frac{1}{b_3}\dot{\alpha} - \frac{1}{\hat{b}_3}\hat{\dot{\alpha}}\right) \tag{24}$$

which can also be expressed as $d_e = u_a - u_0$.

It should be noticed that the matrix in the first term of (23) will be positive definite whenever

$$w_1 w_2 \frac{K_f(x_1)}{K_{fmin}} k_s - w_1^2 \frac{K_f^2(x_1)}{4} > 0 \tag{25}$$

or in another form,

$$k_s > \frac{w_1}{w_2} \frac{K_f(x_1) K_{fmin}}{4}. \tag{26}$$

The term u_s on the other hand will be chosen to reduce the error introduced by the uncertainties in d_e. A possible form for this term is

$$u_s = -\hat{\mathbf{K}}_e \hat{\mathbf{z}} \tag{27}$$

where $\hat{\mathbf{K}}_e = \frac{1}{2}\eta \hat{\mathbf{B}}^T \hat{\mathbf{P}}$, for $\eta > 1$ and $\hat{\mathbf{z}} = [x_3 - \alpha \;\; x_4 - \hat{\beta}]^T$. Here, the symbols $\hat{\mathbf{K}}_e$ and $\hat{\mathbf{z}}$ are used because of their dependence on estimated values. In the same way, the symbols $\mathbf{K}_e$ and $\mathbf{z}$ are used to indicate dependence on exact values.

Then, proceeding as previously done with α_s, leads to

$$\begin{aligned} 2\mathbf{z}^T\mathbf{PB}\left(u_s + d_e\right) &= 2\mathbf{z}^T\mathbf{PB}\left(-\mathbf{K}_e\mathbf{z} + d_e^*\right) \\ &= -2\left(\sqrt{\frac{\eta}{2}}\mathbf{z}^T\mathbf{PB}\right)^2 + 2\mathbf{z}^T\mathbf{PB}d_e^* \\ &\leq \frac{(d_e^*)^2}{\eta}. \end{aligned} \tag{28}$$

Here, d_e^* is given by

$$d_e^*(t,\mathbf{x},\beta,\hat{\beta}) = \left(\frac{a_{33}}{b_3} - \frac{\hat{a}_{33}}{\hat{b}_3}\right)\alpha + \left(\frac{a_{34}}{b_3}\beta - \frac{\hat{a}_{34}}{\hat{b}_3}\hat{\beta}\right) - \left(\frac{1}{b_3}\dot{\alpha} - \frac{1}{\hat{b}_3}\hat{\dot{\alpha}}\right) - \tilde{\mathbf{K}}_e\mathbf{z} - \hat{\mathbf{K}}_e\tilde{\mathbf{z}} \tag{29}$$

where $\tilde{\mathbf{z}} = [0 \; \tilde{\beta}]^T$.

It should be noticed that the function d_e^* is such that there is a function δ_e (not necessarily known) that satisfies $|d_e^*(t,\mathbf{x},\beta,\hat{\beta})| \leq \delta_e(\mathbf{x},\beta,\hat{\beta})$. It is assumed without loss of generality that the bounding function δ_e is continuous.

Based on the prior development, in the following section the stability of the system with the proposed control law is evaluated.

Stability

In this section a general result on the behavior of systems with conditionally bounded states is given first. Then, based on this result, stability of the proposed controller is demonstrated.

Result on the behavior of a system with conditionally bounded states

In order to simplify the analysis of systems in which several state variables might not vanish with time, but could be bounded under certain conditions, the following result is developed.

Theorem 1. *Consider a system which can be represented as*

$$\begin{aligned} \dot{\mathbf{x}}_a &= \mathbf{f}_a(t,\mathbf{x}_a,\mathbf{x}_b) \\ \dot{\mathbf{x}}_b &= \mathbf{f}_b(t,\mathbf{x}_a,\mathbf{x}_b) \end{aligned} \tag{30}$$

where $\mathbf{x}_a \in \mathbb{R}^n$, $\mathbf{x}_b \in \mathbb{R}^m$, and either $\mathbf{f}_a$ and $\mathbf{f}_b$ are piecewise continuous in t and locally Lipschitz in $(\mathbf{x}_a,\mathbf{x}_b)$ on $[0,\infty) \times \mathbb{R}^{n+m}$.

Suppose that the system is such that following conditions hold:

1. Conditional boundedness on $\mathbf{x}_b$. *Whenever $\mathbf{x}_a(\tau) \in D_a$ for all $t_0 \leq \tau \leq t$, then $(\mathbf{x}_a(t),\mathbf{x}_b(t)) \in D$, being $D = D_a \times D_b$, D_a and D_b bounded domains.*
2. Lyapunov-like condition on $\mathbf{x}_a$. *There exists a continuous positive definite function $W : \mathbb{R}^n \to \mathbb{R} : \mathbf{x}_a \mapsto W(\mathbf{x}_a)$ and a continuously differentiable function $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R} : (t,\mathbf{x}_a) \mapsto V(t,\mathbf{x}_a)$ which is positive definite, radially unbounded, and decrescent. There is a constant $B \geq 0$ such that the following holds:*

$$\begin{aligned} \dot{V}(t,\mathbf{x}_a(t)) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}_a}\mathbf{f}_a(t,\mathbf{x}_a,\mathbf{x}_b) \\ &\leq -W(\mathbf{x}_a) \end{aligned} \tag{31}$$

for all $t \geq 0$, $(\mathbf{x}_a,\mathbf{x}_b) \in \bar{D}$, $\|\mathbf{x}_a\| \geq B$.
It should be noticed that being $V(t,\mathbf{x}_a)$ positive definite, radially unbounded and decrescent, there exist class $\mathcal{K}_\infty$ functions α_1, α_2 such that[18]

$$\alpha_1(\|\mathbf{x}_a\|) \leq V(t,\mathbf{x}_a) \leq \alpha_2(\|\mathbf{x}_a\|). \tag{32}$$

Then, it is assumed that the following condition holds for B.

3. Condition on B. *$B < \alpha_2^{-1}(\alpha_1(r))$ for some $r > 0$ such that $B_r = \{\mathbf{x}_a \in \mathbb{R}^n : \|\mathbf{x}_a\| \leq r\} \subseteq D_a$.*

In such a case, for every initial state $\mathbf{x}(t_0) = (\mathbf{x}_a(t_0),\mathbf{x}_b(t_0))$ such that $\|\mathbf{x}_a(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$ and $\mathbf{x}_b(t_0) \in D_b$, there exists a solution $\mathbf{x}(t)$ defined for $t \geq t_0 \geq 0$. This solution is bounded. Furthermore, for $B > 0$ there exists a class $\mathcal{K}_\infty$ function α_B, a class $\mathcal{KL}$ function $\beta(r,s)$ and a constant $T > 0$, dependent on $\mathbf{x}_a(t_0)$ and B, such that $\mathbf{x}_a(t)$ satisfies:

$$\begin{aligned} &\|\mathbf{x}_a(t)\| \leq \beta(\|\mathbf{x}_a(t_0)\|, t - t_0), \qquad \forall t \in [t_0, t_0 + T] \\ &\|\mathbf{x}_a(t)\| \leq \alpha_B(B), \qquad \forall t \in [t_0 + T, \infty). \end{aligned} \tag{33}$$

In the particular case that $B = 0$, the result for $\mathbf{x}_a$ reduces to

$$\|\mathbf{x}_a(t)\| \leq \beta(\|\mathbf{x}_a(t_0)\|, t - t_0), \qquad \forall t \in [t_0, \infty) \tag{34}$$

whereby $\mathbf{x}_a(t) \to 0$ for $t \to \infty$.

Proof: Proof is given in the appendix.

Although the previous result is local in principle, it should be clear that the only limit on the diameter of the sets D_a and D_b is given by the largest set where condition 2 of the theorem and the conditional boundedness of D_b are verified (which could establish a relation between the diameters of D_b and D_a), whenever B is small enough to fulfill condition 3.

Also, it should be clear that if $\mathbf{x}_b$ is unconditionally bounded the result is still valid, as condition 1 is obviously verified.

Stability of the proposed controller

For the system shown with the proposed control law, the following result can be derived as a corollary of Theorem 1.

Corollary 1. *For the given dynamics, with the properties assumed for the reference trajectory and the functions involved, suppose the control law is given by*

$$u = -\hat{\mathbf{K}}_e\hat{\mathbf{z}} - \frac{1}{\hat{b}_3}\left(\hat{a}_{33}\alpha + \hat{a}_{34}\hat{\beta} - \hat{\dot{\alpha}}\right) \tag{35}$$

where $\hat{\mathbf{K}}_e = \frac{1}{2}\eta\hat{\mathbf{B}}^T\hat{\mathbf{P}}$*, for* $\eta > 1$*,* $\hat{\mathbf{z}} = [x_3 - \alpha \ x_4 - \hat{\beta}]^T$*,* $\hat{\beta}$ *is obtained* from*

$$\dot{\hat{\beta}} = \left(\hat{a}_{44} - \frac{\hat{b}_4}{\hat{b}_3}\hat{a}_{34}\right)\hat{\beta} + \left(\hat{a}_{43} - \frac{\hat{b}_4}{\hat{b}_3}\hat{a}_{33}\right)\alpha + \frac{\hat{b}_4}{\hat{b}_3}\dot{\alpha} \tag{36}$$

and α *is given by*

$$\begin{aligned}\alpha = &-\frac{k_s}{K_{fmin}}s - \frac{\delta_m^2(t,\mathbf{x})}{4K_{fmin}\epsilon}s \\ &+ \frac{1}{\hat{K}_f(x_1)}\Big[\hat{F}_{cg}(x_1) + \hat{F}_f(t) \\ &\qquad + \hat{m}\left(\ddot{x}_d - \psi_0 e_1 - \psi_1 e_2\right)\Big]\end{aligned} \tag{37}$$

where $s = m\psi_0 e_0 + m\psi_1 e_1 + m e_2$*, the state* e_0 *is defined as* $\dot{e}_0 = e_1$*, and* k_s *verifies the following condition:*

$$k_s > \frac{w_1}{w_2}\frac{K_f(x_1)K_{fmin}}{4}. \tag{38}$$

In such a case, for ϵ *and* η *properly chosen, the system solution is bounded and there is a class* $\mathcal{K}_\infty$ *function* α_F *such that the error variables* $e_0(t)$*,* $e_1(t)$*, and* $e_2(t)$ *converge to the region* $\|e_i(t)\| \le \alpha_F(B_d)$*, for* $i = 0, 1, 2$*, where*

$$B_d = \sqrt{\frac{w_1\epsilon + w_2\frac{\delta_{e,max}^2}{\eta}}{\lambda_{min}\theta}} \tag{39}$$

for some $\theta \in (0,1)$*, and for a constant* $\delta_{e,max}$ *dependent on the initial conditions.*

Furthermore, in absence of unmodeled disturbances and without model uncertainties, $e_i(t) \to 0$ *for* $t \to \infty$*.*

Proof: Proof is given in the appendix.

It should be noticed that although the result in corollary 1 depends on a set of admissible values for the initial conditions of the system, nothing prevents to choose such a set to contain the maximum values admitted by the physical system, considering the practical limits on displacement, speed, and control action for the device in use. Furthermore, the same set can be considered in order to choose a locally valid bound δ_M for the bounding function $\delta_m(t,\mathbf{x})$. In this way, the term α_s can be expressed in the simpler form $\alpha_s = -K_S s$, where

$$K_S = \frac{k_s + \frac{\delta_M^2}{4\epsilon}}{K_{fmin}}. \tag{40}$$

This particular form of α_s has been chosen to test the proposed controller in the experimental setup.

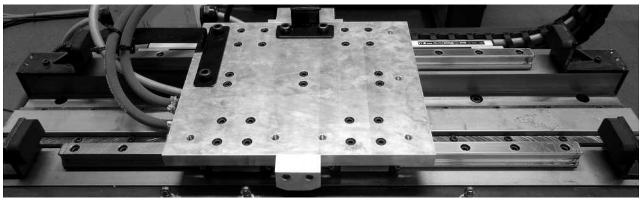

Figure 2. Linear motor stage.

Experimental results

The proposed controller has been tested on the linear motor stage shown in Fig. 2. The experimental setup[17] includes a permanent magnet linear synchronous motor with ball guide rails, and a current amplifier. The setup also includes a linear scale which provides position feedback, both for the current amplifier and for the proposed controller.

The Generalized Maxwell-slip (GMS) friction model[38,39], with the parameters obtained in the work by Villegas et al.[17], has been used for the feedforward compensation of friction in (37). The models used for compensation of cogging force and ripple effect take the following form[17]:

$$\begin{aligned}F_{cg}(x) &= \sum_{k=0}^{3} c_{k,c}x^k + \sum_{k=1}^{3}\left(a_{k,c}\cos\left(\frac{2\pi kx}{37.5}\right)\right. \\ &\qquad \left. + b_{k,c}\sin\left(\frac{2\pi kx}{37.5}\right)\right) \\ K_f(x) &= \sum_{k=0}^{2} c_{k,r}x^k + \sum_{k=1}^{3}\left(a_{k,r}\cos\left(\frac{2\pi kx}{37.5}\right)\right. \\ &\qquad \left. + b_{k,r}\sin\left(\frac{2\pi kx}{37.5}\right)\right)\end{aligned} \tag{41}$$

for x within the physically feasible motion range for this motor. Outside of this range it is considered to be defined in agreement with the assumptions made for both functions in the previous sections.

The contribution of this work comes from the inclusion of the electrical dynamics in the controller design; therefore, the proposed controller is compared with the control law

*Notice that $\hat{\beta}$ is obtained from α through a proper LTI system. Thus, estimation of $\dot{\alpha}$ is not necessary to calculate β.

Table 1. Reference detail.

$n\,[\mathrm{rpm}]$	$v_{max}\left[\frac{\mathrm{mm}}{\mathrm{s}}\right]$	$a_{max}\left[\frac{\mathrm{mm}}{\mathrm{s}^2}\right]$	$j_{max}\left[\frac{\mathrm{mm}}{\mathrm{s}^3}\right]$
5	10.1	29.9	150.4
7.5	15.1	67.2	504.4
10	20.2	119.4	1185
15	30.27	268.6	4077

Table 2. Position error comparison.

Ref.	$e_{max}\,[\mu\mathrm{m}]$		δ_{max}	$e_{rms}\,[\mu\mathrm{m}]$		δ_{rms}
	DRC	DRC+ED	[%]	DRC	DRC+ED	[%]
$5\,\mathrm{rpm}$	20.8	14.2	-31.7	3.5	2.3	-34.3
$7.5\,\mathrm{rpm}$	22.2	20.6	-7.2	4.15	2.9	-30.1
$10\,\mathrm{rpm}$	26.4	23.9	-9.5	5.1	3.8	-25.5
$15\,\mathrm{rpm}$	28.5	20.2	-29.1	6.5	3.95	-39.2

α given at the first stage of the previous section, which constitutes in itself a deterministic robust controller (DRC) in which electrical dynamics has been neglected. This allows to evaluate the improvement on the performance obtained by the inclusion of the electrical dynamics in the control law, and whether such inclusion would justify the increase in complexity for this controller. For the sake of simplicity, the proposed and the reference controllers are referred from here on as DRC+ED and DRC respectively, where ED refers to the inclusion of the electrical dynamics in control design.

In the experiments, the tracking performance for both controllers is tested for a type of reference that includes complex variations in position, velocity, acceleration and jerk. This particular type of reference represents the carriage movement for a cam grinding operation for three turns of the cam. These cam-related references correspond to a process with a grinding wheel diameter of $360\,\mathrm{mm}$, and a particular cam profile of $50\,\mathrm{mm}$ base circle radius and $15\,\mathrm{mm}$ total rise rotating at different rotation speeds. The position trajectory and its derivatives for one of such references, with a rotation speed of $15\,\mathrm{rpm}$, are shown in Fig. 3. Details on the different reference trajectories used in the experiments are shown in table 1 for the different values of the rotation speed n.

The controller parameters in the experiments were tuned to achieve the lowest maximum error in the DRC controller, and the same values were used for the corresponding parameters of the DRC+ED controller.

The tracking errors for the different reference trajectories are shown in table 2, where maximum errors as well as rms errors are given for both controllers. Also, for ease of comparison, the relative error difference between the controllers is shown, given as $\delta = \frac{e_{DRC+ED} - e_{DRC}}{e_{DRC}}$ for both maximum and rms errors. In particular, the tracking error profile for references of $5\,\mathrm{rpm}$ and $15\,\mathrm{rpm}$ can be seen in Figs. 4 and 5 respectively, where the error for both controllers along a period of the reference is shown, as well as the reference itself.

Thus, although the proposed controller gains in complexity, the improvement when considering the electrical dynamics can be important, reducing the maximum error up to $31.7\,\%$ and the rms error up to $39.2\,\%$ in the experiments.

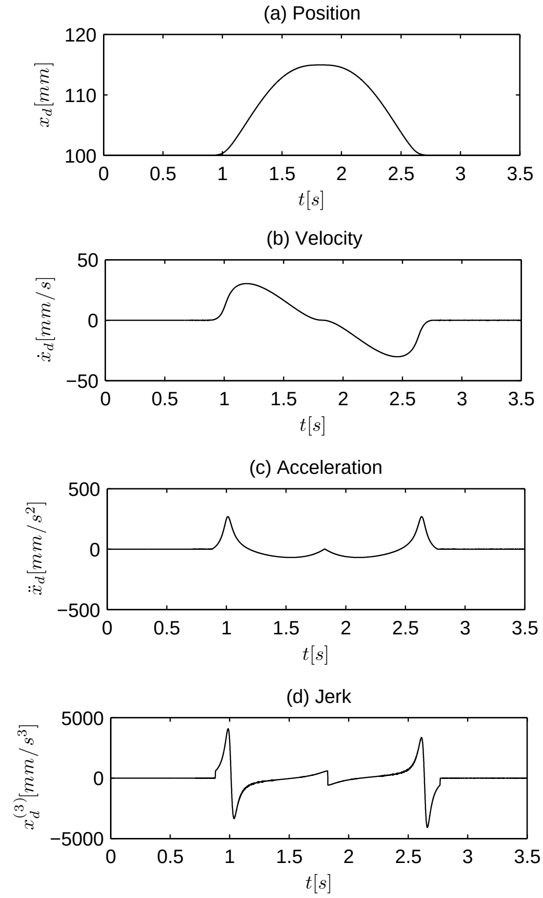

Figure 3. cam-related reference at $15\,\mathrm{rpm}$.

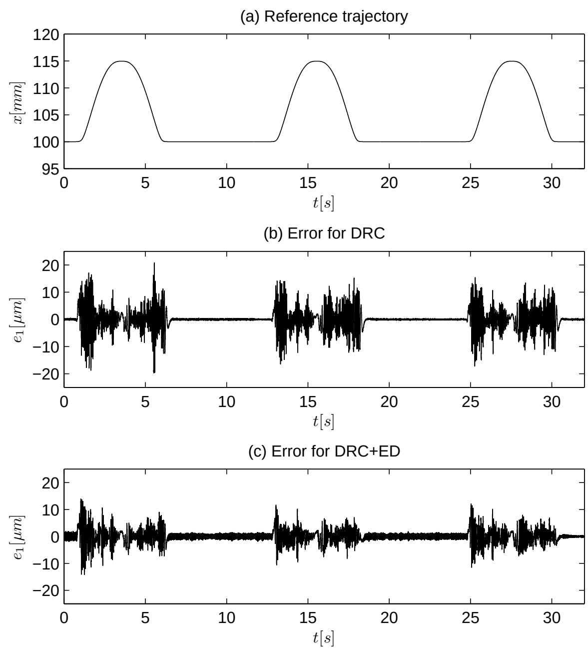

Figure 4. Position error for a cam-related reference at $5\,\mathrm{rpm}$.

Conclusion

The present work has shown the importance of considering the electrical dynamics in a system, even when it has a much

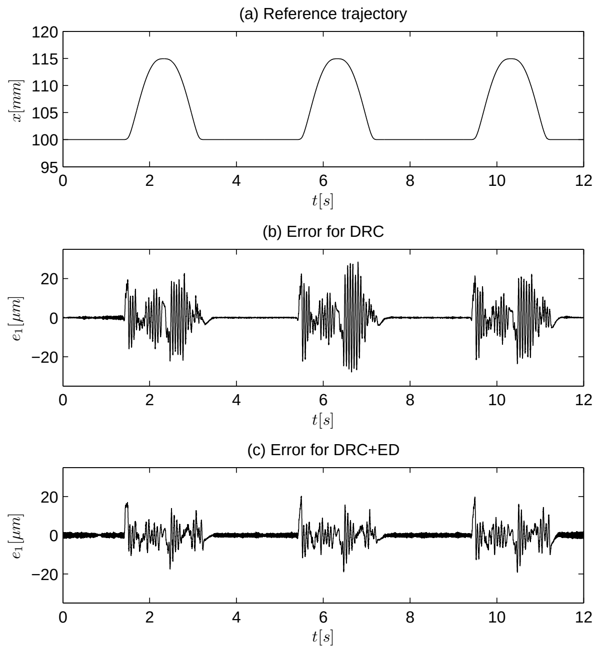

Figure 5. Position error for a cam-related reference at $15\,\mathrm{rpm}$.

faster response than the mechanical dynamics. Starting from a typical deterministic robust controller for the mechanical system, this article has shown a particular way to deal with a second order electrical dynamics, such as the one in the experimental setup.

Stability results for the proposed controller have been presented. Furthermore, a more general result has been shown for the stability analysis of systems in which part of the system can be shown to be bounded or conditionally bounded. Also, this paper has displayed a particular and useful form in which a second order LTI system of relative degree one can be expressed.

Finally, the proposed controller has been tested on the experimental setup and compared with a typical deterministic robust controller for which electrical dynamics is neglected. It has been shown that the proposed controller has achieved a reduction of tracking error up to a $31.7\,\%$ for the maximum values and $39.2\,\%$ for the rms values.

Appendix

Proof of proposition 1

Let $\mathbf{C} = [c_1\ c_2]$ and $\mathbf{B} = [b_1\ b_2]^T$ for the second order SISO LTI system in (2). Let $\mathbf{T}$ be a nonsingular matrix. Then, a state transformation $\mathbf{x} = \mathbf{Tz}$ would take the original system to the form

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{T}^{-1}\mathbf{ATz} + \mathbf{T}^{-1}\mathbf{B}u \\ y &= \mathbf{CTz}. \end{aligned} \tag{42}$$

Let the matrix $\mathbf{T}$ be

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}. \tag{43}$$

Then $\mathbf{T}^{-1}$ is given by

$$\mathbf{T}^{-1} = \frac{1}{\det \mathbf{T}} \begin{bmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{bmatrix} \tag{44}$$

where $\det \mathbf{T} = t_{11}t_{22} - t_{12}t_{21} \neq 0$ as $\mathbf{T}$ is nonsingular.

In order to obtain vectors $\mathbf{CT} = [c_{n,1}\ c_{n,2}]$ and $\mathbf{T}^{-1}\mathbf{B} = [b_{n,1}\ b_{n,2}]^T$, the matrix components t_{ij} should satisfy

$$\begin{bmatrix} c_1 & 0 & c_2 & 0 \\ 0 & c_1 & 0 & c_2 \\ 0 & -b_2 & 0 & b_1 \\ b_2 & 0 & -b_1 & 0 \end{bmatrix} \begin{bmatrix} t_{11} \\ t_{12} \\ t_{21} \\ t_{22} \end{bmatrix} = \begin{bmatrix} c_{n,1} \\ c_{n,2} \\ b_{n,1} \det \mathbf{T} \\ b_{n,2} \det \mathbf{T} \end{bmatrix}. \tag{45}$$

Thus, the proposition will be proved if there is a nonsingular matrix $\mathbf{M}$ with components m_{ij} such that

$$\begin{bmatrix} c_1 & 0 & c_2 & 0 \\ 0 & c_1 & 0 & c_2 \\ 0 & -b_2 & 0 & b_1 \\ b_2 & 0 & -b_1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \tag{46}$$

In such a case, the matrix $\mathbf{T}_d$ of the proposition is given by $\mathbf{T}_d = \mathbf{M}$.

The determinant of the first matrix is

$$\begin{vmatrix} c_1 & 0 & c_2 & 0 \\ 0 & c_1 & 0 & c_2 \\ 0 & -b_2 & 0 & b_1 \\ b_2 & 0 & -b_1 & 0 \end{vmatrix} = (c_1b_1 + c_2b_2)^2 = (\mathbf{CB})^2. \tag{47}$$

Thus, for a system with relative degree one, as $\mathbf{CB} \neq 0$ (see p. 512 in[18]), the determinant of this matrix is nonzero, and the equation in (46) has a unique solution. Thus, having determined $\mathbf{M}$, its nonsingularity has yet to be proved.

Consider the row vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ of matrix $\mathbf{M}$ ($\mathbf{M}^T = \left[\mathbf{M}_1^T\ \mathbf{M}_2^T\right]$). Equation (46) can be expressed with these vectors as

$$\begin{aligned} c_1\mathbf{M}_1 + c_2\mathbf{M}_2 &= \mathbf{C}_d \\ b_2\mathbf{M}_1 - b_1\mathbf{M}_2 &= \mathbf{B}_d^* \end{aligned} \tag{48}$$

where $\mathbf{C}_d = [1\ 0]$ and $\mathbf{B}_d^* = [1\ \ -1]$.

It is clear that if $\mathbf{M}_1$ or $\mathbf{M}_2$ would be the zero vector, then $\mathbf{C}_d$ and $\mathbf{B}_d^*$ would be linearly dependent, which is not the case. Now suppose that $\mathbf{M}_1$ and $\mathbf{M}_2$ were to be nonzero but linearly dependent. Then $\mathbf{M}_1 = \gamma\mathbf{M}_2$ for some scalar γ, and (48) can be put as

$$\begin{aligned} (\gamma c_1 + c_2)\mathbf{M}_2 &= \mathbf{C}_d \\ (\gamma b_2 - b_1)\mathbf{M}_2 &= \mathbf{B}_d^*. \end{aligned} \tag{49}$$

As neither $\mathbf{C}_d$ nor $\mathbf{B}_d^*$ are the zero vector, the scalar $\gamma c_1 + c_2 \neq 0$ and also $\gamma b_2 - b_1 \neq 0$. Then, $\mathbf{M}_1$ and $\mathbf{M}_2$ linearly dependent would imply $\mathbf{C}_d = \frac{\gamma c_1 + c_2}{\gamma b_2 - b_1}\mathbf{B}_d^*$, that is it would imply that $\mathbf{C}_d$ and $\mathbf{B}_d^*$ are also linearly dependent. As $\mathbf{C}_d$ and $\mathbf{B}_d^*$ are not linearly dependent, neither can be $\mathbf{M}_1$ and $\mathbf{M}_2$.

Thus, the matrix $\mathbf{M} = \left[\mathbf{M}_1^T\ \mathbf{M}_2^T\right]^T$ is not singular, the nonsingular transformation matrix $\mathbf{T}_d$ exists and is given by $\mathbf{T}_d = \mathbf{M}$, and the system in (2) can be transformed to the form in (3) through the state transformation $\mathbf{x} = \mathbf{T}_d\mathbf{z}$.

Proof of theorem 1

This theorem is easily proven following a procedure similar to those used for other Lyapunov-like stability results in chapter 4 of the book by Khalil[18]. Such a proof is given next.

Considering the case $B > 0$, let $\eta = \alpha_2(B)$. Then, the time-dependent set $\Omega_{t,B} = \{\mathbf{x}_a \in B_r : V(t, \mathbf{x}_a) \leq \eta\}$ is such that $\{\mathbf{x}_a \in B_r : \|\mathbf{x}_a\| \leq B\} \subseteq \Omega_{t,B}$. On the other hand, $\Omega_{t,B} \subseteq \{\mathbf{x}_a \in B_r : \alpha_1(\|\mathbf{x}_a\|) \leq \eta\}$, being both compact sets, as $V(t, \mathbf{x}_a)$ is radially unbounded and α_1 a class $\mathcal{K}_\infty$ function. Finally, by condition 3, this last set is a subset of $\{\mathbf{x}_a \in \mathbb{R}^n : \alpha_1(\|\mathbf{x}_a\|) \leq \alpha_1(r)\} = B_r \subseteq D_a$.

If $\mathbf{x}_a(t_0) \in \Omega_{t,B}$ (and $\mathbf{x}_b(t_0) \in D_b$), as $\dot{V}(t, \mathbf{x}_a)$ is negative on its boundary, $\mathbf{x}_a(t)$ cannot leave this compact set, and by condition 1 $\mathbf{x}_b(t)$ stays in D_b.

In case $\mathbf{x}_a(t_0) \notin \Omega_{t,B}$ but $\mathbf{x}_a(t_0) \in \Omega_r = \{\mathbf{x}_a \in B_r : \alpha_2(\|\mathbf{x}_a\|) \leq \alpha_1(r)\} \subseteq \Omega_{t,r} = \{\mathbf{x}_a \in B_r : V(t, \mathbf{x}_a) \leq \alpha_1(r)\}$ (and $\mathbf{x}_b(t_0) \in D_b$), $\dot{V}$ is negative until $\mathbf{x}_a(t)$ enters $\Omega_{t,B}$, and therefore $\mathbf{x}_a(t)$ stays in $\Omega_{t,r}$. Furthermore, it stays in the set $\Omega^0_{t,\mathbf{x}_a} = \{\mathbf{x}_a \in B_r : V(t, \mathbf{x}_a) \leq V(t_0, \mathbf{x}_a(t_0))\}$, which is a subset of $\{\mathbf{x}_a \in \mathbb{R}^n : \alpha_1(\|\mathbf{x}_a\|) \leq \alpha_2(\|\mathbf{x}_a(t_0)\|)\} \subseteq B_r$. Considering the properties stated on the definition of $V(t, \mathbf{x}_a)$, these are compact sets and

$$\begin{aligned} \Omega^0_{t,\mathbf{x}_a} &\subseteq \{\mathbf{x}_a \in B_r : V(t, \mathbf{x}_a) \leq \alpha_2(\|\mathbf{x}_a(t_0)\|)\} \\ &\subseteq \{\mathbf{x}_a \in B_r : V(t, \mathbf{x}_a) \leq \alpha_1(r)\} = \Omega_{t,r} \\ &\subseteq \{\mathbf{x}_a \in \mathbb{R}^n : \alpha_1(\|\mathbf{x}_a\|) \leq \alpha_1(r)\} = B_r \subseteq D_a. \end{aligned} \tag{50}$$

Then, $\mathbf{x}_b(t) \in D_b \subseteq \bar{D}_b$; therefore $\mathbf{x}(t) = (\mathbf{x}_a(t), \mathbf{x}_b(t))$ stays in a compact set. Thus, according to Theorem 3.3 in the book by Khalil[18] there is a unique solution $\mathbf{x}(t)$ for this dynamics for $t \geq t_0$.

In this manner, if the initial condition is such that $\mathbf{x}(t_0) = (\mathbf{x}_a(t_0), \mathbf{x}_b(t_0)) \in \Omega_r \times D_b$ the solution will stay in the compact set

$$\Lambda = \Omega_a \times \bar{D}_b \tag{51}$$

where

$$\Omega_a = \{\mathbf{x}_a \in B_r : \alpha_1(\|\mathbf{x}_a\|) \leq \max(\eta, \alpha_2(\|\mathbf{x}_a(t_0)\|))\}. \tag{52}$$

Hence, the solution $\mathbf{x}(t)$ starting in $\Omega_r \times D_b$ is bounded.

It has been previously shown that a trajectory $\mathbf{x}_a(t)$ starting in $\Omega_{t,B}$ will stay in that set $\forall t \geq t_0$. As for the trajectories starting in $\{\Omega_r - \Omega_{t,B}\}$ let $k = \min_{B \leq \|\mathbf{x}_a\| \leq r} W(\mathbf{x}_a)$. As the set $\{\mathbf{x}_a \in \mathbb{R}^n : B \leq \|\mathbf{x}_a\| \leq r\}$ contains $\{\Omega_r - \Omega_{t,B}\}$, in this last set $\dot{V}(t, \mathbf{x}_a(t)) \leq -k < 0$. Thus, as long as such a trajectory stays in this set, by comparison lemma[18],

$$\begin{aligned} V(t, \mathbf{x}_a(t)) &\leq V(t_0, \mathbf{x}_a(t_0)) - k(t - t_0) \\ &\leq \alpha_2(\|\mathbf{x}_a(t_0)\|) - k(t - t_0). \end{aligned} \tag{53}$$

In this manner $V(t, \mathbf{x}_a(t))$ reduces to η within the time interval $[t_0, t_0 + (\alpha_2(\|\mathbf{x}_a(t_0)\|) - \eta)/k]$, with $\mathbf{x}_a(t)$ entering the set $\Omega_{t,B}$ in finite time. Let $t_0 + T$ be the particular instant in which $\mathbf{x}_a(t)$ enters this set (if $\mathbf{x}_a(t_0) \in \Omega_{t,B}$, $T = 0$). Then, as $\Omega_{t,B} \subseteq \{\mathbf{x}_a \in D_a : \alpha_1(\|\mathbf{x}_a\|) \leq \eta = \alpha_2(B)\}$, it results that $\forall t \geq t_0 + T$, $\|\mathbf{x}_a(t)\| \leq \alpha_1^{-1}(\alpha_2(B)) = \alpha_B(B)$, where according to Lemma 4.2 in the book by Khalil[18], $\alpha_B = \alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}_\infty$ function.

Now the transient behavior will be considered. Being $W(\mathbf{x}_a)$ positive definite, there exists a class $\mathcal{K}$ function α_w defined on $[0, \infty)$ such that $W(\mathbf{x}_a) \geq \alpha_w(\|\mathbf{x}_a\|)$[18]. Thus, for all $t \in [t_0, t_0 + T]$

$$\begin{aligned} \dot{V} &\leq -W(\mathbf{x}_a) \leq -\alpha_w(\|\mathbf{x}_a\|) \\ &\leq -\alpha_w\left(\alpha_2^{-1}(V)\right) = -\alpha(V) \end{aligned} \tag{54}$$

where $\alpha = \alpha_w \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function which can be assumed to be Lipschitz (see p. 153 in[18]). From comparison lemma, and Lemma 4.4 in the book by Khalil[18], there exists a class $\mathcal{KL}$ function $\sigma(r, s)$ such that

$$V(t, \mathbf{x}_a(t)) \leq \sigma(V(t_0, \mathbf{x}_a(t_0)), t - t_0) \tag{55}$$

for all $t \in [t_0, t_0 + T]$. Then, considering (32) leads to

$$\begin{aligned} \|\mathbf{x}_a(t)\| &\leq \alpha_1^{-1}(\sigma(V(t_0, \mathbf{x}_a(t_0)), t - t_0)) \\ &\leq \alpha_1^{-1}(\sigma(\alpha_2(\|\mathbf{x}_a(t_0)\|), t - t_0)) \\ &= \beta(\|\mathbf{x}_a(t_0)\|, t - t_0), \qquad \forall t \in [t_0, t_0 + T] \end{aligned} \tag{56}$$

where according to Lemma 4.2 in the book by Khalil[18], $\beta(r, s) = \alpha_1^{-1}(\sigma(\alpha_2(r), s))$ is a $\mathcal{KL}$ function.

In the particular case that $B = 0$, the set $\Omega_{t,B}$ is no longer considered, and the same argument presented for the set $\Omega^0_{t,\mathbf{x}_a}$ is followed to conclude that $\mathbf{x}(t)$ exists and is bounded. In this case the same definition of Ω_a can be used, as $\alpha_2(B) = 0$. On the other hand, the procedure used to show the entrance on $\Omega_{t,B}$ in finite time cannot be used, as the minimum of $W(\mathbf{x}_a)$ will be zero. However, the same argument followed to analyze the transient behavior can be applied, leading to

$$\|\mathbf{x}_a(t)\| \leq \beta(\|\mathbf{x}_a(t_0)\|, t - t_0), \qquad \forall t \in [t_0, \infty) \tag{57}$$

Thus, being $\beta(r, s)$ a class $\mathcal{KL}$ function, $\mathbf{x}_a(t) \to 0$ as $t \to \infty$

Proof of Corollary 1

Consider the vectors $\mathbf{x}_a = [s\ z_3\ z_4]^T$ and $\mathbf{x}_b = [e_0\ e_1\ \beta\ \hat{\beta}]^T$, and express the system equations as

$$\begin{aligned} \dot{\mathbf{x}}_a &= \mathbf{f}_a(t, \mathbf{x}_a, \mathbf{x}_b) \\ \dot{\mathbf{x}}_b &= \mathbf{f}_b(t, \mathbf{x}_a, \mathbf{x}_b). \end{aligned} \tag{58}$$

Given the assumptions on x_d and its derivatives, K_f, F_{cg} and its estimates, $\hat{F}_f$ and f_{dis}, the functions $\mathbf{f}_a$ and $\mathbf{f}_b$ are piecewise continuous in t and locally Lipschitz in $(\mathbf{x}_a, \mathbf{x}_b)$ on $[0, \infty) \times \mathbb{R}^7$. That is, functions $\mathbf{f}_a$ and $\mathbf{f}_b$ behave as functions $\mathbf{f}_a$ and $\mathbf{f}_b$ of Thm. 1.

Variables e_0 and e_1 can be obtained from s through a strictly proper and stable LTI system. Thus, for a bounded s these variables are also bounded. Under these conditions and the above assumptions on the functions involved, α is bounded. Therefore, β and $\hat{\beta}$, which are obtained from α through proper and stable LTI systems, are also bounded. Hence, a bounded $\mathbf{x}_a$ results in a bounded $\mathbf{x}_b$. Thus, for a given bounded domain D_a (which will be chosen to ensure the assumptions on initial conditions in Thm. 1) and a properly chosen D_b, condition 1 of Thm. 1 is verified.

Consider the auxiliary function $V(\mathbf{x}_a) = \frac{1}{2}w_1 s^2 + w_2 \mathbf{z}^T \mathbf{P} \mathbf{z}$. This is a continously differentiable function, which is also positive definite, radially unbounded, and decrescent. Therefore, there are class $\mathcal{K}_\infty$ functions α_1 and α_2 such that $\alpha_1(\|\mathbf{x}_a\|) \le V(\mathbf{x}_a) \le \alpha_2(\|\mathbf{x}_a\|)$.

Considering (23), (11) and (28), the derivative of V satisfies

$$\begin{aligned}\dot{V} \le -\mathbf{x}_a^T &\begin{bmatrix} w_1\frac{K_f(x_1)}{K_{fmin}}k_s & -w_1\frac{K_f(x_1)}{2} & 0 \\ -w_1\frac{K_f(x_1)}{2} & w_2 & 0 \\ 0 & 0 & w_2 \end{bmatrix} \mathbf{x}_a \\ &+ w_1\epsilon + w_2\frac{\delta_e^2}{\eta} \\ &\le -\lambda_{min}\|\mathbf{x}_a\|^2 + w_1\epsilon + w_2\frac{\delta_e^2}{\eta}\end{aligned} \tag{59}$$

where λ_{min} is the smallest eigenvalue of the matrix in the first term. Due to the assumption on k_s in (38), this matrix is positive definite, and thus λ_{min} is positive.

Since δ_e is continuous, δ_e^2 will reach a maximum $\delta_{e,max}^2$ in the compact set $\bar{D}$, the closure of the set $D = D_a \times D_b$. Thus, considering a value $\theta \in (0, 1)$ and a value r such that $B_r = \{\mathbf{x}_a \in \mathbb{R}^3 : \|\mathbf{x}_a\| \le r\} \subseteq D_a$, there are values of η and ϵ such that

$$B_d = \sqrt{\frac{w_1\epsilon + w_2\frac{\delta_{e,max}^2}{\eta}}{\lambda_{min}\theta}} < \alpha_2^{-1}(\alpha_1(r)). \tag{60}$$

Then, as

$$\begin{aligned}\dot{V}(\mathbf{x}_a) \le -\lambda_{min}(1-\theta)\|\mathbf{x}_a\|^2 - \lambda_{min}\theta\|\mathbf{x}_a\|^2 \\ + w_1\epsilon + w_2\frac{\delta_e^2}{\eta}\end{aligned} \tag{61}$$

considering (60) leads to

$$\dot{V}(\mathbf{x}_a) \le -W(\mathbf{x}_a), \qquad \forall\|\mathbf{x}_a\| \ge B_d \tag{62}$$

for all $(\mathbf{x}_a, \mathbf{x}_b) \in \bar{D}$, where $W(\mathbf{x}_a) = \lambda_{min}(1-\theta)\|\mathbf{x}_a\|^2$ and $B_d < \alpha_2^{-1}(\alpha_1(r))$, verifying conditions 2 and 3 of Thm. 1.

Then, from Thm. 1, for every initial state $\mathbf{x}(t_0) = (\mathbf{x}_a(t_0), \mathbf{x}_b(t_0))$ such that $\|\mathbf{x}_a(t_0)\| \le \alpha_2^{-1}(\alpha_1(r))$ and $\mathbf{x}_b(t_0) \in D_b$, there is a solution $\mathbf{x}(t)$ defined for $t \ge t_0 \ge 0$ which is bounded. Furthermore, there is a class $\mathcal{K}_\infty$ function α_B, and a constant $T > 0$ dependent on $\mathbf{x}_a(t_0)$ and B_d, such that $\|\mathbf{x}_a(t)\| \le \alpha_B(B_d)$ for all $t \in [t_0 + T, \infty)$.

Variables e_0, e_1 and e_2 can be obtained from s through strictly stable LTI systems. As shown in Sec. 4.9 of the book by Khalil[18], the state for such a kind of system with state equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ is bounded as

$$\|\mathbf{x}(t)\| \le ke^{-\lambda(t-t^*)}\|\mathbf{x}(t^*)\| + \frac{k\|\mathbf{B}\|}{\lambda}\sup_{t^* \le \tau \le t}\|u(\tau)\| \tag{63}$$

for positive constants k and λ, from which a similar relation can be shown for the output. Therefore, considering $t^* \ge t_0 + T$, and that in such conditions

$$\sup_{t^* \le \tau \le t}\|s(\tau)\| \le \sup_{t^* \le \tau \le t}\|\mathbf{x}_a(\tau)\| \le \alpha_B(B_d) \tag{64}$$

these variables will be ultimately bounded by a class $\mathcal{K}_\infty$ function of B_d.

In the absence of model uncertainties and unmodeled disturbances, that is $d_m = d_e = 0$, $\hat{\mathbf{K}}_e = \mathbf{K}_e$ and $\hat{\mathbf{z}} = \mathbf{z}$, from (23) results an expression similar to (59), except for the absence of the last two terms. Then, from Thm. 1 results $\mathbf{x}_a(t) \to 0$ and therefore $s(t) \to 0$ for $t \to \infty$. Following the same reasoning for the ultimate bound on e_0, e_1 and e_2 it can be shown that this errors also decay to zero.

Declaration of conflicting interests

The Authors declare that there is no conflict of interest.

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